Let Γ be an $n \times n$ game.

$$\Gamma := \begin{bmatrix} (\gamma_{11}^1, \gamma_{11}^2) & \cdots & (\gamma_{1n}^1, \gamma_{1n}^2) \\ \vdots & \ddots & \vdots \\ (\gamma_{n1}^1, \gamma_{n1}^2) & \cdots & (\gamma_{nn}^1, \gamma_{nn}^2) \end{bmatrix}$$

The i −th row denotes the i −th pure strategy of Player 1 (from now on denoted P1). The j −th column denotes the j −th pure strategy of Player 2 (from now on denoted P2). Hence, $(\gamma_{ij}^1, \gamma_{ij}^2) \in \Gamma$ denotes the payoff for P1 & P2 when P1 adopts his i −th pure strategy and P2 chooses her j −th pure strategy.

Without loss of generality, we assume that the utility function of both players is the identity function, in other words, the payoff already represents his and her preferences accurately.

Convention. We will call any $(\gamma_{ij}^1, \gamma_{ij}^2) \in \Gamma$ a *pivot entry of* Γ.

Observation: Every element of Γ will eventually be considered a pivot entry.

Definition 1. Consider any pivot entry. We will say that a pivot entry *strictly dominates* another entry $(\gamma_{kj}^1, \gamma_{ik}^2) \in \Gamma$ if and only if:

i) $\gamma_{ij}^1 > \gamma_{kj}^1 \quad (i \neq k)$ or
ii) $\gamma_{ij}^2 > \gamma_{ik}^2 \quad (j \neq k)$

Definition 2. We will say that a pivot entry is *strictly dominant* if and only if:

i) $\gamma_{ij}^1 > \gamma_{kj}^1 \quad (\forall i \neq k)$ or
ii) $\gamma_{ij}^2 > \gamma_{ik}^2 \quad (\forall j \neq k)$

Definition 3. We will say that a pivot entry is *strictly dominated* by some other entry $(\gamma_{kj}^1, \gamma_{ik}^2) \in \Gamma$ if and only if:

i) $\gamma_{ij}^1 < \gamma_{kj}^1 \quad (i \neq k)$ or
ii) $\gamma_{ij}^2 < \gamma_{ik}^2 \quad (j \neq k)$

Definition 4. We will say that a pivot entry is *strictly dominated* if and only if:

i) $\gamma_{ij}^1 < \gamma_{kj}^1 \quad (\forall i \neq k)$ or
ii) $\gamma_{ij}^2 < \gamma_{ik}^2 \quad (\forall j \neq k)$

Theorem. If all pivot entries meet the following condition, there will be a unique Nash Equilibrium in mixed strategies. Condition: The number of times a pivot entry strictly dominates another entry is the same number of times it is strictly dominated by another entry, and that number is exactly $n - 1$.

Proof of the Theorem.

We will first enunciate two Lemmas that will imply the inexistence of a pure strategy equilibrium. Following Nash's existence theorem (Nash, 1950) we will then focus on the mixed strategy equilibrium and prove it is unique.

Lemma 1. If the condition of the Theorem holds, then given any strictly dominant pivot entry, either condition i) holds or condition ii) holds, but not both simultaneously.

Proof of Lemma 1. Let us suppose this is not the case. Then there must be some pivot entry $(\gamma_{ij}^1, \gamma_{ij}^2) \in \Gamma$ such that:

i) $\gamma_{ij}^1 > \gamma_{kj}^1 \quad (\forall i \neq k)$ and
ii) $\gamma_{ij}^2 > \gamma_{ik}^2 \quad (\forall j \neq k)$

Then the number of times said pivot entry strictly dominates other entries is $2(n-1)$!!!!

Therefore, it must be the case that for every strictly dominant pivot entry, conditions i) and ii) of the definition are mutually exclusive.

Lemma 2. If the condition of the Theorem holds, then one player's best response implies another player's worst response.

Proof of Lemma 2. Let us first point out that any pivot entry makes $2(n-1)$ comparisons. Let $(\gamma_{ij}^1, \gamma_{ij}^2)$ be an arbitrary pivot entry. We first hold constant P2's j −th pure strategy and count the number of times $\gamma_{ij}^1 > \gamma_{kj}^1$ $(i \neq k)$ and the number of times $\gamma_{ij}^1 < \gamma_{kj}^1$ $(i \neq k)$, let's say they occur $\alpha_{ij}^{>}$ and $\alpha_{ij}^{<}$ times respectively. Then we hold constant P1's i −th strategy and count the number of times $\gamma_{ij}^2 > \gamma_{ik}^2$ $(j \neq k)$ and the number of times $\gamma_{ij}^2 < \gamma_{ik}^2$ $(j \neq k)$, let's say they occur $\beta_{ij}^{>}$ and $\beta_{ij}^{<}$ times respectively. The condition can be summed up by saying that for every pivot entry, it must be the case that $\alpha_{ij}^{>} + \beta_{ij}^{>} = \alpha_{ij}^{<} + \beta_{ij}^{<} = n-1$. It follows that for both players, no indifference between payoffs is allowed.

