Algorithmic Differentiation

Manuel Alberto Parra Díaz
MSc. Mathematics

Faculty of Economics
Universidad del Rosario

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Presentation Outline

1. Options
2. Black-Scholes-Merton Formulas
3. Black-Scholes-Merton Formulas with Adjoint Algorithmic Differentiation

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The **European Oprions** can be exercised only on the expiration date itself.

Example: A trader instructs a broker to buy one December call option contract on Apple with a strike price of \$340.

Strike price	*June 2020*		*September 2020*		*December 2020*	
(\$)	*Bid*	*Ask*	*Bid*	*Ask*	*Bid*	*Ask*
290	29.80	30.85	39.35	40.40	46.20	47.60
300	21.55	22.40	32.50	33.90	40.00	41.15
310	14.35	15.30	26.35	27.25	34.25	35.65
320	8.65	9.00	20.45	21.70	28.65	29.75
330	4.20	5.00	15.85	16.25	23.90	24.75
340	1.90	2.12	11.35	12.00	19.50	20.30

Figure 1: Prices of call options on Apple, May 21,2020.
Stock price: bid \$316.23, ask \$316.50.

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- If the price of Apple does not rise above \$340 by December 18, 2020, the option is not exercised and the trader loses $\$2,030$.
- The option is exercised when the stock price is \$400, the trader is able to buy 100 shares at \$340 and immediately sell them for \$400 for a profit of \$3.970.

So, there are four types of participants in the options markets:

- Buyers of calls
- Sellers of calls
- Buyers of puts
- Sellers of puts

Buyers are referred to as having long positions and sellers are referred to as having short positions.

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How is related the option price with the change of one of these factors?

Stock Price and Strike Price

- **Call Option**
  The payoff will be the amount by which the stock price exceeds the strike price. *The more high stock price is, the more valuable the option is*.
- **Put Option**
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Time of Expiration

- **American Options**
  They become more valuable as the time to expiration increases.
- **European Options**
  They usually become more valuable as the time to expiration increases, but it is not always the case. (A large dividend expected to be paid between to expiration dates will cause the stock price to decline, so that a call short-life option could be worth more than the long-life option).

Volatility

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Risk-Free Interest Rate

The risk-free interest rate affects the price of an option in a less clear-cut way. If we assume that interest rates increase while all other variables stay the same then, the value of call options increases and the value of put options decreases.

Dividens

Dividends have the effect of reducing the stock price on the ex-dividend date[1]. So, if the ex-dividend date is during the life of an option, then the value of the option is negatively related to the size of the dividend if the option is a call and positively related to the size of the dividend if the option is a put.

[1]When a dividend is declared, an ex-dividend date is specified. Investors who own shares of the stock just before the ex-dividend date receive the dividend.

Black-Scholes-Merton Formulas

The formulas for the prices of European call and put options with no dividens are:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{1}$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{2}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Example:

Assume that a financial institution has sold for \$300.000 a European call option on 100.000 shares of a non-dividend-paying stock with

$$S_0 = 49, \quad K = 50, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 0.3846 \text{ (20 weeks)}.$$

Analyze this situation.

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \\
&= \frac{\ln\left(\frac{49}{50}\right) + \left(0.05 + \frac{0.2^2}{2}\right) 0.3846}{0.2\sqrt{0.3846}} \\
d_1 &\approx 0.054 \\
N(d_1) &\approx 0.52
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T}. \\
&\approx 0.054 - 0.2\sqrt{0.3846} \\
d_2 &\approx -0.07 \\
N(d_2) &\approx 0.47
\end{aligned}
$$

$$\begin{aligned} C &= S_0 N(d_1) - Ke^{-rT} N(d_2) \\ &\approx 49 \cdot 0.52 - 50 \cdot e^{-0.05 \cdot 0.3846} \cdot 0.47 \\ C &\approx \$2.40 \end{aligned}$$

Then, the price based on the Black-Scholes-Merton model is \$240.000. Therefore, the financial institution has sold a product for \$60.000 more than its theoretical value.

Greeks

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They are measures that quantify different aspects of the risk in an option position.

A trader naturally feels confident if the risks of changes in all the variables of the pricing model have been adequately hedged.

Delta: It is the rate of change of the option price with respect to the price of the underlying asset.

$$\Delta = \frac{\partial P}{\partial S}$$

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Then, under the BSM model,

$$\Delta_{call} = N(d_1)$$

$$\Delta_{put} = N(d_1) - 1$$

Theta: It is the rate of change of the option price with respect to the time. "Time decay".

