Local volatility

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Black Scholes model

The classical Black-Scholes model assumes that the market consists of one risky asset and one riskless asset.

The price process of the risky asset follows a geometric Brownian motion of the form

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 > 0. \tag{1}$$

where W is a standard Brownian motion. The continuously-compounded rate of return of the risk-less assset is $r > 0$. We also assume

- There are no arbitrage opportunities i.e., there is no way to make a riskless profit.
- Traders can borrow and lend any amount, even fractional, of cash at rate r.
- Traders can buy and sell any amount, even fractional, of the risky asset. This includes short selling.
- No transaction costs.

BS model and risk-neutral valuation

By the fundamental theorem of asset pricing, we have

- BS market model is complete,
- There exists an unique equivalent martingale (risk-neutral) measure Q under which the risky asset price process follows the dynamics

$$dS_t = S_t[(r - \delta)\, dt + \sigma\, dW_t]$$

  where δ is the continuously-compounded dividend payout rate

Therefore, the price at time t of a derivative with payoff function $g(S_T)$ at time $T > 0$ is given by $C(t, S_t)$, where

$$C(t, s) := e^{-r(T-t)}\mathbb{E}^Q[g(S_T) \,|\, S_t = s].$$

Here $\mathbb{E}^Q$ denotes expectation under the measure Q.

BS model and BSM - PDE

Main takeaway: the value of the risky asset at time T is unknown, yet the price of the derivative can be determined at the current time $t < T$.

Moreover, the price function $C(t, s)$ satisfies the **Black-Scholes-Merton PDE**

$$\frac{\partial C}{\partial t} + (r - \delta)s\frac{\partial C}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} = rC \tag{2}$$

with boundary (final) condition $C(T, s) = g(s)$.

BS Formula

For an European call option with payoff

$$g(S_T) = (S_T - K)^+$$

we have a closed-form solution for the price function, known as **Black-Scholes formula**

$$C(t,s) = C(t,s;\sigma,T,K) = se^{-\delta(T-t)}\Phi(d_1(t,s)) - Ke^{-r(T-t)}\Phi(d_2(t,s))$$

with

$$d_{1,2}(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r - \delta \pm \frac{1}{2}\sigma^2\right)(T-t)\right].$$

Remark Note that volatility is one of six inputs $\sigma, r, \delta, s, T, K$, used for option pricing in the BS model, but is the only one that is not directly observable in the market.

Implied volatility

Implied volatility $\sigma_{IV}(T, K)$ of an option contract is the value of the volatility parameter that satisfies

$$C(0, S_0; \sigma_{IV}(T, K), T, K) = C^{\mathrm{Mkt}}(T, K)$$

where S_0 is the current spot price of the underlying and $C^{\mathrm{Mkt}}(T, K)$ is the market price of an option with maturity $T > 0$ and strike $K > 0$.

- IV is forward-looking, it helps describe the market perception on the underlying's volatility.
- For $T > 0$ fixed, $\sigma_{IV}(T, K)$ may be different for different strikes $K > 0$, a phenomenon commonly known as "smile" or "smirk".
- It is a standard market practice quoting options in terms of the IV. Traders, practitioners and market makers have developed over the years intuition on this quantity.

Local volatility

The existence of the smile has forced quants to move from the simple Black-Scholes model to more sophisticated ones that would be able to describe this pattern.

Local volatility models Volatility coefficient of underlying asset is no longer a constant value, but a deterministic function of time and spot price of underlying asset itself: $\sigma = \sigma_{LV}(t, s)$

Under the risk neutral measure Q, we have

$$dS_t = S_t[(r - \delta)\, dt + \sigma_{LV}(t, S_t)\, dW_t^Q] \tag{3}$$

Risk-neutral density

Let $p(S,T)$ denote the risk-neutral density function of S_T. Then the value of a call option $C(K,T)$ as a function of K and T satisfies

$$C(K,T) = e^{-rT}\int_0^{+\infty}(S-K)^+ p(S,T)\,dx = e^{-rT}\int_K^{+\infty}(S-K)p(S,T)\,dS \tag{4}$$

