DIGITAL SIGNAL PROCESSING

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211EC228

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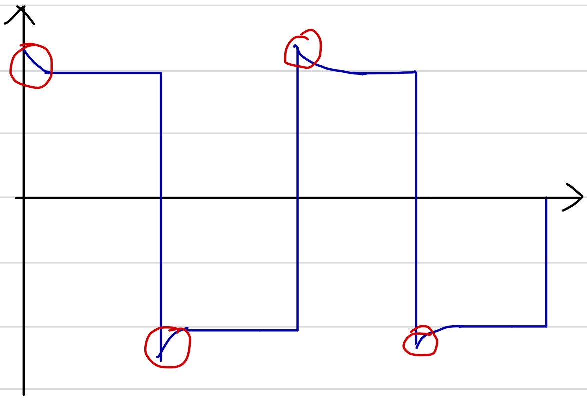
→ Digital source interact with digital system to create digital signal.

Gibb's phenomenon:

These distortion occur due to Gibb's phenomenon.



Because of limitation of bandwidth in oscilloscope, all the harmonics of fourier series are not added, which create these distortion.

- Let $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$, here $|C_k|$ is continuously decreasing. Also continuous series, $|C_i| \neq |C_j|$

- In discrete domain, $x(n) = x(n+KN)$
  But $|C_k| = |C_{k+N}|$

Cosine waveform: $x_1(t) = A\cos(\omega t + \phi)$
Switch cosine waveform: $x_2(t) = x_1(t)\left(u(t) - u(t-t_0)\right)$

Mathematical models used in DSP -

1> Difference equations
2> Impulse response → $h[n]$
3> state variable model.
4> DTFS, DTFT, DFT, FFT
5> Z transform

$$\delta(n) = \begin{cases} u(n) - u(n-1) = 1, & \text{for only } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Also $\delta(t) = \frac{d}{dt}\big(u(t)\big)$

Signal classification:

Energy v/s Power

$$\text{Power} = \frac{V^2}{R} \text{ or } I^2 R$$

if $R = 1$, $\text{Power} = V^2$ or I^2

Let $x = V$ or I

If x was complex, $x = |x| e^{j\phi}$

$$E \triangleq \lim_{N \to \infty} \sum_{n=-N}^{N} |x(n)|^2$$

For finite length signal, $P_{avg} = \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

For infinite signal,

$$P_{av} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

- If E is finite then the signal is energy signal with $P_{avg} = 0$.
- If E is infinite & P_{avg} is finite then it is power signal

DSD lab had Energy signal.

Periodic vs Aperiodic signal

Periodic: $x(n) = x(n+N)$

$N \to$ fundamental period

If $x_1(n)$ has period N_1
& $x_2(n)$ has period N_2
& if $x(n) = x_1(n) + x_2(n)$

then $x(n)$ is periodic with period N if $N = pN_1 = qN_2$ where p, q are integers

Also $N = LCM(N_1, N_2)$

Eg 1: $x(n) = \cos\left(\frac{\pi}{9}n\right) + \frac{1}{2}\sin\left(\frac{n\pi}{7} + 30^\circ\right)$. Find period

$\cos\left(\frac{\pi}{9}(n+N_1)\right) = \cos\left(\frac{\pi}{9}n\right)$

$\Rightarrow \frac{\pi N_1}{9} = 2\pi \quad \Rightarrow N_1 = 18$

$$\sin\left(30^\circ + \frac{\pi(n+N_2)}{7}\right) = \sin\left(30^\circ + \frac{\pi n}{7}\right)$$

$$\Rightarrow \quad \frac{\pi N_2}{7} = 2\pi \qquad \therefore \quad N_2 = 14$$

$\therefore \quad N = \text{LCM}(18, 14) = 126$

$\therefore$ Fundamental period of $x(n) = 126$

NOTE: $y(n) = \alpha x_1(n) \pm \beta x_2(n)$ is periodic

$z(n) = x_1^2(n) + 2x_2^2(n) \rightarrow$ not periodic

Even & odd signals:

Even $\Rightarrow \quad x(n) = x(-n)$

odd $\Rightarrow \quad x(n) = -x(-n)$

$$x(n) = x_e(n) + x_o(n) \longrightarrow (1)$$

$$x(-n) = x_e(n) - x_o(n) \longrightarrow (2)$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \rightarrow (3)$$

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right] \rightarrow (4)$$

$$x_o = \left[x(n) + (-1)x(-n)\right] \times \tfrac{1}{2}$$

$$x_{e/o}(n) = \frac{1}{2}\left[x(n) + e^{\pm j\pi} x(-n)\right]$$

Ex. 1) $x(n) = e^{-n/2} u(n)$

$y(n) = 2x\left(\frac{5n}{3}\right)$ $\qquad z(n) = x(2n)$

2) $x(n) = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$

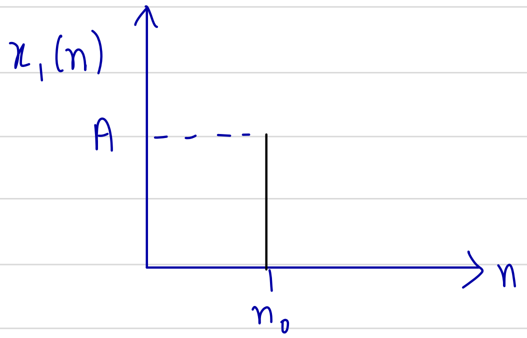
Delayed impulse & step sequence

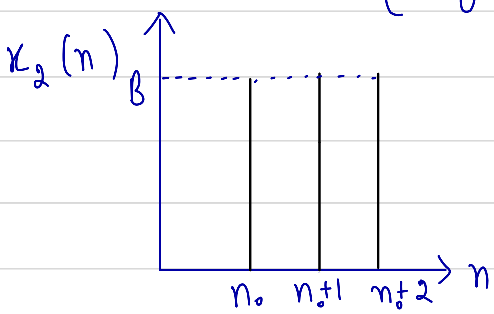
Assume impulse of amplitude A, happens at $n = n_0$ instead of $n = 0$,

$\Rightarrow \quad x_1(n) = A\delta(n - n_0)$

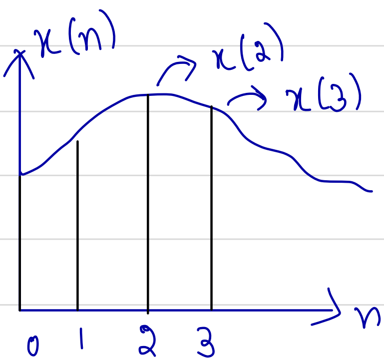
Let the step sequence happen from n_0 with amplitude B

