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Preconditioning the time-harmonic elastic wave equation at multiple frequencies

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Problem statement

The discretized **time-harmonic elastic wave equation** yields,

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b}, \quad k = 1, \ldots, N, \tag{1}$$

where $\{\omega_1, \ldots, \omega_N\}$ are a range of (angular) frequencies.

1. Linearization:

$$\left\{ \begin{bmatrix} iC & K \\ I & 0 \end{bmatrix} - \omega_k \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} \omega_k \mathbf{x}_k \\ \mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \tag{2}$$

2. Matrix equation:

$$K\mathbf{X} - iC\mathbf{X}\Omega + M\mathbf{X}\Omega^2 = B, \quad \Omega := \text{diag}(\omega_1, \ldots, \omega_N). \tag{3}$$

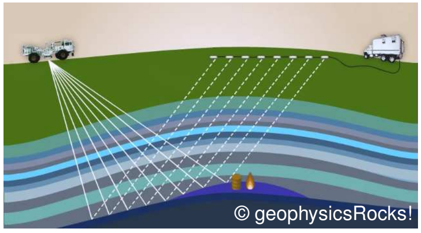

Fig. 1: Full-waveform inversion

Preconditioning multi-shift wave propagation problems

Right-preconditioning of (2) yields, $\quad \mathcal{A} := \begin{bmatrix} iC & K \\ I & 0 \end{bmatrix}, \ \mathcal{B} := \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$

$$(\mathcal{A} - \omega_k \mathcal{B})\mathcal{P}_k^{-1}\mathbf{y}_k = \mathbf{b} \quad \Leftrightarrow \quad (\mathcal{A}(\mathcal{A} - \tau\mathcal{B})^{-1} - \eta_k I)\mathbf{y}_k = \mathbf{b},$$

where $\eta_k = \omega_k/(\omega_k - \tau)$ and $\mathcal{P}_k = (1 - \eta_k)(\mathcal{A} - \tau\mathcal{B})$.

- Use MSSS matrix computations to approximately apply $(\mathcal{A} - \tau\mathcal{B})^{-1}$, cf. Figure 2.
- Choose τ optimally in the sense of,

$$\tau^* = \min_{\tau \in \mathbb{C}} \max_{k=1,\ldots,N} \left(\frac{R_k(\tau)}{|c_k(\tau)|} \right), \quad \text{cf. Figure 3.}$$

Theorem: GMRES convergence bound [Y. Saad, 2003]

Let the eigenvalues of the preconditioned matrix be enclosed by a circle of radius R and center c. Then the GMRES-residual norm after i iterations $\|r^i\|$ satisfies,

$$\frac{\|r^i\|}{\|r^0\|} \leq c_2(X) \left(\frac{R}{|c|} \right)^i,$$

where X is the matrix of eigenvectors, and $c_2(X)$ its condition number in 2-norm.

MSSS matrix structure

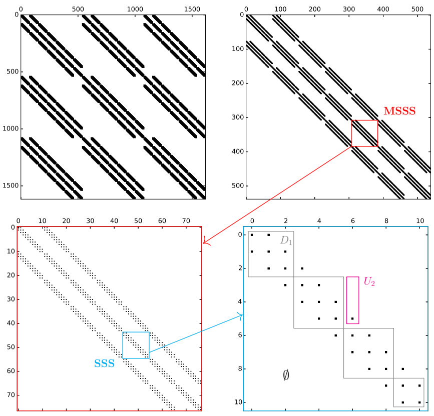

Fig. 2: MSSS structure of the 3D matrix (1)

Simulation results in 2D and 3D

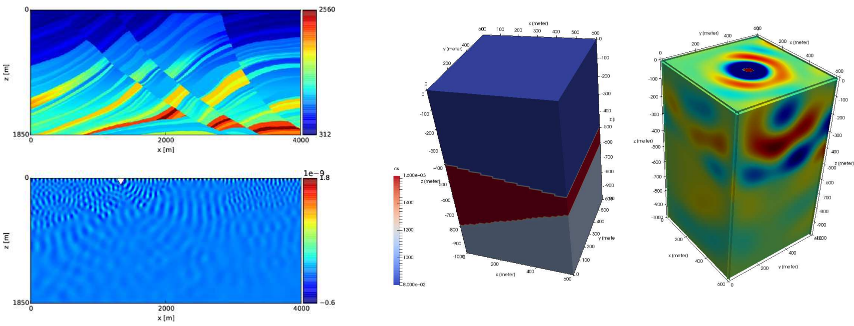

Fig. 4: Simulation results for the 2D `Marmousi-II` problem, and a 3D `wedge` problem

Preconditioned spectra

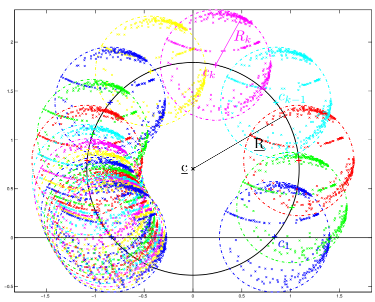

Fig. 3: Spectra after Möbius transformation

References

[1] M. Baumann, R. Astudillo, Y. Qiu, E. Ang, M.B. Van Gijzen, and R.-E. Plessix. *An MSSS-Preconditioned Matrix Equation Approach for the Time-Harmonic Elastic Wave Equation at Multiple Frequencies.* DIAM Technical Report 16-04, Delft University of Technology, 2016

[2] M. Baumann and M.B. Van Gijzen. *Nested Krylov methods for shifted linear systems.* SIAM Journal of Scientific Computing, **37**(5), S90-S112 (2015).