Hence a best response of a player is his or her maximum payoff, holding fixed a given column in the case of P1 and holding fixed a given row in the case of P2. We define a worst response as the minimum payoff of a player, holding fixed a given column in the case of P1 and holding fixed a row in the case of P2.

It is easy to see why one player's best response implies the other's worst response. Consider P1's best response to P2's $j-th$ strategy. Let's say that his best response is given by γ_{ij}^1 for some i. Thus $\gamma_{ij}^1 > \gamma_{kj}^1$ $(i \neq k)$ exactly $n-1$ times. The condition requires that $\alpha_{ij}^{>} + \beta_{ij}^{>} = \alpha_{ij}^{<} + \beta_{ij}^{<} = n-1$, but if $\alpha_{ij}^{>} = n-1$ that implies that $\alpha_{ij}^{<} = 0$ and the only way that the condition can be met is if $\beta_{ij}^{>} = 0$ and $\beta_{ij}^{<} = n-1$, which implies that $\gamma_{ij}^2 < \gamma_{ik}^2$ $(j \neq k)$ exactly $n-1$ times, making j P2's worst response to P1's i −th pure strategy.

It follows from Lemmas 1 and 2 that there cannot be a Nash Equilibrium in pure strategies. If there were one, given that there are no indifferences between payoffs for both players, the Equilibrium would necessarily require both payoffs to be the best response of both players, but we saw that one player's best response implies another player's worst response. Ergo, there cannot be a Nash Equilibrium in pure strategies.

We now turn to Nash's theorem which guarantees the existence of *at least* one equilibrium in mixed strategies for any non-cooperative finite game. We now know that the only possible equilibriums must be in mixed strategies, but we're left to prove the unicity of the equilibrium. First, we will provide a third Lemma.

Lemma 3. If the condition of the Theorem holds, then every strategy for P1 and P2 is a best response exactly one time.

Proof of Lemma 3. Let us assume this is not the case for P1 (the same proof will apply for P2). Then at least one of P1's pure strategies is a best response to two or more pure strategies of P2 (when held constant). Let´s call this strategy for P1 i. Since there are no indifferences between payoffs, that implies that for at least two strategies of P2 (i.e. $j_1, \dots, j_m$ with $m < n$) the strategy i yields the highest payoff for P1 holding constant the strategies $j_1, \dots, j_m$, in other words, $\max\{\gamma^1_{ij_h}, \gamma^1_{kj_h}\} = \gamma^1_{ij_h} \forall h \in \{1, \dots, m\}$. But from Lemma 2, we know that one player's best response is the other player's worst response, therefore, if $\gamma^1_{ij_h}$ is the maximum of the j_h −th column, then $\gamma^2_{ij_h}$ must be the minimum of the i −th row. However, we assumed that there were at least two strategies of P2 for which i was a best response, that means that the i −th row has at least two minimums. Nevertheless, we also assumed that indifferences between payoffs were not allowed!!!!

Therefore, it must be the case that for P1, no strategy is a best response to two or more strategies of P2. Now we must show that every strategy of P1 is a best response once. This is straightforward since every column must have a maximum (recall that no indifferences between payoffs are allowed). Therefore, every strategy for P1 is a best response to one of P2's strategies. Similarly, every strategy of P2 is a best response to one of P1's strategies (the proof is done with the same assumptions for P2 and P1 instead of P1 and P2 respectively). That implies that out of the $n \times n$ pivot entries, n are strictly dominant for P1, n are strictly dominant for P2, no pivot entry is both strictly dominant for P1 and P2 simultaneously, so $n^2 - 2n$ pivot entries are not strictly dominant.

A convenient way to sum up Lemmas 1-3 is to think of Γ as a $n \times n$ square grid. Each pivot entry will be considered a cell of the square grid. Let a strictly dominant pivot entry due to P1's payoff be colored red. Let a strictly dominant pivot entry due to P2's payoff be colored blue. A Nash Equilibrium would occur if a cell of the grid was both colored red and blue (in other words, if we had a purple cell). Lemmas 1 and 2 imply that no purple cells exist. Lemma 3 implies that there will be exactly n red cells, one for each row and n blue cells, one for each column. Furthermore, if we make permutations among columns, it is possible to make the red cells fall in the main diagonal. Likewise, for the blue cells if we made permutations among rows instead.

