MAT251 Differential Geometry

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Theoretical summary of the course. Instructor: Jaime Cuadros Valle.

Contents

1 Curves in the plane and in space

1.1 What is a curve?

Definition 1. A parametrized curve in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

Definition 2. If γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is called the tangent vector of γ at the point $\gamma(t)$.

Proposition 1. *If $\dot{\gamma}(t) = a$ for all t, where a is a constant vector, we have, integrating componentwise,*

$$\gamma(t) = \int \frac{d\gamma}{dt} dt = \int a\, dt = ta + b,$$

where b is another constant vector. If $a \neq 0$, this is the parametric equation of the straight line parallel to a and passing through the point b. If $a = 0$, the image of γ is a single point (namely, b).

1.2 Arc-length

We recall that, if $v = (v_1, \ldots, v_n)$ is a vector in $\mathbb{R}^n$, its length is

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Definition 3. The arc-length of a curve γ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du.$$

Definition 4. If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, its speed at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be unit-speed curve if $\dot{\gamma}(t)$ us unit vector for all $t \in (\alpha, \beta)$.

We recall that the dot product of vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ is

$$a \cdot b = \sum_{i=1}^{n} a_i b_i.$$

Proposition 2. *Let $n(t)$ be a unit vector that is a smooth function of a parameter t. Then, the dot product*

$$\dot{n} \cdot n(t) = 0$$

for all t. In particular, if γ is a unit-speed curve, then $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.

1.3 Reparametrization

Definition 5. A parametrized curve $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a reparametrization of a parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ (the reparametrization map) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \text{ for all } \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}).$$

Note. Since ϕ has a smooth inverse, γ is a reparametrization of $\tilde{\gamma}$:

$$\tilde{\gamma}(\phi^{-1}(t)) = \gamma(\phi(\phi^{-1}(t))) = \gamma(t) \text{ for all } t \in (\alpha, \beta).$$

Two curves that are reparametrizations of each other have the same image, so they should have the same geometric properties

Definition 6. A point $\gamma(t)$ of a parametrized curve γ is called a regular point if $\dot{(\gamma)}(t) \neq 0$; otherwise $\gamma(t)$ is a singular point of γ. A curve is regular if all of its points are regular.

Proposition 3. *Any reparametrization of a regular curve is regular.*

Proposition 4. *If $\gamma(t)$ is a regular curve, its arc-length s, starting at any point of γ, is a smooth function of t.*

Proposition 5. *A parametrized curve has a unit-speed reparametrization if and only if it is regular.*

Corollary 7. *Let γ be a regular curve and let $\tilde{\gamma}$ be a unit-speed reparametrization of γ:*

$$\tilde{\gamma}(u(t)) = \gamma(t) \textit{ for all } t,$$

where u is a smooth function of t. Then, if s is the arc-length of γ (starting at any point), we have

$$u = \pm s + c,$$

where c is a constant. Conversely, if u is given by the last equation for some value of c and with either sign, then $\tilde{\gamma}$ is a unit-speed reparametrization of γ.

2 How much does a curve curve?

2.1 Curvature

Definition 8. If γ is a unit-speed curve with parameter t, its curvature $\kappa(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$.

Proposition 6. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then its curvature is*

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3},$$

where the $\times$ indicates the cross product.

2.2 Plane curves

Suppose that $\gamma(t)$ is a unit-speed vector curve in $\mathbb{R}^2$. Let

$$t = \dot{\gamma}$$

be the tangent vector of γ, note that t is a unit vector. There are two unit vectors perpendicular to t; we make a choice by defining n_s, the signed unit normal of γ, to be the unit vector obtained by rotating t anticlockwise by $\pi/2$. $\dot{t} = \ddot{\gamma}$ is perperndicualr to t, and hence parallel to n_s. Thus, there is a scalar κ_s such that

$$\ddot{\gamma} = \kappa_s n_s.$$

κ_s is called the signed curvature of γ. Since $\|n_s\| = 1$, we have

$$\kappa = \|\ddot{\gamma}\| = \|\kappa_s n_s\| = |\kappa_s|,$$

so the curvature of γ is the absolute value of its signed curvature.

Corollary 9. *The total signed curvature of a closed plane curve is an integer multiple of 2π.*

Theorem 10. *Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then, there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k. Further, if $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a direct isometry M of $\mathbb{R}^2$ such that*

$$\tilde{\gamma}(s) = M(\gamma(s)) \text{ for all } s \in (\alpha, \beta).$$

2.3 Space curves

Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^3$, and let $t = \dot{\gamma}$ be its unit tangent vector. If the curvature $\kappa(s)$ is non-zero, we define the principal normal of γ at the point $\gamma(s)$ to be the vector

$$n(s) = \frac{1}{\kappa(s)} \dot{t}(s).$$

Since $\|\dot{t}\| = \kappa$, n is a unit vector. Further, $t \cdot \dot{t} = 0$ so t and n are actually perpendicular unit vectors. It follows that

$$b = t \times n$$

is a unit vector perpendicular to both t and n. The vector $b(s)$ is called the binormal vector of γ at the point $\gamma(s)$. This, $\{t, n, b\}$ is an orthonormal basis of $\mathbb{R}^3$ and is right-handed

$$b = t \times n, n = b \times t, t = n \times b.$$

Being perpendicular to both t and b, $\dot{b}$ must be parallel to n, so

$$\dot{b} = -\tau n,$$

for some scalar τ, which is called the torsion of γ.

Proposition 7. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Then, its torsion is given by*

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}.$$

Proposition 8. *Let γ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Then, the imagen of γ is contained in a plane if and only if τ is zero at every point of the curve.*

Theorem 11. *Let γ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then*

$$\begin{aligned}
\dot{t} &= \kappa n \\
\dot{n} &= -\kappa t + \tau b \\
\dot{b} &= -\tau n.
\end{aligned}$$

These equations are called the Frenet-Serret equations.

Note. The matrix

$$\begin{pmatrix} \dot{t} \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

which expresses $\dot{\tau}, \dot{n}, \dot{b}$ in terms of t, n and b is skew-symmetric.

