P 3.1

$$V_{avg} = \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\left[(z+R)-(R-z)\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

If there is more than 1 charge, then V_{avg} due to internal charges $= \frac{1}{4\pi\varepsilon_0}\frac{q_{enc}}{\varepsilon_0}$

Average potential (V_{avg}) due to external charges is V_{centre}

So, $\boxed{V_{avg} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}}$ ✓

P 3.2 A stable equilibrium is a point of local minima in the potential energy. Here, the potential energy is qV. But Laplace equation allows no local minima for V. What looks like a minimum is a saddle point. The box leaks through the center of each face

P 3.3 $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \rightarrow r^2 \frac{dV}{dr} = c$

$\rightarrow \frac{dV}{dr} = \frac{c}{r^2} \rightarrow \boxed{V = \frac{-c}{r} + k}$

Example: potential of uniformly charged sphere.

Cylindrical Coordinates

$\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s \frac{dV}{ds}\right) = 0 \rightarrow s\frac{dV}{ds} = c$

$\rightarrow \frac{dV}{ds} = \frac{c}{s} \rightarrow \boxed{V = c \ln s + k}$

Example: Potential of a long wire.

P3.4 Lets assume 2 ~~wrong~~ fields E_1 and E_2 for a given charge density ρ and either V or $\partial V/\partial n$ on boundary surface

Case (a): given ρ, and V_0 at boundary surface.
Let $E_3 = E_2 - E_1$

$$\int_V \nabla \cdot (V_3 E_3)\, d\tau = \oint_S V_3 E_3 \cdot da = -\int (E_3)^2 d\tau$$

$$\Rightarrow \oint V_3 E_3 \cdot da = -\int E_3^2 \, d\tau$$

$V_3 = 0$ since $\boxed{V_3 = V_0 - V_0 = 0}$

$$\Rightarrow -\int E_3^2 \, d\tau = 0 \qquad \Rightarrow E_3 = 0$$

$$\Rightarrow E_2 - E_1 = 0$$

$$\Rightarrow \boxed{E_1 = E_2} \checkmark$$

Case (b): given ρ, and $\frac{dV}{dn}$ is uniquely determined on boundary surface

$\cancel{E_1 = \frac{dV}{dn}}$ Let $E_3 = E_2 - E_1$

Now, $E_1 = \frac{dV_1}{dn} = \frac{dV}{dn}$, $E_2 = \frac{dV_2}{dn} = \frac{dV}{dn}$

$$\Rightarrow E_3 = E_2 - E_1 = \frac{dV}{dn} - \frac{dV}{dn} = 0 \quad \Rightarrow E_3 = E_2 - E_1 = 0$$

$$\Rightarrow \boxed{E_1 = E_2} \checkmark$$

P 3.5 Putting $U = T = V_3$ in Green's identity, we get

$$\int_V (V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3)\, d\tau = \oint_S V_3 \nabla V_3 \cdot da$$

But $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = \frac{-\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$

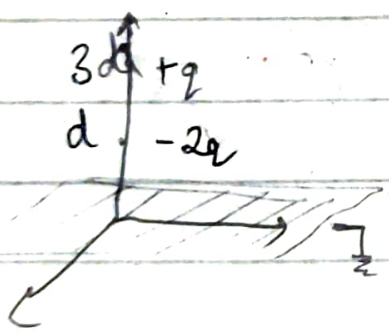
$\nabla V_3 = -E_3$

So, $\int_V E_3^2\, d\tau = \oint_S V_3 E_3 \cdot da \Rightarrow \int_V E_3^2\, d\tau = 0$

