

Binary Search Trees

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Introduction

A **binary search tree** (or *bst*) is a more organized binary tree. In it each node contains a key value by which it can be searched, additional satellite data, and pointers *left*, *right*, and *p* which point to the left child, right child, and parent node, respectively. The pointers may be populated, otherwise they will have the value NIL (coming from Latin, meaning *nothing*). The root is the top node from which all nodes are descended.

These keys are stored in such a way that they satisfy the ***binary-search-tree property***:

Let x be a node in a binary search tree. If y is a node in the left subtree of x , then $y.key \leq x.key$. If y is a node in the right subtree of x , then $y.key \geq x.key$.

Three ways to walk, or traverse, binary search trees:

Inorder tree walk: traverse left subtree, visit current node, and then traverse right subtree.

This prints out all keys in sorted order, and traversing every node takes $\Theta(n)$ time when starting from the root.

```
INORDER-TREE-WALK( $x$ )
1  if  $x \neq \text{NIL}$ 
2      INORDER-TREE-WALK( $x.left$ )
3      print  $x.key$ 
4      INORDER-TREE-WALK( $x.right$ )
```

Preorder tree walk: visit current node, traverse left subtree, and traverse right subtree.

```
PREORDER-TREE-WALK( $x$ )
1  if  $x \neq \text{NIL}$ 
2      print  $x.key$ 
3      INORDER-TREE-WALK( $x.left$ )
4      INORDER-TREE-WALK( $x.right$ )
```

Postorder tree walk: traverse left subtree, traverse right subtree, and visit current node.

```

POSTORDER-TREE-WALK( $x$ )
1  if  $x \neq \text{NIL}$ 
2      INORDER-TREE-WALK( $x.\text{left}$ )
3      INORDER-TREE-WALK( $x.\text{right}$ )
4      print  $x.\text{key}$ 

```

Querying a binary search tree

Each of the following operations will run in $O(h)$ time for a bst of height h . The procedures discussed are, SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, and SUCCESSOR.

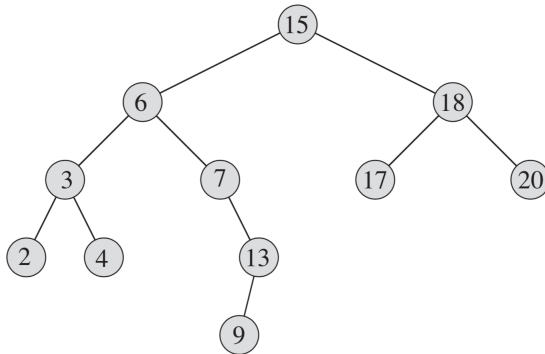


Figure 1: Referring to the above binary search tree, here are example cases of queries.

i. To **search** for the lucky number 7, we always start searching at the root (15 at the top). Compare $k=7$ with the key of the root, $7 \nless 15$, so go down the left subtree. Now compare $k=7$ with 6, $7 \ngtr 6$, so go down the right subtree. Now compare $k=7$ with 7, $7 = 7$, duh, so in this simple procedure we return the value 7. **ii.** The **minimum** key of this bst is 2, you simply follow the *left* pointers. **iii.** The **maximum** key of this bst is 20, you simply follow the *right* pointers. **iv.** The **predecessor** of key 6 is 4, if the node has a left subtree, find the maximum of that left subtree. The predecessor of 9 is 7, if no left subtree exists, simply find the parent of the first node that is a right child (in this case 13 is 7's right child). **v.** The **successor** of 15 is 17, if the node has a right subtree, find the minimum of that subtree. The successor of 13 is 15, if no right subtree exists, simply find the parent of the first node that is a left child (in this case 6 is the right child of 15).

