Further Notes

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Comparing Growth Rates

The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if a_n appears before b_n in the list, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{a_n} = \infty :$$

$$\{\ln^q(n)\} << \{n^p\} << \{n^p \ln^r(n)\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$$

The ordering applies for positive numbers, p, q, r, s, and $b > 1$. [1]

example:

$$\lim_{n\to\infty} \frac{\ln(n^{10})}{0.00001n} = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

example:

$$\lim_{n\to\infty} \frac{n^8 \ln(n)}{n^{8.001}} = \lim_{n\to\infty} \frac{n^p \ln^r(n)}{n^{p+s}} = 0$$

example:

$$\lim_{n\to\infty} \frac{n!}{10^n} = \lim_{n\to\infty} \frac{n!}{b^n} = \infty$$

(Some) Logarithm Rules

For all real $a > 0$, $b > 0$, $c > 0$, and n.

$$
\begin{aligned}
a &= b^{\log_b a} \\
\log_c ab &= \log_c a + \log_c b \\
\log_b a^n &= n * \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
\log_b (1/a) &= -\log_b a \; (= \log_b a^{-1}) \\
\log_b a &= \frac{1}{\log_a b} \\
a^{\log_b c} &= c^{\log_b a}
\end{aligned}
$$

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots$ when $|x| < 1$.

Also, for $x > -1$
$\frac{x}{x+1} \leq \ln(1 + x) \leq x$.

References

[1] B. Gillett W. L. Briggs, L. Cochran. *Calculus: Early Transcendentals, 2nd Edition.* Pearson, 2015.