

Further Notes

Manuel Serna-Aguilera

Comparing Growth Rates

The following sequences are ordered according to increasing growth rates as $n \rightarrow \infty$; that is, if a_n appears before b_n in the list, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \infty :$$

$$\{\ln^q(n)\} \ll \{n^p\} \ll \{n^p \ln^r(n)\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

The ordering applies for positive numbers, p, q, r, s , and $b > 1$. [1]

example:

$$\lim_{n \rightarrow \infty} \frac{\ln(n^{10})}{0.00001n} = \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$$

example:

$$\lim_{n \rightarrow \infty} \frac{n^8 \ln(n)}{n^{8.001}} = \lim_{n \rightarrow \infty} \frac{n^p \ln^r(n)}{n^{p+s}} = 0$$

example:

$$\lim_{n \rightarrow \infty} \frac{n!}{10^n} = \lim_{n \rightarrow \infty} \frac{n!}{b^n} = \infty$$

(Some) Logarithm Rules

For all real $a > 0$, $b > 0$, $c > 0$, and n .

$$a = b^{\log_b a}$$

$$\log_c ab = \log_c a + \log_c b$$

$$\log_b a^n = n * \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a (= \log_b a^{-1})$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \text{ when } |x| < 1.$$

Also, for $x > -1$

$$\frac{x}{x+1} \leq \ln(1+x) \leq x. \quad [1]$$

References

- [1] B. Gillett W. L. Briggs, L. Cochran. *Calculus: Early Transcendentals, 2nd Edition*. Pearson, 2015.