Efficient iterative methods for multi-frequency wave propagation problems: A comparison study

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Abstract
In this paper we present a comparison study for three different iterative Krylov methods that we have recently developed for the simultaneous numerical solution of wave propagation problems at multiple frequencies. The three approaches have in common that they require the application of a single shift-and-invert preconditioner at a suitable *seed* frequency. The focus of the present work, however, lies on the performance of the respective iterative method. We conclude with numerical examples that provide guidance concerning the suitability of the three methods.

Keywords: Time-harmonic elastic wave equation, global GMRES, nested multi-shift GMRES, shifted polynomial preconditioners

1 Introduction

After spatial discretization, for instance using the finite element method [5] with N degrees of freedom, the time-harmonic wave equation has the form,

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b}, \quad \omega_k := 2\pi f_k, \quad k = 1, ..., n_\omega, \tag{1}$$

with stiffness matrix K, mass matrix M, and C consisting of non-trivial boundary conditions [2]. Note that (1) yields a sequence of n_ω linear systems of equations. One way to solve the systems (1) simultaneously is to define the block matrix of unknowns, $\mathbf{X} := [\mathbf{x}_1, ..., \mathbf{x}_{n_\omega}] \in \mathbb{C}^{N \times n_\omega}$, and note that (1) can be rewritten as,

$$\mathcal{A}(\mathbf{X}) := K\mathbf{X} + iC\mathbf{X}\Omega - M\mathbf{X}\Omega^2 = B, \quad \text{with } \Omega := \operatorname{diag}(\omega_1, ..., \omega_{n_\omega}) \text{ and } B := \mathbf{b}\mathbf{1}^\mathsf{T}. \tag{2}$$

The matrix equation (2) can then be solved using a global Krylov method. A second approach is a linearization of the form,

$$\left(\begin{bmatrix} iC & K \\ I & 0 \end{bmatrix} - \omega_k \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \omega_k \mathbf{x}_k \\ \mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}, \quad k = 1, ..., n_\omega, \tag{3}$$

where the angular frequencies $\omega_1, ..., \omega_{n_\omega}$ appear as a (linear) shift. For short-hand notation, we define the block matrices,

$$\mathcal{K} := \begin{bmatrix} iC & K \\ I & 0 \end{bmatrix} \in \mathbb{C}^{2N \times 2N} \quad \text{and} \quad \mathcal{M} := \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \in \mathbb{C}^{2N \times 2N},$$

and write (3) as $(\mathcal{K} - \omega_k \mathcal{M})\mathbf{x}_k = \mathbf{b}, \text{for } k = 1, ..., n_\omega$. We will consider the case $C \equiv 0$ independently. The matrix equation (2) then reduces to two terms, and we can identify $\mathcal{K} = K$ as well as $\mathcal{M} = M$ and avoid doubling of dimensions in (3). In this paper, we review and compare the following recently developed algorithms:

- Global GMRES [11] for the matrix equation approach [5] (cf. Algorithm 1),
- Polynomial preconditioners [1, 7] for multi-shift GMRES (cf. Algorithm 2),
- Nested multi-shift FOM-FGMRES as presented in [6] (cf. Algorithm 3-4).

This list does not consider a comparison with the algorithms suggested by [4, 13] and by [15, 16]. Moreover, we restrict ourselves to GMRES-variants of the respective methods, and refer to [3] for global IDR(s) and [6] for the more memory-efficient combination nested IDR-QMRIDR(s). In [1] a polynomial preconditioner is used within multi-shift BiCG. The derivations in Section 2 emphasize that the cost-per-iteration of each proposed algorithm is comparable. In Section 3, we evaluate the three approaches for a numerical benchmark problem of the discretized time-harmonic elastic wave equation.

2 Iterative Krylov methods for multi-frequency wave propagation problems

The review of the subsequent algorithms is based on our works [5, 6, 7].

