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Sensitivity analysis of Matching Pennies game

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ABSTRACT

In this paper, we have discussed the results of sensitivity analysis in a payoff matrix of the Matching Pennies game. After representing the game as a LP model, the sensitivity analysis of the elements of the payoff matrix is presented. The game value and the optimal strategies for different values of parameters are determined and compared.

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1. Introduction

In many strategic situations, one person prefers a "match", whereas the other prefers a "mismatch". As an example, a soccer goalie prefers to dive in the same direction as the penalty kick goes, whereas the kicker has opposite preferences. As a result, there is no Nash equilibrium in pure strategies. Similarly, while a business manager hopes that an audit occurs only in cases when preparations have been made, the auditor prefers that audits catch sloppy record keepers. These games can be approximated as zero-sum Matching Pennies games (MP)[1] [1]. Indeed, the Matching Pennies game, a simplified version of the more popular Rock, Paper, Scissors, schematically represents competitions between organisms which endeavor to predict each other's behavior [2].

As in Rock, Paper, Scissors, in order for a player to play optimally in iterated MP competitions, he/she should produce unpredictable sequences of choices and attempt to detect nonrandomness in the opponent's choices. This is the Nash equilibrium of MP: "Against a choice-randomizing opponent, the best strategy is to randomize one's own choices" [2].

The Matching Pennies game has been studied and analyzed for decades. For instance, in the literature on evolutionary games, the instability of the equilibrium point (which is a mixed strategy equilibrium) in MP is well known[2] [3]. Also this game has been analyzed in the realm of behavioral game theory.[3] It has been shown that humans [4] and other primates [5] can learn to compete efficiently in MP [2].

Although suboptimal sensitivity to one's own payoffs in mixed strategy games has been observed in human players [6,7], the Nash theorem [8–10] guarantees that there exists a unique optimal solution (or equivalently a unique equilibrium point) for any n-person game, including MP.

In this paper, by introducing an analytical method, we will apply a sensitivity analysis to this special class of zero-sum matrix games. Although we have focused on a particular version of MP, introduced first by vos Savant [11], to avoid

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1 Examples also abound in everyday life and even literature, as documented in [15], which cites passages from the works of William Faulkner, Edgar Allen Poe, and Sir Arthur Conan Doyle.

2 See [3] for a thorough analysis of the dynamics of the Matching Pennies game.

3 For example see [2].

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complication in more general cases, the method used in this paper can be easily generalized to any matrix game. In fact, this method is an effective tool to deal with practical applications of game theory. By using this method, one can realize whether he/she should or should not participate in a game, even when the parameters are random variables with known probability distributions.

Our most practical finding is the calculation of a critical value for each entry in the payoff matrix. This value determines an upper (lower) bound for the increase (decrease) in each of the payoffs so that the game favors the same player.[4]

The MP introduced by vos Savant [11] can be described as follows. "The game has two players, each one given a fair coin labeled with a head and a tail. Then, each player will show either a head or a tail. If both players show a head, then the second player (Player #2) wins \$3. If both players show a tail, then Player #2 wins \$1. If one player shows a head and the other shows a tail, then Player #1 wins \$2". According to the Fundamental Theorem of Matrix Games, this game has at least one optimal strategy for each player which can be found by linear programming (LP).

We found the optimal strategies as well as the value of the game after changing the payoff of several outcomes in the payoff matrix. Then we compared the results with those of Jacobson and Shyryayev [12] and found a close similarity in the results, although ours were more general.

2. Formulating MP as a matrix game

According to the above-mentioned description, MP can be modeled as a zero-sum matrix game by the following payoff matrix[5]:

		Player #2 H	Player #2 T
Player #1	H	−3	2
	T	2	−1

where **H** represents a head and **T** represents a tail. Each number refers to player #1's gain. A negative number indicates a loss for player #1 or equivalently a gain for player #2, since the game is zero-sum.

3. Formulating MP as a LP problem and finding optimal strategies

Each matrix game can be formulated as a LP problem in several ways. We have used the procedure discussed in [13]:

3.1. Solving a matrix game

Imagine a zero-sum matrix game as follows:

		Column Player c_1	c_2	…	c_m
Row Player	r_1	a_{11}	a_{12}	…	a_{1m}
	r_2	a_{21}	a_{22}	…	a_{2m}
	…	…	…	…	…
	r_n	a_{n1}	a_{n2}	…	a_{nm}

where $c_j(r_i)$ is the probability of choosing strategy $j(i)$ by the Column Player (Row Player).

First, we check for the existence of saddle points. If there is one, you can solve the game by selecting each player's optimal pure strategy. Otherwise, continue with the following steps:

Step 1: Reduce the payoff matrix by dominance.

Step 2: Add a fixed number k to each of the entries. Therefore, all entries become non-negative.

Step 3: Solve the corresponding LP problem using the simplex method.[6]

$$\textit{maximize } p = x_1 + x_2 + \cdots + x_m$$
subject to:
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \le 1$$

4 See Section 5 for an accurate definition of the "critical value".

5 See [14] for the detailed procedure of representing a matrix game.

6 See [16] for a complete discussion of the simplex method.

Table 1
Final simplex tableau in solving MP.

	p	x_1	x_2	s_1	s_2	
p	1	0	0	0.12	0.2	0.32
x_2	0	0	1	0.2	0	0.2
x_1	0	1	0	−0.08	0.2	0.12

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \leq 1$$
$$\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \leq 1$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad \ldots \quad x_m \geq 0.$$

We know that the decision variable x_j is related to c_j and e (value of the game) as follows:

$$x_j = \frac{c_j}{e}.$$

Step 4: Find the optimal strategies and the expected value as follows:

Column strategy:

1. Express the solution to the LP problem as a column vector (we call this vector $\boldsymbol{X}$).
2. Normalize the solution by dividing each entry of the solution vector by the optimal value of p, say p^* (which is also the sum of the values of the decision variables). It means that $c_j = \frac{x_j}{p^*}$.
3. Insert zeros in positions corresponding to the columns deleted during reduction.

