CHAPTER 1 | THE WAVE FUNCTION

P1.1 (a) $\langle j^2 \rangle = \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[14^2 + 15^2 + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2)\right]$

$= \frac{1}{14}(196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14}$

$= \boxed{459.571}$ Ans. , $\langle j \rangle^2 = 21^2 = \boxed{441}$ Ans

(b) $j = 14$, $\Delta j = j - \langle j \rangle = 14 - 21 = -7$

$j = 15$, $\Delta j = j - \langle j \rangle = 15 - 21 = -6$

$j = 16$, $\Delta j = j - \langle j \rangle = 16 - 21 = -5$

$j = 22$, $\Delta j = j - \langle j \rangle = 22 - 21 = 1$

$j = 24$, $\Delta j = j - \langle j \rangle = 24 - 21 = 3$

$j = 25$, $\Delta j = j - \langle j \rangle = 25 - 21 = 4$

$\sigma^2 = \frac{1}{N}\sum (\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2 + (-6)^2 + (-5)^2 \cdot 3 + (1)^2 \cdot 2 + (4)^2 \cdot 5\right]$

$= \frac{1}{14}\left[49 + 36 + 75 + 2 + 18 + 80\right] = \frac{260}{14} = \boxed{18.571}$ Ans

$\sigma = \sqrt{18.571} = \boxed{4.309}$ Ans

[Eq 1.2]

(c) $\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571 \approx \sigma^2$ ✓ JED

$\boxed{P\ 1.2}$ (a) $\langle x^2\rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}}\,dx = \frac{1}{2\sqrt{h}}\left(\frac{2}{5}x^{5/2}\right)\Big|_0^h = \frac{h^2}{5}$

$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3}\right)^2 = \frac{4}{45}h^2 \Rightarrow \boxed{\sigma = \frac{2h}{3\sqrt{5}}}$ Ans

(b) $P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}}\,dx = 1 - \frac{1}{2\sqrt{h}}(2\sqrt{x})\Big|_{x_-}^{x_+}$

$= 1 - \frac{1}{\sqrt{h}}(\sqrt{x_+} - \sqrt{x_-})$

$x_+ = \langle x\rangle + \sigma = 0.33h + 0.29h = 0.62h$

$x_- = \langle x\rangle - \sigma = 0.33h - 0.29h = 0.04h$

$P = 1 - \sqrt{0.62} + \sqrt{0.04} = \boxed{0.393}$ Ans

$\boxed{P\ 1.3}$ $\int_{-\infty}^{+\infty} \rho(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{+\infty} A e^{-\lambda(x-a)^2}\,dx = 1$

$u = x - a \quad du = dx$

Pg 1.3

$$\Rightarrow A\int_{-\infty}^{+\infty} e^{-\lambda u^2}\,du = 1$$

$$\Rightarrow A\sqrt{\frac{\pi}{\lambda}} = 1 \Rightarrow \boxed{A = \sqrt{\frac{\lambda}{\pi}}}$$ Ans

(b) $$\langle x\rangle = A\int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2}\,dx = A\int_{-\infty}^{+\infty}(u+a)\,e^{-\lambda u^2}\,du$$

$$= A\left[\int_{-\infty}^{\infty} u\, e^{-\lambda u^2}\,du + a\int e^{-\lambda u^2}\,du\right]$$

$$= A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = Aa\sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\cancel{\lambda}}{\cancel{\pi}}}\,a\sqrt{\frac{\cancel{\pi}}{\cancel{\lambda}}} = \boxed{a}$$ Ans

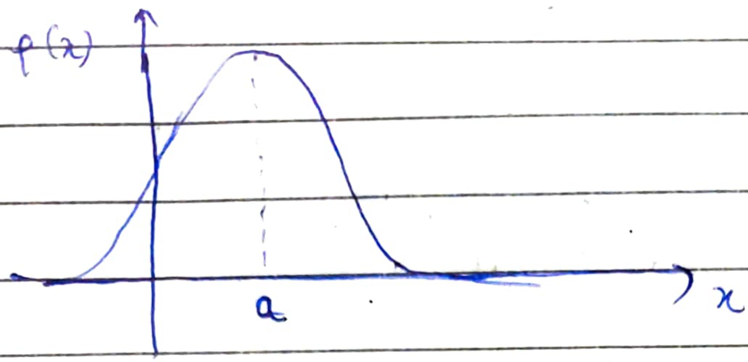
$$\langle x^2\rangle = A\int_{-\infty}^{+\infty} x^2 e^{-\lambda(x-a)^2}\,dx = A\int_{-\infty}^{\infty}(u+a)^2 e^{-\lambda u^2}\,du$$

$$A\left[\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} + 2a\int_{-\infty}^{+\infty} u\, e^{-\lambda u^2}\,du + a^2\int_{-\infty}^{+\infty} e^{-\lambda u^2}\,du\right]$$

$$= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \sqrt{\frac{\lambda}{\pi}}\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + a^2\sqrt{\frac{\pi}{\lambda}}\right]$$

$$= \boxed{\frac{1}{2\lambda} + a^2}$$ Ans

(c)