$\Rightarrow \quad x_2(n) = Bu(n - n_0)$ $\qquad \Rightarrow \quad x_2(n) = \begin{cases} B, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$





NOTE: Usually Periodic signals are power signal



$x(n) = x(0)\delta(n-0) + x(1)\delta(n-1) + x(2)\delta(n-2) + \dots$

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Causal signals: Practical signals which can be generated in lab

Non - causal signals: Non practical signals which cannot be generated in lab. (Usually starting from $-\infty$ and/or going upto $+\infty$)

$$x(k)\delta(k) \longrightarrow \boxed{\text{LTI system}} \longrightarrow x(k)\,h(k)$$

$$x(2)\delta(n-2) \longrightarrow \boxed{\text{LTI system}} \longrightarrow x(2)\,h(n-2)$$

$h \Rightarrow$ impulse response

Convolution sum:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = x(n) * h(n) \quad \rightarrow (1)$$

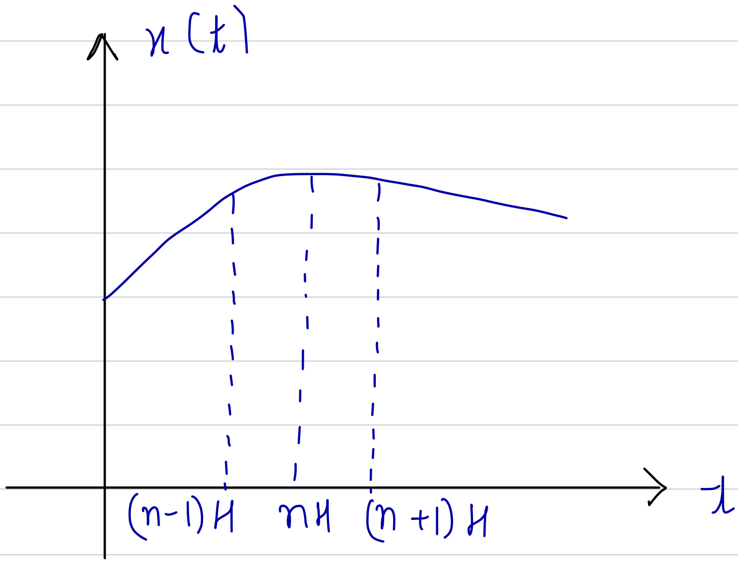
It is response of a LTI system excited by scaled and delayed arbitrary signal.

Let $n-k = m \implies k \in (-\infty, \infty) \Rightarrow m \in (-\infty, \infty)$

$\Rightarrow k = n-m$

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m) \quad \longrightarrow (2)$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$$



$$y(t) = \int_0^t x(\tau)\, d\tau$$

$$y(nH) = \int_0^{nH} x(\tau)\, d\tau$$

$$= \int_0^{(n-1)H} x(\tau)\, d\tau + \int_{(n-1)H}^{nH} x(\tau)\, d\tau$$

$$y(nH) = y\left[(n-1)H\right] + x\left[(n-1)\, H\right] H$$

$\Rightarrow y(n) = y(n-1) + H\,x(n-1)$ //
This is first order difference equation.

→ This is also known as Euler's rule of integration.

<u>Note:</u> • $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 T}$ in continuous time domain.

• But in discrete time domain,

$\omega_0 T = \Omega_0$

If $x_1(n)$ has period N,

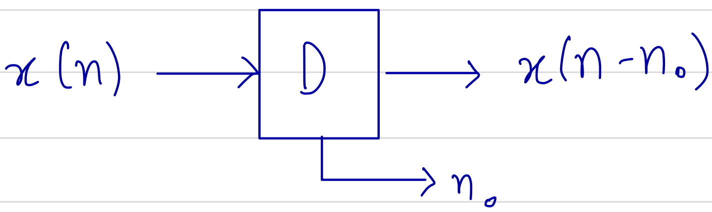
$$e^{j\Omega_0 n} = e^{j\Omega_0 (n+N)} = e^{j\Omega_0 n} \cdot e^{j\Omega_0 N}$$

$$\Rightarrow \Omega_0 N = 2\pi m$$

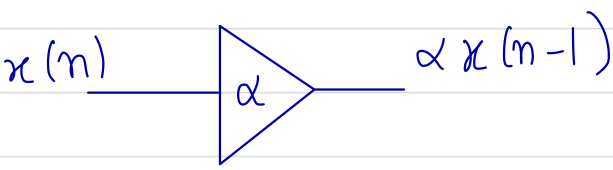
$$\therefore \quad N = \frac{2\pi m}{\Omega_0}$$

Hardwares used in DSP

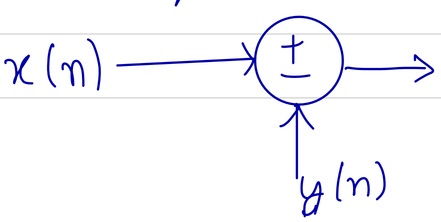
1) Delay unit:



2) Scaler or multiplier



3) Adder / Subtractor



$z(n) = x(n) + y(n)$

LTI - DTS (Linear time invariant - Discrete time system) of order N.

$$Y(n) = Y(n-1) + x(n-1) \rightarrow (1)$$

$$Y(n) = a_1 Y(n-1) + a_2 Y(n-2) + \dots + a_N Y(n-N) + b_0 x(n) + b_1 x(n-1) + \dots + b_N x(n-N) \rightarrow (2)$$

$$Y(n) = \sum_{i=1}^{\infty} a_i y(n-i) + \sum_{k=0}^{\infty} b_k x(n-k)$$

Properties of even and odd symmetric sequences

Given: $x_1(n) = \cos \omega_1 n$ $\quad x_2(n) = \cos \omega_2 n$

$x_3(n) = \sin \omega_3 n$ $\quad x_4(n) = \sin \omega_4 n$

- Delaying a signal by integer multiple of period, it does not change symmetry of signal.

- Let $y_1(n) = x_1(n) + x_2(n)$.

Here both are of even symmetry so $y_1(n)$ is even.

$y_2(n) = x_1(n) + x_3(n)$.

$y_2(n)$ is neither odd nor even.