The importance of Lemma 3 has to do with the fact that if the condition of the Theorem is met, no pure strategy will be strictly (nor weakly) dominated by any other pure strategy, thus avoiding iterated elimination of strictly dominated strategies.

We will now focus on the unicity of the Mixed Strategy Nash Equilibrium (MSNE from now on).

Let us consider the two matrices we can form from Γ regarding P1 and P2's payoffs:

$$\Gamma^1 := \begin{bmatrix} \gamma^1_{11} & \cdots & \gamma^1_{1n} \\ \vdots & \ddots & \vdots \\ \gamma^1_{n1} & \cdots & \gamma^1_{nn} \end{bmatrix}$$

$$\Gamma^2 := \begin{bmatrix} \gamma^2_{11} & \cdots & \gamma^2_{1n} \\ \vdots & \ddots & \vdots \\ \gamma^2_{n1} & \cdots & \gamma^2_{nn} \end{bmatrix}$$

Let us also define two column vectors, each of size $n \times 1$ containing the probability that P1 and P2 plays his or her i −th or j −th strategy:

$$\sigma^1 := \begin{bmatrix} \sigma_1^1 \\ \vdots \\ \sigma_n^1 \end{bmatrix}$$

$$\sigma^2 := \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_n^2 \end{bmatrix}$$

We wish to obtain the expected utility of each strategy for each player, since that is the first step to compute a MSNE. Using our notation:

$$\Gamma^1 \cdot \sigma^2 = \mathbb{E}U^1$$

Where $\mathbb{E}U^1$ is an $n \times 1$ column vector containing the expected utility of each strategy for P1.

We also have

$$(\sigma^2)' \cdot \Gamma^2 = \mathbb{E}U^2$$

Where $(\mathbb{E}U^2)'$ is the transpose of $\mathbb{E}U^2$, which is a $1 \times n$ row vector containing the expected utility of each strategy for P2.

The next step in the standard procedure to compute a MSNE is to form a linear system and solve for the unknowns, i.e. the probabilities.

To take a better look at the coefficients involved (since we will eventually equate expected utilities and form an $(n-1) \times (n-1)$ linear system to be solved), we will expand the notation, first for P1, then for P2.

P1 faces the following scenario:

$$\begin{array}{ccccc} \sigma_1^2 & \sigma_2^2 & \cdots & \sigma_{n-1}^2 & 1 - \sum_{i=1}^{n-1} \sigma_i^2 \\ \hline \gamma_{11}^1 & \gamma_{12}^1 & \cdots & \gamma_{1,n-1}^1 & \gamma_{1n}^1 \\ \gamma_{21}^1 & \gamma_{22}^1 & \cdots & \gamma_{2,n-1}^1 & \gamma_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{n-1,1}^1 & \gamma_{n-1,2}^1 & \cdots & \gamma_{n-1,n-1}^1 & \gamma_{n-1,n}^1 \\ \gamma_{n1}^1 & \gamma_{n2}^1 & \cdots & \gamma_{n,n-1}^1 & \gamma_{nn}^1 \end{array}$$

We can now represent his expected utility for each of his n strategies as:

$$\mathbb{E}U_1^1 = \gamma_{11}^1 \cdot \sigma_1^2 + \gamma_{12}^1 \cdot \sigma_2^2 + \cdots + \gamma_{1,n-1}^1 \cdot \sigma_{n-1}^2 + \gamma_{1n}^1 \cdot \left(1 - \sum_{i=1}^{n-1} \sigma_i^2\right)$$

$$\mathbb{E}U_2^1 = \gamma_{21}^1 \cdot \sigma_1^2 + \gamma_{22}^1 \cdot \sigma_2^2 + \cdots + \gamma_{2,n-1}^1 \cdot \sigma_{n-1}^2 + \gamma_{2n}^1 \cdot \left(1 - \sum_{i=1}^{n-1} \sigma_i^2\right)$$

$$\vdots$$

$$\mathbb{E}U_{n-1}^1 = \gamma_{n-1,1}^1 \cdot \sigma_1^2 + \gamma_{n-1,2}^1 \cdot \sigma_2^2 + \cdots + \gamma_{n-1,n-1}^1 \cdot \sigma_{n-1}^2 + \gamma_{n-1,n}^1 \cdot \left(1 - \sum_{i=1}^{n-1} \sigma_i^2\right)$$

$$\mathbb{E}U_n^1 = \gamma_{n1}^1 \cdot \sigma_1^2 + \gamma_{n2}^1 \cdot \sigma_2^2 + \cdots + \gamma_{n,n-1}^1 \cdot \sigma_{n-1}^2 + \gamma_{nn}^1 \cdot \left(1 - \sum_{i=1}^{n-1} \sigma_i^2\right)$$