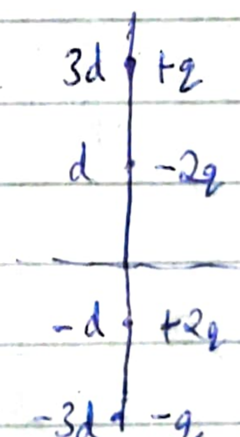
$\Rightarrow E_3 = 0$

$\Rightarrow E_2 - E_1 = 0 \Rightarrow \boxed{E_1 = E_2}$

P 3.6



$\xrightarrow[\text{of images}]{\text{Method}}$



$$F_{+q} = F_{(-2q)} + F_{(+2q)} + F_{(-q)}$$

$$= \frac{-2q^2}{4\pi\varepsilon_0 (2d)^2} + \frac{2q^2}{4\pi\varepsilon_0 (4d)^2} + \frac{-q^2}{4\pi\varepsilon_0 (6d)^2} = \frac{q^2}{4\pi\varepsilon_0 d}\left(\frac{-2}{2^2} + \frac{2}{4^2} - \frac{1}{6^2}\right)$$

$$= \cancel{\frac{q^2}{4\pi\varepsilon_0 d}\left(\frac{-18 + (-2)}{72}\right) = \frac{-8}{12}\frac{q^2}{4\pi\varepsilon_0 d}}$$

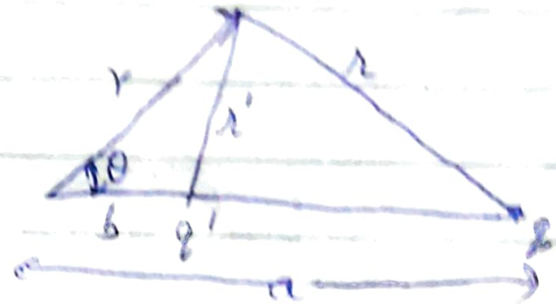
$$= \frac{q^2}{4\pi\varepsilon_0 d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right) = \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{-36+9-2}{72}\right) = \boxed{-\frac{29 q^2}{72(4\pi\varepsilon_0)d^2}}$$

Ans

P 3.7

(a) $\mathscr{r} = \sqrt{r^2+a^2-2ar\cos\theta}$

$\mathscr{r}' = \sqrt{r^2+b^2-2br\cos\theta}$



$$\frac{q'}{\mathscr{r}'} = -\frac{R}{a}\,\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}}$$

$$= -\frac{R}{a}\,\frac{q}{\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}} = \frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2+R^2-2ra\cos\theta}}$$

Now, $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$

$$= \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right\}$$

(b) At $r = R$, $\frac{\partial V}{\partial n} = \frac{dV}{dr}$

$$\sigma = -\varepsilon_0 \frac{dV}{dn} = -\varepsilon_0 \frac{dV}{dr}$$

$$\Rightarrow \sigma(\theta) = -\varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0}\right) \left\{ -\frac{1}{2}(r^2+a^2-2ra\cos\theta)^{-3/2}(2r-2a\cos\theta) + \frac{1}{2}\left(R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r-2a\cos\theta\right)\right\}\Bigg|_{r=R}$$

$$\Rightarrow \sigma(\theta) = \frac{-q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta) + (R^2+a^2-2Ra\cos\theta)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}$$

$$= \frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cancel{\cos\theta}-\frac{a^2}{R}+a\cancel{\cos\theta}\right]$$

$$= \boxed{\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}}$$

$$q_{induced} = \int \sigma \, da = \frac{q}{4\pi R}(R^2-a^2)\int (R^2+a^2-2Ra\cos\theta)^{-3/2} R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q}{4\pi R}(R^2-a^2)\, 2\pi R^2 \left[\frac{-1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2} \right]_0^{\pi}$$

$$= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}} - \frac{1}{\sqrt{R^2+a^2-2Ra}} \right]$$

$$= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{a+R} - \frac{1}{a-R} \right] = \frac{q\cancel{(a^2-R^2)}}{\cancel{2a}}\left[\frac{\cancel{2a}(-2R)}{\cancel{(a^2-R^2)}} \right]$$

$$= \boxed{\frac{-qR}{a} = q'} \quad \text{Ans}$$

(c) $F = \frac{1}{4\pi\varepsilon_0}\frac{q q'}{(a-b)^2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{-R}{a}q^2\right)\frac{1}{\left(a-\frac{R^2}{a}\right)^2} = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{q^2 Ra}{(a^2-R^2)^2}}$