$y_3(n) = x_3(n) + x_4(n)$. $\quad y_3(n) \rightarrow$ odd

Let $y_4(n) = x_1(n) \cdot x_2(n) \Rightarrow y_4(n) = \text{even}$
$y_5(n) = x_1(n) \cdot x_3(n) \Rightarrow y_5(n) = \text{odd}$
$y_6(n) = x_3(n) \cdot x_4(n) \Rightarrow y_6(n) = \text{even}$

NOTE: odd $\times$ even = odd
odd $\times$ odd = even
even $\times$ even = even

$$x_0(t) = e^{j\omega_0 t} \xrightarrow[t=nT]{} e^{j\omega_0 nT} = e^{jn\Omega_0}$$

$$\Omega_0 = \omega_0 T \qquad x(n) = e^{jn\omega_0}$$

$$T \triangleq \frac{\Omega_0}{\omega_0}$$

$x_k(t) = e^{jk\omega_0 t}$ & $x_k(n) = e^{jk\Omega_0 n}$ are kth harmonics

$$x_k(t+T) = e^{jk\omega_0(t+T)} = e^{jk\omega_0 T} \cdot e^{jk\omega_0 t}$$
$$= e^{jk\omega_0 \frac{2\pi}{\omega_0}} \cdot e^{jk\omega_0 t}$$
$$= e^{jk\omega_0 t}$$
$$= x_k(t)$$

$$x_{(k+N)}(n) = e^{j(k+N)\Omega_0 n}$$
$$= e^{jkn\Omega_0} \cdot e^{jN\Omega_0 n}$$
$$= e^{jkn\Omega_0} \cdot e^{jN \times \frac{2\pi m}{N} n}$$
$$= x_k(n)$$

$$\therefore x_k(n) = x_{k+mN}(n)$$

Discrete time signals

$x_1(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

$x_k(t) = e^{jk\omega_0 t}$

$x_k(n) = e^{jk\omega_0 T_s n} = e^{jk\Omega_0 n}$

$\Rightarrow x(n) = e^{j\Omega_0 n}$

$x_{k+N}(n) = x_k(n)$ N is period of x_k

i.e $N = \dfrac{2m\pi}{\Omega_0}$

Ex: Given $x(t) = \sum_{k=-2}^{2} C_k e^{jk 2\pi/3 t}$ with $T_s = 4s$

Find appropriate values of m & N.

Ans $x(n) = \sum_k d_k e^{jk(2\pi/3) 4n}$

$= \sum_k d_k e^{jk\left(\frac{8\pi}{3}\right)n}$

$N = \dfrac{2\pi m}{\Omega_0}$

$N = \dfrac{2\pi m}{8\pi} \times 3 = \dfrac{3}{4} m$

For $m = 4$, $N = 3$

Discrete time systems

$$y(n) = x(n) * h(n) = \sum_{k=0}^{n} x(k) \cdot h(n-k)$$

$$\delta(n) \longrightarrow \boxed{\text{LTI-DTS}} \longrightarrow h(n)$$

When $h(n) = \delta(n)$

$$Y(n) = x(n) * h(n) = x(n) * \delta(n)$$

$$Y(n) = \sum_{k=0}^{\infty} x(k)\, \delta(n-k)$$

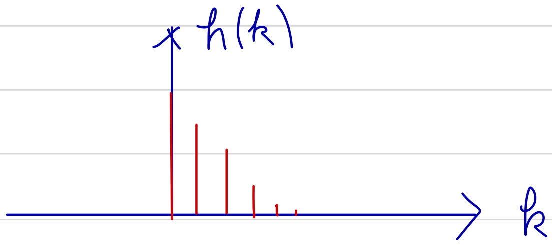
$$Y(n) = \sum_{k=0}^{n-1} x(k)\, \delta(n-k) + x(0)\, \delta(0) + \sum_{n+1}^{\infty} x(k) \cdot \delta(n-k)$$

$$= x(n)$$

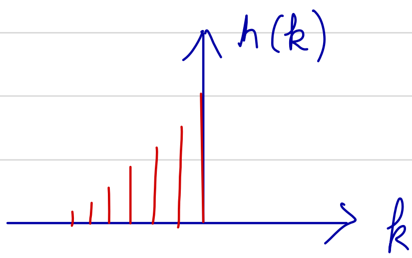
Eg: Given: $x(n) = \{1, -1, \underset{\uparrow}{1}, 2, 0.5, -0.5, 1\}$

$h(n) = \alpha^{n} u(n)$ $\quad \alpha = 0.9$. Find $y(n)$

Ans $h(k) = \alpha^{k} u(k)$



$h(-k) = \alpha^{-k} u(-k)$



NOTE · If x is non zero for $[N_1, N_2]$ and h is non zero for $[N_3, N_4]$.

Then to get non-zero convolution:

$$N_1 + N_3 \leq n \leq N_2 + N_4$$

∴ If x has m samples & h has l samples then, number of samples in $y = m + l - 1$