Row strategy:

1. List the entries under the slack variables in the row representing objective function in the final simplex tableau (these are called shadow prices) in vector form (we call this vector $\boldsymbol{Y}$).
2. Normalize by dividing each entry of the solution vector by the sum of the entries.
3. Insert zeros in positions corresponding to the rows deleted during reduction.

Value of the game: $e = \frac{1}{p^*} - k$

It must be noted that the corresponding LP problems for column and row players are dual to each other. This fact has been used in the above procedure.

Now we aim to solve MP by LP. If we add a number k, say 3, to the entries of the MP matrix, we have

		Player #2	
		H	T
Player #1	H	0	5
	T	5	2

and the corresponding LP problem is formulated as follows:

$$\textit{maximize } p = x_1 + x_2$$
$$\textit{subject to}:$$
$$0x_1 + 5x_2 \leq 1$$
$$5x_1 + 2x_2 \leq 1$$
$$x_1 \geq 0 \quad x_2 \geq 0.$$

The final simplex tableau after applying simplex method to solve this LP problem is presented in Table 1.

Note that s_1 and s_2 are the slack variables of the two constraints respectively.[7]

Therefore:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.20 \end{bmatrix}, \qquad Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.20 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.12/0.32 \\ 0.20/0.32 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}, \qquad \begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \frac{0.12}{0.32} & \frac{0.20}{0.32} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

and the value of the game is calculated as follows:

$$e = \frac{1}{p^*} - k = \frac{1}{0.32} - 3 = 0.125 = \frac{1}{8}.$$

7 See [16] for a complete discussion of the simplex method.

4. Sensitivity analysis of MP

In this section, we change the elements of the MP payoff matrix to examine the effects of these changes to the value of the game as well as its optimal strategies.

Case 1: Change the payoff of outcome **H**–**H**

After adding a number α to the payoff of outcome **H**–**H**, the new payoff matrix would become as follows:

		Player #2 H	Player #2 T
Player #1	H	$-3+\alpha$	2
	T	2	-1

First, we should note a very important point. If $-3+\alpha \geq 2$, which means $\alpha \geq 5$, then strategy **H** would dominate strategy **T** for Player #1. Additionally, in this case strategy **T** would dominate strategy **H** for Player #2. As a result, both players would have pure optimal strategies and the outcome of the game would be **H**–**T**, with a gain of 2 to Player #1. In other words, the entry associated with the outcome **H**–**T** would become a saddle entry for $\alpha \geq 5$, which in turn means that strategies **H** and **T** would be optimal pure strategies for Player #1 and Player #2, respectively [14]. Therefore, we only discuss the cases in which $\alpha < 5$.[8] Therefore we have excluded the cases in which there is a dominated strategy and, as a result, we can skip step 1, which is the reduction by dominance.

Now we consider two situations based on the value of $(-3+\alpha)$ in comparison with -1, since we need to choose an appropriate value for k to be added to all entries and make them non-negative.

Case 1.1: $-3+\alpha \leq -1$

$$-3+\alpha \leq -1 \Rightarrow \alpha \leq 2.$$

For $\alpha \leq 2$ we can choose k to be

$$k = -(-3+\alpha) = 3-\alpha.$$

Now we add $k = 3-\alpha$ to all entries of the game matrix so that the entries become all non-negative. The resulting game matrix would be:

		Player #2 H	Player #2 T
Player #1	H	0	$5-\alpha$
	T	$5-\alpha$	$2-\alpha$

and the corresponding LP problem would be:

$$\begin{aligned}
&\textit{maximize } p = x_1 + x_2\\
&\textit{subject to}:\\
&0x_1 + (5-\alpha)x_2 \leq 1\\
&(5-\alpha)x_1 + (2-\alpha)x_2 \leq 1\\
&x_1 \geq 0 \quad x_2 \geq 0.
\end{aligned}$$

Now, we can directly calculate the new coefficients in the last simplex tableau as follows:

$$x_1 \textit{ coefficients}: \bar{x}_1 = \begin{bmatrix}0\\5\end{bmatrix} \to \begin{bmatrix}0\\5-\alpha\end{bmatrix} \qquad x_2 \textit{ coefficients}: \bar{x}_2 = \begin{bmatrix}5\\2\end{bmatrix} \to \begin{bmatrix}5-\alpha\\2-\alpha\end{bmatrix}.$$

Therefore

$$\Delta\bar{x}_1 = \begin{bmatrix}0\\-\alpha\end{bmatrix} \qquad \Delta\bar{x}_2 \begin{bmatrix}-\alpha\\-\alpha\end{bmatrix}.$$

According to the following notation, we will have:

General notation						
	p	x_1	x_2	s_1	s_2	
p	1	$p_1 - c_1$	$p_2 - c_2$	y_1	y_2	y_0
x_1	0	a_{11}	a_{12}	s_{11}	s_{12}	b_1
x_2	0	a_{21}	a_{22}	s_{21}	s_{22}	b_2

8 Also note that if $\alpha \geq -3$, then the definition of MP no longer holds; because according to the definition, Player #2 should win if **H**–**H** is the outcome, as discussed in the introduction section. But this deviation from the definition does not affect our method for analyzing the game.