Search

The binary search procedure is straightforward, given a key k , we must search the bst for a node with a key that matches k . We always start at the root, if k is less than $x.key$, then we must go down the left subtree, if k is larger than $x.key$, then we must go down the right subtree, if k is equal to $x.key$, then we return a pointer to the node x . We repeat this process until we find the matching key, otherwise we return NIL.

```
TREE-SEARCH( $x, k$ )
1  if  $x == \text{NIL}$  or  $k == x.key$ 
2      return  $x$ 
3  if  $k < x.key$ 
4      return TREE-SEARCH( $x.left, k$ )
5  else
6      return TREE-SEARCH( $x.right, k$ )
```

Here is the iterative version, which is more efficient on computers.

```
ITERATIVE-TREE-SEARCH( $x, k$ )
1  while  $x \neq \text{NIL}$  and  $k \neq x.key$ 
2      if  $k < x.key$ 
3           $x = x.left$ 
4      else
5           $x = x.right$ 
6  return  $x$ 
```

Minimum and maximum

Finding the minimum simply means going left on the bst until a NIL value on the *left* field is encountered.

```
TREE-MINIMUM( $x$ )
1  while  $x.left \neq \text{NIL}$ 
2       $x = x.left$ 
3  return  $x$ 
```

Symmetrically, the right operates in the same manner, but we go all right instead.

```
TREE-MAXIMUM( $x$ )
1  while  $x.right \neq \text{NIL}$ 
2       $x = x.right$ 
3  return  $x$ 
```

Predecessor and successor

Sometimes we want to find the node, based on the inorder tree walk, that comes before or after node x . The **predecessor** of a node x is the node with the *largest* key smaller than $x.key$.

TREE-PREDECESSOR(x)

```
1  if  $x.left \neq \text{NIL}$ 
2      return TREE-MAXIMUM( $x.left$ )
3   $y = x.p$ 
4  while  $y \neq \text{NIL}$  and  $x == y.left$ 
5       $x = y$ 
6       $y = y.p$ 
7  return  $y$ 
```

We have two cases to consider when searching for a predecessor.

1. If x has a left subtree, find the maximum of that subtree.
2. If no left subtree exists, find the parent of the first node that is a right child.

The **successor** of a node x is the node with the *smallest* key larger than $x.key$.

TREE-SUCCESSOR(x)

```
1  if  $x.right \neq \text{NIL}$ 
2      return TREE-MINIMUM( $x.right$ )
3   $y = x.p$ 
4  while  $y \neq \text{NIL}$  and  $x == y.right$ 
5       $x = y$ 
6       $y = y.p$ 
7  return  $y$ 
```

We have two cases to consider when searching for a successor, which is very much symmetrical to predecessor.

1. If x has a right subtree, find the minimum of that subtree.
2. If no right subtree exists, find the parent of the first node that is a left child.

A good way to remember these procedures is to perform the reverse operations after you think you got the right node.

Insertion

Inserting a new node into a binary search tree T starts with a binary search in order to find where to insert node z , this can be to the left or right of some parent node y , or the existing tree may be empty! When we insert z , $z.key$ is some value v , while $z.left$ and $z.right$ will be NIL. The following procedure will take $O(h)$ time for a tree T of height h .

```
TREE-INSERT( $T, z$ )
1   $y = \text{NIL}$ 
2   $x = T.root$ 
3  while  $x \neq \text{NIL}$ 
4       $y = x$ 
5      if  $z.key < x.key$ 
6           $x = x.left$ 
7      else
8           $x = x.right$ 
9   $z.p = y$ 
10 if  $y == \text{NIL}$ 
11      $T.root = z$  // tree  $T$  was empty
12 elseif  $z.key < y.key$ 
13      $y.left = z$ 
14 else
15      $y.right = z$ 
```

Deletion

Deleting a node z from a bst T involves three cases, we will reference figure 12.4 from the book.

- Node z has no children, remove it from the tree. In terms of pointers, modify the parent so that it refers not to z but NIL instead (which is the value of z 's children).
- Node z has one child, replace z with its non-NIL child. Modify the pointers of the parent and non-NIL child of z to point "around" it (part (a) and (b) where l is z 's left child and r is the right).
- Node z has two children, this one is a tad tricky. We replace z with either its predecessor or its successor, but we will go with the successor. There are two cases with this one.
 - Right child r is z 's successor, in which case we just replace z with r .
 - Right child r is not z 's successor, in which case we must find the minimum of the right subtree.