i.e. $m = N_2 - N_1 + 1$

$l = N_4 - N_3 + 1$

Ex: If $h(n) = \{\underset{\uparrow}{1}, 2, 0, -1, 1\}$ and

$x(n) = \{\underset{\uparrow}{1}, 3, -1, -2\}$. Find $Y(n)$

Ans

$$Y(n) = \sum_k h(k)\, x(n-k)$$

$Y(0) = 1$

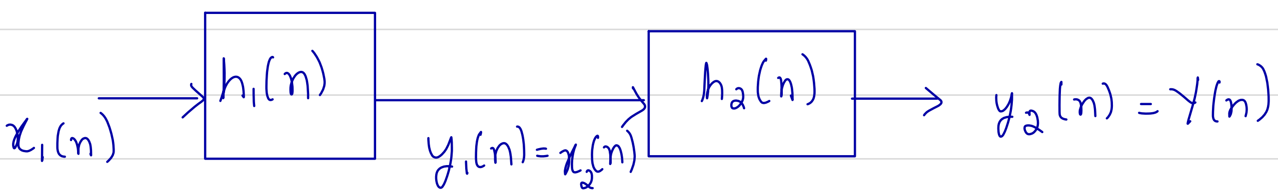
$Y(1) = -1 + 6 = 5$

$\vdots$

$Y(7) =$ some value

$Y(8) \rightarrow Y(\infty) = 0$

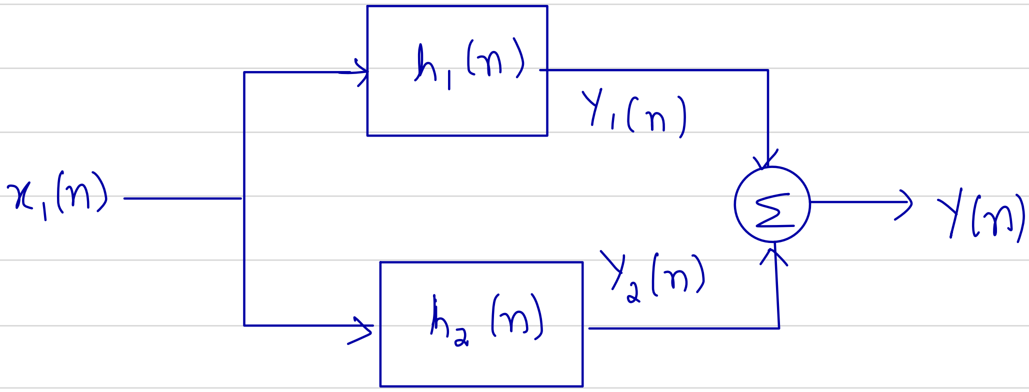
Cascade of parallel arrangements of DT-LTI systems



$y_1(n) = h_1(n) * x_1(n)$

$$Y(n) = x_1(n) * h_1(n) * h_2(n)$$

$$\therefore h_{eq}(n) = h_1(n) * h_2(n) * h_3(n) * \dots * h_m(n)$$



$$h_{eq}(n) = h_1(n) + h_2(n)$$

Periodic Convolution:

$\left.\begin{array}{l} x_1(n) \\ x_2(n) \end{array}\right\}$ Period $= N$ but $x_1(n) \neq x_2(n)$

$$Y(n) = \sum_{k=-\infty}^{\infty} x_1(k) * x_2(n-k)$$

For periodic function, k can be written as $k = rN + l$

$l \rightarrow$ Sample value

$N \rightarrow$ Period $\qquad l = 0, 1, \dots, N-1$

$$\therefore Y(n) = \sum_{r=-\infty}^{\infty} \left[\sum_{l=0}^{N-1} x_1(rN+l)\, x_2(n-rN-l) \right]$$

If [ ] block is positive then $Y(n)$ diverges. So this convolution can't be applied on periodic signals.

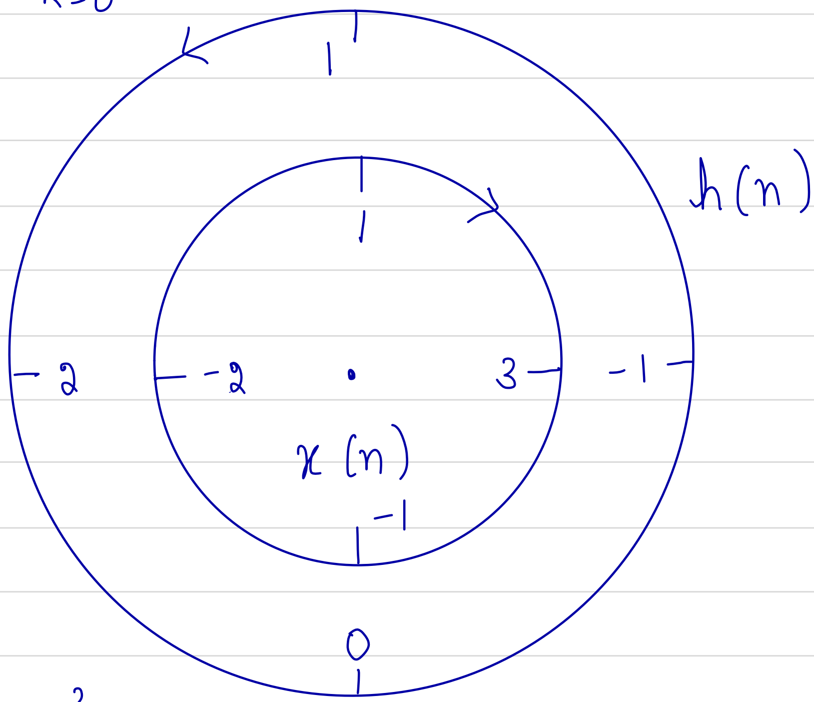
So

$$y_p(n) = \sum_{k=0}^{N-1} x_1(k) \cdot x_2(n-k)$$ is called circular or periodic convolution.

Eg 3: $x(n) = [\underset{\uparrow}{1} \quad 3 \quad -1 \quad -2]$ $h(n) = [\underset{\uparrow}{1} \quad 2 \quad 0 \quad -1]$

Both $x(n)$ & $h(n)$ are periodic with period 4. Find $y_p(n)$

Ans $$y_p(n) = \sum_{k=0}^{3} x(k) \cdot h(n-k)$$



$$y(0) = \sum_{k=0}^{3} x(k)\, h(-k)$$

$$= 1 - 4 + 0 - 3 = -6$$

For $y(1)$ rotate outercycle by 1 unit in clockwise direction:

$$y(1) = \sum_{k=0}^{3} x(k)\, h(1-k)$$

$$y(1) = 2 + 3 + 1 = 6$$

$Y(2) = -1 + 2 + 0 + 6 = 7$

$Y(3) = -2 - 1 + 0 - 2 = -5$

$Y(n) = [\underset{\uparrow}{-6} \quad 6 \quad 7 \quad -5]$

Zero padding

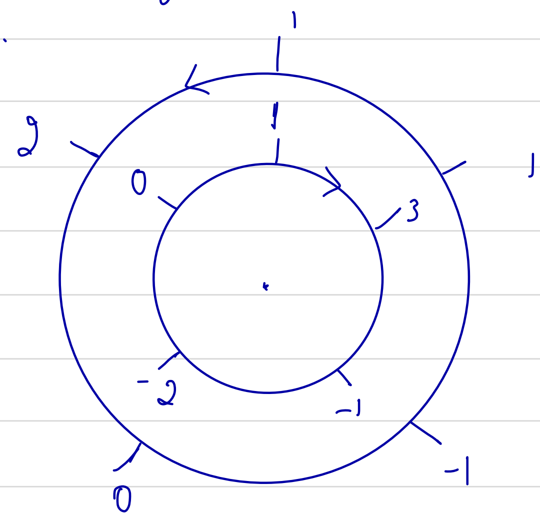
Repeating $\begin{cases} x(n) \rightarrow \{\ \}_{4\times1} \\ h(n) \rightarrow \{\ \}_5 \end{cases} \rightarrow$ LCM = 2, $N \rightarrow 2$

$x(n) = \{1, 3, -1, -2, 0\}_{5\times1}$ $N=5$

$h(n) = \{1, 2, 0, -1, 1\}_{5\times1}$

NOTE: By adding any numbers of zeros the energy wont change.