Proposition 9. *Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then, γ is a parametrization of (part of) a circle.*

Theorem 12. *Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$ for all s. Then, there is a direct isometry M of $\mathbb{R}^3$ such that*

$$\tilde{\tau}(s) = M(\gamma(s)) \textit{ for all } s.$$

Further, if k and t are smooth functions with $k > 0$ everywhere, there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t.

3 Global properties of curves

3.1 Simple closed curves

Definition 13. A simple closed curve in $\mathbb{R}^2$ is a closed curve in $\mathbb{R}^2$ that has no self-intersections.

Theorem 14 (Hopf's Umlaufsatz)**.** *The total signed curvature of a simple closed curve in $\mathbb{R}^2$ is $\pm 2\pi$.*

3.2 The isoperimetric inequality

The area contained by a simple closed curve γ is

$$A(\gamma) = \int_{\mathrm{int}(\gamma)} dxdy.$$

Theorem 15 (Green). *Let $f(x,y)$ and $g(x,y)$ be smooth functions and let γ be a positively oriented simple closed curve. Then,*

$$\int_{int(\gamma)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_{\gamma} f(x,y)\, dx + g(x,y)\, dy.$$

Proposition 10. *If $\gamma(t) = (x(t), y(t))$ is a positively-oriented simple closed curve in $\mathbb{R}^2$ with period T, then*

$$A(\gamma) = \frac{1}{2} \int_0^T (x\dot{y} - y\dot{x})\, dt.$$

Theorem 16 (Isoperimetric Inequality). *Let γ be a simple closed curve, le t $l(\gamma)$ be its length and let $A(\gamma)$ be the area contained by it. Then,*

$$A(\gamma) \leq \frac{1}{4\pi} l^2(\gamma),$$

and equality holds if and only if γ is a circle.

Proposition 11 (Wirtinger's Inequality). *Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function such that $F(0) = F(\pi) = 0$. Then,*

$$\int_0^{\pi} \left(\frac{dF}{dt} \right)^2 dt \geq \int_0^{\pi} F(t)^2\, dt,$$

and equality holds if and only if $F(t) = D \sin t$ for all $t \in [0, \pi]$, where D is a constant.

3.3 The four vertex theorem

Definition 17. A vertex of a curve $\gamma(t)$ in $\mathbb{R}^2$ is a point where its signed curvature κ_s has a stationary point (where $d\kappa_s/dt = 0$).

Theorem 18. *Every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.*

4 Surfaces

Definition 19. A subset S of $\mathbb{R}^3$ is a surface if, for every point $p \in S$ there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing p such that $S \cap W$ is homeomorphic to U. A subset of a surface S of the form $S \cap W$, where W is an open subset of $\mathbb{R}^3$, is called an open subset of S. A homeomorphism $\sigma : U \to S \cap W$ as in this definition is called a surface patch or parametrization of the open subset $S \cap W$ of S. A collection of such surface patches whose images cover the whole of S is called an atlas of S.

Example 20. Every plane in $\mathbb{R}^3$ is a surface with an atlas consisting of a single surface patch.

Example 21. A circular cylinder is the set of points of $\mathbb{R}^3$ that are at a fixed distance (the radius of the cylinder) from a fixed straight line (its axis). For instance, the circular cylinder of radius 1 and axis the z-axis is

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}/$$

The simplest parametrization of S is

$$\sigma(u, v) = (\cos u, \sin u, v).$$

4.1 Smooth surfaces

Definition 22. A surface patch $\gamma : U \to \mathbb{R}^3$ is called regular if it is smooth and the vectors γ_u and γ_v are linearly independent at all points $(u, v) in U$. Equivalently, γ should be smooth and the vector product $\gamma_u \times \gamma_v$ should be non-zero at every point of U.

Definition 23. If S is a surface, and allowable surface patch for S is a regular surface patch $\gamma : U \to \mathbb{R}^3$ such that γ is a homeomorphism from U to an open subset of S. A smooth surface is a surface S such that, for any point $p \in S$, there is an allowable surface patch γ as above such that $p \in \gamma(U)$. A collection A of allowable surface patches for a surface S such that every point of S is in the image of at least one patch in A is called an atlas for the smooth surface S.

Proposition 12. *The transition maps of a smooth surface are smooth.*

4.2 Tangents and derivatives

Definition 24. A tangent vector to a surface S at a point $p \in S$ is the tangent vector at p of a curve in S passing through p. The tangent space T_pS of S at p is the set of all tangent vectors to S at p.

Proposition 13. *Let $\gamma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. The tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors γ_u and γ_v.*

Definition 25. With the above notation, the derivative D_pf of f at the point $p \in S$ is the map $D_pf : T_pS \to T_{f(p)}\tilde{S}$ such that $D_pf(w) = \tilde{w}$ for any tangent vector $w \in T_pS$.

Proposition 14. *If $f : S \to \tilde{S}$ is a smooth map between surfaces and $p \in S$, the derivative $D_pf : T_ps \to T_{f(p)}\tilde{S}$ is a linear map.*

Let $\gamma : U \to \mathbb{R}^3$ be a surface patch of S containing p, say $p = \gamma(u_0, v_0)$, and let α, β be the smooth functions on U such that

$$f(\gamma(u, v)) = \tilde{\gamma}(\alpha(u, v), \beta(u, v)).$$

Then,
$$D_p f(w) = (\lambda \alpha_u + \mu \beta_v)\tilde{\sigma}_{\tilde{u}} + (\lambda \beta_u + \mu \beta_v)\tilde{\sigma}_{\tilde{v}}.$$

The matrix of the linear map $D_p f$ with respect to the basis $\{\sigma_u, \sigma_v\}$ of $T_p f$ and the basis $\{\tilde{\sigma}_{\tilde{u}}, \tilde{\sigma}_{\tilde{v}}\}$ of $T_{f(p)}\tilde{S}$ is the Jacobian matrix

$$\begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$$

of the smooth map $(u, v) \mapsto (\alpha(u, v), \beta(u, v))$.

Proposition 15.

