Lyapunov exponents

Our starting point is the nonlinear mapping: $x_{n+1} = M(x_n), \quad x_n \in \mathbb{R}^d$

We want to quantify the influence of small errors in measuring the initial conditions on predicting future states of the system. To this end we analyze the evolution of small errors around a reference trajectory representing the true realization of system evolution.

Assume that the solution is chaotic. As the period-doubling route to chaos suggests, there is not a finite period of the motion, but it never repeats itself. Nevertheless, the chaotic attractor is made up of an infinite set of periodic orbits. (The number of orbits of a certain time period exponentially grows with the time period; the scaling exponent is called the topological entropy.) These are all unstable (periodic) orbits (UPO), and so in numerics they are never realized – a 'random walk' is performed among them. If the fixed points of the mapping associated with the UPOs are all of the saddle-type, then we speak about a hyperbolic dynamics, in which case many central results of chaos theory apply/can be proven.

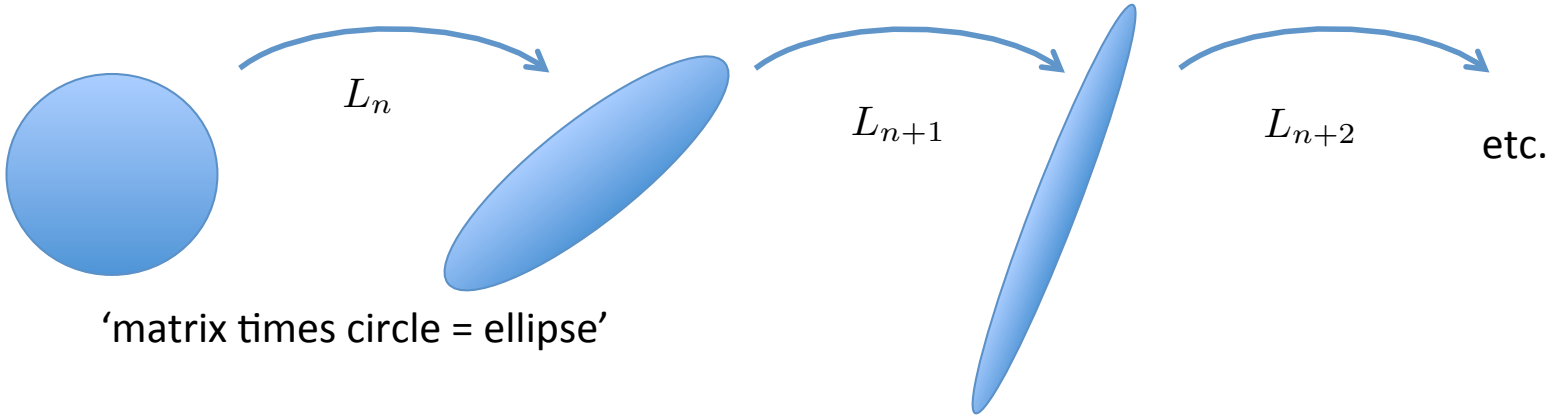
If the error is small, its evolution is approximately governed by the linearized mapping: $\Delta x_{n+1} = L_n \cdot \Delta x_n$

in which matrix L_n is time-dependent through the time-dependent solution x_n where the differentials are substituted: $L_n = \left. \frac{\partial M(x)}{\partial x} \right|_{x=x_n}$

Although at any time the trajectory infinitely approximates a hyperbolic periodic orbit, the eigenvalues of L_n are not necessarily all real, because L_n governs the evolution over a single iterate only, not over the complete hyperbolic periodic orbit, whose period might be very long (>> 1).

Nevertheless, specific directions in which errors grow or diminish can be identified. Consider in 2D (*d* = 2) a small circular vicinity of the true trajectory where the erroneous measurement places that trajectory. Under the mapping the circle turns into an ellipse, which has specific semi-major and -minor axes. Further iterations modify the directions and size of the semi-major and -minor axes, since:

$$\Delta x_{n+m} = L_n^{(m)} \cdot \Delta x_n \qquad \text{where} \qquad L_n^{(m)} = L_n \cdot L_{n+1} \cdot \cdots \cdot L_{n+m}$$



'matrix times circle = ellipse'

The semi-principal axes of the error ellipsoid are given by $s_{n,i}^{(m)} u_{n,i}^{(m)}$, where $(s_{n,i}^{(m)})^2$ and $u_{n,i}^{(m)}$ $i = 1, \ldots, d$ are eigenvalues and normalized unit-length eigenvectors of $L_n^{(m)} \cdot (L_n^{(m)})^T$. $s_{n,i}^{(m)}$ are also called the singular values of $L_n^{(m)}$, which are clearly non-negative.