$Y_p(n) = \sum_{k=0}^{4} x(k)\, h(n-k)$



$Y_p(0) = 5$

$Y_p(1) = 2 + 0 + 2 - 1 + 3 = 6$

$Y_p(2) = 0 + 0 - 2 - 1 + 6 = 3$

$Y_p(3) = -1 + 0 - 2 - 2 + 0 = -5$

$Y_p(4) = 1 + 0 + -4 + 0 - 3 = -6$

In linear convolution

$Y(0) = 1$; $y(5) = 4$ $\Rightarrow$ ($Y(5) = 5$ in circular)

$Y(1) = 5$; $y(6) = 1$ $\Rightarrow$ ($Y(7) = 6$ in circular)

$\rightarrow x(n) = \{1, 3, -1, -2, 0, 0, 0, 0\}_{8\times1}$

$h(n) = \{1, 2, 0, -1, 1, 0, 0, 0\}_{8\times1}$

For above sequences $Y_p(n) \rightarrow 0-7 = Y_l(n)$

NOTE: Adding zeros towards the end, also preserves spectral behaviour of sequence as well as the energy.

Difference equations

$$\sum_{k=0}^{N} a_k Y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Linear constant differential equation of order N

$$\Rightarrow Y(n) = \frac{1}{a_0}\left[-\sum_{k=1}^{N} a_k Y(n-k) + \sum_{k=0}^{M} b_k x(n-k)\right]$$

$$Y(0) = \frac{1}{a_0}\left[-\sum_{k=1}^{N} a_k Y(-k) + \sum_{k=0}^{M} b_k x(-k)\right]$$

For solving above equation we require $Y(-1)$, $Y(-2)$, $Y(-3)$ .... $Y(-N)$ so the given equation is N^{th} order difference equation.

$Y(n) = Y_h(n) + Y_p(n)$

where $Y_h(n)$ i.e when $x(n) = 0$
$Y_p(n)$ i.e for particular $x(n)$

Eg 1> Given $y(n) = x(n) - 3y(n-1)$ with $y(-1) = 0$
& $x(n) = 5n+3$ & $2n^2+n$

Ans Consider $x(n) = 0$

$\Rightarrow y(n) + 3y(n-1) = 0$ || $\alpha \frac{dy}{dt} + \beta y = 0$ $\Rightarrow y = Ae^{\alpha t}$

Similarly let $y(n) = A\alpha^n$

$\therefore A\alpha^n + 3A\alpha^{n-1} = 0$

$\Rightarrow A\alpha^{n-1}[\alpha + 3] = 0$

$A\alpha^{n-1} \neq 0$ $\therefore$ $\alpha + 3 = 0 \Rightarrow \alpha = -3$

$\therefore y_h(n) = A(-3)^n$

Now let $x(n) = 2n^2+n$

$\Rightarrow y(n) + 3y(n-1) = 2n^2 + n + 0$

$\because y(n)$ is linear transform, output should be quadratic as input is quadratic

$\therefore y_p(n) = Bn^2 + Cn + D$

$\therefore Bn^2 + C(n-1) + D + 3[B(n-1)^2 + C(n-1) + D]$
$= 2n^2+n$

$\Rightarrow n^2[B + 3B] + n[C + 3(-2B) + 3C] + D - 3C$
$+ 3D + 3B = 2n^2+n$

$4B = 2 \Rightarrow B = 1/2$

$4C - 6B = 1$

$$\Rightarrow \quad C = \frac{1+6B}{4} = 1$$

$$D = \frac{3(C-B)}{4} = \frac{3}{8} \qquad \therefore \; y_p(n) = \frac{n^2}{2} + n + \frac{3}{8}$$

$$\therefore \quad y(n) = y_h(n) + y_p(n)$$

But $\quad y(-1) = 0$

$$\Rightarrow \quad \frac{n^2}{2} + n + 8 + A(-3)^n = 0 \qquad \text{At } n = -1,$$

$$\frac{1}{2} - 1 + 8 + A(-3)^{-1} = 0$$

$$\Rightarrow \quad \frac{A}{3} = \frac{15}{2} \qquad \Rightarrow \quad A = \frac{45}{2}$$

$$\therefore \quad y(n) = \left(\frac{45}{2}\right)(-3)^n + \frac{n^2}{2} + n + 8$$

<u>NOTE:</u> General form of the particular solution for several types of input signals:

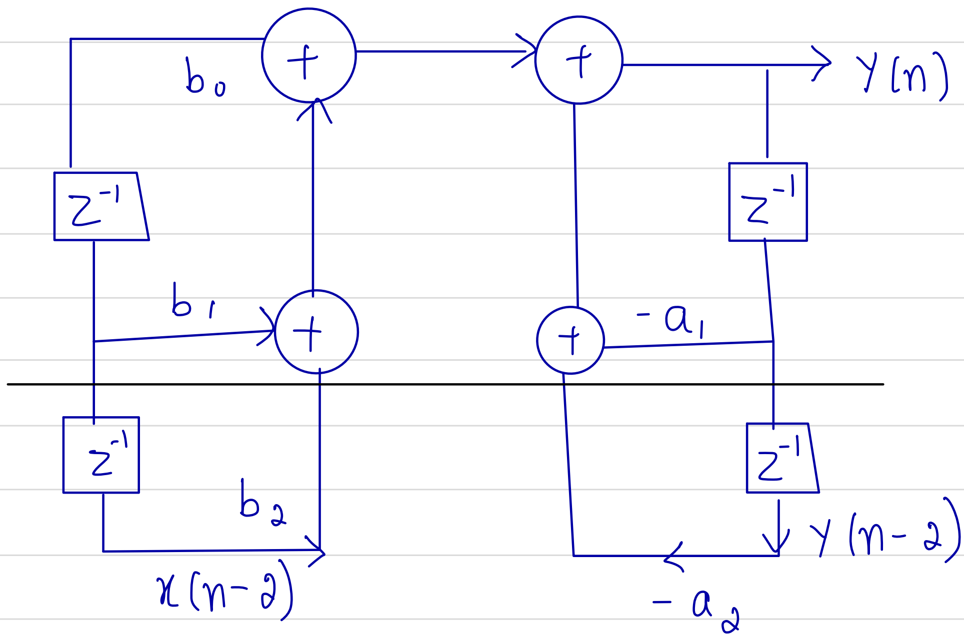
Input	Particular solution:
A	K
AM^n	KM^n
An^M	$K_M n^M + K_{M-1} n^{M-1} + \dots + K_0$
$A\cos\omega_0 n$	$K_1\cos\omega_0 n + K_2 \sin\omega_0 n$
$A\sin\omega_0 n$	$K_1\cos\omega_0 n + K_2 \sin\omega_0 n$