- *If S is a surface and $p \in S$, the derivative at p of the identity map $S \to S$ is the identity map $T_pS \to T_pS$.*
- *If S_1, S_2 and S_3 are surfaces and $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are smooth maps, then for all $p \in S_1$,*
$$D_p(f_2 \circ f_1) = D_{f_1(p)} f_2 \circ D_p f_1.$$
- *If $f : S_1 \to S_2$ is a diffeomorphism, then for all $p \in S_1$ the linear map $D_p f : T_pS_1 \to T_{f(p)}S_2$ is invertible.*

Proposition 16. *Let S and $\tilde{S}$ be surfaces and let $f : S \to \tilde{S}$ be a smooth map. Then, f is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_p f : T_pS \to T_{f(p)}\tilde{S}$ is invertible.*

4.3 Normals and orientability

Since the tangent plane T_pS of a surface S at a point $p \in S$ passes through the origin of $\mathbb{R}^3$, it is completely determined by giving a unit vector perpendicular to it, called a unit normal to S at p. We choose one of two such vectors

$$N_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

called the standard unit normal of the surface patch σ at p.

Definition 26. A surface S is orientable if there exists an atlas A for S with the property that, if Φ is the transition map between any two surface patches in A, then $\det(J(\Phi)) > 0$ where Φ is defined.

Proposition 17. *Let S be an orientable surface equipped with an atlas A. Then, there is a smooth choice of unit normal at any point of S: take the standard unit normal of any surface patch in A.*

An oriented surface is a surface S together with a smooth choice of unit normal N at each point, i.e., a smooth map $N : S \to \mathbb{R}^3$ such that, for all $p \in S, N(p)$ is a unit vector perpendicular to T_pS.

5 The first fundamental form

5.1 Lengths of curves on surfaces

Definition 27. Let p be a point of a surface S. The first fundamental form of S at p associates to tangent vectors $v, w \in T_pS$ the scalar

$$\langle v, w\rangle_{p,S} = v \cdot w.$$

Remark. The first fundamental form is an example of inner product on $\mathbb{R}^3$.

Suppose that $sigma(u, v)$ is a surface patch of S. Then, any tangent vector to S at a point p in the image of σ can be expressed uniquely as a linear combination of σ_u and σ_v. Define maps $du : T_pS \to \mathbb{R}$ and $dv : T_pS \to \mathbb{R}$ by

$$du(v) = \lambda,\ du(v) = \mu,\ v = \lambda\sigma_u + \mu\sigma_v.$$

We have

$$\langle v, v\rangle = \lambda^2\langle \sigma_u, \sigma_v\rangle + 2\lambda\mu\langle \sigma_u, \sigma_v\rangle + \mu^2\langle \sigma_v, \sigma_v\rangle.$$

Writing

$$E = \|\sigma_u\|^2, F = \sigma_u \cdot \sigma_v, G = \|\sigma_v\|^2,$$

this becomes

$$\langle v, v\rangle = Edu^2 + 2Fdu\,dv + Gdv^2$$

which is called the first fundamental form of the surface patch $\sigma(u, v)$.

If γ is a curve lying in the image of a surface patch σ, we have $\gamma(t) = \sigma(u(t), v(t))$ and $\dot\gamma = \dot u\sigma_u + \dot v\sigma_v$, so

$$\langle \dot\gamma, \dot\gamma\rangle = E\dot u^2 + 2F\dot u\dot v + G\dot v^2,$$

and the length of γ is given by

$$\int (E\dot u^2 + 2F\dot u\dot v + G\dot v^2)^{1/2}dt.$$

Example 28. For the plane

$$\sigma(u, v) = a + up + vq$$

with p and q being perpendicular unit vectors, we have $\sigma_u = p, \sigma_v = q$, so $E = \|p\|^2 = 1, F = p \cdot q = 0, G = \|q\|^2 = 1$, and the first fundamental form is simply $du^2 + dv^2$.

5.2 Isometries of surfaces

Definition 29. If S_1 and S_2 are surfaces, a smooth map $f : S_1 \to S_2$ is called a local isometry if it takes any curve in S_1 to a curve of the same length in S_2. If a local isometry $f : S_1 \to S_2$ exists, we say that S_1 and S_2 are locally isometric.

Remark. Every local isometry is a local diffeomorphism. Any composite of local isometries is a local isometry, and the inverse of any isometry is an isometry.

Theorem 30. *A smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle\, , \rangle_p$ and $f^*\langle\, , \rangle_p$ on T_pS_1 are equal for all $p \in S_1$.*

Thus, f is a local isometry if and only if

$$\langle D_pf(v), D_pf(w)\rangle_{f(p)} = \langle v, w\rangle_p$$

for all $p \in S_1$ and all $v, w \in T_pS_1$.

Corollary 31. *A local diffeomorphism $f : S_1 \to S_2$ is a local isometry if and only if, for any surface patch σ_1 and $f \circ \sigma_1$ of S_1 and S_2, respectively, have the same first fundamental form.*

Definition 32. A tangent developable is the union of the tangent lines to a curve in $\mathbb{R}^3$. The tangent line to a curve γ at a point $\gamma(u)$ is the straight line passing through $\gamma(u)$ and parallel to the tangent vector $\dot\gamma(u)$.

Proposition 18. *Any tangent developable is locally isometric to a plane.*

5.3 Conformal mappings of surfaces

Suppose that two curves γ and $\tilde\gamma$ on a surface S intersect at a point p. The angle θ of intersection of γ and $\tilde\gamma$ at p is defined to be the angle between the tangent vectors $\dot\gamma$ and $\dot{\tilde\gamma}$. Using the dot product formula for the angle between vectors, we see that θ is given by

$$\begin{aligned}\cos\theta &= \frac{\dot\gamma \cdot \dot{\tilde\gamma}}{\|\dot\gamma\|\|\dot{\tilde\gamma}\|} \\ &= \frac{E\dot u\dot{\tilde u} + F(\dot u\dot{\tilde v} + \dot{\tilde v}\dot v) + G\dot v\dot{\tilde v}}{(E\dot u^2 + 2F\dot u\dot v + G\dot v^2)^{1/2}(E\dot{\tilde u}^2 + 2F\dot{\tilde u}\dot{\tilde v} + G\dot{\tilde v}^2)^{1/2}}.\end{aligned}$$

Definition 33. If S_1 and S_2 are surfaces, a conformal map $f : S_1 \to S_2$ is a local diffeomorphism such that, if γ_1 and $\tilde\gamma_1$ are any two curves on S_1 that intersect, say at a point $p \in S_1$, and if γ_2 and $\tilde\gamma_2$ are their images under f, the angle of intersection of γ_1 and $\tilde\gamma_1$ at p is equal to the angle of intersection of γ_2 and $\tilde\gamma_2$ at $f(p)$.