Table 2
The initial revised simplex tableau in solving the perturbed MP.

	p	x_1	x_2	s_1	s_2	
p	1	-0.2α	-0.32α	0.12	0.2	0.32
x_2	0	0	$1-0.2\alpha$	0.2	0	0.2
x_1	0	$1-0.2\alpha$	-0.12α	-0.08	0.2	0.12

Table 3
The final revised simplex tableau in solving the perturbed MP.

	p	x_1	x_2	s_1	s_2	
p	1	0	0	$\frac{0.12}{(1-0.2\alpha)^2}$	$\frac{0.2}{(1-0.2\alpha)}$	$\frac{-0.04\alpha+0.32}{(1-0.2\alpha)^2}$
x_2	0	0	1	$\frac{0.2}{(1-0.2\alpha)}$	0	$\frac{0.2}{(1-0.2\alpha)}$
x_1	0	1	0	$\frac{0.04\alpha-0.08}{(1-0.2\alpha)^2}$	$\frac{0.2}{(1-0.2\alpha)}$	$\frac{0.12}{(1-0.2\alpha)^2}$

$$\Delta(p_i - c_i) = -\Delta c_i + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \Delta\bar{x}_i \quad i = 1, 2$$
$$\Delta a_{i,j} = \begin{bmatrix} s_{i1} & s_{i2} \end{bmatrix} \Delta\bar{x}_j \quad i, j = 1, 2.$$

In order to apply sensitivity analysis, we need to calculate the above values. Hence, we have:

$$\Delta(p_1 - c_1) = -0 + \begin{bmatrix} 0.12 & 0.2 \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} = -0.2\alpha$$
$$\Delta(p_2 - c_2) = -0 + \begin{bmatrix} 0.12 & 0.2 \end{bmatrix} \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix} = -0.32\alpha$$
$$\Delta a_{11} = \begin{bmatrix} 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} = 0$$
$$\Delta a_{21} = \begin{bmatrix} -0.08 & 0.2 \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} = -0.2\alpha$$
$$\Delta a_{12} = \begin{bmatrix} 0.2 & 0 \end{bmatrix} \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix} = -0.2\alpha$$
$$\Delta a_{22} = \begin{bmatrix} -0.08 & 0.2 \end{bmatrix} \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix} = -0.12\alpha.$$

The revised simplex tableau is presented in Table 2.

Now we examine this revised tableau to see under which values of α we can improve p. First, we should compose the unique vector under the basic variables x_1 and x_2. The resulting tableau is presented in Table 3.

Since $1 - 0.2\alpha > 0$, it is easy to see that all coefficients in the objective function row and all right-hand side coefficients are non-negative, and we have:

$$p^* = \frac{-0.04\alpha + 0.32}{(1 - 0.2\alpha)^2}$$

and the value of the game would be

$$e = \frac{1}{p^*} - k = \frac{(1 - 0.2\alpha)^2}{-0.04\alpha + 0.32} - (3 - \alpha) = \frac{0.04 + 0.04\alpha}{-0.04\alpha + 0.32} = \frac{1 + \alpha}{8 - \alpha}.$$

It is easy to investigate that if $\alpha > 0$, then the value of the game would be greater than its value before changing the payoff matrix. In other words, we have

$$new\ value = \frac{1 + \alpha}{8 - \alpha} > \frac{1}{8} = old\ value \Leftrightarrow \alpha > 0.$$

It must be noted that this conclusion is valid for $\alpha \leq 2$.

Since $\alpha > 0$, it means that Player #1 will gain more if a particular outcome (i.e. **H**–**H**) arises. Hence the greater value of the game, which is the expected value that Player #1 will gain.

Also we have

$$new\ value = \frac{1 + \alpha}{8 - \alpha} > 0 \Leftrightarrow \alpha > -1.$$

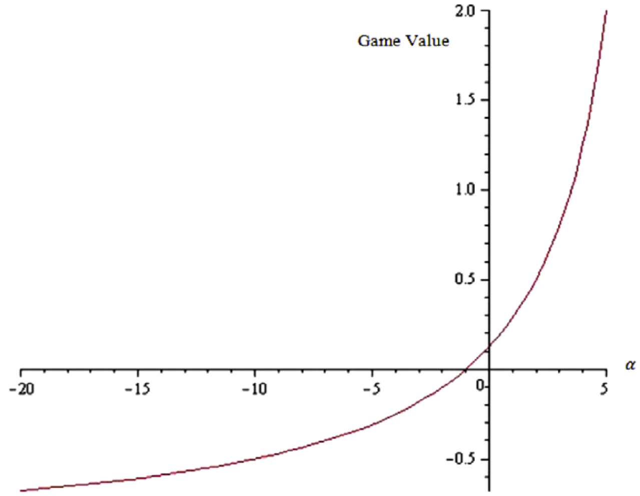

Fig. 1. Game value for different values of $\alpha < 5$ $\left(e = \frac{1+\alpha}{8-\alpha}\right)$.

For $\alpha \leq 2$, the game favors Player #1 if $\alpha > -1$. It means that for $-1 < \alpha \leq 2$, the game favors Player #1.

Now we can calculate the new optimal strategies as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1/p^* \\ x_2/p^* \end{bmatrix} = \begin{bmatrix} 0.12/(-0.04\alpha + 0.32) \\ (-0.04\alpha + 0.2)/(-0.04\alpha + 0.32) \end{bmatrix} = \begin{bmatrix} 3/(8-\alpha) \\ (5-\alpha)/(8-\alpha) \end{bmatrix}$$

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \dfrac{y_1}{y_1+y_2} & \dfrac{y_2}{y_1+y_2} \end{bmatrix} = \begin{bmatrix} \dfrac{0.12}{-0.04\alpha + 0.32} & \dfrac{-0.04\alpha + 0.2}{-0.04\alpha + 0.32} \end{bmatrix} = \begin{bmatrix} \dfrac{3}{8-\alpha} & \dfrac{5-\alpha}{8-\alpha} \end{bmatrix}.$$

It is easy to see that in this case and the following similar cases, all c_1, c_2, r_1 and r_2 are non-negative (for $\alpha \leq 2$ in Case 1.1); and since $c_1 + c_2 = 1$ and $r_1 + r_2 = 1$, each one must be less than 1. It makes sense, since they are probabilities with which Player #1 and Player #2 show head and tail, respectively.

It is clear that for $\alpha = 0$ the above results are the same as the results obtained before changing the payoff matrix.

Now we return to our game matrix to examine the second situation in comparing $(-3 + \alpha)$ with -1.