P1.4 Normalized wave form obeys below rule

$$\int_{-\infty}^{+\infty} \psi(x,t)^2 dx = 1$$

$$\Rightarrow \cancel{0} + \frac{A^2}{a^2}\int_0^a x^2 + \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2 dx + 0^2 = 1$$

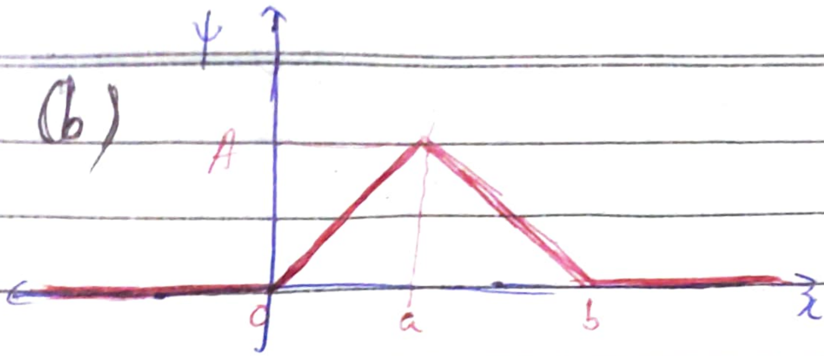
($x = b - t$)

$$\rightarrow \frac{A^2}{\cancel{a^2}} \frac{\cancel{a^3}\, a}{3} + \frac{A^2}{(b-a)^2}\int_{b-a}^{0} t^2 (-dt) = 1$$

$$\frac{A^2}{(b-a)^2}\left[\frac{t^3}{3}\right]_0^{b-a} = 1 - \frac{a^2 A}{3} \Rightarrow \frac{A^2(b-\cancel{a})}{3} = 1 - \frac{a^2 \cancel{A}}{\cancel{3}}$$

$$\Rightarrow \frac{A^2 b}{3} = 1 \rightarrow \boxed{A = \sqrt{\frac{3}{b}}}$$ Ans

(b)



(c) From graph, its clear that particle has high chanses of being found at $\boxed{x=a}$.

(d) Probability of particle being found in left of a, is:

$$P = \int_0^a \psi^2\,dx = \frac{A^2}{a^2}\int_0^a x^2\,dx = \frac{A^2}{\cancel{a^2}}\frac{\cancel{a^3}}{3} = \frac{\cancel{3}}{b}\times\frac{a}{\cancel{3}} = \frac{a}{b}$$

If $\boxed{b=a}$, $P = \frac{a}{a} = \boxed{1}$ Ans

If $\boxed{b=2a}$ $P = \frac{a}{2a} = \frac{1}{2}$ Ans

(e) $$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi^2|\,dx = 0 + |A|^2\left\{\frac{1}{a^2}\int_0^a x^3\,dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2\,dx\right\}$$

$x = b-t$

$$\langle x\rangle = \left(\frac{3}{b}\right)\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Bigg|_0^a + \frac{1}{(b-a)^2}\int_{b-a}^{0}(b-t)t^2(-dt)\right\}$$

$\rightarrow \frac{bt^3}{3} - \frac{t^4}{4}$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\frac{a^4}{4} + \frac{1}{(b-a)^2}\left(\frac{b(b-a)^3}{3} - \frac{(b-a)^4}{4}\right)\right\}$$

$$= \frac{3}{b}\left[\frac{a^2}{4} + \frac{4b(b-a) - 3(b-a)^2}{3\times 4}\right] = \frac{3}{b}\left[\frac{3a^2 + 4b^2 - 4ab - 3(a^2+b^2-2ab)}{12}\right]$$

$$= \frac{3}{b}\left[\frac{b^2+2ab}{12}\right] = \frac{3(b+2a)}{12} = \boxed{\frac{b+2a}{4}}$$ Ans

