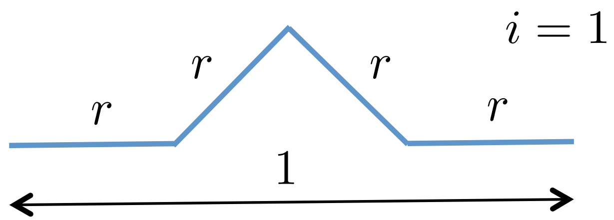
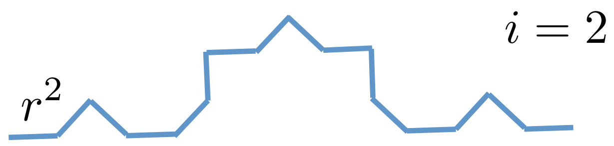
Fractal geometry

Observed surface or perimeter: $S(\varepsilon),\ P(\varepsilon) \sim \varepsilon^{-\gamma},\quad \varepsilon \ll 1$ is the linear size or resolution of observation

It is more meaningful to provide γ, than *S*, *P* for any ε. In practice $\varepsilon > 0$, and the scaling $\varepsilon^{-\gamma}$ should hold for several order of magnitude to call something a fractal. Examples are: beer froth, coast lines, Moon's surface.

Self-similarity. Consider a simple model of a coast line, the Koch curve. Its 'building block' is seen on the right:
The construction takes an infinity of iterations (i) of 1. taking 4 copies of the basic shape 2. rescaling them by *r,* and 3. rejoining them. The length/perimeter *P* can be calculated as follows:





i	L	N_r	$P = LN_r$
1	$4r$	$1 = 4^0$	$4r$
2	$4r^2$	$4 = 4^1$	4^2r^2
n	$4r^n$	4^{n-1}	$(4r)^n$

L – Length of basic shape
N_r – Number of repetitions of basic shape

Define resolution as: $\varepsilon = r^n \ \rightarrow\ n = \ln\varepsilon/\ln r \ \rightarrow\ P(\varepsilon) = 4^n r^n = 4^{\ln\varepsilon/\ln r}\varepsilon = \varepsilon^{\ln 4/\ln r + 1}$

$\rightarrow \quad \gamma = \ln 4/\ln(1/r) - 1 > 0 \quad \rightarrow \quad \lim_{\varepsilon \to 0} P(\varepsilon)$ does not exist. If, e.g.

$r \rightarrow 1/2 \ \rightarrow$ Koch curve gets more ramified

$r = 1/4 \ \rightarrow$ * is a straight line, $\gamma = 0$, and so the perimeter is independent of the resolution

Fractal dimension

Nontraditional lines, surfaces have increasing observational length, area with increasing resolution: they penetrate the dimensions above their own. Let's assign a fractional/noninteger number D_0 to their dimension, such that a more ramified object has a higher dimension! To this end cover the object with boxes of linear size ε, and define the dimension as the scaling exponent in

$$N_b(\varepsilon) \sim \varepsilon^{-D_0}$$

where N_b is the minimum number of boxes (of dimension $\lceil D_0 \rceil$) needed. With this

$$P(\varepsilon) = \varepsilon N_b(\varepsilon), \quad S(\varepsilon) = \varepsilon^2 N_b(\varepsilon),$$

In case of the Koch curve $\varepsilon = r^n$, and

$$N_b(\varepsilon) = 4N_r = 4^n = 4^{\ln \varepsilon / \ln r} = \varepsilon^{-\ln 4/\ln(1/r)} \sim \varepsilon^{-D_0} \quad \rightarrow \quad D_0 = 1 + \gamma = \frac{\ln 4}{\ln(1/r)}$$

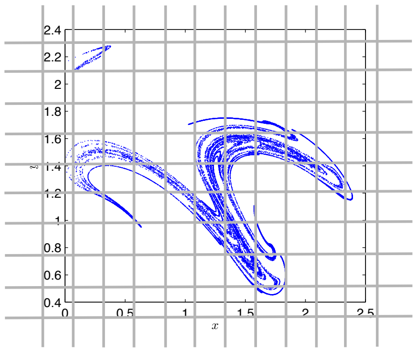
And so for the triadic ($r = 1/3$) Koch curve $D_0 = \ln 4/\ln 3 = 1.262...$, and

$S(\varepsilon) \sim \varepsilon^{2-D_0}$ decreases, so it is not area-filling.