Case 1.2: $-3 + \alpha \geq -1$

$$-3 + \alpha \geq -1 \Rightarrow \alpha \geq 2.$$

For $\alpha \geq 2$ we can choose k to be

$$k = -(-1) = 1.$$

After adding k to all entries of the payoff matrix, the resulting game matrix would be:

		Player #2	
		H	T
Player #1	H	$\alpha - 2$	3
	T	3	0

By applying the same procedure as Case 1.1, the new value of the game and the new optimal strategies will not change.

As a result, we conclude that in general, the game favors Player #1 for $\alpha > -1$, and favors Player #2 for $\alpha < -1$, and is fair for $\alpha = -1$.

Interestingly enough, in both cases $\alpha \leq 2$ and $\alpha \geq 2$, the value of p is optimal in the simplex tableau. In fact this is due to the fact that all necessary coefficients in the simplex tableau are non-negative. Now assume that one of these coefficients is negative. It means that the basic variables would change and so at least one of x_1 or x_2 would become non-basic and therefore would have a value of 0. But it means that the corresponding probability (i.e. c_1 or c_2) would equal to 0. Thus Player #2 would never choose that particular strategy (i.e. **H** or **T**), which in turn means that that strategy would be dominated by the other strategy (it is easy to see that in Case 1, it is only possible that strategy **T** dominates strategy **H** for Player #2 for $\alpha \geq 5$), and it generates a contradiction, since we have excluded the cases in which strategy **T** could dominate strategy **H** for Player #2 (by choosing $\alpha < 5$). So it is reasonable that none of the necessary coefficients in the simplex tableau be negative.

The game value and the optimal strategies for both players for different values of $\alpha < 5$ are shown in Figs. 1–3.

It must be noted that as α approaches $-\infty$, Player #1, naturally avoiding the high risk of encountering outcome **H**–**H**, tends to choose his/her second strategy (i.e. r_2 approaches 1, as shown in Fig. 2) and conclusively, Player #2 also tends to

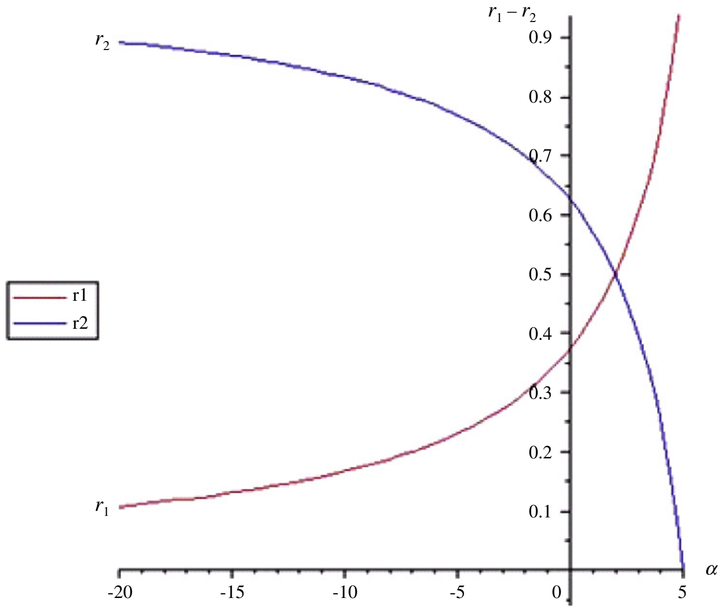

Fig. 2. Optimal strategies for Player #1 for different values of $\alpha < 5$ $\left(r_1 = \frac{3}{8-\alpha}, r_2 = \frac{5-\alpha}{8-\alpha}\right)$.

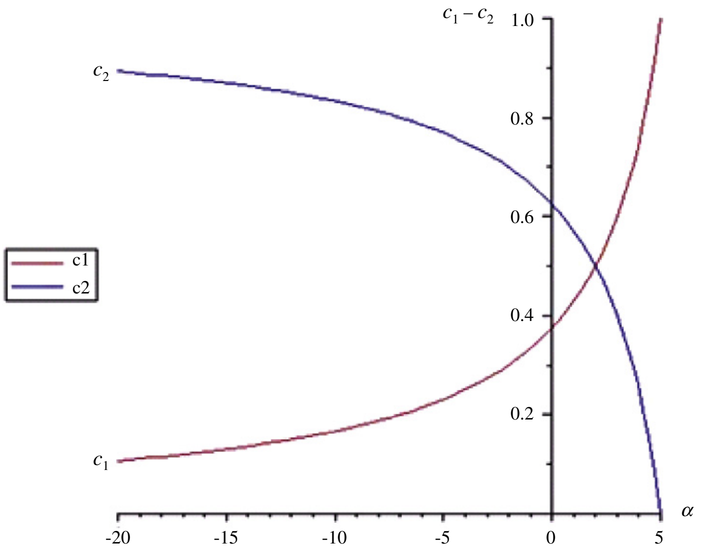

Fig. 3. Optimal strategies for Player #2 for different values of $\alpha < 5$ $\left(c_1 = \frac{3}{8-\alpha}, c_2 = \frac{5-\alpha}{8-\alpha}\right)$.

choose his/her second strategy to maximize his/her gain (i.e. c_2 approaches 1, as shown in Fig. 3), and as a result, the game value approaches -1 (as shown in Fig. 1), which is the payoff of outcome **T**–**T**.

Also by increasing α, Player #1 hopes to minimize his/her loss, with a conservative approach, by choosing his/her first strategy (i.e. r_1 approaches 1, as shown in Fig. 2) and Player #2, since $-3+\alpha < 2$ for all $\alpha < 5$, would in this case be more likely to choose his/her first strategy (i.e. $c_1$approaches 1, as shown in Fig. 3), and therefore, the game value would approach the payoff of outcome **H**–**H** (as shown in Fig. 1).