Note 1: Because $L_n^{(m)} \cdot (L_n^{(m)})^T$ is symmetric, it has only real eigenvalues and eigenvectors.

Note 2: $s_{n,i}^{(m)} \neq \Pi_{\nu=n}^{(n+m)} s_{\nu,i}$

With this the Lyapunov exponents are defined as long-term average stretching/shrinking exponents:

$$\lambda_i = \lim_{m \to \infty} \frac{\ln s_{n,i}^{(m)}}{m}$$

Lyapunov exponents for time-continuous systems $\dot{x} = f(x), \quad x = x(x_0, t)$

Let us introduce the deformation gradient: $X = \dfrac{\partial x}{\partial x_0}$

By differentiating both sides of the ODE wrt. x_0 (after some easy manipulation) we arrive at the (linear) variational equation:

$$\dot{X} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x(x_0,t)} \cdot X, \quad X_0 = X(x_0, t=0) = I$$

Notice that $\Delta x(x_0, t) = X(x_0, t) \cdot \Delta x_0$, which provides a linear mapping of initial small errors. (Note: the validity of this for finite small Δx_0 breaks down quickly.) This establishes a connection of flows *f* with discrete-time mappings *M*:

$L_n^{(m)} = X(t_{n+m}) \cdot X^{-1}(t_n), \; t_n = nT$ (therefore $L_0^{(m)} = X(t_m)$ since $X_0^{-1} = X_0 = I$)

Otherwise, the LEs can be put in terms of the singular values $s_i(t)$ of $X(t)$:

$$\lambda_i = \lim_{t\to\infty} \frac{\ln s_i(t)}{t}$$

In numerics a long stretch of integration would be needed to calculate the LEs as the above formula dictates; the integration cannot be broken down according to Note 2 above. However, over time the order of magnitude of entries of X are growing apart. Because of this, and the finite-size representation of numbers on the computer, some singular values are more and more poorly estimated, up to the point that the estimates are useless.

There exists, fortunately, an indirect method, the **Gram-Schmidt (orthonormalization) procedure**

0. It starts with a set of orthonormal basis of vectors: $\{w_1^0, \ldots, w_d^0\}$

1. Computes the new vectors: $z_1 = L_0 \cdot w_1^0, \ldots, z_d = L_0 \cdot w_d^0$

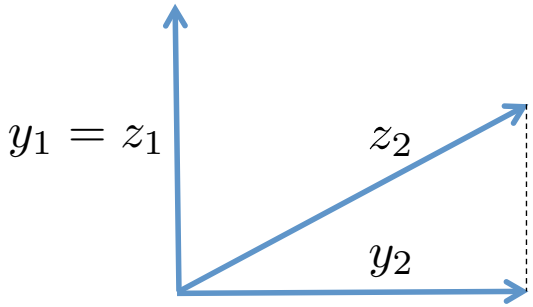
2. Orthogonalizes them:

$$y_1 = z_1$$

$$y_2 = z_2 - \frac{z_2 \cdot y_1}{\|y_1\|^2} y_1$$

$$y_3 = z_3 - \frac{z_3 \cdot y_1}{\|y_1\|^2} y_1 - \frac{z_3 \cdot y_2}{\|y_2\|^2} y_2$$

$$y_d = z_d - \sum_{i=1}^{d-1} \frac{z_d \cdot y_i}{\|y_i\|^2} y_i$$



3. And reinitializes: $w_1^1 = y_1, \ldots, w_d^1 = y_d$

The orthogonal set $\{w_1^1, \ldots, w_d^1\}$ spans an ellipsoid the same volume as $L_1 \cdot B$ where *B* is the unit-volume ball.

Continue iteratively such as: $z_1 = L_1 \cdot w_1^1, \ldots, z_d = L_1 \cdot w_d^1$ etc.