2.1 Preconditioned matrix equation approach

The matrix equation (2) with right preconditioning reads,

$$\mathcal{A}(P(\tau)^{-1}\mathbf{Y}) = B, \quad \mathbf{X} = P(\tau)^{-1}\mathbf{Y}, \quad \text{where } P(\tau) := (K + i\tau C - \tau^2 M)^{-1}, \tag{4}$$

and $\mathcal{A}(\cdot)$ as in (2). A similar reformulation has been suggested in [15]. We note that, for instance, the preconditioner $P(\tau)$ can be applied inexactly using an incomplete LU factorization. The (possibly complex) parameter τ is called the *seed* frequency. In Algorithm 1, we state the global GMRES method [11]. Note that in the block Arnoldi method the trace inner product is used, and norms are replaced by the Frobenius norm $\|\cdot\|_{\mathsf{F}}$ for block matrices. After m iterations, an approximate solution to (2) in the block Krylov subspace $\mathcal{K}_m(\mathcal{A}P(\tau)^{-1}, B)$ is obtained.

2.2 Preconditioners for shifted linear systems

The methods presented in this section are both two-level preconditioning approaches. As a first-level preconditioner, a shift-and-invert preconditioner of the form,

$$\begin{aligned} \mathcal{P}(\tau)^{-1} = (\mathcal{K} - \tau\mathcal{M})^{-1} &= \left(\begin{bmatrix} iC & K \\ I & 0 \end{bmatrix} - \tau \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & \tau I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (K + i\tau C - \tau^2 M)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -iC + \tau M \end{bmatrix}, \end{aligned} \tag{5}$$

Algorithm 1 Right-preconditioned global GMRES for the matrix equation (2), cf. [11]

Set $R_0 = B,\ V_1 = R_0/\|R_0\|_{\mathsf{F}}$ ▷ Initialization (when $X_0 = 0$)
for $j = 1$ to m **do**
  Apply $W = \mathcal{A}(P(\tau)^{-1}V_j)$ ▷ Preconditioner might be inexact
  for $i = 1$ to j **do** ▷ Block-Arnoldi method
    $h_{i,j} = \mathrm{tr}(W^{\mathsf{H}}V_i)$
    $W = W - h_{i,j}V_i$
  end for
  Set $h_{j+1,j} = \|W\|_{\mathsf{F}}$ and $V_{j+1} = W/h_{j+1,j}$
end for
Set $\underline{\mathrm{H}}_m = [h_{i,j}]_{i=1,\ldots,m}^{j=1,\ldots,m+1}$ and $\mathbf{V}_m = [V_1, ..., V_m]$ ▷ $\mathbf{V}_m$ is basis of block Krylov space
Solve $\mathbf{y}_m = \mathrm{argmin}_{\mathbf{y}} \|\underline{\mathrm{H}}_m\mathbf{y} - \|B\|_{\mathsf{F}}\mathbf{e}_1\|_2$
Compute $\mathbf{X}_m = P(\tau)^{-1}(\mathbf{V}_m * \mathbf{y}_m)$ ▷ '$*$' denotes the star product

is applied. Based on the decomposition (5) we note that $P(\tau)^{-1} = (K + i\tau C - \tau^2 M)^{-1}$ as defined in (4) is the main computational work and, hence, the work-per-iteration is comparable to Algorithm 1. For the block systems (3), the following equivalence holds,

$$(\mathcal{K} - \omega_k\mathcal{M})\mathcal{P}_k^{-1}\mathbf{y}_k = \mathbf{b} \quad \Leftrightarrow \quad (\mathcal{K}\mathcal{P}(\tau)^{-1} - \eta_k I)\mathbf{y}_k = \mathbf{b}, \tag{6}$$

where $\eta_k := \omega_k/(\omega_k - \tau)$, and $\mathcal{P}_k^{-1} := (1 - \eta_k)\mathcal{P}(\tau)^{-1} = (1 - \eta_k)(\mathcal{K} - \tau\mathcal{M})^{-1}$. Note that the latter is a preconditioned shifted linear system with (complex) shifts η_k and system matrix $\mathcal{C} := \mathcal{K}\mathcal{P}(\tau)^{-1} = \mathcal{K}(\mathcal{K} - \tau\mathcal{M})^{-1}$. Due to the equivalence in (6), the preconditioner (5) needs to be applied exactly.

