A Fast Iterative Solution of the Time-harmonic Elastic Wave Equation with MSSS-preconditioned IDR(s)

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Collaborators

This is joint work with:

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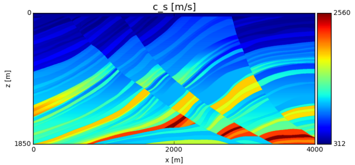
- Y. Qiu, Max Plack Institute Magdeburg,
- E. Ang, Nanyang Technological University,
- M.B. Van Gijzen, Delft Univ. of Technology,
- R.-E. Plessix, Shell Global Solutions International B.V.

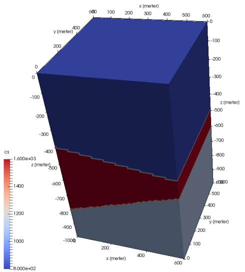
Motivation

The time-harmonic elastic wave equation



We want to solve:

- elastic wave equation,
- modeling in frequency-domain,
- preconditioner for 2D and 3D problems,
- multiple frequencies and multiple RHSs via matrix equation formulation.



Outline

1. FEM discretization
2. MSSS preconditioner for 2D problems
3. SSOR-MSSS preconditioner for 3D problems
4. Numerical results

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The time-harmonic elastic wave equation

Continuous setting

Elastic wave equation

$$-\omega_k^2 \rho(\mathbf{x})\mathbf{u}_k - \nabla \cdot \sigma(\mathbf{u}_k) = \mathbf{s}$$

with boundary conditions

$$\begin{aligned} i\omega_k \rho(\mathbf{x}) B \mathbf{u}_k + \sigma(\mathbf{u}_k)\hat{\mathbf{n}} &= \mathbf{0}, \\ \sigma(\mathbf{u}_k)\hat{\mathbf{n}} &= \mathbf{0}, \end{aligned}$$

on $\partial\Omega_a \cup \partial\Omega_r$.

Discrete setting

Solve

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b}$$

with FEM matrices

$$\begin{aligned} K_{ij} &= \int_\Omega \sigma(\varphi_i) : \nabla \varphi_j \, d\Omega, \\ M_{ij} &= \int_\Omega \rho(\mathbf{x}) \varphi_i \cdot \varphi_j \, d\Omega, \\ C_{ij} &= \int_{\partial\Omega_a} \rho(\mathbf{x}) B \varphi_i \cdot \varphi_j \, d\Gamma. \end{aligned}$$

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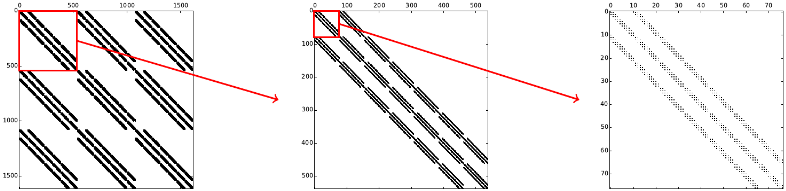
Preconditioned Krylov methods

$$\mathcal{P}(\tau) \equiv K + i\tau C - \tau^2 M$$

"Standard Krylov methods require preconditioning."



- incomplete block LU factorization (SSOR)
- regular grid $\rightarrow$ structured matrices
- new: use MSSS matrix computations

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The time-harmonic elastic wave equation

A multiple-frequency approach

We suggest to re-write the discrete problem

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b}, \quad k = 1, ..., N_\omega$$

as a matrix equation

$$\mathcal{A}(\mathbf{X}) \equiv K\mathbf{X} + iC\mathbf{X}\Sigma - M\mathbf{X}\Sigma^2 = B,$$

where

- $\mathbf{X} := [\mathbf{x}_1, ..., \mathbf{x}_{N_\omega}]$
- $\Sigma := \mathrm{diag}(\omega_1, ..., \omega_{N_\omega})$
- $B := [\mathbf{b}, ..., \mathbf{b}]$, or $B = [\mathbf{b}_1, ..., \mathbf{b}_{N_\omega}]$