Direct form I

Consider a 1st order system:

$$Y(n) = \sum_{i=0}^{M-1} b_i x(n-i) - \sum_{i=1}^{M-1} a_i y(n-i) \qquad \text{i.e } M=1$$

$$\therefore Y(n) = b_0 x(n) + b_1 (n-1) - a_1 Y(n-1) - a_2 Y(n-2)$$



↓ 2nd order

Stability of LTI - DTS

$$Y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

for any n, $|x(n)| \leq p < \infty$

$$|Y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |x(k)| \, |h(n-k)|$$

$$\leq p \sum_{k=-\infty}^{\infty} |h(n-k)| \leq p \sum_{k=-\infty}^{\infty} |h(k)| \leq M$$

$\therefore$ The system is stable if

$$\sum_{k=-\infty}^{\infty} |h(k)| \leq M < \infty$$

Discrete time fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$\downarrow$

(continuous fourier series)

DTFT:
$$x(n) = \sum_{k} a_k \, e^{jk\Omega_0 n}$$

$$\Omega_0 = \frac{2m\pi}{N}$$

$$\therefore \quad x(n) = \sum_{k} a_k \, e^{j\Omega_k N}$$

$$= \sum_{k} a_k \, x_k$$

$$\therefore \quad x_{k+N}(n) = e^{j(k+N)\cdot\frac{2\pi m}{N} n}$$

$$= e^{jk\frac{2\pi m}{N}}$$

$$= x_k(n)$$

We have,

$$x(n) = \sum_{m=\langle N\rangle} a_m e^{jm\Omega_0 n}$$

$$x(n)\, e^{-jk\Omega_0 n} = \sum_{m=\langle N\rangle} a_m e^{j(m-k)\Omega_0 n}$$

$$\Rightarrow \sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n} = \sum_{n=0}^{N-1} \sum_{m=\langle N\rangle} a_m e^{j(m-k)\Omega_0 n}$$

$$= \sum_{m=\langle N\rangle} a_m \left[\sum_{n=0}^{N-1} \left[e^{j(m-k)\frac{2\pi}{N}} \right]^n \right]$$

Now if $m-k \neq rN$

then $$\sum_{n=0}^{N-1} \left[e^{j(m-k)\frac{2\pi}{N}} \right]^n = \frac{1 - e^{j(m-k)\frac{2\pi N}{N}}}{1 - e^{j(m-k)\frac{2\pi}{N}}} = 0$$

Now if $m-k = rN$

then $$\sum_{n=0}^{N-1} \left[e^{j(m-k)\frac{2\pi}{N}} \right]^n = \sum_{n=0}^{N-1} 1 = N,$$

$\therefore$ [ ] term can be written as $N \cdot \delta(m-k-rN)$ i.e $\begin{cases} N & \text{if } m-k = rN \\ 0 & \text{if } m-k \neq rN \end{cases}$

$$\therefore \sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n} = \sum_{m=\langle N\rangle} N\, a_m\, \delta(m-k-rN)$$

But to remain within fundamental block, $r = 0$

$$\therefore \sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n} = N \sum_{m=0}^{N-1} a_m\, \delta(m-k)$$

But $\delta(m-k) = 1$ only when $m = k$

$$\therefore N a_k = \sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n}$$

$$\boxed{\therefore a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n}}$$

NOTE: • Fourier transform coefficients are either real valued or they are complex conjugate pairs.

• $C_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

Eg 1> Given $x(n) = \{2, -1, 1, 2\}$, $N = 4$. Find a_k

Ans

$$a_k = \frac{1}{4} \sum_{n=0}^{3} x(n) \left(e^{-j\frac{2\pi}{4}}\right)^{kn}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n)\, (-j)^{kn}$$

$$a_0 = \frac{1}{4} \sum_{n=0}^{3} x(n) = \frac{2+2-1+1}{4} = 1$$

$$a_1 = \frac{1}{4} \sum_{n=0}^{3} x(n)\,(-j)^n = \frac{2+j-1+2j}{4} = \frac{1+3j}{4}$$

$$a_2 = \frac{1}{4} \sum_{n=0}^{3} x(n) (-j)^{2k}$$

$$= \frac{1}{4} [2 + 1 + 1 - 2] = \frac{1}{2}$$

$a_3 = \frac{1-3j}{4}$ ($\because a_2 \in \mathbb{R}$, a_3 should be conjugate of a_1)

$$\therefore a_k = \left\{ 1, \frac{1+3j}{4}, \frac{1}{2}, \frac{1-3j}{4} \right\}$$

NOTE $a_k = a^*_{N-k}$ where N is period

DTFS properties:

1> Linearity: $x(n) \longleftrightarrow a_k$, $h(n) \longleftrightarrow b_k$, $y(n) \longleftrightarrow c_k$ } period N

$$y(n) = \alpha x(n) + \beta h(n) \Rightarrow c_k = \alpha a_k + \beta b_k$$

2> Time shift: $x(n) \xrightarrow[\text{delay } (m)]{} y(n) = x(n-m)$,

$x(n)$ is periodic with period N, $x_n \leftrightarrow a_k$ then $y(n)$ is also periodic with period N and $y(n) \longleftrightarrow b_k$

$$b_k = \frac{1}{N} \sum_{n=0}^{N-1} y(n) e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n-m) e^{-jk\Omega_0 n}$$

let $n - m = l$, $n = 0 \Rightarrow l = -m$

$n = N-1 \Rightarrow l = N-1-m$

$$b_k = \frac{1}{N} \sum_{l=-m}^{-m+N-1} x(l)\, e^{-jk\Omega_0 (l+m)}$$

$$= \left(\frac{e^{-jk\Omega_0 m}}{N} \right) \sum_{l=0}^{N-1} x(l)\, x^{-jk\Omega_0 l}$$

$$\boxed{b_k = e^{-jk\Omega_0 m}\, a_k}$$

Also $\quad |b_k| = |a_k|$

3> <u>Modulation</u>: Consider $x(n)$ & $h(n)$ with period N, and $x(n) \neq h(n)$