2.2.1 Optimal polynomial preconditioners

After applying the shift-and-invert preconditioner (5) to (3), we remain with solving,

$$(\mathcal{C} - \eta_k I)\mathbf{y}_k = \mathbf{b}, \quad \mathbf{x}_k = \mathcal{P}_k^{-1}\mathbf{y}_k, \tag{7}$$

where $\mathcal{C} = \mathcal{K}\mathcal{P}(\tau)^{-1}$, and with (complex) shifts $\eta_k = \omega_k/(\omega_k - \tau)$. Efficient algorithms for shifted systems (7) rely on the shift-invariance property, $\mathcal{K}_m(\mathcal{C}, \mathbf{b}) \equiv \mathcal{K}_m(\mathcal{C} - \eta I, \mathbf{b})$, for any shift $\eta \in \mathbb{C}$ when the base vector is at least collinear; cf. [10, 14]. The (preconditioned) spectrum of $\mathcal{C}$ is known to be enclosed by a circle of radius R and center c [7, 17]. Therefore, the Neumann preconditioner [12, Chapter 12.3] of degree n,

$$\mathcal{C}^{-1} \approx \sum_{i=0}^{n} (I - \xi\mathcal{C})^i =: p_n(\mathcal{C}), \quad \text{with } \xi = \frac{1}{c} = -\frac{\tau - \bar{\tau}}{\bar{\tau}}, \tag{8}$$

has optimal spectral radius. The polynomial preconditioner (8) can also be represented as $p_n(\mathcal{C}) = \sum_{i=0}^{n} \alpha_i\mathcal{C}^i$. Shift-invariance can be preserved if the following holds,

$$(\mathcal{C} - \eta_k I)p_{n,k}(\mathcal{C}) = \mathcal{C}p_n(\mathcal{C}) - \tilde{\eta}_k I, \tag{9}$$

where $p_{n,k}(\mathcal{C}) = \sum_{i=0}^{n} \alpha_{i,k}\mathcal{C}^i$ is a polynomial preconditioner for $(\mathcal{C} - \eta_k I)$. Substitution yields,

$$\sum_{i=0}^{n} \alpha_{i,k}\mathcal{C}^{i+1} - \sum_{i=0}^{n} \eta_k\alpha_{i,k}\mathcal{C}^i - \sum_{i=0}^{n} \alpha_i\mathcal{C}^{i+1} + \tilde{\eta}_k I = 0. \tag{10}$$

The latter (10) is a difference equation and can be solved in closed form [1]:

$$\alpha_{n,k} = \alpha_n \tag{11a}$$
$$\alpha_{i-1,k} = \alpha_{i-1} + \eta_k \alpha_{i,k}, \quad \text{for } i = n, ..., 1 \tag{11b}$$
$$\tilde{\eta}_k = \eta_k \alpha_{0,k} \tag{11c}$$

Algorithm 2 Multi-shift GMRES with polynomial preconditioner for (9), cf. [1, 7]

1: Set $\mathbf{r}_0 = \mathbf{b},\ \mathbf{v}_1 = \mathbf{r}_0/\|\mathbf{r}_0\|$ ▷ Initialization
2: **for** $j = 1$ to m **do**
3: Apply $\mathbf{w} = \mathcal{C}p_n(\mathcal{C})\mathbf{v}_j$ ▷ Polynomial preconditioner (8) of degree n
4: **for** $i = 1$ to j **do** ▷ Arnoldi method
5: $h_{i,j} = \mathbf{w}^{\mathsf{H}}\mathbf{v}_i$
6: $\mathbf{w} = \mathbf{w} - h_{i,j}\mathbf{v}_i$
7: **end for**
8: Set $h_{j+1,j} = \|\mathbf{w}\|$ and $\mathbf{v}_{j+1} = \mathbf{w}/h_{j+1,j}$
9: **end for**
10: Set $\underline{\mathrm{H}}_m = [h_{i,j}]_{i=1,...,m}^{j=1,...,m+1}$ and $V_m = [\mathbf{v}_1, ..., \mathbf{v}_m]$
11: **for** $k = 1$ to n_ω **do**
12: Solve $\mathbb{C}^m \ni \mathbf{z}_k = \mathrm{argmin}_{\mathbf{z}} \|(\underline{\mathrm{H}}_m - \tilde{\eta}_k \underline{\mathrm{I}}_m)\mathbf{z} - \|\mathbf{r}_0\|\mathbf{e}_1\|$ ▷ Shifts $\tilde{\eta}_k$ according to (11c)
13: Resubstitute $\mathbf{y}_k = p_{n,k}(\mathcal{C})V_m\mathbf{z}_k$ ▷ Coefficients of $p_{n,k}$ according to (11a)-(11b)
14: **end for**