The above description is a logical reason for why $r_1 = c_1$ and $r_2 = c_2$ for all $\alpha < 5$, since Player #2 would change his/her strategies, in the same direction, according to his/her perception of the behavior of Player #1.[9] This conclusion (which is in

[9] Note that "conservative behavior of all players" is a basic assumption in finding the equilibrium point in a noncooperative game, which results in the max–min and min–max strategies for the row player and column player, respectively.

the realm of behavioral game theory) is to some extent supported by the results of the experimental research by Sanabria and Thrailkill [2].[10]

Case 2: Change the payoff of outcome H–T (or T–H)

Based on the payoff of the game, the analysis of changing the payoffs of outcomes **H**–**T** and **T**–**H** are similar, and we will only examine a change in the payoff of outcome **H**–**T**.

After adding a number β to the payoff of outcome **H**–**T**, the new payoff matrix would become as follows:

		Player #2 H	Player #2 T
Player #1	H	-3	$2+\beta$
	T	2	-1

First, we should note that if $2+\beta \leq -1$, which means $\beta \leq -3$, then strategy **T** would dominate strategy **H** for Player #1. Additionally if $2+\beta \leq -3$, which means $\beta \leq -5$, then strategy **T** would dominate strategy **H** for Player #2. As a result, for $\beta \leq -5$, both players would have pure optimal strategies and the outcome of the game would be **T**–**T**, with a gain of -1 to Player #1. For $-5 \leq \beta \leq -3$, Player #1 would choose his/her optimal strategy, which of course is strategy **T**, and Player #2 logically would choose strategy **T**. Thus, the outcome of the game for $\beta \leq -3$ would always be outcome **T**–**T**. Therefore, we only discuss the cases in which $\beta > -3$.[11] As a result, we have excluded the cases in which there is a dominated strategy.

Now for $\beta > -3$ we have $2+\beta > -1 > -3$ and we can choose $k = 3$.

After adding k, the resulting game matrix would be:

		Player #2 H	Player #2 T
Player #1	H	0	$5+\beta$
	T	5	2

By solving the LP model, we have

$$p^* = \frac{0.32+0.04\beta}{1+0.2\beta}$$

and the value of the game would be

$$e = \frac{1}{p^*} - k = \frac{1+0.2\beta}{0.32+0.04\beta} - 3 = \frac{0.04+0.08\beta}{0.32+0.04\beta} = \frac{1+2\beta}{8+\beta}.$$

It is easy to investigate that if $\beta > 0$, then the value of the game would be greater than its value before changing the payoff matrix for $\beta > -3$.

Also for $\beta > -3$ we have

$$new\ value = \frac{1+2\beta}{8+\beta} > 0 \Leftrightarrow \beta > -0.5.$$

For $\beta > -0.5$, the game is unfair and favors Player #1; and for $-3 < \beta < -0.5$, the game is unfair and favors Player #2 as for $\beta \leq -3$. *We conclude that in general, the game favors Player #1 for* $\beta > -0.5$, *and favors Player #2 for* $\beta < -0.5$, *and is fair for* $\beta = -0.5$.

Now the new optimal strategies are as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (3+\beta)/8+\beta \\ 5/(8+\beta) \end{bmatrix}$$

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{8+\beta} & \frac{5+\beta}{8+\beta} \end{bmatrix}.$$

It is clear that for $\beta = 0$ the above results are the same as the results obtained before changing the payoff matrix.

Again interestingly enough, for $\beta > -3$, the value of p is optimal in the simplex tableau.

In Figs. 4–6, the value of the game and the optimal strategies of both players are shown for different values of $\beta > -3$.

It must be noted that although by increasing β, the gain of outcome **H**–**T** for Player #1 increases unlimitedly (see Fig. 4), the conservative approach of Player #1 accompanied with the negative gain of outcome **H**–**H**, -3, prevents him/her from sticking to his/her first strategy. Therefore, he/she prefers to minimize his/her loss by turning to his/her second strategy

10 Note that an obvious reason for why $r_1 = c_1$ and $r_2 = c_2$ is the symmetry in the game in Case 1.

11 Also note that if $\beta \leq -2$, then the definition of MP no longer holds; but this problem does not affect our method for analyzing the game.

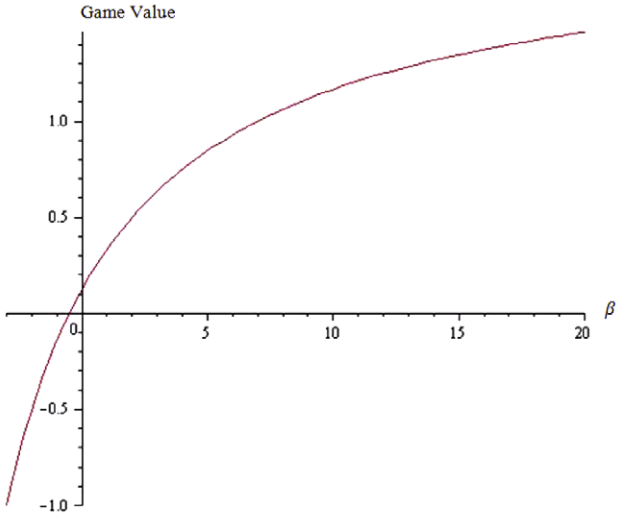

Fig. 4. Game value for different values of $\beta > -3 \left(e = \frac{1+2\beta}{8+\beta}\right)$.