In short, f is conformal if and only if it preserves angles.

Theorem 34. *A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if there is a function $\lambda : S_1 \to \mathbb{R}$ such that*

$$f^*\langle v, w\rangle_p = \lambda(p)\langle v, w\rangle_p \text{ for all } p \in S_1 \text{ and } v, w \in T_pS_1.$$

Corollary 35. *A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if, for any surface patch σ of S_1, the first fundamental forms of the patches σ of S_1 and $f \circ \sigma$ of S_2 are proportional.*

Theorem 36. *Every surface has an atlas consisting of conformal surface patches.*

5.4 Equiareal maps and a theorem of Archimedes

Definition 37. The area $A_\sigma(R)$ of the part $\sigma(R)$ of a suface patch $\sigma : U \to \mathbb{R}^3$ corresponding to a region $R \subset U$ is

$$A_\sigma(R) = \int_R \|\sigma_u \times \sigma_v\| du\, dv.$$

Proposition 19. $\|\sigma_u \times \sigma_v\| = (EG - F^2)^{1/2}$.

Proposition 20. *The area of a surface patch is unchanged by reparametrization.*

Definition 38. Let S_1 and S_2 be two surfaces. A local diffeomorphism $f : S_1 \to S_2$ is said to be equiareal if it takes any region in S_1 to a region of the same area in S_2.

Theorem 39. *A local diffeomorphism $f : S_1 \to S_2$ is equiareal if and only if, for any surface patch $\sigma(u, v)$ on S_1, the first fundamental forms*

$$E_1du^2 + 2F_1dudv + G_1dv^2 \text{ and } E_2du^2 + 2F_2dudv + G_2dv^2$$

of the patches σ on S_1 and $f \circ \sigma$ on S_2 satisfy

$$E_1G_2 - F_1^2 = E_2G_2 - F_2^2.$$

Theorem 40 (Archimedes' Theorem)**.** *The map f is an equiareal diffeomorphism.*

A spherical triangle is a triangle on a sphere whose sides are arcs of great circles.

Theorem 41. *The area of spherical triangle of the unit sphere S^2 with internal angles α, β and γ is*

$$\alpha + \beta + \gamma - \pi.$$

6 Curvature of surfaces

6.1 The second fundamental form

Suppose that σ is a surface patch in $\mathbb{R}^3$ with standard unit normal $\boldsymbol{N}$. Writing

$$L = \sigma_{uu} \cdot \boldsymbol{N}, M = \sigma_{uv} \cdot \boldsymbol{N}, N = \sigma_{vv} \cdot \boldsymbol{N}.$$

One calls the expression

$$Ldu^2 + 2Mdudv + Ndv^2$$

the second fundamental form of the surface of patch σ.

Example 42. Consider the plane

$$\sigma(u, v) = a + up + vq.$$

Since $\sigma_u = p$ and $\sigma_v = q$, we have $\sigma_{uu} = \sigma_{uv} = \sigma_{uv} = 0$. Hence, the second fundamental form of a plane is zero.

6.2 The Gauss and Weingarten maps

Consider $\boldsymbol{N}$ the unit normal of an oriented surface S. The values of $\boldsymbol{N}$ at the points of S are recorded by its Gauss map $\mathcal{G}_S$. This is the map from S to the unit sphere S^2 that assigns to any point $p \in S$ the point $\boldsymbol{N}_p \in S^2$, where $\boldsymbol{N}_p$ is the unit normal of S at p.

Definition 43. Let p be a point of a surface S. The Weingarten map $\mathcal{W}_{p,S}$ of S at p is defined by

$$\mathcal{W}_{p,S} = -D_p\mathcal{G}.$$

The second fundamental form of S at $p \in S$ is the bilinear form on T_pS given by

$$\langle v, w\rangle_{p,S} = \langle \mathcal{W}_{p,S}(v), w\rangle_{p,s}, v, w \in T_pS.$$

Proposition 21. *Let p be a point of a surface S, let $\sigma(u,v)$ be a surface patch of S with p in its image, and let $Ldu^2+2Mdudv+Ndv^2$ be the second fundamental form of σ. Then, for any $\boldsymbol{v}, \boldsymbol{w} \in T_pS$,*

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = Ldu(\boldsymbol{v})du(\boldsymbol{w}) + M(du(\boldsymbol{v})dv(\boldsymbol{w}) + du(\boldsymbol{w})dv(\boldsymbol{v})) + Ndv(\boldsymbol{v})dv(\boldsymbol{w}).$$

Lemma 44. *Let $\sigma(u,v)$ be a surface patch with standard unit normal $\boldsymbol{N}(u,v)$. Then,*

$$\boldsymbol{N}_u \cdot \sigma_u = -L, \; \boldsymbol{N}_u \cdot \sigma_v = \boldsymbol{N}_v \cdot \sigma_u = -M, \; \boldsymbol{N}_v \cdot \sigma_v = -N.$$

Corollary 45. *The second fundamental form is a symmetric bilinear form. Equivalently, the Weingarten map is self-adjoint.*

6.3 Normal and geodesic curvatures

If γ is a unit-speed curve on an oriented surface S, then

$$\ddot{\gamma} = \kappa_n \boldsymbol{N} + \kappa_g \boldsymbol{N} \times \dot{\gamma}.$$

Definition 46. The scalars κ_n and κ_g are called the normal curvature and the geodesic curvature of γ, respectively.

