9.5.11

AI24BTECH11021 - Manvik Muthyapu

Question:
Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by sides $x = 0, x = 4, y = 4,$ and $y = 0$ into three equal parts. (12, 2018)
Solution: Area of the Square $= 4 \times 4 = 16$.
To find area under the curve $y^2 = 4x$ between $y = 0$ and $y = 4$, we integrate the expression for x with respect to y:

$$\text{Area} = \int_0^4 \frac{y^2}{4}\,dy = \frac{1}{4}\int_0^4 y^2\,dy \tag{1}$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} \tag{2}$$

$$= \frac{16}{3} \tag{3}$$

Now, for area under the $x^2 = 4y$ between $x = 0$ and $x = 4$, we integrate the expression for y with respect to x:

$$\text{Area} = \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{4}\int_0^4 x^2\,dx \tag{4}$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} \tag{5}$$

$$= \frac{16}{3} \tag{6}$$

The area enclosed by both the curves together is the area of square remaining after removing the area enclosed by the curves and the coordinate axes:

$$16 - \frac{16}{3} + \frac{16}{3} = \frac{16}{3} \tag{7}$$

Thus, from the results (3), (6), (7) we get that the parabolas divide the the square into three regions of equal area, each having an area of $\frac{16}{3}$.