2.2.2 Inner-outer Krylov methods

We modify (7) by the substitution, $\bar{\mathcal{C}} := \mathcal{C} - \eta_1 I$, and solve the equivalent systems,

$$(\bar{\mathcal{C}} - (\eta_k - \eta_1)I)\mathbf{y}_k = \mathbf{b}, \quad \mathbf{x}_k = \mathcal{P}_k^{-1}\mathbf{y}_k, \tag{12}$$

with the advantage that for $k = 1$ we solve the *base* system $\bar{\mathcal{C}}\mathbf{y}_1 = \mathbf{b}$ (unshifted). The nested multi-shift Krylov algorithm consists in general of m_i inner iterations and m_o outer iterations. The nested FOM-FGMRES algorithm [6] is a combination of inner multi-shift FOM (Algorithm 3) with outer flexible multi-shift GMRES (Algorithm 4). In [6] we derive that if the inner method yields collinear residuals in the sense,

$$\mathbf{r}_j^{(k)} = \gamma_j^{(k)}\mathbf{r}_j, \quad \gamma_j^{(k)} \in \mathbb{C} \text{ for } k = 1, ..., n_\omega, \tag{13}$$

for $\mathbf{r}_j$ being the residual of the base system after m_i inner iterations, we can preserve shift-invariance in the outer method. The consecutive collinearity factors of the inner method then appear on a diagonal matrix Γ_k of a modified Hessenberg matrix in the outer loop (see line 13 in Algorithm 4 and [6], respectively). More precisely, after m_o outer iterations, the solution to,

$$\mathbf{z}_k = \underset{\mathbf{z}\in\mathbb{C}^{m_o}}{\mathrm{argmin}} \left\| \left((\underline{\mathrm{H}}_{m_o} - \underline{\mathrm{I}}_{m_o})\Gamma_k + \underline{\mathrm{I}}_{m_o} \right) \mathbf{y} - \|\mathbf{r}_0\|\mathbf{e}_1 \right\|, \quad \mathbf{y}_k = Z_{m_o}^{(k)}\mathbf{z}_k, \tag{14}$$

yields approximate solutions to (12) in the search spaces $Z_{m_o}^{(k)}$ that minimize the 2-norm of the residual of the k-th shifted system, cf. [6]. In (14), the Hessenberg matrix $\underline{\mathrm{H}}_{m_o}$ corresponds to the base system, and $\Gamma_k := \mathrm{diag}(\gamma_1^{(k)}, ..., \gamma_{m_o}^{(k)})$ is constructed from the collinearity factors in (13).

Algorithm 3 Inner multi-shift FOM for (12), cf. [14]

Set $\mathbf{r}_0 = \mathbf{b},\ \mathbf{v}_1 = \mathbf{r}_0/\|\mathbf{r}_0\|$ ▷ Initialization
for $j = 1$ to m_i **do**
  Apply $\mathbf{w} = \mathcal{K}(\mathcal{K} - \tau\mathcal{M})^{-1}\mathbf{v}_j$ ▷ Apply matrix $\bar{\mathcal{C}}$, cf. definition in (12)
  for $i = 1$ to j **do** ▷ Arnoldi method
    $h_{i,j} = \mathbf{w}^{\mathsf{H}}\mathbf{v}_i$
    $\mathbf{w} = \mathbf{w} - h_{i,j}\mathbf{v}_i$
  end for
  Set $h_{j+1,j} = \|\mathbf{w}\|$ and $\mathbf{v}_{j+1} = \mathbf{w}/h_{j+1,j}$
end for
Set $H_{m_i} = [h_{i,j}]_{i=1,\dots,m_i}^{j=1,\dots,m_i}$ and $V_{m_i} = [\mathbf{v}_1, ..., \mathbf{v}_{m_i}]$
for $k = 1$ to n_ω **do**
  Solve $\mathbb{C}^{m_i} \ni \mathbf{y}_k = (H_{m_i} - \eta_k I_{m_i})^{-1}(\|\mathbf{r}_0\|\mathbf{e}_1)$ ▷ Solve shifted Hessenberg systems
  Compute $\gamma_k = \mathbf{y}_k(m_i)/\mathbf{y}_1(m_i)$ ▷ Collinearity factors, cf. [6]
  Compute $\mathbf{x}_k = V_{m_i}\mathbf{y}_k$
end for