Proposition 22. *With the above notation, we have*

$$\kappa_n = \ddot{\gamma} \cdot \boldsymbol{N},$$
$$\kappa_g = \ddot{\gamma} \cdot (\boldsymbol{N} \times \dot{\gamma}),$$
$$\kappa^2 = \kappa_n^2 + \kappa_g^2, \kappa_n = \kappa \cos\psi, \kappa_g = \pm\kappa \sin\psi,$$

where κ is the curvature of γ and ψ is the angle between $\boldsymbol{N}$ and the principal normal $\boldsymbol{n}$ of γ.

Proposition 23. *If γ is a unit-speed curve on an oriented surface S, its normal curvature is given by*

$$\kappa_n = \langle \dot{\gamma}, \dot{\gamma}\rangle.$$

If σ is a surface patch of S and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ, then

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2.$$

Proposition 24 (Meusnier's Theorem)**.** *Let p be a point of a surface S and let v be a unit tangent vector to S at p. Let Π_θ be the plane containing the line through p parallel to v and making an angle θ with the tangent plane T_pS, and assume that Π_θ is not parallel to T_pS. Suppose that Π_θ intersects S in a cruve with curvature κ_θ. Then, $\kappa_\theta \sin\theta$ is independent of θ.*

Corollary 47. *The curvature κ, normal curvature κ_n and geodesic curvature κ_g of a normal section of a surface are related by*

$$\kappa_n = \pm\kappa, \ \kappa_g = 0.$$

6.4 Parallel transport and convariant derivative

Definition 48. Let γ be a curve on a surface S and let v be a tangent vector field along γ. The covariant derivative of v along γ is the orthogonal projection $\nabla_\gamma v$ of dv/dt onto the thangent plane $T_{\gamma(t)}S$ at a point $\gamma(t)$.

Definition 49. v is said to be parallel along γ if $\nabla_\gamma v = 0$ at every point of γ.

Proposition 25. *A tangent vector field v is parallel along a curve γ on a surface S if and only if $\dot{v}$ is perpendicular to the tangent plane of S at all points of γ.*

Proposition 26 (Gauss Equations)**.** *Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$. Then*

$$\begin{aligned}
\sigma_u u &= \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + L\boldsymbol{N},\\
\sigma_u v &= \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + M\boldsymbol{N},\\
\sigma_v v &= \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + N\boldsymbol{N},
\end{aligned}$$

where

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)},\ \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)},$$
$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)},\ \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)},$$
$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)},\ \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}.$$

The six Γ coefficients in these formulas are called Christoffel symbols.

Note. Christoffel symbols depend only on the first fundamental form of σ.

Proposition 27. *Let $\gamma(t) = \sigma(u(t), v(t))$ be a curve on a surface patch σ, and let $\boldsymbol{v}(t) = \alpha(t)\sigma_u + \beta(t)\sigma_v$ be a tangent vector field along γ, where α and β are smooth functions of t. Then $\boldsymbol{v}$ is parallel along γ if and only if the following equations are satisfied*

$$\begin{aligned}
\dot{\alpha} + (\Gamma^1_{11}\dot{u} + \Gamma^1_{12})\alpha + (\Gamma^1_{12}\dot{u} + \Gamma^1_{22}\dot{v})\beta &= 0\\
\dot{\beta} + (\Gamma^2_{11}\dot{u} + \Gamma^2_{12})\alpha + (\Gamma^2_{12}\dot{u} + \Gamma^2_{22}\dot{v})\beta &= 0.
\end{aligned}$$

Corollary 50. *Let γ be a curve on a surface S and let v_0 be a tangent vector of S at the point $\gamma(t_0)$. Then, there is exactly one tangent vector field v that is parallel along γ and is such that $v(t_0) = v_0$.*

If p and q are two points on a curve γ on a surface S, the covariant derivative enables us to associate to any vector in the tangent plane T_pS a vector in the tangent plane T_qS.

Definition 51. With the above notation, the map $\Pi_\gamma^{pq} : T_pS \to T_qS$ that takes $v_0 \in T_pS$ to $v_1 \in T_qS$ is called parallel transport from p to q along γ.

Proposition 28. *With the above notation,*

- $\Pi_\gamma^{pq} : T_pS \to T_qS$ *is a linear map.*
- Π_γ^{pq} *is an isometry.*

7 Gaussian, mean and principal curvatures

7.1 Gaussian and mean curvatures

Definition 52. Let $\mathcal{W}$ be the Weingarten map of an oriented surface S at a point $p \in S$. The Gaussian curvature K and mean curvature H of S at p are defined by

$$K = \det(\mathcal{W}),\, H = \frac{1}{2}\mathrm{trace}(\mathcal{W}).$$

Define symmetric 2×2 matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ by

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix},\, \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Proposition 29. *Let σ be a surface patch of an oriented surface S. Then, with the above notation, the matrix of $\mathcal{W}_{p,S}$ with respect to the basis $\{\sigma_u, \sigma_v\}$ of T_pS is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$.*

Corollary 53. *We have*

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)},\, K = \frac{LN - M^2}{EG - F^2}.$$

7.2 Principal curvatures of a surface

Proposition 30. *Let p be a point of a surface S. There are scalars κ_1, κ_2 and a basis $\{t_1, t_2\}$ of the tangent plane T_pS such that*

$$\mathcal{W}(t_1) = \kappa_1 t_1,\, \mathcal{W}(t_2) = \kappa_2 t_2.$$

Moreover, if $\kappa_1 \neq \kappa_2$, then $\langle t_1, t_2 \rangle = 0$.

Corollary 54. *If p is a point of a surface S, there is an orthonormal basis of the tangent plane T_pS consisting of principal vectors.*

Proposition 31. *If κ_1 and κ_2 are the principal curvatures of a surface, the mean and Gaussian curvatures are given by*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2),\ K = \kappa_1\kappa_2.$$

Theorem 55 (Euler's Theorem)**.** *Let γ be a curve on an oriented surface S, and let κ_1 and κ_2 be the principal curvatures of σ, with non-zero principal vectors t_1 and t_2. Then, the normal curvature of γ is*

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where θ is the oriented angle $\widehat{t_1\dot{\gamma}}$.

