Exercício I: $Y_T = \left(\frac{1}{3}\right) \cdot (e_{T-1} + e_T + e_{T+1})$

b) Sabemos que $e_T \sim N(0, \sigma^2)$, ~~sendo~~ e_T independentes;

Sabemos também que:

$$\begin{cases} E[Y_T] = \left(\frac{1}{3}\right)\{E[e_{T-1}] + E[e_T] + E[e_{T+1}]\} = 0; \forall T \\ \text{e} \\ Var(Y_T) = \left(\frac{1}{9}\right) Var(e_{T-1} + e_T + e_{T+1}) = \left(\frac{1}{9}\right) \cdot (\sigma^2 + \sigma^2 + \sigma^2) = \end{cases}$$

$$= \frac{1}{3}\sigma^2$$

• Para $s = T-1$, temos:

$$\to Cov(Y_T, Y_{T-1}) = E[(Y_T - E[Y_T]) \cdot (Y_{T-1} - E[Y_{T-1}])] =$$

$$= E[Y_T \cdot Y_{T-1}] = \frac{1}{9} \cdot E[(e_{T-1} + e_T + e_{T+1}) \cdot (e_{T-2} + e_{T-1} + e_T)] =$$

$$= \frac{1}{9}\{E[e_{T-1} \cdot e_{T-2}] + E[e_{T-1}^2] + E[e_{T-1} \cdot e_T] + E[e_T \cdot e_{T-2}] +$$

$$E[e_T \cdot e_{T-1}] + E[e_T^2] + E[e_{T+1} \cdot e_{T-2}] + E[e_{T+1} \cdot e_{T-1}] +$$

$$E[e_{T+1} \cdot e_T]\} =$$

$$= \frac{1}{9}\{0 + \sigma^2 + 0 + 0 + 0 + \sigma^2 + 0 + 0 + 0\} = \frac{2}{9}\sigma^2$$

$$\to Corr(Y_T, Y_{T-1}) = \frac{Cov(Y_T, Y_{T-1})}{Var(Y_T)} = \frac{\frac{2}{9}\cancel{\sigma^2}}{\frac{1}{3}\cancel{\sigma^2}} = \frac{2}{\cancel{9}} \cdot \cancel{3} = \frac{2}{3}$$

• Repare que, se $s = T-2$, temos:

$$Cov(Y_T, Y_{T-2}) = \frac{1}{9} E[(e_{T-1} + e_T + e_{T+1}) \cdot (e_{T-3} + e_{T-2} + e_{T-1})] =$$

$$= \frac{1}{9}\{E[e_{T-1}^2]\} = \frac{1}{9}\sigma^2$$

$$\therefore Corr(Y_T, Y_{T-2}) = \frac{\cancel{Cov(Y_T, Y_{T-2})}}{Var(Y_T)} = \frac{\frac{1}{9}\cancel{\sigma^2}}{\frac{1}{3}\cancel{\sigma^2}} = \frac{3}{9} = \frac{1}{3}$$

b) (Continuando)

- Para $s = T-3$, temos:

$$Cov(Y_T, Y_{T-3}) = \frac{1}{9} E[(e_{T-1} + e_T + e_{T+1})(e_{T-4} + e_{T-3} + e_{T-2})] =$$

$$= \frac{1}{9} \cdot \{0\} = 0$$

$$\therefore Corr(Y_T, Y_{T-3}) = \frac{Cov(Y_T, Y_{T-3})}{Var(Y_T)} = \frac{0}{\frac{1}{3}\sigma^2} = 0 \quad \blacksquare$$

- Para $s = T+1$, temos:

$$Cov(Y_T, Y_{T+1}) = \frac{1}{9} \cdot E[(e_{T-1} + e_T + e_{T+1})(e_T + e_{T+1} + e_{T+2})] =$$

$$= \frac{1}{9} \{E[e_T^2] + E[e_{T+1}^2]\} = \frac{1}{9} \cdot 2\sigma^2 = \frac{2}{9}\sigma^2$$

$$Corr(Y_T, Y_{T+1}) = \frac{Cov(Y_T, Y_{T+1})}{Var(Y_T)} = \frac{\frac{2}{9}\cancel{\sigma^2}}{\frac{1}{3}\cancel{\sigma^2}} = \frac{2}{9} \cdot 3 = \frac{2}{3} \quad \blacksquare$$

- Para $s = T+2$, temos:

$$Cov(Y_T, Y_{T+2}) = \frac{1}{9} E[(e_{T-1} + e_T + e_{T+1}) \cdot (e_{T+1} + e_{T+2} + e_{T+3})] =$$

$$= \frac{1}{9} \cdot \{E[e_{T+1}^2]\} = \frac{1}{9}\sigma^2 \quad \blacksquare$$

$$Corr(Y_T, Y_{T+2}) = \frac{Cov(Y_T, Y_{T+2})}{Var(Y_T)} = \frac{\frac{1}{9}\cancel{\sigma^2}}{\frac{1}{3}\cancel{\sigma^2}} = \frac{1}{9} \cdot 3 = \frac{1}{3} \quad \blacksquare$$

- Para $s = T+3$, temos:

$$Cov(Y_T, Y_{T+3}) = \frac{1}{9} E[(e_{T-1} + e_T + e_{T+1}) \cdot (e_{T+2} + e_{T+3} + e_{T+4})] =$$

$$= \frac{1}{9} \cdot \{0\} = 0 \quad \blacksquare$$

$$Corr(Y_T, Y_{T+3}) = \frac{Cov(Y_T, Y_{T+3})}{Var(Y_T)} = \frac{0}{\frac{1}{3}\sigma^2} = 0 \quad \blacksquare$$

b) (Continuando)

• Para $s = T$, temos:

$$Cov(Y_T, Y_T) = E[Y_T \cdot Y_T] = E[(Y_T - 0)^2] =$$
$$= Var(Y_T) = \frac{1}{3}\sigma^2 \quad \blacksquare$$

$$Corr(Y_T, Y_T) = \frac{Cov(Y_T, Y_T)}{Var(Y_T)} = \frac{Var(Y_T)}{Var(Y_T)} = 1 \quad \blacksquare$$

Por fim, podemos obter, por indução matemática, que:

$$Cov(Y_T, Y_{T-K}) = \begin{cases} \frac{1}{3}\sigma^2, & \text{se } K = 0 \\ \frac{2}{9}\sigma^2, & \text{se } |K| = 1 \\ \frac{1}{9}\sigma^2, & \text{se } |K| = 2 \\ 0, & \text{se } |K| > 2 \end{cases}$$
; sendo $\forall K \in \mathbb{Z}$

e

$$Corr(Y_T, Y_{T-K}) = \begin{cases} 1, & \text{se } K = 0 \\ 2/3, & \text{se } |K| = 1 \\ 1/3, & \text{se } |K| = 2 \\ 0, & \text{se } |K| > 2 \end{cases} \quad \blacksquare \quad \boxed{3}$$

• Perceba que:
$$Cov(Y_T, Y_{T-1}) = Cov(Y_T, Y_{T+1}) \equiv \frac{2}{9}\sigma^2$$
$$Corr(Y_T, Y_{T-1}) = Corr(Y_T, Y_{T+1}) \equiv \frac{2}{3}$$
$$Cov(Y_T, Y_T) = \frac{1}{3}\sigma^2 \text{ e } Corr(Y_T, Y_T) = 1 \quad \blacksquare$$

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