Algorithm 4 Outer multi-shift FGMRES for (12), cf. [6, 10]

Set $\mathbf{r}_0 = \mathbf{b},\ \mathbf{v}_1 = \mathbf{r}_0/\|\mathbf{r}_0\|$ ▷ Initialization
for $j = 1$ to m_o **do**
  $[\mathbf{z}_j^{(k)},\ \{\gamma_j^{(k)}\}_{k=1}^{n_\omega}] = \texttt{msFOM}(\mathcal{C}, \{\eta_k\}_{k=1}^{n_\omega}, \mathbf{v}_j, \text{maxit}{=}m_i)$ ▷ Inner method (Algorithm 3)
  Apply $\mathbf{w} = \mathcal{K}(\mathcal{K} - \tau\mathcal{M})^{-1}\mathbf{z}_j^{(k=1)}$ ▷ Apply matrix $\bar{\mathcal{C}}$ to base system
  for $i = 1$ to j **do** ▷ Arnoldi method
    $h_{i,j} = \mathbf{w}^{\mathsf{H}}\mathbf{v}_i$
    $\mathbf{w} = \mathbf{w} - h_{i,j}\mathbf{v}_i$
  end for
  Set $h_{j+1,j} = \|\mathbf{w}\|$ and $\mathbf{v}_{j+1} = \mathbf{w}/h_{j+1,j}$
end for
Set $\underline{\mathrm{H}}_{m_o} = [h_{i,j}]_{i=1,\dots,m_o}^{j=1,\dots,m_o+1}$ and $Z_{m_o}^{(k)} = [\mathbf{z}_1^{(k)}, ..., \mathbf{z}_{m_o}^{(k)}]$ ▷ Search spaces
for $k = 1$ to n_ω **do**
  Set $\underline{\mathrm{H}}_{m_o}^{(k)} = \left(\underline{\mathrm{H}}_{m_o} - \underline{\mathrm{I}}_{m_o}\right)\Gamma_k + \underline{\mathrm{I}}_{m_o}$, where $\Gamma_k := \mathrm{diag}(\gamma_1^{(k)}, ..., \gamma_{m_o}^{(k)})$
  Solve $\mathbb{C}^{m_o} \ni \mathbf{y}_k = \mathrm{argmin}_{\mathbf{y}} \left\|\underline{\mathrm{H}}_{m_o}^{(k)}\mathbf{y} - \|\mathbf{r}_0\|\mathbf{e}_1\right\|$ ▷ Hessenberg systems as in (14)
  Compute $\mathbf{x}_k = Z_{m_o}^{(k)}\mathbf{y}_k$
end for

3 Numerical experiments

We focus our numerical experiments on linear systems (1) that stem from a finite element discretization[1] of the time-harmonic elastic wave equation [2, 5]:

$$-\omega_k^2\rho\mathbf{u}_k - \nabla\cdot\sigma(\mathbf{u}_k) = \mathbf{s}, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^{d=\{2,3\}}, \tag{15a}$$

$$i\omega_k\rho\ B(c_p, c_s)\mathbf{u}_k + \sigma(\mathbf{u}_k)\hat{\mathbf{n}} = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega_a, \tag{15b}$$

$$\sigma(\mathbf{u}_k)\hat{\mathbf{n}} = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega_r. \tag{15c}$$

[1] For the finite element discretization we use the Python package `nutils` (`http://nutils.org`).

The Stress tensor in (15a) fulfills Hooke's law, $\sigma(\mathbf{u}_k) = \lambda(\mathbf{x})\,(\nabla\cdot\mathbf{u}_k\ I_d) + \mu(\mathbf{x})\big(\nabla\mathbf{u}_k + (\nabla\mathbf{u}_k)^{\mathsf{T}}\big)$, and we consider Sommerfeld radiation boundary conditions on $\partial\Omega_a$ that model absorption, and a free-surface boundary condition on $\partial\Omega_r$ (reflection). A finite element discretization with basis functions that are B-splines [9, Chapter 2] of degree $p \in \mathbb{N}_{>0}$ yields,

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{u}_k = \mathbf{s}, \quad k = 1, ..., n_\omega,$$

where $\mathbf{u}_k$ contains FEM coefficients of the k-th displacement vector, and $\mathbf{s}$ models a source term. In the case of purely reflecting boundary conditions, $\partial\Omega_a = \emptyset$, we obtain $C = 0$; cf. [5]. The inhomogeneous set of parameters $\{\rho, c_p, c_s\}$ is described in Figure 1. Here, we prescribe material-air boundary conditions at the upper boundary only.