To move subtrees around, the book defines a neat subroutine called TRANSPLANT. This TRANSPLANT procedure replaces the subtree rooted at u with another subtree rooted at a different node v , and u 's parent becomes v 's parent. The pointers to the children of v are not modified, so keep that in mind.

```

TRANSPLANT( $T, u, v$ )
1  if  $u.p == \text{NIL}$ 
2       $T.root = v$ 
3  elseif  $u == u.p.left$ 
4       $u.p.left = v$ 
5  else
6       $u.p.right = v$ 
7  if  $v \neq \text{NIL}$ 
8       $v.p = u.p$ 

```

Having TRANSPLANT defined makes deleting much shorter.

```

TREE-DELETE( $T, z$ )
1  if  $z.left == \text{NIL}$ 
2      TRANSPLANT( $T, z, z.right$ )
3  elseif  $z.right == \text{NIL}$ 
4      TRANSPLANT( $T, z, z.left$ )
5  else
6       $y = \text{TREE-MINIMUM}(z.right)$ 
7      if  $y.p \neq z$ 
8          TRANSPLANT( $T, y, y.right$ )
9           $y.right = z.right$ 
10          $y.right.p = y$ 
11     TRANSPLANT( $T, z, y$ )
12      $y.left = z.left$ 
13      $y.left.p = y$ 

```

The TREE-INSERT and TREE-DELETE procedure take $O(h)$ time for a bst of height h . For insertion, we must perform binary search, which takes $O(h)$ time while rearranging pointers takes constant time. For deletion, everything but TREE-MINIMUM takes constant time, finding the minimum takes $O(h)$ time again. Figure 12.4 from the book gives a good visualization for the delete cases.

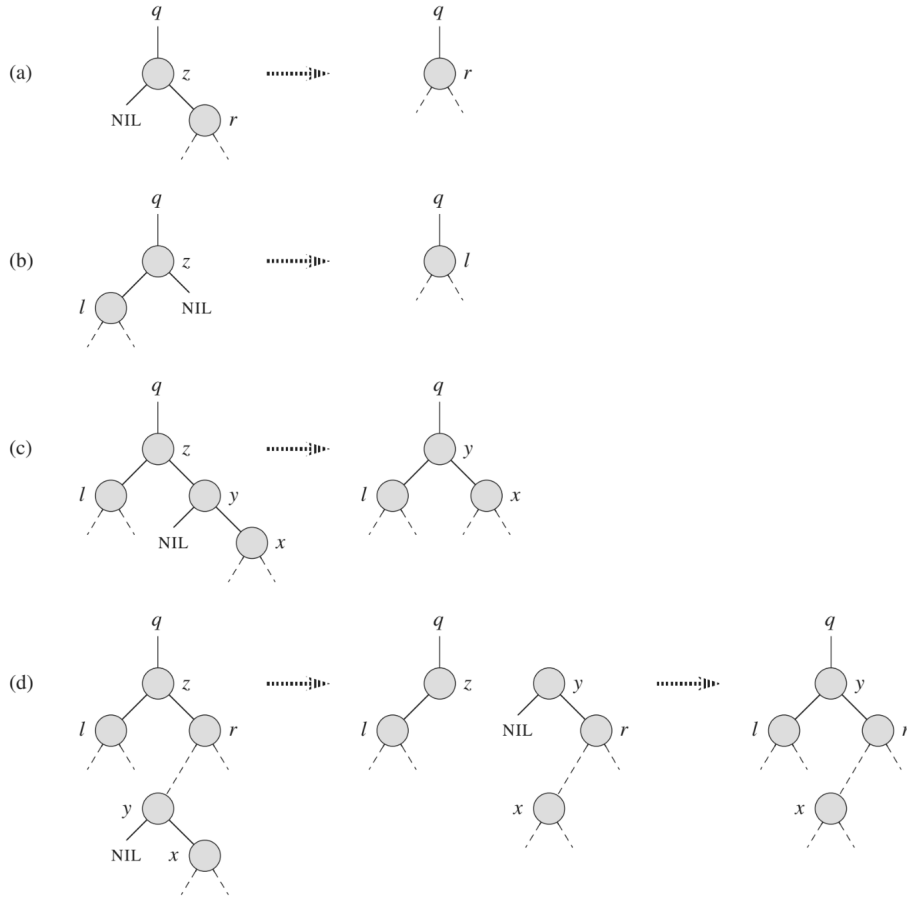


Figure 2: A visualization of the delete cases. **(a)** shows the single-child case for a right child r , and **(b)** for a left child l . **(c)** Two children case, y is z 's successor, so replace z with y . **(d)** Two children case where z 's successor is not its right child, so we must find the smallest key larger than z , key —the minimum of the right subtree.