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Then, under the BSM model,

$$\Theta_{call} = -\frac{S_0 n(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$\Theta_{put} = -\frac{S_0 n(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Gamma: It is the rate of change of the option delta with respect to the price of the underlying asset.

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$$\Gamma_{call} = \Gamma_{put} = \frac{n(d_1)}{S_0 \sigma \sqrt{T}}$$

Vega: It is the rate of change of the option price with respect to the volatility of the underlying asset.

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Then, under the BSM model,

$$\mathcal{V}_{call} = \mathcal{V}_{put} = S_0\sqrt{T}n(d_1)$$

Rho: It is the rate of change of the option price with respect to the interest rate.

$$\mathrm{P} = \frac{\partial P}{\partial r}$$

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Then, under the BSM model,

$$\mathrm{P}_{call} = KTe^{-rT}N(d_2)$$

$$\mathrm{P}_{put} = -KTe^{-rT}N(-d_2)$$

Greek	Definition	BSM	Value
Δ	$\frac{\partial P}{\partial S}$	$N(d_1)$	0.522
Θ	$\frac{\partial P}{\partial T}$	$-\frac{S_0 n(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$	-4.31
Γ	$\frac{\partial^2 P}{\partial S^2}$	$\frac{n(d_1)}{S_0\sigma\sqrt{T}}$	0.066
$\mathcal{V}$	$\frac{\partial P}{\partial \sigma}$	$S_0\sqrt{T}n(d_1)$	12.1
P	$\frac{\partial P}{\partial r}$	$KTe^{-rT}N(d_2)$	8.91

Table 1: Example's Greeks

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Volatility of the underlying set non-constant.

- Vega hedging involves making a position vega neutral, this is achieved by taking an offsetting position in a traded option.

BSM with Adjoint Algorithmic Differentiation

We will show that the implementation of the Black-Scholes-Merton formulas can be done through a SAC[2] program.

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

[2]The program that represents the algorithm to compute F is a simple linear program

```python
def Price(z):
  d1 = (log(z[0]/z[1]) + ((z[2]+0.5*pow(z[4],2))*z[3]))/(z[4]*sqrt(z[3]))
  d2 = d1 - (z[4]*sqrt(z[3]))
  n1 = norm.cdf(d1)
  n2 = norm.cdf(d2)
  price = ((z[0]*n1) - (z[1]*exp(-1*z[2]*z[3])*n2))
  return [d1,d2,n1,n2,price]
```

Listing 1: Python example for call option prices

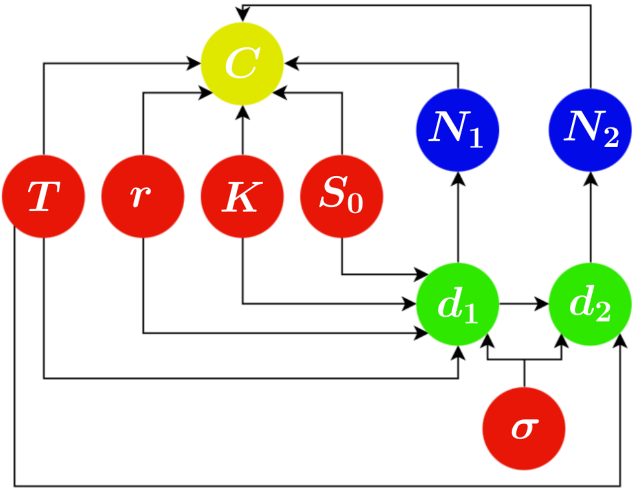

Figure 2: Diagram of the price function

Python's code: AAD Example for BSM

It can be proved that $\overline{d_1}$ is always zero, so the code can be reduced and optimized with this observation.

The BSM for hedging shows that what is often required is not all of the sensitivities, but only a limited number of them. In this case, maybe just Δ is desired for delta hedging.

The philosophy of AD is to use the composition. Each part of the composition need to have a derivative version. It should be mainly irrelevant if each block of the composition is written automatically or manually (for example, finite differences).

Marc Henrard

List of References

[1] M. Henrard. *Algorithmic differentiation in finance explained*. 2017.

[2] J. C. Hull. *Options futures and other derivatives*. 2022.