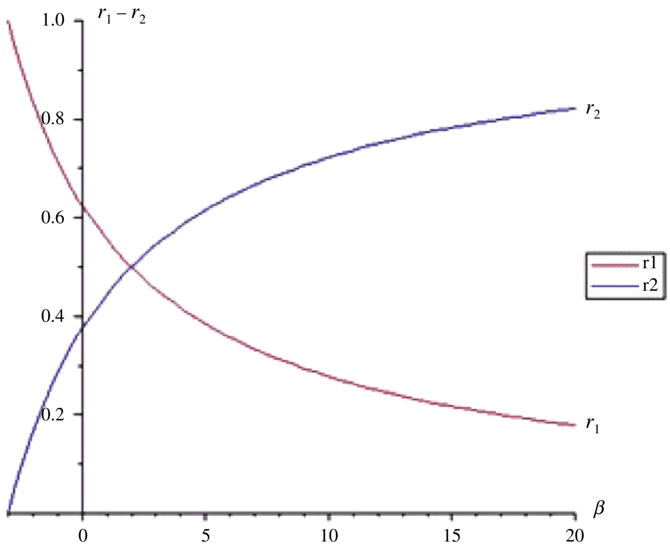

Fig. 5. Optimal strategies for Player #1 for different values of $\beta > -3 \left(r_1 = \frac{3}{8+\beta}, r_2 = \frac{5+\beta}{8+\beta}\right)$.

(i.e. r_2 approaches 1, as shown in Fig. 5). By the same conclusion for Player #2, as β increases, he/she avoids the high risk of encountering outcome **H**–**T** and he/she will return to his/her first strategy (i.e. c_1 approaches 1, as shown in Fig. 6).

Also as β decreases, Player #2 conservatively prefers to guarantee a negative payoff (which means a gain for him/her) by turning to his/her second strategy (i.e. c_2 approaches 1, as shown in Fig. 6) and Player #1, since $2 + \beta > -1$ for all $\beta > -3$, would choose his/her first strategy (i.e. r_1 approaches 1, as shown in Fig. 5) and the game value would approach the payoff of outcome **H**–**T** (see Fig. 4).

Case 3: Change in the payoff of outcome T–T

After adding a number γ to the payoff of outcome **T**–**T**, the new payoff matrix would become as follows:

		Player #2: H	Player #2: T
Player #1	H	-3	2
	T	2	$-1+\gamma$

First, we should note that if $-1 + \gamma \geq 2$, which means $\gamma \geq 3$, then strategy **T** would dominate strategy **H** for Player #1. Furthermore, in this case, strategy **H** would dominate strategy **T** for Player #2. As a result, for $\gamma \geq 3$, both players would

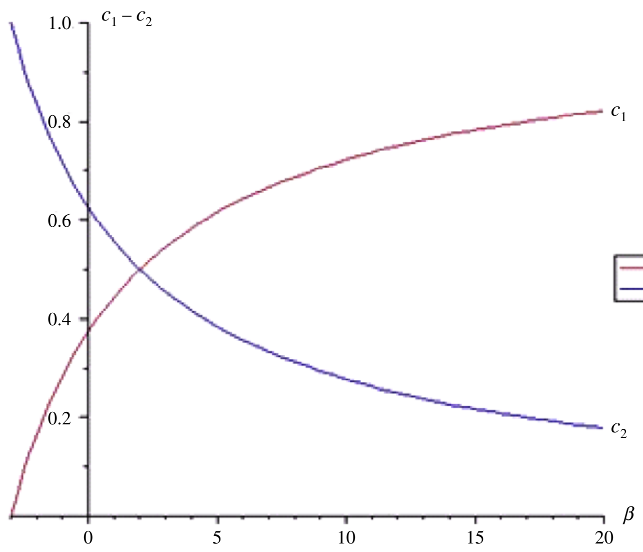

Fig. 6. Optimal strategies for Player #2 for different values of $\beta > -3 \left(c_1 = \frac{3+\beta}{8+\beta}, c_2 = \frac{5}{8+\beta}\right)$.

have pure optimal strategies and the outcome of the game would be **T**–**H**, with a gain of **2** to Player #1. Therefore we only discuss the cases in which $\gamma < 3$.[12] Similar to the first two cases, we have excluded the cases in which there is a dominated strategy.

Now we consider two situations based on the value of $(-1+\gamma)$ in comparison with -3.

Case 3.1: $-1+\gamma \le -3$

$$-1+\gamma \le -3 \Rightarrow \gamma \le -2$$

and we can choose k to be

$$k = -(-1+\gamma) = 1-\gamma.$$

After adding k, the resulting game matrix would be:

		Player #2 H	Player #2 T
Player #1	H	$-2-\gamma$	$3-\gamma$
	T	$3-\gamma$	0

After solving the corresponding LP model, we have

$$p^* = \frac{0.32 - 0.04\gamma}{(0.6-0.2\gamma)^2}$$

and the value of the game would be

$$e = \frac{1}{p^*} - k = \frac{(0.6-0.2\gamma)^2}{0.32-0.04\gamma} - (1-\gamma) = \frac{0.04+0.12\gamma}{0.32-0.04\gamma} = \frac{1+3\gamma}{8-\gamma}.$$

It is easy to investigate that since $\gamma \le -2$, which means that γ is negative, the value of the game will be less than its value before changing the payoff matrix. In other words, we have

$$\textit{new value} = \frac{1+3\gamma}{8-\gamma} < \frac{1}{8} = \textit{old value} \Leftrightarrow \gamma < 0.$$

Also we have

$$\textit{new value} = \frac{1+3\gamma}{8-\gamma} < 0 \quad \textit{for all } \gamma \le -2.$$

It means that for $\gamma \le -2$, the game is unfair and favors Player #2.

[12] Also note that if $\gamma \ge 1$, then the definition of MP no longer holds; but this problem does not affect our method for analyzing the game.

The new optimal strategies are as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (3-\gamma)/8-\gamma \\ 5/(8-\gamma) \end{bmatrix}$$
$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \dfrac{3-\gamma}{8-\gamma} & \dfrac{5}{8-\gamma} \end{bmatrix}.$$

It is clear that for $\gamma = 0$, the above results are the same as the results obtained before changing the payoff matrix.

Now we return to our game matrix to examine the second situation in comparing $(-1+\gamma)$ with -3.