Corollary 56. *The principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectores are the tangent vectors of the curves giving these maximum and minimum values.*

Proposition 32. *The principal curvatures are the roots of the equation*

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0,$$

and the principal vectors corresponding to the principal curvature κ are the tangent vectors $t = \xi\sigma_u + \eta\sigma_v$, such that

$$\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proposition 33. *Let S be a (connected) surface of which every point is umbilic. Then, S is an open subset of a plane or a sphere.*

7.3 Gaussian curvature of compact surfaces

The relative signs of the principal curvatures at a point p of a surface S determine the shape of S near p. In fact, since the Gaussian curvature K of S is the product of its principal curvatures, then

- If $K > 0$ at p, then p is an elliptic point.
- If $K < 0$ at p, then p is a hyperbolic point.
- If $K = 0$ at p, then p is either a parabolic point or a planar point.

Proposition 34. *If S is a compact surface, there is a point of S at which its Gaussian curvature K is greater than 0.*

8 Geodesics

8.1 Definition and basic properties

Definition 57. A curve γ on a surface S is called a geodesic if $\ddot{\gamma}(t)$ is zero or perpendicular to the tangent plane of the surface at the point $\gamma(t)$, i.e., parallel to its unit noirmal, for all values of the parameter t.

Remark. Equivalently, γ is geodesic if and only if its tangent vector $\dot{\gamma}$ is parallel along γ.

Proposition 35. *Any geodesic has constant speed.*

Proposition 36. *A unit-speed curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.*

Proposition 37. *Any (part of a) straight line on a surface is a geodesic.*

Proposition 38. *Any normal section of a surface is a geodesic.*

8.2 Geodesic equations

Theorem 58. *A curve γ on a surface S is a geodesic if and only if, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ of S, the following two equations are satisfied:*

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),$$
$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2).$$

These differential equations are called the geodesic equations.

Proposition 39. *A curve γ of a surface S is a geodesic if and only if, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch γ of S, the following two equations are satisfied:*

$$\ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 = 0$$
$$\ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 = 0$$

Proposition 40. *Let p be a point of a surface S, and let t be a unit tangent vector to S at p. Then, there exist a unique unit-speed geodesic γ on S which passes through p and has tangent vector t there.*

In short, there is a unique geodesic through any given point of a surface in any given tangent direction.

Corollary 59. *Any local isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.*

8.3 Geodesics on a surface of revolution

We parametrize the surface of revolution in the usual way

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

We assume that $f > 0$ and unit speed geodesics, so that

$$\dot{u}^2 + f^2(u)\dot{v}^2 = 1.$$

Proposition 41. *On the surface of revolution*

- *Every meridian is a geodesic.*
- *A parallel $u = u_0$ is a geodesic if and only if $df/du = 0$ when $u = u_0$.*

Theorem 60 (Clairaut's Theorem)**.** *Let γ be a unit-speed curve on a surface of revolution S, let $\rho : S \to \mathbb{R}$ be the distance of a point of S from the axis rotation, and let ψ be the angle between $\dot{\gamma}$ and the meridians of S. If γ is a geodesic, then $\rho \sin\psi$ is constant along γ. Conversely, if $\rho\sin\psi$ is constant along γ, and if no part of γ is part of some parallel of S, then γ is a geodesic.*

9 Gauss' Theorema Egregium

9.1 The Gauss and Codazzi-Mainardi equations

Proposition 42 (Codazzi-Mainardi Equations)**.** *Given the first and second fundamental forms of a surface patch $\sigma(u, v)$, then*

$$\begin{aligned} L_v - M_u &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}, \\ M_v - N_u &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}. \end{aligned}$$

Proposition 43 (Gauss Equations)**.** *If K is the Gaussian curvature of the surface patch $\gamma(u, v)$ in the preceding proposition, then*

$$\begin{aligned} EK &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2, \\ FK &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}, \\ FK &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11}, \\ GK &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}. \end{aligned}$$

Theorem 61. *Let $\sigma : U \to \mathbb{R}^3$ and $\tilde{\sigma} : U \to \mathbb{R}^3$ be surface patches with the same first and second fundamental form. Then, there is a direct isometry M of $\mathbb{R}^3$ such that $\tilde{\sigma} = M(\sigma)$.*

9.2 Gauss' remarkable theorem

Theorem 62 (Gauss' Theorema Egregium)**.** *The Gaussian curvature of a surface is preserved by local isometries.*

Corollary 63. *The Gaussian curvature is given by*

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$

Corollary 64. *With the above notation*

- *If $F = 0$, we have*

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right).$$

- *If $E = 1$ and $F = 0$, we have*

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

Proposition 44. *Any map of any region of the earth's surface must distort distances.*

9.3 Surfaces of constant Gaussian curvature

Theorem 65. *Any point of a surface of constant Gaussian curvature is contained in a patch that is isometric to an open subset of a plane, a sphere or a pseudosphere.*

Proposition 45. *Let S be a surface of constant Gaussian curvature -1. If $p \in S$, there is a surface patch of S containing p whose first and second fundamental forms are*

$$du^2 + 2\cos\theta du dv + dv^2 \text{ and } 2\sin\theta du dv,$$

respectively, where $\theta(u, v)$ is a smooth function such that $0 < \theta < \pi$ for all u, v.

Lemma 66. *Let $\sigma : U \to \mathbb{R}^3$ be a surface patch containing a point p that is not an umbilic. Let $\kappa_1 \geq \kappa_2$ be the principal curvatures of σ and suppose that κ_1 has a local maximum at p and κ_2 has a local minimum there. Then, the Gaussian curvature of σ at p is not positive.*

Theorem 67. *Every connected compact surface whose Gaussian curvature is constant is a sphere.*

9.4 Geodesic mappings

Definition 68. If S and $\tilde{S}$ are surfaces, a local diffeomorphism $F : S \to \tilde{S}$ is said to be geodesic if f takes every pre-geodesic on S to a pre-geodesic on $\tilde{S}$.