P1.5 $\psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t}$

At $t=0$, $\psi(x,0) = A e^{-\lambda|x|} e^{0} = A e^{-\lambda|x|}$

(a) Normalization: $\int_{-\infty}^{\infty} \psi^2(x,0)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} A^2 e^{-2\lambda x} dx = 1 \quad \cancel{\Rightarrow 2\int_0^{\infty} A^2 e^{-2\lambda x} dx}$$

$$\Rightarrow 2A^2 \left[\frac{e^{-2\lambda x}}{-2\lambda}\right]_0^{\infty} = 1$$

$$\Rightarrow \frac{A^2}{\lambda} = 1 \quad \Rightarrow \boxed{A = \sqrt{\lambda}}$$ Ans

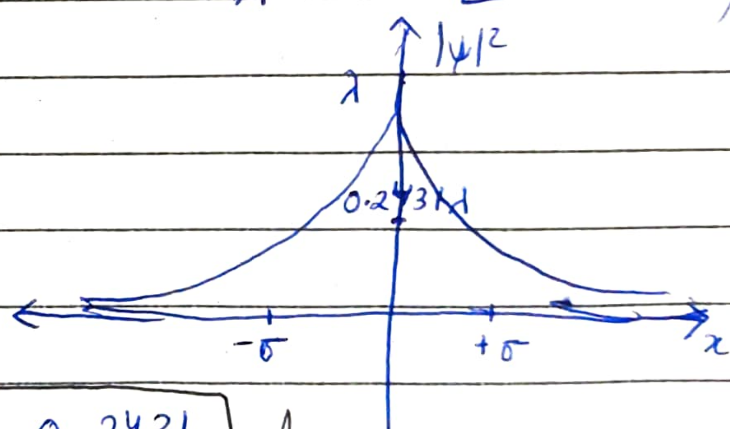
(b) $\langle x \rangle = \int x \psi^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda |x|} dx = 0$

$\langle x^2 \rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3}\right] = \boxed{\frac{1}{2\lambda^2}}$ Ans

(c) $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2} - 0 = \boxed{\frac{1}{2\lambda^2}}$ Ans

$\Rightarrow \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}}$ Ans

$|\psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda \frac{1}{\sqrt{2}\lambda}} = \lambda e^{-\sqrt{2}} = \boxed{0.2431\lambda}$



Probability Outside.

$2\int_{\sigma}^{\infty} \psi^2 dx = 2|A|^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx$

$= 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)_{\sigma}^{\infty} = e^{-2\lambda\sigma}$

$= \boxed{e^{-\sqrt{2}} = 0.2431}$ Ans

$\boxed{\text{P1.6}}$ For integration by parts, the differentiation has to be with respect to integration variable (in this case integration variable is x and differentiation is wrt t)

P1.7 ~~From~~ $\frac{d\langle p\rangle}{dt} = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial x}\right)dx$

Now, $\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial x}\right) = \frac{\partial\psi^*}{\partial t}\cdot\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial t}\right)$

$$= \left[\frac{-i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*\right]\frac{\partial\psi}{\partial x}$$

$$+ \psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi\right]$$

$$= \frac{i\hbar}{2m}\left(\psi^*\frac{\partial^3\psi}{\partial x^3} - \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\right) + \frac{i}{\hbar}\left[V\psi^*\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial}{\partial x}(V\psi)\right]$$

The first term integrates to zero, by using integration by parts twice, and the second term can be simplified to $V\psi^*\frac{\partial\psi}{\partial x} - \psi^* V\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial V}{\partial x}\psi$

$$= -|\psi|^2\frac{\partial V}{\partial x}$$

Now, $\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\psi|^2\frac{\partial V}{\partial x}dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle \Rightarrow$ QED, Hence, proved.

Problem 1.8

Lets suppose ψ satisfies Schrödinger Equation without V_0.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi \rightarrow \text{I}$$

Also, let ψ_0 be a solution to Schrödinger Equation with V_0

$$i\hbar \frac{\partial \psi_0}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi_0}{\partial x^2} + (V+V_0)\psi_0 \rightarrow \text{II}$$

CLAIM $\psi_0 = \psi e^{-iV_0 t/\hbar}$.

PROOF. $$i\hbar \frac{\partial \psi_0}{\partial t} = i\hbar \frac{\partial \psi}{\partial t} e^{-iV_0 t/\hbar} + i\hbar\psi\left(\frac{-iV_0}{\hbar}\right)e^{-iV_0 t/\hbar}$$

From I

$$\Rightarrow i\hbar \frac{\partial \psi_0}{\partial t} = \left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right]e^{-iV_0 t/\hbar} + V_0\psi e^{-iV_0 t/\hbar}$$

$$= \left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + (V+V_0)\psi\right]e^{iV_0 t/\hbar}$$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2 \psi_0}{\partial x^2} + (V+V_0)\psi_0 \quad \boxed{\text{QED}}.$$

NOTE: This has no effect on expectation value of a dynamical variable, since extra phase factor is independant of x, cancels out.