Differentiating twice with respect to K

$$\frac{\partial C}{\partial K} = e^{-rT}\int_K^{+\infty} p(S,T)\,dS \tag{5}$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} p(K,T) \tag{6}$$

The risk-neutral density function of the underlying satisfies the so-called **Breeden-Litzenberger** formula

$$p(S,T) = e^{rT}\frac{\partial^2 C}{\partial K^2}\Big|_{K=S} \tag{7}$$

Risk-neutral density

Differentiating (4) with respect to T

$$\frac{\partial C}{\partial T} = -rC + e^{-rT} \int_K^{+\infty} (S-K) \frac{\partial p}{\partial T}\, dS.$$

Now, $p(S, T)$ satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial T} = \frac{1}{2} \frac{\partial^2}{\partial S^2} \left[\sigma_{LV}^2(S,T) S^2 p(S,T)\right] - (r-\delta) \frac{\partial}{\partial S}[Sp(S,T)] \tag{8}$$

Combining the last two expressions, we get

$$\begin{aligned} \frac{\partial C}{\partial T} = -rC + e^{-rT} \int_K^{+\infty} (S-K) \Bigg\{ & \frac{1}{2} \frac{\partial^2}{\partial S^2} \left[\sigma_{LV}^2(S,T) S^2 p(S,T)\right] \\ & - (r-\delta) \frac{\partial}{\partial S}[Sp(S,T)] \Bigg\}\, dS \end{aligned}$$

Risk-neutral density

Integrating the right hand side by parts

$$\begin{aligned}\frac{\partial C}{\partial T} + rC = {} & e^{-rT}\frac{1}{2}\left[(S-K)\frac{\partial}{\partial S}\left[\sigma_{LV}^2(S,T)S^2 p(S,T)\right]\right]_{S=K}^{S=+\infty} \\ & - e^{-rT}\int_K^{+\infty}\frac{\partial}{\partial S}\left[\sigma_{LV}^2(S,T)^2 S^2 p(S,T)\right]\, dS \\ & - e^{-rT}(r-\delta)\left[\left[(S-K)Sp(S,T)\right]_{S=K}^{S=+\infty} - \int_K^{+\infty} Sp(S,T)\, dS\right]\end{aligned}$$

Since $p(S,T)$ decays exponentially fast as $S \to \infty$, and so does its derivative with respect to S, we obtain

$$\frac{\partial C}{\partial T} + rC = \frac{1}{2}e^{-rT}\sigma_{LV}^2(K,T)^2 K^2 p(K,T) + (r-\delta)\left[C + e^{-rT}K\int_K^{+\infty} p(S,T)\, dS\right]$$

Dupire's Equation

Using expressions (5) and (6), we get the forward **Dupire's** PDE for the price of the call option $C(T, K)$ as function of T and K

$$\frac{\partial C}{\partial T} - \frac{1}{2}\sigma_{LV}^2(K, T)K^2\frac{\partial^2 C}{\partial K^2} + (r - \delta)K\frac{\partial C}{\partial K} - \delta C = 0, \quad \text{on } (0, \infty)^2$$

$$\begin{aligned}\lim_{K\to\infty} C(K; T) &= 0, \quad \forall T \in (0, \infty) \\ C(0; T) &= S, \quad \forall T \in (0, \infty) \\ C(K; 0) &= (S - K)^+, \quad \forall K \in (0, \infty)\end{aligned}$$

Local Volatility

From Dupire's equation, it is possible to derive a formula to evaluate the local volatility function σ_{LV}

$$\sigma_{LV}^2(K, T) = 2\frac{\partial_T C + (r-\delta)K\partial_K C + \delta C}{K^2 \partial_K^2 C} \tag{9}$$

- This formula seems straightforward: by observing a continuum of plain vanilla prices we could obtain σ_{LV}^2.
- However, only some option prices can be derived from direct observation of market data, so prices of other options (and their volatilities) have to be interpolated and extrapolated from the observations.