Proposition 46. *The following are geodesic local diffeomorphisms:*

- *Every local isometry.*
- *Every dilation of* $\mathbb{R}^3$.
- *Every composite of geodesic local diffeomorphisms.*

Theorem 69. *Let S be a connected surface. If there is a geodesic local diffeomorphism from S to a plane, then S has constant Gaussian curvature. Conversely, if S has constant Gaussian curvature, then for any point $p \in S$ there is a surface patch $\sigma : U \to S$ of S such that $p \in \sigma(U)$ and $\sigma^{-1} : \sigma(U) \to U$ is a geodesic diffeomorphism.*

10 The Gauss-Bonnet theorem

10.1 Gauss-Bonnet for simple closed curves

Definition 70. A curve $\gamma(t) = \sigma(u(t), v(t))$ on a surface patch $\sigma : U \to \mathbb{R}^3$ is called a simple closed curve with period T if $\pi(t) = (u(t), v(t))$ is a simple closed curve in $\mathbb{R}^2$ with period T such that region $\mathrm{int}(\pi)$ of $\mathbb{R}^2$ enclosed by π is entirely contained in U. The curve γ is said to be positively-oriented if π is positively oriented. Finally, the image of $\mathrm{int}(\pi)$ under the map σ is defined to be the interior of γ.

Theorem 71. *Let $\gamma(s)$ be a unit-speed simple closed curve on a surface patch σ of length $l(\gamma)$, and assume that γ is positively-oriented. Then,*

$$\int_0^{l(\gamma)} \kappa_g \, ds = 2\pi - \int_{int(\gamma)} K dA_\sigma,$$

where κ_g is the geodesic curvature of γ, K is the Gaussian curvature of σ and dA_σ is the area element of σ.

We shall make use of a smooth orthonormal basis $\{e', e''\}$ of the tangent plane at each point of the surface patch.

Lemma 72. *With the above notation, we have*

$$e'_u \cdot e''_v - e''_u \cdot e'_v = \frac{LN - M^2}{(EG - F^2)^{1/2}}.$$

and

$$\int_0^{l(\gamma)} \dot{\theta} ds = 2\pi.$$

10.2 Gauss-Bonnet for curvilinear polygons

Definition 73. A curvilinear polygon in $\mathbb{R}^2$ is a continuous map $\pi : \mathbb{R} \to \mathbb{R}^2$ such that, for some real number T and some points $0 = t_0 < t_1 < \cdots < t_n = T$:

- $\pi(t) = \pi(t')$ if and only if $t' - t$ is an integer multiple of T.
- π is smooth on each of the open intervals $(t_0, t_1), \ldots, (t_{n-1}, t_n)$.
- The one-sided derivative exist for $i = 1, \ldots, n$ and are non-zero and not parallel.

The points if $\gamma(t_i)$ for $i = 1, \ldots, n$ are called the vertices of the curvilinear polygon π, and the segments of it corresopnding to the open intervals (t_{i-1}, t_i) are called its edges.

Theorem 74. *Let γ be a positively-oriented unit-speed curvilinear polygon with n edges on a surface σ, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the interior angles at its vertices. Then,*

$$\int_0^{l(\gamma)} \kappa_g \, ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \int_{int(\gamma)} K \, dA_\sigma.$$

Corollary 75. *If γ is a curvilinear polygon with n edges each of which is an arc of a geodesic, then the internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the polygon satisfy the equation*

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \int_{int(\gamma)} K \, dA_\sigma.$$

10.3 Gauss-Bonnet for compact surfaces

Definition 76. Let S be a surface, with atlas consisting of the patches $\sigma_i : U_i \to \mathbb{R}^3$. A triangulation of S is a collection of curvilinear polygons, each of which is contained, together with its interior, in one of the $\sigma_i(U_i)$, such that:

- Every point of S is in at least one of the curvilinear polygons.
- Two curvilinear polygons are either disjoint, or their intersection is a common edge or a common vertex.
- Each edge is an edge of exactly two polygons.

Theorem 77. *Every compact surface has a triangulation with finitely many polygons.*

Definition 78. The Euler Number χ of a triangulation of a compact surface S with finitely many polygons is

$$\chi = V - E + F,$$

where

$$\begin{aligned} V &= \text{the total number of vertices of the triangulation,} \\ E &= \text{the total number of edges of the triangulation,} \\ F &= \text{the total number of polygons of the triangulation.} \end{aligned}$$

Theorem 79. *Let S be a compact surface. Then, for any triangulation of S,*

$$\int_S K dA = 2\pi\chi,$$

where χ is the Euler number of the triangulation.

Corollary 80. *The Euler number χ of a triangulation of a compact surface S depends only on S and not on the choice of triangulation.*

Theorem 81. *The Euler number of the compact surface T_g of genus g is $2-2g$.*

Corollary 82. *We have*

$$\int_{T_g} K dA = 4\pi(1-g).$$

10.4 Map colouring

Map Colouring Problem For a given compact surface S, what is the smallest positive integer n such that every map on S can be n-coloured?

This smallest integer n is called the chromatic number of S.

The Four Colour Conjecture The chromatic number of a sphere is 4.

Let $h(\chi)$ be the largest integet $\leq N(\chi)$.

Heawood's Conjecture The chromatic number of a compact surface of Euler number $\chi \leq 0$ is $h(\chi)$.

Theorem 83. *Any compact surface of Euler number $\chi \leq 0$ can be $h(\chi)$-coloured.*

Theorem 84. *The chromatic number of a torus is 7.*

Proposition 47 (Five Neighbours Theorem)**.** *Ever map on a sphere has at least one country with five or fewer neighbours.*

Corollary 85 (Six Colour Theorem)**.** *Every map on a sphere can be six-coloured.*

10.5 Holonomy and Gaussian curvature

Proposition 48. *Let γ be a unit-speed curve on a surface patch σ and let $\boldsymbol{v}$ be a non-zero parallel vector field along γ. Let φ be the oriented angle $\widehat{\dot{\gamma}\boldsymbol{v}}$ from $\dot{\gamma}$ to $\boldsymbol{v}$. Then, the geodesic curvature of γ is*

$$\kappa_g = -\frac{d\varphi}{ds}.$$

Proposition 49. *Let γ be a positively-oriented unit-speed simple closed curve on a surface σ, let κ_g be the geodesic curvature of γ, and let v be a non-zero parallel vector field along γ. Then, on going once around γ, v rotates through an angle*

$$2\pi - \int_0^{l(\gamma)} \kappa_g ds.$$

Definition 86. If γ is a unit-speed closed curve on a surface S, the angle of the previous proposition is called the holonomy around γ, and is denoted by h_γ.