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The time-harmonic elastic wave equation

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where

- $\mathbf{X} := [\mathbf{x}_1, ..., \mathbf{x}_{N_\omega}]$
- $\Sigma := \text{diag}(\omega_1, ..., \omega_{N_\omega})$ $\leftarrow$ multiple frequencies
- $B := [\mathbf{b}, ..., \mathbf{b}]$, or $B = [\mathbf{b}_1, ..., \mathbf{b}_{N_\omega}]$ $\leftarrow$ multiple RHSs

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MSSS preconditioner for 2D problems (1/2)

Definition: SSS matrix [Chandrasekaran et al. 2005]

Let A be an $n \times n$ block matrix with **s**equentially **s**emi-**s**eperable structure. Then A can be written in the following block partitioned form

$$A_{i,j} = \begin{cases} U_i W_{i+1} \cdots W_{j-1} V_j^T, & \text{if } i < j; \\ D_i, & \text{if } i = j; \\ P_i R_{i-1} \cdots R_{j+1} Q_j^T, & \text{if } i > j. \end{cases}$$

- linear computational complexity for $(\cdot)^{-1}$
- limit off-diagonal rank with MOR
- MSSS: constructors are SSS matrices

$$\begin{array}{|c|c|c|c|} \hline D_1 & U_1 V_2^T & \cdots & \\ \hline P_2 Q_1^T & D_2 & \ddots & \\ \hline \vdots & \ddots & \ddots & \\ \hline & & & D_n \\ \hline \end{array}$$

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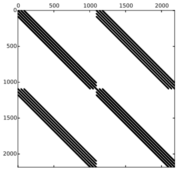
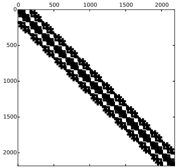
MSSS preconditioner for 2D problems (2/2)

MSSS permutation:

$$\Psi^T \mathcal{P} \Psi = \begin{bmatrix} P_{1,1} & P_{1,2} & & \\ P_{2,1} & P_{2,2} & P_{2,3} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & P_{n_x,n_x} \end{bmatrix}$$





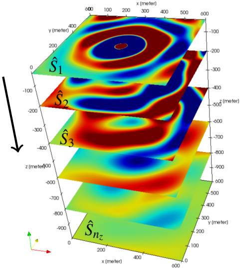
LSU factorization with Schur complements,

$$S_i = \begin{cases} P_{i,i} & \text{if } i = 1 \\ P_{i,i} - P_{i,i-1} S_{i-1}^{-1} P_{i-1,i} & \text{if } 2 \leq i \leq n_x. \end{cases}$$

SSOR-MSSS preconditioner for 3D problems

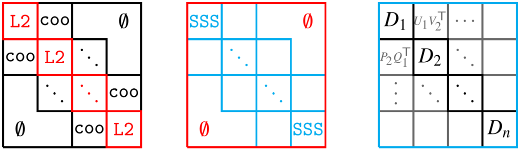
Consider the splitting,

$$\mathcal{P}(\tau) = \underline{L} + \hat{S} + \overline{U}.$$



An SSOR preconditioner is given by,

$$\mathcal{P}(\tau) = (\underline{L}\hat{S}^{-1} + I)\hat{S}(\hat{S}^{-1}\overline{U} + I).$$



(a) 3D level (b) 2D level (c) 1D level

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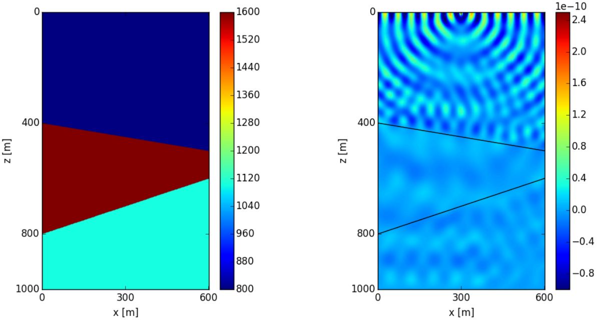
Numerical results

