★ Master theorem :-

1) $T(n) = a\,T(n/b) + f(n)$ $a \geq 1\,;\, b > 1$

Case 1 :- If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$.

then $T(n) = \Theta(\cancel{n\log}\, n^{\log_b a})$

Ex :- $T(n) = 9\,T(n/3) + n$.

so $a = 9$ $b = 3$.   so $f(n) = O(n^{2-\epsilon})$.

$\log_3 9 = 2$.   let $\epsilon = 1$.

so $T(n) = \Theta(n^2)$. $f(n) = O(n)$

$n = O(n)$.

Case 2 :- If $f(n) = \Theta(n^{\log_b a})$

then $T(n) = \Theta(n^{\log_b a} \log n)$.

Ex :- $T(n) = 1\,T\left(\frac{2n}{3}\right) + 1$.

$a = 1$ $b = 3/2$. $f(n) = 1$

$n^{\log_{3/2} 1} = n^0 = 1$.

$f(n) = 1 = \Theta(1)$

so $T(n) = \Theta(1 . \log n)$.

$T(n) = \Theta(\log n)$.

Case 3 :- $T(n) = a\,T(n/b) + f(n)$ $a \geq 1$, $b > 1$ ✓

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\epsilon > 0$ ✓

(and) $a\,f(n/b) \leq c . f(n)$ $c < 1$ $\forall n$ ✓

then $T(n) = \Theta(f(n))$

Ex :- $T(n) = 3T\left(\frac{n}{4}\right) + n \log n$.

$a = 3$ $b = 4$ $f(n) = n \log n$.

$\log_b a \Rightarrow \log_4 3 = 0.793$

Set $\epsilon = 0.2$.

So $0.793 + 0.2 \approx 1$.

$f(n) = \Omega(n^{\log_b a + \epsilon})$

$= \Omega(n^{(1)})$

$f(n) = \Omega(n)$

$n\log n = \Omega(n)$.

So if $a f(\frac{n}{b}) \leq c \cdot f(n)$.

$3 \cdot (\frac{n}{4}) \log(\frac{n}{4}) \leq c\, n\log n$. Set $c = \frac{3}{4}$.

$\cancel{\frac{3}{4}}\, \cancel{n} \log(\frac{n}{4}) \leq \cancel{\frac{3}{4}}\, \cancel{n} \log n$.

$\log(\frac{n}{4}) \leq \log n \quad \forall n \,;\, c \leq 1$

So $T(n) = \Theta(f(n))$.

$= \Theta(n\log n)$.

★ *Extended Master theorem*

$T(n) = aT(\frac{n}{b}) + \Theta(n^k \log^p n)$, $a \geq 1$, $b > 1$, $k > 0$

p is a real no:-

Case 1:- if $a > b^k$ then $T(n) = \Theta(n^{\log_b a})$

Case 2:- if $a = b^k$ then

a) $p > -1 \rightarrow T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

b) $p = -1 \rightarrow T(n) = \Theta(n^{\log_b a} \log\log n)$.

c) $p < -1 \rightarrow T(n) = \Theta(n^{\log_b a})$

Case 3:- if $a < b^k$.

a) $p \geq 0 \rightarrow T(n) = \Theta(n^k \log^p n)$.

b) $p < 0 \rightarrow T(n) = \Theta(n^k)$

Ex:- $T(n) = 2T\left(\frac{n}{2}\right) + n\log^3 n$.

$a = 2$ $\quad$ $k = 1$ $\quad$ $a = b^k$

$b = 2.$ $\quad$ $p = 2.$ $\quad$ $2 = 2.$

$\Rightarrow p > -1 \rightarrow T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$.

$T(n) = \Theta(n^{\log_2 2} \log^{2+1} n)$

$T(n) = \Theta(n^1 \log^3 n)$

★ Inadmissible (Failure cases of Master theorem Equat.

- $T(n) = 2^n T(n/2) + n^n$ ✗.
  $\downarrow$
  a is not a constant

- $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$ → Master theorem ✗.
  → Extended theorem ✓.

  $\Rightarrow 2T\left(\frac{n}{2}\right) + n\log^{-1} n \Rightarrow aT\left(\frac{n}{b}\right) + \Theta(n^k \log^p n)$

  where P is real.
  $k > 0.$
  $a \geq 1$ ✓
  $b > 1.$

- $T(n) = 0.5T\left(\frac{n}{2}\right) + n$ ✗.
  $\Rightarrow 0.5 \Rightarrow a$ should be greater then 1.
  $a < 1$

- $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \log n$.
  $\Rightarrow f(n)$, which is the combination time is not positive.

- $T(n) = T(n/2) + n(2 - \cos n)$ ✗.
  $\downarrow$
  regularity violation
  (Sin n, cos θ)
  all
  trigonometric functions.

*) Special cases.

1) $T(n) = T(n/2) + 2^n$

$\Rightarrow O(2^n)$.

2) $T(n) = 2T(n/2) + n!$

$\Rightarrow O(n!)$.

(guess / assume)

Substitution Method (Mathematical induction)

$\Rightarrow$ $T(n) = 2T(n/2) + n$ ← Merge sort. $\exists\, C > 0$

✓ Guess $\Rightarrow T(n) = O(n\log n) \Rightarrow T(n) \leq Cn\log n \quad \forall\, n \geq n_0$.

assume that it is true $m < n$

To prove $T(n) \leq Cn\log n$ assuming $T(m) \leq Cm\log m \quad \forall\, m < n$

so let $m = \boxed{\frac{n}{2} < n}$

$\Rightarrow T(n/2) = O(\frac{n}{2}\log n/2) \Rightarrow T(n/2) \leq C\frac{n}{2}\log n/2$.

$T(n) = 2T(n/2) + n$.

$\rightarrow \quad \leq 2C\frac{n}{2}\log n/2 + n$.

$= Cn(\log n - \log_2 2) + n$.

$= Cn\log n - Cn + n$.

$\leq Cn\log n$. ✓

$T(n/2) \leq C\frac{n}{2}\log\frac{n}{2}$

$\log\frac{a}{b} = \log a - \log b$

$\log_2 2 = 1$

× Guess $\Rightarrow T(n) = O(n) \Rightarrow T(n) \leq C(n)$.

assume $T(m) = O(m) \leq C(m) \quad \forall\, m < n$.

To prove $T(n) \leq Cn$ assuming $T(m) \leq Cm \quad \forall\, m < n$.

$\boxed{m = n/2 < n}$ so $T(n/2) = Cn/2$.

$T(n) = 2T(n/2) + n$.

$= 2C\frac{n}{2} + n$.

$= C(n) + n$

$T(n) \leq C(n) + n$. ×