Theorem 87. *Let γ be positively-oriented simple closed curve on a surface patch σ, let h_γ be the holonomy around γ, and let K be the Gaussian curvature of σ. Then,*

$$h_\gamma = \int_{int(\gamma)} K dA_\gamma.$$

Proposition 50. *Suppose that a surface S has the property that, for any two points $p, q \in S$, the parallel transport Π_γ^{pq} is independent of the curve γ joining p and q. Then S is flat.*

11 Hyperbolic geometry

11.1 Upper half-plane model

The pseudosphere is parametrized as

$$\tilde{\sigma}(v, w) = \left(\frac{1}{w} \cos v, \frac{1}{2} \sin v, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1} w \right).$$

with first fundamental form

$$\frac{dv^2 + dw^2}{w^2}.$$

Proposition 51. *Hyperbolic angles in $\mathcal{H}$ are the same as Euclidean angles.*

Proposition 52. *The geodesics in $\mathcal{H}$ are the half-lines parallel to the imaginary axis and the semicircles with centres on the real axis.*

Proposition 53.

- *There is a unique hyperbolic line passing through any tow distinct points of $\mathcal{H}$.*
- *The parallel axiom does not hold in $\mathcal{H}$.*

Proposition 54. *THe hyperbolic distance between two points $a, b \in \mathcal{H}$ is*

$$d_{\mathcal{H}}(a, b) = 2 \tanh^{-1} \frac{|b - a|}{|b - \bar{a}|}.$$

Theorem 88. *Let $\mathcal{P}$ be a n-sided hyperbolic polygon in $\mathcal{H}$ with internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, the hyperbolic area of the polygon is given by*

$$A(\mathcal{P}) = (n-2)\pi - \alpha_1 - \ldots \alpha_n.$$

Lemma 89. *Let a and b be the endpoints of a segment l of a hyperbolic line in $\mathcal{H}$ that forms part of a semicircle with centre p on the real axis, and suppose that the radius vectors joining p to a and p to b make angles φ and ψ, respectively, with the positive real axis. Then,*

$$\int_l \frac{dv}{w} = \varphi - \psi.$$

11.2 Isometries of $\mathcal{H}$

It is easy to define some isometries of $\mathcal{H}$:

- Translations parallel to the real axis, given by

$$T_a(z) = z + a, a \in \mathbb{R}.$$

- Reflections in lines parallel to the imaginary axis, given by

$$R_a(z) = 2a - \overline{z}, a \in \mathbb{R}.$$

- Dilations by a factor $a > 0$, given by

$$D_a(z) = az.$$

- Inversions in circles with centres on th e real axis. The inversion in the circle with centre $a \in \mathbb{R}$ and radius $r > 0$ is

$$\mathcal{I}_{a,r}(z) = a + \frac{r^2}{\overline{z} - a}.$$

Proposition 55. *Any composite of a finite number of maps of the types defined above is an isometry of $\mathcal{H}$.*

Proposition 56. *The inversion $\mathcal{I}_{a,r}$ in the circle with centre $a \in \mathbb{R}$ and radius $r > 0$ takes hyperbolic lines that intersect the real axis perpendicularly at a to half-lines, and all other hyperbolic lines to semicircles.*

Proposition 57. *Let l_1 and l_2 be hyperbolic lines in $\mathcal{H}$, and let z_1 and z_2 be points on l_1 and l_2, respectively. Then, there is an isometry of $\mathcal{H}$ that takes l_1 to l_2 and z_1 to z_2.*

Theorem 90. *In hyperbolic geometry, similar triangles are congruent.*

11.3 Poincaré disc model

We consider the transformation

$$\mathcal{P}(z) = \frac{z-i}{z+i}.$$

It defines a bijection between the complex plane with the point $-i$ removed and the complex plane with the point 1 removed, its inverse being

$$\mathcal{P}^{-1}(z) = \frac{z+1}{i(z-1)}.$$

Let the unit disc be $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 91. The Poincaré disc model $\mathcal{D}_P$ of hyperbolic geometry is the disc $\mathcal{D}$ equipped with the first fundamental form for which $\mathcal{P} : \mathcal{H} \to \mathcal{D}_P$ is an isometry.

Proposition 58. *The first fundamental form of $\mathcal{D}_P$ is*

$$\frac{4(dv^2 + dw^2)}{(1 - v^2 - w^2)^2}.$$

In particular, $\mathcal{D}_P$ is a conformal model of hyperbolic geometry.

Proposition 59.

- *Let Γ be a circle that intersects $\mathcal{C}$ perpendicularly. Then, inversion in Γ is an isometry of $\mathcal{D}_P$.*
- *Let l be a line passing through the origin (and so perpendicular to $\mathcal{C}$). Then, (Euclidean) reflection in l is an isometry of $\mathcal{D}_P$.*

Proposition 60. *For $a, b \in \mathcal{D}_P$, we have*

$$d_{\mathcal{D}_P}(a,b) = 2\tanh^{-1}\frac{|b-a|}{|1-\overline{a}b|}.$$

Proposition 61. *The hyperbolic lines in $\mathcal{D}_P$ are the lines and circles that intersect $\mathcal{C}$ perpendicularly.*

Theorem 92. *Consider a hyperbolic triangle with angles α, β, γ and sides of length A, B, C. Then,*

$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma.$$

This formula is called the hyperbolic cosine rule.

Corollary 93. *Suppose that a hyperbolic triangle has sides of lengths A, B and C and that the angle opposite the side of length C is a right angle. Then,*

$$\cosh C = \cosh A \cosh B.$$