1. **Parameter studies**
2. MSSS preconditioner for 2D problems
3. SSOR-MSSS preconditioner for 3D problems



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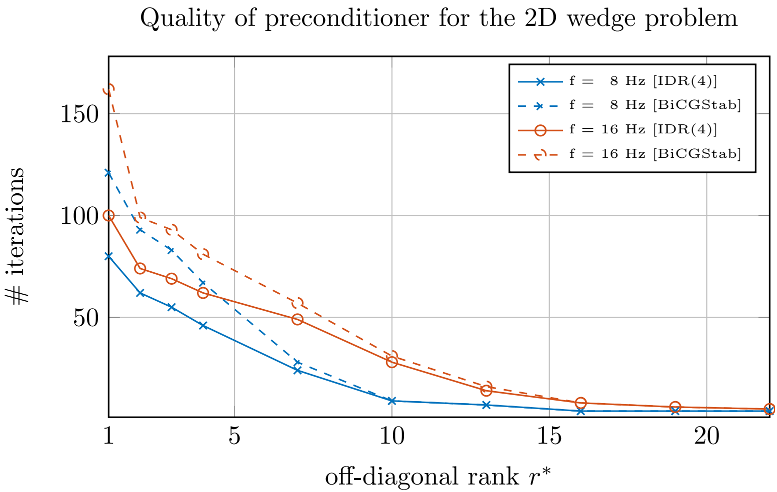
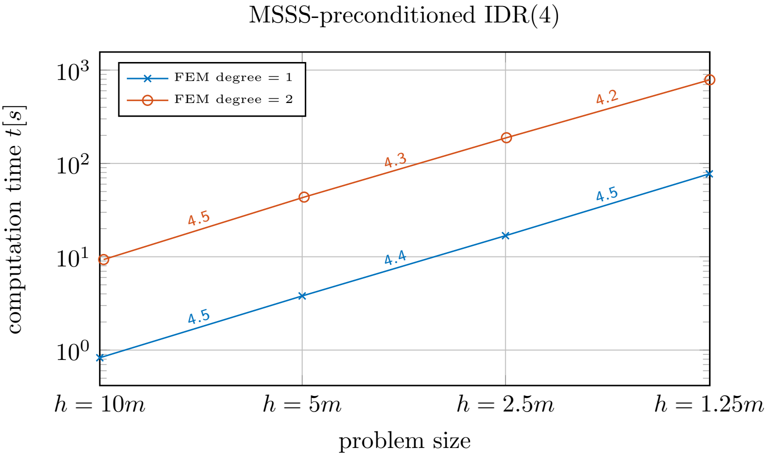
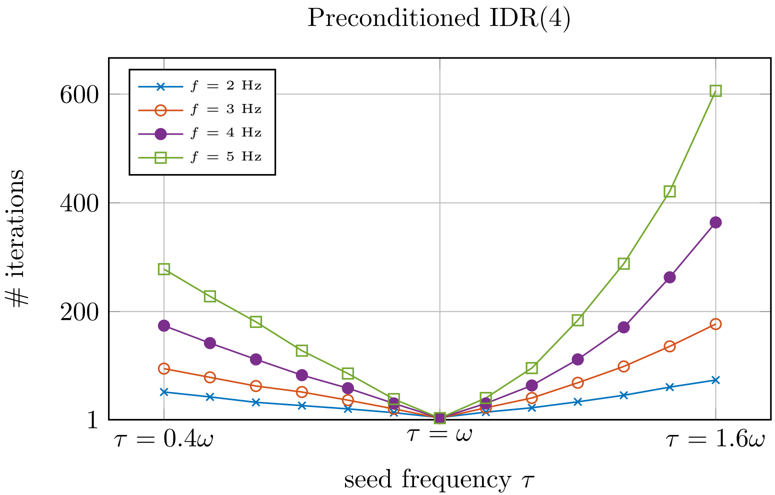
Parameter studies (1/3)