$x(n) \longleftrightarrow a_k$, $\quad h(n) \longleftrightarrow b_k$, $\quad y(n) \longleftrightarrow C_k$

$y(n) = x(n) \cdot h(n) \Rightarrow C_k = a_k \circledast b_k$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot h(n) \cdot e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} a_k\, e^{jm\Omega_0 n} \right] h(n)\, e^{-jk\Omega_0 n}$$

$$= \sum_{m=0}^{N-1} a_m \cdot \left[\frac{1}{N} \sum_{n=0}^{N-1} h(n)\, e^{-j(k-m)\Omega_0 n} \right]$$

But, $\displaystyle \frac{1}{N} \sum_{n=0}^{N-1} h(n)\, e^{-j(k-m)\Omega_0 n} = b_{k-m}$

$$\therefore C_k = \sum_{m=0}^{N-1} a_k \cdot b_{k-m}$$

$\therefore C_k = a_k \circledast b_k$ $\qquad$ (Periodic convolution)

NOTE $\quad \circledast \rightarrow$ Circular $\qquad * -$ linear

Sampling: Consider impulse train:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \int_{-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{n=-\infty}^{\infty} \delta(t-nT)\right) e^{jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} 1 \cdot e^{jk\omega_0 t}\, dt = \frac{1}{T}$$

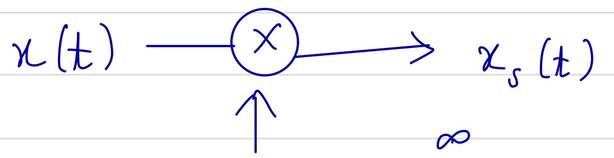
$$\therefore \delta_T(t) = \int_{-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t} = 2\pi\delta(\omega - k\omega_0)$$

Discrete time Fourier Transform (DTFT)

$$x(t) \xrightarrow{CTFT} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) \xrightarrow{\text{sampling}} x_s(t)$$



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) \xrightarrow{CTFT} X_s(\omega)$$

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$$

$$\therefore X_s(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

We can represent $x(nT_s)$ as $x(n)$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n T_s}$$

$\omega T_s = \Omega$

$$\therefore X_s(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega T_s n}$$

→ Ω is continuous frequency variable defined for sequence.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

Now replace Ω with $\Omega \pm 2m\pi$

$$X(\Omega \pm 2m\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega \pm 2m\pi) n}$$

$$= X(\Omega)$$

Inverse DTFT

$X(\Omega) = g_1(n)$

$x(n) = g_2(\Omega)$

$$X(\Omega) = \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\Omega p}$$

$$\int_{\langle 2\pi \rangle} X(\Omega)\, e^{j\Omega n}\, d\Omega = \int_{\langle 2\pi \rangle} \left[\sum_{p=-\infty}^{\infty} x(p)\, e^{-j\Omega p}\right] e^{j\Omega n}\, d\Omega$$

$$\Rightarrow \int_0^{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega = \sum_{p=-\infty}^{\infty} x(p) \int_0^{2\pi} e^{(n-p)\Omega}\, d\Omega$$

if $n = p$, $\int_0^{2\pi} e^{j(n-p)\Omega} = 2\pi$

else $\int_0^{2\pi} e^{j(n-p)\Omega} = 0$

$\therefore \int_0^{2\pi} e^{j(n-p)\Omega} = 2\pi\, \delta(n-p)$

$$\therefore \int_0^{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega = \sum_{p=-\infty}^{\infty} x(p) \cdot 2\pi\, \delta(n-p)$$

$$= x(n) \cdot 2\pi$$

$$\therefore \boxed{x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega}$$

Eg: Given: $x(n) = a^n\, u(n)$ with $|a| < 1$.
Find $X(\Omega)$ and sketch it.

Ans

$$x(n) = \begin{cases} a^n & ; \quad n \geq 0 \\ 0 & ; \quad n < 0 \end{cases}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

Here $|a e^{-j\Omega}| < 1$

$$\therefore X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}$$

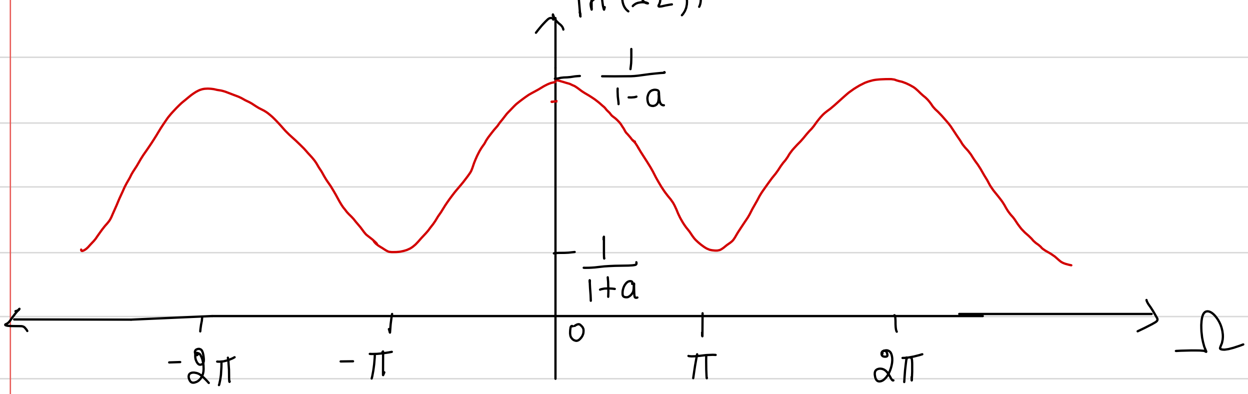
$$\therefore X(\Omega) = \frac{1}{1 - a\cos\Omega + ja\sin\Omega}$$