Case 3.2: $-1+\gamma \geq -3$

$$-1+\gamma \geq -3 \Rightarrow \gamma \geq -2.$$

So for $\gamma \geq -2$ we can choose k to be

$$k = -(-3) = 3.$$

After adding k, the resulting game matrix would be:

		Player #2 H	Player #2 T
Player #1	H	0	5
	T	5	$2+\gamma$

By solving the LP model, we have

$$p^* = 0.32 - 0.04\gamma$$

and the value of the game would be

$$e = \frac{1}{p^*} - k = \frac{1}{0.32-0.04\gamma} - 3 = \frac{0.04+0.12\gamma}{0.32-0.04\gamma} = \frac{1+3\gamma}{8-\gamma}.$$

It is easy to investigate that if $\gamma > 0$, then the value of the game would be greater than its value before changing the payoff matrix.

It must be noted that this conclusion is valid for $-2 \leq \gamma < 3$.

Also we have

$$\textit{new value} = \frac{1+3\gamma}{8-\gamma} > 0 \Leftrightarrow \gamma > -\frac{1}{3}.$$

For $-\frac{1}{3} < \gamma < 3$, the game is unfair and favors Player #1, as for $\gamma \geq 3$. *Therefore, we conclude that in general, the game favors Player #1 for $\gamma > -\frac{1}{3}$, and favors Player #2 for $\gamma < -\frac{1}{3}$, and is fair for $\gamma = -\frac{1}{3}$.*

The new optimal strategies are as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (3-\gamma)/8-\gamma \\ 5/(8-\gamma) \end{bmatrix}$$
$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \dfrac{3-\gamma}{8-\gamma} & \dfrac{5}{8-\gamma} \end{bmatrix}.$$

It is clear that for $\gamma = 0$, the above results are the same as the results obtained before changing the payoff matrix.

Again note that for $\gamma < 3$, the value of p is optimal in the simplex tableau.

The descriptions of Figs. 7–9 are similar to the descriptions of Figs. 1–3, since the behavior of both groups of curves is alike.

By decreasing γ, Player #1, avoiding the high risk of encountering outcome **T**–**T**, tends to choose his/her first strategy (i.e. r_1 approaches 1, as shown in Fig. 8) and logically, Player #2 also tends to choose his/her first strategy to maximize his/her gain (i.e. c_1 approaches 1, as shown in Fig. 9) and as a result, the game value approaches the payoff of outcome **H**–**H**, -3 (see Fig. 7).

Additionally, by increasing γ, Player #1 hopes to minimize his/her loss by choosing his/her second strategy (i.e. r_2 approaches 1, as shown in Fig. 8) and Player #2, since $-1+\gamma < 2$ for all $\gamma < 3$, would prefer to choose his/her second strategy (i.e. c_2 approaches 1, as shown in Fig. 9). Therefore, the game value would approach the payoff of outcome **T–T** (see Fig. 7).

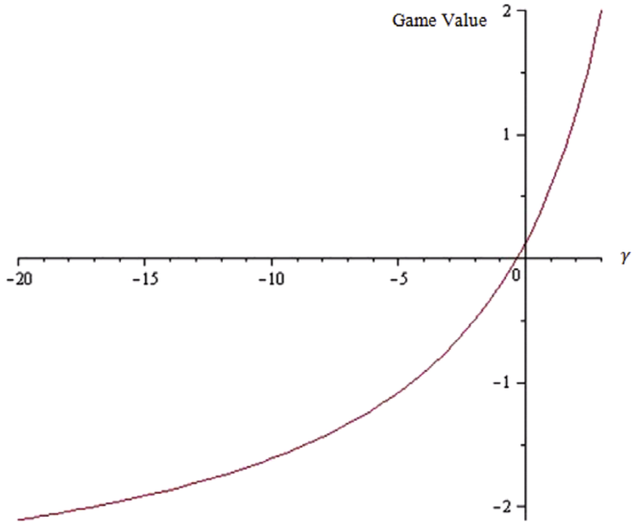

Fig. 7. Game value for different values of $\gamma < 3$ $\left(e = \frac{1+3\gamma}{8-\gamma}\right)$.

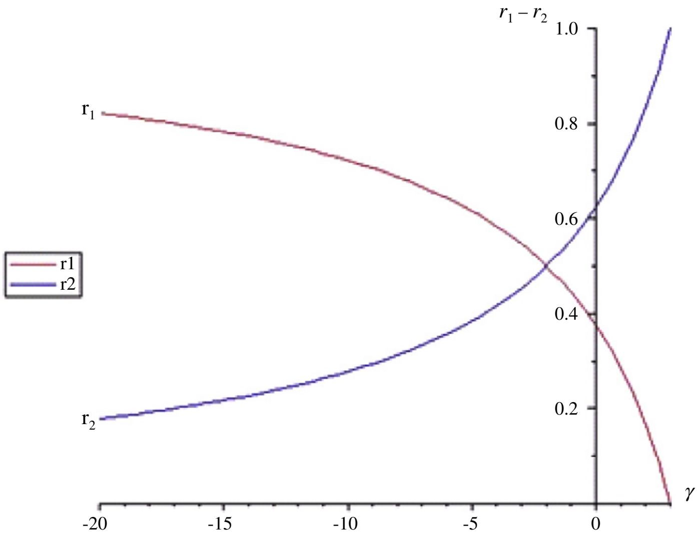

Fig. 8. Optimal strategies for Player #1 for different values of $\gamma < 3$ $\left(r_1 = \frac{3-\gamma}{8-\gamma}, r_2 = \frac{5}{8-\gamma}\right)$.

5. Analysis of results

In the previous section, we analyzed the game by adding numbers α, β and γ to the outcomes of MP to see the effects of these changes on the value of the game as well as on the optimal strategies of both players. The payoff of the game in these cases is shown in the following matrix:

		Player #2	
		H	T
Player #1	H	$-3+\alpha$	$2+\beta$
	T	2	$-1+\gamma$

The final results of sensitivity analysis of MP in each case are presented in Table 4.