Ans

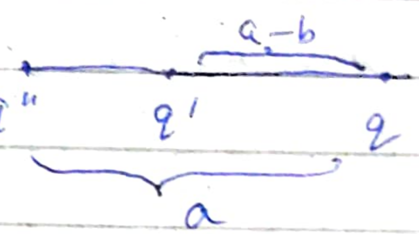
$$W = \frac{q^2 R}{4\pi\varepsilon_0}\int_{\infty}^{a} \frac{\bar{a}}{(\bar{a}^2-R^2)^2}\, d\bar{a} = \frac{q^2 R}{4\pi\varepsilon_0}\left[-\frac{1}{2}\,\frac{1}{(\bar{a}^2-R^2)}\right]_{\infty}^{a}$$

$$= \boxed{\frac{-1}{4\pi\varepsilon_0}\,\frac{q^2 R}{2(a^2-R^2)}} \quad \text{Ans}$$

P 3.8 For this, we need to place an additional charge q'' at the centre of the sphere such that V_0 is due to q''.

So, $V_0 = \frac{q''}{4\pi\varepsilon_0 R} \Rightarrow \boxed{q'' = \cancel{V_0}(4\pi\varepsilon_0)V_0 R}$

For neutral sphere, $\boxed{q' + q'' = 0}$



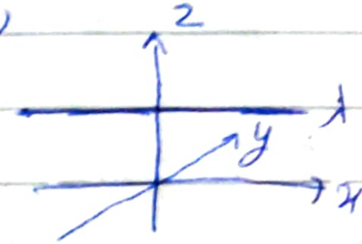
Now, $F = \frac{q}{4\pi\varepsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right)$

$$= \frac{qq''}{4\pi\varepsilon_0}\left(-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right) = \frac{qq'}{4\pi\varepsilon_0}\left(\frac{\cancel{a^2}+b^2-2ab-\cancel{a^2}}{a^2(a-b)^2}\right)$$

$$= \frac{qq'}{4\pi\varepsilon_0}\,\frac{b(2a-b)}{a^2(a-b)^2} = \frac{q\left(\frac{-Rq}{a}\right)}{4\pi\varepsilon_0}\,\frac{\frac{R^2}{a}\left(2a-\frac{R^2}{a}\right)}{a^2\left(a-\frac{R^2}{a}\right)^2}$$

$$\boxed{F = -\frac{q^2}{4\pi\varepsilon_0}\left(\frac{R}{a}\right)^3 \frac{(2a^3 - R^2)}{(a^2-R^2)^2}}$$ Ans

[P 3.9] Image problem: λ above, $-\lambda$ ~~d~~ below



(a) $$V(y,z) = \frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\frac{s_-}{s_+}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{s_-^2}{s_+^2}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}$$

(b) $$\sigma = -\varepsilon_0\frac{dV}{dn} \quad ; \quad \frac{dV}{dn} = \frac{dV}{dz}, \text{ at } z=0$$

$$\sigma(y) = -\varepsilon_0\frac{\lambda}{4\pi\varepsilon_0}\left\{\frac{1}{y^2+(z+d)^2}2(z+d) - \frac{1}{y^2+(z-d)^2}2(z-d)\right\}\Bigg|_{z=0}$$

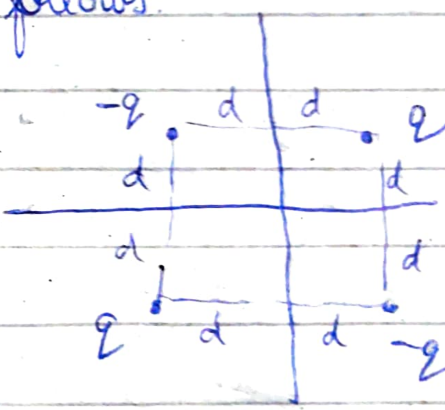
$$= \frac{-2\lambda}{4\pi}\left\{\frac{d}{y^2+d^2} - \frac{-d}{y^2+d^2}\right\} = \boxed{\frac{-\lambda d}{\pi(y^2+d^2)}}$$

$$q_{in} = -\frac{\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{dy}{y^2+d^2} = -\frac{\lambda d}{\pi}\left[\frac{1}{d}\tan^{-1}\frac{y}{d}\right]_{-\infty}^{\infty}$$

$$= -\frac{\lambda d}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \boxed{-\lambda l = q_{ind}}$$