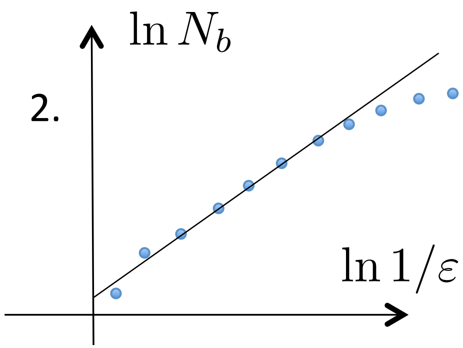
In numerics, naively we can really just:

1. Count the boxes, and
2. Fit a straight line

1.



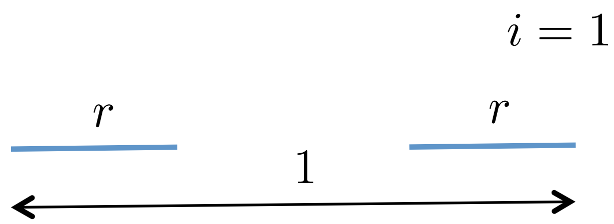
2.



Because we can rearrange the scaling such as:

$$D_0 \sim \frac{\ln N_b}{\ln 1/\varepsilon}$$

Another example: Cantor set. Its basic shape:
After the *i* = *n*-th iterate:



$$P = \varepsilon N_b = r^n 2^n$$

$$N_b = 2^n = 2^{\ln \varepsilon / \ln r} = \varepsilon^{-\ln 2 / \ln(1/r)} \quad \rightarrow$$

$$D_0 = \frac{\ln 2}{\ln(1/r)}$$

In general, for a self-similar ‘linear’ object whose basic shape consists of *M* units:

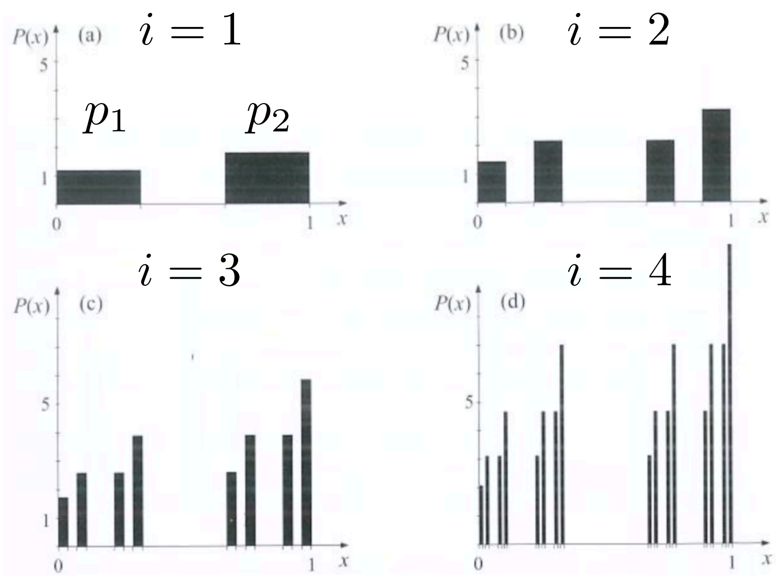
$$D_0 = \frac{\ln M}{\ln(1/r)}$$

Therefore, when $M > r^{-2}$ the object will be area-filling.