As indicated previously, the calculation of the critical value is the most practical application of this method. For instance, by knowing this value, each player would know under which circumstances he/she should or should not participate in the game.

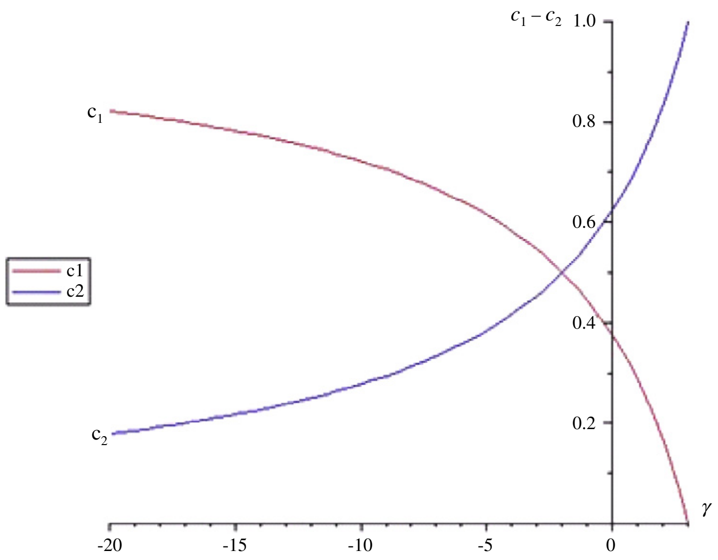

Fig. 9. Optimal strategies for Player #2 for different values of $\gamma < 3 \left(c_1 = \frac{3-\gamma}{8-\gamma}, c_2 = \frac{5}{8-\gamma}\right)$.

Table 4
Results of sensitivity analysis of MP.

Variable	Perturbed outcome	Range	Game value	P#1 Optimal strategy		P#2 Optimal strategy		Critical value[a]
				r_1	r_2	c_1	c_2	
Original game	–	–	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	–
α	H–H	$\alpha < 5$	$\frac{1+\alpha}{8-\alpha}$	$\frac{3}{8-\alpha}$	$\frac{5-\alpha}{8-\alpha}$	$\frac{3}{8-\alpha}$	$\frac{5-\alpha}{8-\alpha}$	-1
β	H–T	$\beta > -3$	$\frac{1+2\beta}{8+\beta}$	$\frac{3}{8+\beta}$	$\frac{5+\beta}{8+\beta}$	$\frac{3+\beta}{8+\beta}$	$\frac{5}{8+\beta}$	-0.5
γ	T–T	$\gamma < 3$	$\frac{1+3\gamma}{8-\gamma}$	$\frac{3-\gamma}{8-\gamma}$	$\frac{5}{8-\gamma}$	$\frac{3-\gamma}{8-\gamma}$	$\frac{5}{8-\gamma}$	$-\frac{1}{3}$

[a] Critical value is the value of the variables α, β or γ for which the game is fair, for values more than which, the game favors Player #1, and for values less than which, the game favors Player #2.

Another interesting application of this method arises when the parameters α, β or γ have a probability distribution. In such cases, one can easily find the *expected* value of the game by knowing the game value in terms of α, β or γ, and then conditioning on the value of these random variables.

Jacobson and Shyryayev [12] have shown in their paper that in MP, both players can guarantee a specific gain by choosing their optimal strategies. They obtained the following results, which are the same as our results:

$$\textit{Value of the game} = e = \frac{1}{8}$$

$$\textit{Player \#1 optimal strategy} = \begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

$$\textit{Player \#2 optimal strategy} = [c_1\ c_2] = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}.$$

Moreover, they have claimed that if the game is shown parametrically as follows (U_1, U_2, V_1 and V_2 are all non-negative):

		Player #2 H	Player #2 T
Player #1	H	$-U_2$	U_1
	T	V_1	$-V_2$

and if $U_1V_1 > U_2V_2$, then the game is unfair and favors Player #1 (i.e. the value of the game is positive) and the optimal strategy for Player #1 is

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} = \left[\frac{V_1 + V_2}{U_1 + U_2 + V_1 + V_2} \ \frac{U_1 + U_2}{U_1 + U_2 + V_1 + V_2} \right].$$

Table 5
Validation of results according to Jacobson and Shyryayev [12].

Case	$U_1V_1 > U_2V_2$	Player #1 optimal strategy		Critical value
		r_1	r_2	
Case 1	$\alpha > -1$	$\frac{3}{8-\alpha}$	$\frac{5-\alpha}{8-\alpha}$	-1
Case 2	$\beta > -0.5$	$\frac{3}{8+\beta}$	$\frac{5+\beta}{8+\beta}$	-0.5
Case 3	$\gamma > -\frac{1}{3}$	$\frac{3-\gamma}{8-\gamma}$	$\frac{5}{8-\gamma}$	$-\frac{1}{3}$

According to our results, the above conclusion is also valid in all cases discussed in this paper; as it is shown in Table 5.

Moreover, the critical values of all of the above cases are equal to the lower bounds of parameters presented in Jacobson and Shyryayev [12].

It must be noted that our conclusions in this paper are more general than the results of Jacobson and Shyryayev [12], since they have obtained optimal strategies under a particular condition (i.e. $U_1V_1 > U_2V_2$), but our results are valid even when their particular condition is not satisfied. More generally, as indicated previously, our method can be used to analyze *any* matrix game (since all matrix games can be solved by the simplex method).

Finally our method will remain valid if we change the initial game matrix (i.e. we change the initial payoffs). We can even assume four parameters as the payoffs of different outcomes. In this case, although the procedure would be the same, this parametric representation of the game would exponentially increase the complication of the procedure, since we need to compare the value of different payoffs in both step 1 and step 2 of the procedure (i.e. in the reduction step and also for determining an appropriate value for k). This level of complexity is beyond the scope of this paper whose main goal is to introduce an analytical method for the sensitivity analysis of matrix games.

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