$\boxed{\lambda_{ind} = -\lambda}$ Ans

P3.10 The image configuration is as follows.



$$V(x,y) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right\}$$

$$F = \frac{q^2}{4\pi\varepsilon_0}\left\{\frac{-1}{(2a)^2}\hat{x} - \frac{1}{(2b)^2}\hat{y} + \frac{1}{\left(2\sqrt{a^2+b^2}\right)^2}\left[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right]\right\}$$

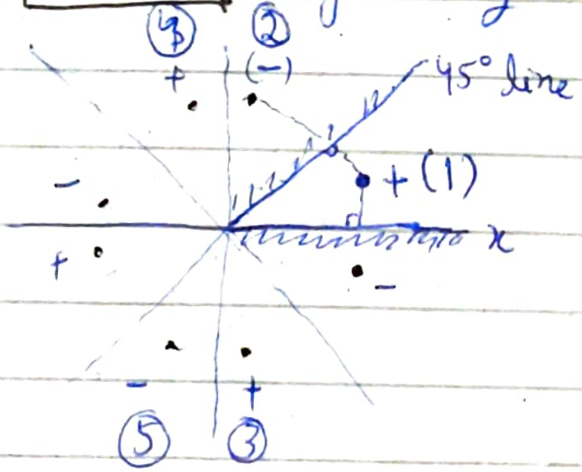
where, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

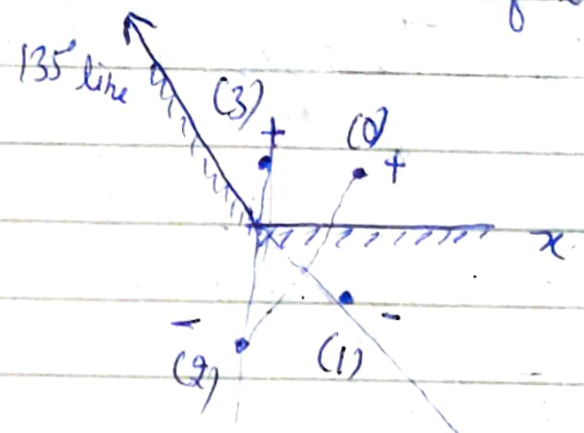
$$\boxed{F = \frac{q^2}{16\pi\varepsilon_0}\left\{\left[\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right]\hat{x} + \left[\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right]\hat{y}\right\}}$$

Ans

For this to work, $\boxed{\theta \text{ must be an integer divisor of } 180}$

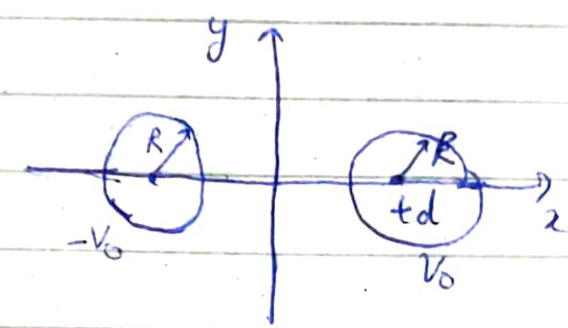
Reason Symmetry allows us to use method of images





Doesn't work for 135°

P 3.11 From Prob 2.52 ($y_0 \to d$)



$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

$$\left.\begin{array}{l} a\coth(2\pi\varepsilon_0 V_0/\lambda) = d \\ a\,\mathrm{cosech}(2\pi\varepsilon_0 V_0/\lambda) = R \end{array}\right\} \text{dividing} \to \frac{d}{R} = \cosh\left(\frac{2\pi\varepsilon_0 V_0}{\lambda}\right)$$

$$\boxed{\lambda = \frac{2\pi \varepsilon_0 V_0}{\cosh^{-1}\left(\frac{d}{R}\right)}}$$ Ans