The LEs are:

$$\lambda_i = \lim_{m \to \infty} \frac{\ln \|w_i^m\|}{m}$$

However, this still suffers from the increasing inaccuracy. To cater for that in each step we can normalize:

3*. $w_i^j = \dfrac{y_i^j}{\|y_i^j\|}$

And so the LEs become: $\lambda_i = \lim\limits_{m\to\infty} \dfrac{\sum_{j=1}^{m} \ln \|y_i^j\|}{m}$

Note 1: This is not an averaging of the ln of singular values, because we 'follow the turning of the vectors'. A transient is better to be discarded, before the vectors reach their own attractor.

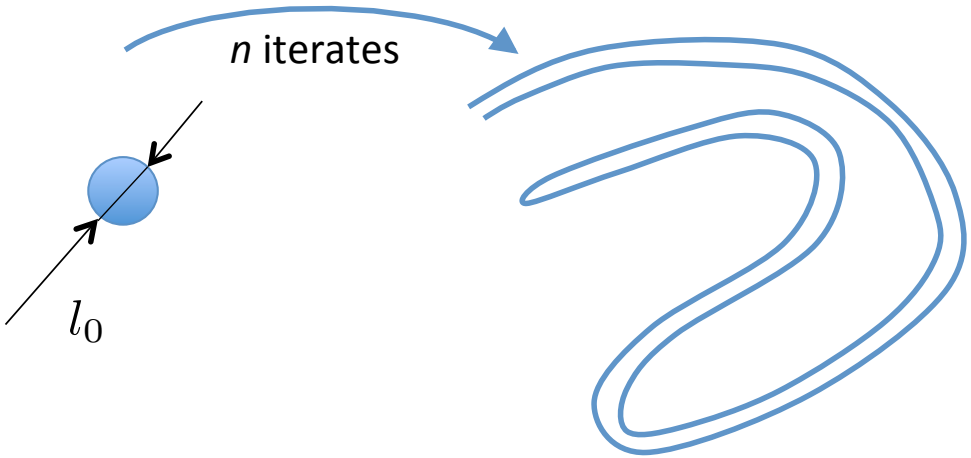
Note 2: The calculation of $\{y_1, \ldots, y_d\}$ can be written with a matrix formalism.

Note 3: The LEs are conventionally presented in a decreasing order. The largest one λ_1 is the easiest to calculate, in fact independently of the rest of the LEs. It can be considered as a measure of predictability, since it gives the 'rate of exponentially fast' separation of two trajectories in the phase space. This actually allows us to calculate it without having to integrate the variational eq.

The Kaplan-Yorke relation…

… establishes a connection between dynamics and attractor geometry.

In 2D start with a 'blob of initial conditions' and let them evolve over *n* iterations of the map:



Length: $l_0 e^{\bar{\lambda} n}$

Width: $l_0 e^{\bar{\lambda}' n}$

Area: $A_n = l_0^2 e^{(\bar{\lambda}+\bar{\lambda}')n}$

Make the width the resolution of observation: $\varepsilon_n = l_0 e^{\bar{\lambda}' n}$

Number of boxes to cover iterated shape: $N(\varepsilon_n) = A_n/\varepsilon_n^2 = e^{(\bar{\lambda}-\bar{\lambda}')n} \sim \varepsilon_n^{\bar{\lambda}/\bar{\lambda}'-1}$ →

The information dimension: $D_1 = 1 + \bar{\lambda}/|\bar{\lambda}'|$ (considering typical behaviour)

Conjecture for higher dimensions: $D_1 = k + \dfrac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \quad \sum_{i=1}^{k+1} \lambda_i \leq 0 \leq \sum_{i=1}^{k} \lambda_i$

Exercise: Calculate the information dimension via the Kaplan-Yorke relation and compare it to the result of the previous exercise (obtained via a direct box-counting calculation)

- You can look for existing codes on the internet for calculating the spectrum of Lyapunov exponents. For example in Matlab the following is a reliable script: http://de.mathworks.com/matlabcentral/fileexchange/4628-calculation-lyapunov-exponents-for-ode
- You would then need to code only the variational eq.
- (Feel free to check your result for the maximal Lyapunov exponent (MLE) λ_1 by calculating it in the alternative way through the separation of two very closely initialized trajectories: Plot the ln of the distance against time, and fit a straight line in the range of nice scaling. The slope of the line gives you the MLE. For a smooth scaling you should average ln(distance) over dozens of trajectory-pair realizations. These realizations can be generated by considering points on the trajectory as initial conditions well and equally separated in time, and the pair of each IC can be taken by a tiny random perturbation.)