Recall that in the procedure to determine a MSNE, we next equate two expected utilities at a time, thus forming a $(n-1) \times (n-1)$ linear system:

$$\mathbb{E}U_1^1 = \mathbb{E}U_2^1 \coloneqq$$

$$\sigma_1^2(\gamma_{11}^1 - \gamma_{21}^1 + \gamma_{2n}^1 - \gamma_{1n}^1) + \sigma_2^2(\gamma_{12}^1 - \gamma_{22}^1 + \gamma_{2n}^1 - \gamma_{1n}^1) + \cdots + \sigma_{n-1}^2\left(\gamma_{1,n-1}^1 - \gamma_{2,n-1}^1 + \gamma_{2n}^1 - \gamma_{1n}^1\right) = \gamma_{2n}^1 - \gamma_{1n}^1$$

$$\vdots$$

$$\mathbb{E}U_{n-1}^1 = \mathbb{E}U_n^1 \coloneqq$$

$$\sigma_1^2\left(\gamma_{n-1,1}^1 - \gamma_{n1}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1\right) + \sigma_2^2\left(\gamma_{n-1,2}^1 - \gamma_{n2}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1\right) + \cdots + \sigma_{n-1}^2\left(\gamma_{n-1,n-1}^1 - \gamma_{n,n-1}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1\right) = \gamma_{nn}^1 - \gamma_{n-1,n}^1$$

Now we can sum up the $(n-1) \times (n-1)$ linear system as:

$$\Sigma_\Gamma^1 \cdot \sigma_{-n}^2 = K^1$$

Where $\Sigma_\Gamma^1 \coloneqq$

$$\begin{array}{|cccc|}
\hline
(\gamma_{11}^1 - \gamma_{21}^1 + \gamma_{2n}^1 - \gamma_{1n}^1) & (\gamma_{12}^1 - \gamma_{22}^1 + \gamma_{2n}^1 - \gamma_{1n}^1) & \cdots & (\gamma_{1,n-1}^1 - \gamma_{2,n-1}^1 + \gamma_{2n}^1 - \gamma_{1n}^1) \\
\vdots & \vdots & \ddots & \vdots \\
(\gamma_{n-1,1}^1 - \gamma_{n1}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1) & (\gamma_{n-1,2}^1 - \gamma_{n2}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1) & \cdots & (\gamma_{n-1,n-1}^1 - \gamma_{n,n-1}^1 + \gamma_{nn}^1 - \gamma_{n-1,n}^1) \\
\hline
\end{array}$$

$K^1 \coloneqq \begin{bmatrix} \gamma_{2n}^1 - \gamma_{1n}^1 \\ \vdots \\ \gamma_{nn}^1 - \gamma_{n-1,n}^1 \end{bmatrix}$ and $\sigma_{-n}^2 = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{n-1}^2 \end{bmatrix}$

One way to prove unicity would be to prove that Σ_Γ^1 is a nonsingular matrix. If that were the case, then an analytic solution would exist, for instance, using Cramer's rule (although there are more efficient computational methods to solve the system). What's left now is to prove that Σ_Γ^1 is a nonsingular matrix.

We refer to the invertible matrix theorem to prove this is the case. To help visualize and simplify the notation of Σ_Γ^1, let us note first that the entries of Σ_Γ^1 consist of the sum of two differences:

$$
\begin{array}{|cccc|}
\hline
\left(\left(\gamma^1_{11} - \gamma^1_{21}\right) + \left(\gamma^1_{2n} - \gamma^1_{1n}\right)\right) & \left(\left(\gamma^1_{12} - \gamma^1_{22}\right) + \left(\gamma^1_{2n} - \gamma^1_{1n}\right)\right) & \cdots & \left(\left(\gamma^1_{1,n-1} - \gamma^1_{2,n-1}\right) + \left(\gamma^1_{2n} - \gamma^1_{1n}\right)\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\left(\gamma^1_{n-1,1} - \gamma^1_{n1}\right) + \left(\gamma^1_{nn} - \gamma^1_{n-1,n}\right)\right) & \left(\left(\gamma^1_{n-1,2} - \gamma^1_{n2}\right) + \left(\gamma^1_{nn} - \gamma^1_{n-1,n}\right)\right) & \cdots & \left(\left(\gamma^1_{n-1,n-1} - \gamma^1_{n,n-1}\right) + \left(\gamma^1_{nn} - \gamma^1_{n-1,n}\right)\right) \\
\hline
\end{array}
$$

Note that each row of Σ^1_Γ only has payoffs of the two strategies whose expected utilities are being equated. Thus, if the expected utility of strategies 1 and 2 for P1 are being equated (as is the case of the first row of Σ^1_Γ), then only payoffs from rows 1 and 2 of Γ^1 are being involved in the computation of the coefficients of $\sigma^2_1, \dots, \sigma^2_{n-1}$