Fractal distributions

Consider a chaotic process in a dissipative system. The trajectory is performing motion bound to an attractor. Consider the occupation frequency in boxes covering the attractor based on finite time series. With the increasing length of the time series, these frequencies become time-independent. By normalizing the frequencies with the time series length, we get probability distributions. If the support of the distribution, the attractor, is a fractal set, then the distribution itself is also a fractal of some kind. But its fractal dimension does not have to be the same as that of the attractor!

Consider the following simple construction. Take the triadic Cantor set as a support, and define a probability distribution on it. The definition proceeds with the building of the Cantor set from its basic shape (see figure below). At any resolution, the probability that the trajectory visits the left (right) part of the basic shape: p_1 (p_2), conditionally that the particular copy of basic shape has been visited.



$i=1$	$p_1 \qquad p_2$
$i=2$	$p_1^2 \qquad \underbrace{p_1p_2 \qquad p_2p_1}_{2p_1p_2} \qquad p_2^2$
$i=3$	$p_1^3 \quad \underbrace{p_1^2p_2 \quad p_1^2p_2 \quad p_1p_2^2 \quad p_2p_1^2 \quad p_2^2p_1 \quad p_2^2p_1}_{3p_1^2p_2+3p_1p_2^2} \quad p_2^3$
$i=n$	$\underbrace{\binom{n}{m}}_{N_m}\underbrace{p_1^m p_2^{n-m}}_{p_m}, \qquad m=1,\dots,n$

Question: What is the typical interval for a given *n*?
Typical intervals: carry the same p_m , and in total provide the largest weight: $\max_m N_m p_m$

m = 0, *n* are atypical, clearly. Otherwise, in fact: $\lim_{n\to\infty} \max_m N_m p_m = 1$

Denote the dimension of the fractal set made up of typical boxes by D_1, which can be considered the fractal dimension of the probability distribution. That is, the number of boxes $N^*(\varepsilon)$ of size ε (3^{-n} for the triadic Cantor set) to cover typical boxes scales as:

$$N^*(\varepsilon) \sim \varepsilon^{-D_1}$$

$D_1 \leq D_0$ as typical boxes make a subset of all the boxes. Equality holds only for uniform distributions.

For multi-fractals we can define the typical boxes by $\max_m N_m p_m^q$ by which we have a nontrivial dimension spectrum D_q .

Alternative definition of D_1 more suitable for numerics

Consider the (normalized) occupation frequency: $P_i(\varepsilon),\ i = 1, \ldots, N_b(\varepsilon),\ \sum_{i=1}^{N_b(\varepsilon)} P_i(\varepsilon) = 1$

With this we can define the information as: $$I(\varepsilon) = - \sum_{i=1}^{N_b(\varepsilon)} P_i(\varepsilon) \ln P_i(\varepsilon)$$

which characterizes the inhomogeneity of the distribution P_i. Of course the larger the resolution the more information we have. Experience shows that the information is proportional to $\ln(1/\varepsilon)$, with a prefactor that is called the information dimension:

$$I(\varepsilon) \sim D_1 \ln(1/\varepsilon)$$

The numerics gives a reliable estimate of D_1 with much fewer data than needed for D_0.

It is instructive to apply the definition of typical boxes:

$$I(\varepsilon) = \sum_{i=1}^{N^*(\varepsilon)} \frac{1}{N^*(\varepsilon)} \ln N^*(\varepsilon) = \boxed{\ln N^*(\varepsilon) \sim D_1 \ln(1/\varepsilon)}$$

which reveals that the information dimension D_1 is really the fractal dimension of typical boxes.

Exercise: Calculate the information dimension D_1 numerically for the winter chaotic attractor of L84

- Set $F = 8$
- Impose the sectioning surface $z = 0$ (like in case of constructing the bifurcation diagram), and produce the Poincaré section on the x-y plane -- a scatter diagram made up of distinct points (see figure below). Run a long enough simulation that produces at least 20.000 data points on the Poincaré section.
- The correct value should be about $D_1 = 1.46$.
- (The dimension of the full attractor is one plus that of the Poincaré section.)