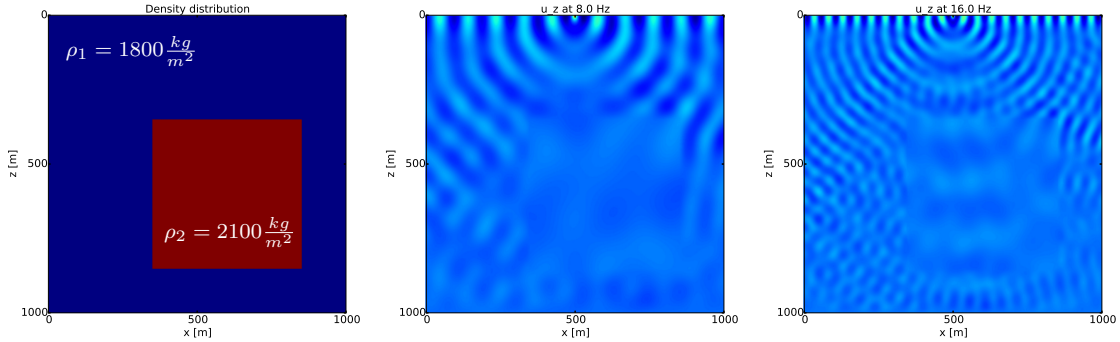

Figure 1: Set-up of the 2D numerical experiments: Density distribution (left), and real part of z-component of the displacement at $f = 8$Hz (middle) and $f = 16$Hz (right). The speed of pressure waves and shear waves are $c_p = \{2000, 3000\}\frac{m}{s}$ and $c_s = \{800, 1600\}\frac{m}{s}$, respectively, and the Lamé parameters $\{\lambda, \mu\}$ in Hooke's law can be calculated accordingly.

When comparing convergence behavior of the matrix equation approach (2) with the shifted system re-formulation (3), we use the identity,

$$\|\mathbf{R}_m\|_{\mathsf{F}} = \sqrt{\sum_{k=1}^{n_\omega} \left\|\mathbf{r}_m^{(k)}\right\|_2^2}, \qquad \text{for } \mathbf{R}_m := [\mathbf{r}_m^{(1)}, ..., \mathbf{r}_m^{(n_\omega)}] \in \mathbb{C}^{N\times n_\omega},$$

as a *fair* stopping criteria.

Experiment 1: Inexact solves in Algorithm 1

`scipy.sparse.linalg.spilu` or `PyAMG` [8] in Algorithm 1.

Experiment 2: Memory requirements

Experiment 3: Computational performance

4 Conclusions

We have compared three GMRES-based algorithms for the simultaneous iterative solution of frequency-domain wave propagation problems at multiple frequencies (1). The three approaches

Table 1: Comparison regarding memory requirements and costs-per-iteration when (1) has fixed problem size N and n_ω frequencies. Note that a single `MatVec` requires a solve for the shift-and-invert preconditioner.

Algorithm	leading memory requirement	# `MatVec`'s
`Gl-GMRES(m)`	$N \cdot n_\omega \cdot m$ for $\mathbf{V}_m$ (in Alg. 1, line 10)	$n_\omega \cdot m$
`poly-msGMRES(m,n)`	$2N \cdot m$ for V_m (in Alg. 2, line 10)	$(n+1) \cdot m$
`FOM(m_i)-FmsGMRES(m_o)`	$2N \cdot n_\omega \cdot m_o$ for $Z_{m_o}^{(k)}$ (in Alg. 4, line 11)	$m_i \cdot m_o$

share that they require the application of a single shift-and-invert preconditioner at a so-called *seed* frequency. Our observations are the following: Krylov space needs to appropriate union of preconditioned spectra. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

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Code Availability

The source code of the implementations used to compute the presented numerical results can be obtained from:

`https://github.com/ManuelMBaumann/freqdom_compare`

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Please contact Manuel Baumann for licensing information.