Parameter studies (2/3)



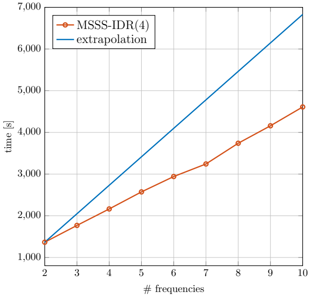
Parameter studies (3/3)



2D examples – `marmousi2`

Multiple frequencies

- $f_k \in [2.4, 2.8]Hz$
- complex τ

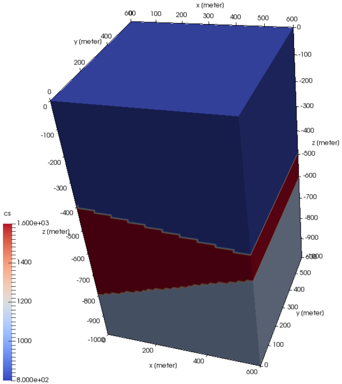


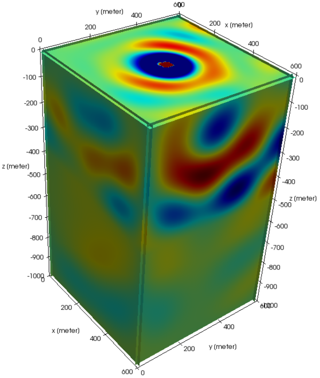
Multiple right-hand sides

- multiple sources
- efficient block MatVec's
- single MSSS preconditioner

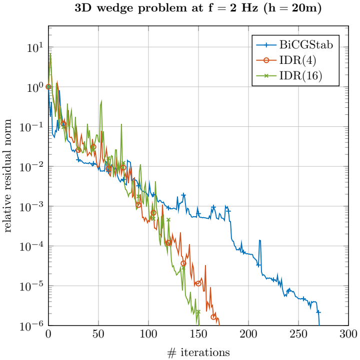
# rhs	MSSS [sec]	PIDR(4) [sec]
1	60.2	8.18 (8 iter.)
5	60.2	25.0 (8 iter.)
10	60.1	43.5 (8 iter.)
20	60.3	108.3 (8 iter.)

3D examples – an elastic wedge problem

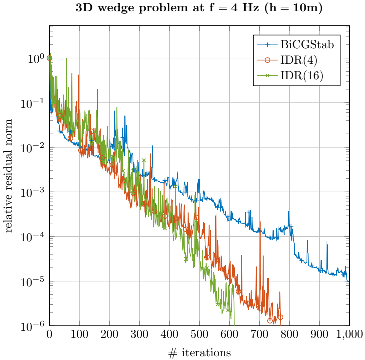




3D examples – an elastic wedge problem



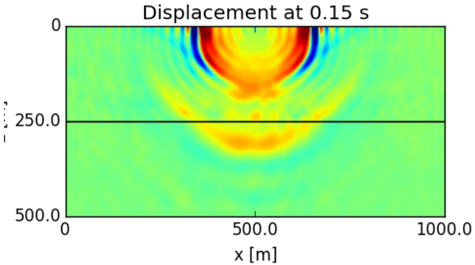
$3 \times 30 \times 30 \times 50$



$3 \times 60 \times 60 \times 100$

Conclusions

- ✓ MSSS preconditioner for 2D problems
- ✓ SSOR-MSSS preconditioner for 3D problems
- ✓ IDR(s) outperforms BiCGStab



Thank you for your attention!

Further readings:

M. Baumann and M. B. van Gijzen. *Nested Krylov methods for shifted linear systems.* SIAM J. Sci. Comput., **37**(5), S90-S112 (2015).

M. Baumann, R. Astudillo, Y. Qiu, E. Ang, M.B. Van Gijzen, and R.-E. Plessix. *A Preconditioned Matrix Equation Approach for the Time-Harmonic Elastic Wave Equation at Multiple Frequencies.* Technical report, TU Delft [in preparation].

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