P1.9

(a) $2|A|^2 \int_0^\infty e^{-2amx^2/\hbar} dx = 1$

$\Rightarrow 2|A|^2 \frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = 1$

$\Rightarrow \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}}$ Ans

(b) $\frac{\partial\psi}{\partial t} = -ia\psi$, $\frac{\partial\psi}{\partial x} = \frac{-2amx}{\hbar}\psi$

$, \frac{\partial^2\psi}{\partial x^2} = \frac{-2am}{\hbar}\left(\psi + x\frac{\partial\psi}{\partial x}\right) = \frac{-2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\psi$

Plugging into Schrödinger's equation, we get:

$V\psi = i\hbar(-ia)\psi + \frac{\hbar^2}{2m}\left(\frac{-2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\psi$

$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\psi = 2a^2mx^2\psi$

$\boxed{V(x) = 2ma^2x^2}$

$\boxed{\text{Pg } 1.11}$

(c) ~~$\langle x \rangle = \int_{-\infty} x|\psi|^2 d$~~

(c) $\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2 dx = \boxed{0}$ Ans [Odd Integrand]

$$\langle x^2 \rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2amx^2/\hbar} dx$$

$$= 2|A|^2 \frac{1}{2^2(2am/\hbar)} \sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}} \text{ Ans}$$

$$\langle p \rangle = m\frac{\partial \langle x \rangle}{\partial t} = \boxed{0} \text{ Ans}$$

$$\langle p^2 \rangle = \int \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi \, dx = -h^2 \int \psi^* \frac{\partial^2 \psi}{\partial x^2} \partial x$$

$$= -\hbar^2 \int \psi^* \left[-\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\psi\right] \partial x$$

$$= 2am\hbar \left\{ \int |\psi|^2 \partial x - \frac{2am}{\hbar} \int x^2 |\psi|^2 \partial x \right\}$$

$$= 2am\hbar \left(1 - \frac{2am}{\hbar}\langle x^2 \rangle\right) = 2am\left(1 - \frac{2am}{\hbar}\frac{\hbar}{4am}\right)$$

$$2am\hbar\left(\frac{1}{2}\right) = am\hbar$$

(d) $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \dfrac{\hbar}{4am} \Rightarrow \sigma_x = \sqrt{\dfrac{\hbar}{4am}}$

$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Rightarrow \sigma_p = \sqrt{am\hbar}$

$\sigma_x \sigma_p = \sqrt{\dfrac{\hbar}{4am}}\sqrt{am\hbar} = \dfrac{\hbar}{2} \geq \dfrac{\hbar}{2} \rightarrow \boxed{QED}$

$\boxed{P1.20}$ $\pi = 3.14159265358979323846264 3 \ldots$

(a) $P(0) = \frac{0}{25} = 0.00$, $P(4) = \frac{3}{25} = 0.12$, $P(8) = \frac{2}{25} = 0.08$

$P(1) = \frac{2}{25} = 0.08$, $P(5) = \frac{3}{25} = 0.12$, $P(9) = \frac{3}{25} = 0.12$

$P(2) = \frac{3}{25} = 0.12$, $P(6) = \frac{3}{25} = 0.12$,

$P(3) = \frac{5}{25} = 0.20$, $P(7) = \frac{1}{25} = 0.04$

(b) Most Probable digit = 3 with $\boxed{P(3) = 0.20}$ Ans

Median digit = digit at $\frac{25+1}{2} = 13^{th}$ position = $\boxed{4}$ Ans

if numbers are written in ordered array.

$$\text{Average} = \frac{0(0)+1(2)+2(3)+3(5)+4(3)+5(3)+6(3)+7(1)+8(2)+9(3)}{25}$$

$$= \frac{2+6+15+12+15+18+7+16+27}{25} = \frac{118}{25} = \boxed{4.75} \text{ Ans}$$

$\boxed{Pg\ 1.14}$

$$\langle j^2 \rangle = \frac{\sum_j N(j)\, j^2}{\sum_j N(j)} = \frac{0(0^2) + 2(1^2) + 3(2^2) + 5(3^2) + 3(4^2) + 3(5^2) + 3(6)^2 + 1(7^2) + 2(8^2) + 3(9^2)}{0+2+3+5+3+3+3+1+2+3}$$

$$= \frac{710}{25} = \boxed{28.4} \text{ Ans}$$

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = \sqrt{28.4 - (4.72)^2} = \boxed{2.47} \text{ Ans}$$