$$|X(\Omega)| = \frac{1}{\sqrt{(1-a\cos\Omega)^2 + (a\sin\Omega)^2}}$$

$$= \frac{1}{\sqrt{1+a^2 - 2a\cos\Omega}}$$

at $\Omega = 0$, $|X(0)| = \dfrac{1}{1-a}$

at $\Omega = \pm\pi$, $|X(\pm\pi)| = \dfrac{1}{1+a}$



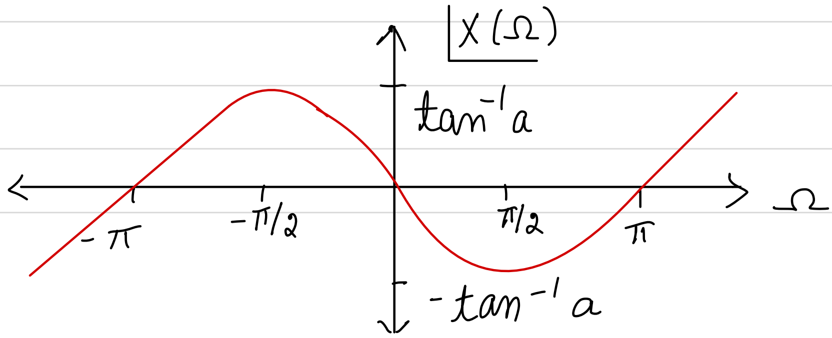
$$\angle X(\Omega) = -\tan^{-1}\left(\frac{a\sin\Omega}{1-a\cos\theta}\right)$$

at $\Omega = 0$, $\angle X(0) = 0$

at $\Omega = \pm\pi$, $\angle X(\pm\pi) = 0$

at $\Omega = +\pi/2$ $\angle X(\pi/2) = -\tan^{-1} a$

at $\Omega = -\pi/2$ $\angle X(-\pi/2) = \tan^{-1} a$



NOTE: If $x(n) = e^{j\Omega_0 n} \Rightarrow X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$

If $x(n)$ is periodic,
$X_p(\Omega) = 2\pi\delta(\Omega - \Omega_0 - 2m\pi)$

Consider $X_p(\Omega)$,

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X_p(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} \delta(\Omega - \Omega_0 - 2m\pi) \times 2\pi \cdot e^{j\Omega n}\, d\Omega$$

But $m = 0$ for range $(-\pi, \pi)$

So $x(n) = \frac{1}{2\pi} \times 2\pi \int_{-\pi}^{\pi} \delta(\Omega - \Omega_0)\, e^{j\Omega n}\, d\Omega$

$x(n) = e^{j\Omega_0 n}$ //

DTFT properties

1) Periodicity: $X(\Omega) = X(\Omega \pm 2m\pi)$

2) Linearity: $x_1(n) \longleftrightarrow X_1(\Omega)$
$x_2(n) \longleftrightarrow X_2(\Omega)$
$y(n) = \alpha x_1(n) + \beta x_2(n)$
$\Rightarrow Y(\Omega) = \alpha X_1(\Omega) + \beta X_2(\Omega)$

3> <u>Time shift</u>; If $Y(n) = x(n-m)$

$$\Rightarrow \quad Y(\Omega) = \sum_{n=-\infty}^{\infty} x(n-m)\, e^{-j\Omega n}$$

let $n-m = l$

$$\Rightarrow \quad Y(\Omega) = \sum_{l=-\infty}^{\infty} x(l)\, e^{-j\Omega(l+m)}$$

$$= e^{-j\Omega m} \sum_{l=-\infty}^{\infty} x(l)\, e^{-j\Omega l}$$

$$\therefore \quad Y(\Omega) = e^{-j\Omega m}\, X(\Omega)$$

$$\Rightarrow \quad \operatorname{Arg} Y = \operatorname{Arg} X - (m\Omega)$$

4> <u>Frequency shift</u>: $Y(n) = x(n)\, e^{j\Omega_0 n}$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\Omega_0 n}\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega-\Omega_0)n}$$

$$\therefore Y(\Omega) = X(\Omega - \Omega_0)$$

5> <u>Gradient in frequency domain</u>:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n)\, \frac{d}{d\Omega}\left(e^{-j\Omega n}\right)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\, (-jn)$$

$$\Rightarrow \quad j\frac{dX}{d\Omega} = \sum_{n=-\infty}^{\infty} n\,x(n)\,e^{-j\Omega n}$$

$$\therefore \quad j\frac{dX}{d\Omega} = \text{DTFT}[n\,x(n)]$$ //

Eg: Given: $x(n) = n a^n u(n)$

Ans We have $\text{DTFT}[n\,x(n)] = j\dfrac{dX}{d\Omega}$

$$X(\Omega) = \frac{1}{1 - a\cos\Omega + ja\sin\Omega}$$

$$\frac{dX}{d\Omega} = \frac{-1\,(a\sin\Omega + ja\cos\Omega)}{(1 - a\cos\Omega + ja\sin\Omega)^2}$$ //

$$\therefore \quad \text{DTFT}[n\,x(n)] = \frac{-j\,(a\sin\Omega + ja\cos\Omega)}{(1 - a\cos\Omega + ja\sin\Omega)^2}$$ //

NOTE: $\text{DTFT}[n^k x(n)] = (j)^k \dfrac{d^k(X(\Omega))}{d\Omega^k}$ //

6> <u>Convolution in time domain</u>:

$$x(n) \longleftrightarrow X(\Omega)$$
$$h(n) \longleftrightarrow H(\Omega)$$

CS: (Convolution sum) $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-jn\Omega} \rightarrow (2)$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)\, e^{-jn\Omega}$$

let $n-k=l \Rightarrow n=k+l$

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} h(l)\, e^{-j(k+l)\Omega}$$

$$= \left[\sum_{k=-\infty}^{\infty} x(k)\, e^{-jk\Omega}\right] \sum_{l=-\infty}^{\infty} h(l)\, e^{-jl\Omega}$$

$$= X(\Omega) \cdot H(\Omega)$$

$$\Rightarrow H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

$$\Rightarrow H(\Omega) = \frac{|Y(\Omega)|\, e^{j\angle Y(\Omega)}}{|X(\Omega)|\, e^{j\angle X(\Omega)}}$$

$$= \frac{|Y(\Omega)|}{|X(\Omega)|}\, e^{j\Delta\theta} \quad \text{where } \Delta\theta = \angle Y(\Omega) - \angle X(\Omega)$$

NOTE:

1, 5 → Low frequency region

2, 4 → Medium frequency region

3 → High frequency region

Eg: Given: $H(\Omega) = \begin{cases} 1 & 0 \leq |\Omega| \leq \Omega_c \\ 0 & \Omega_c < |\Omega| < \pi \end{cases}$. Find $h(n)$

Ans

$$h(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\Omega) \, e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega) \, e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} \, d\Omega = \frac{1}{2\pi} \left. \frac{e^{j\Omega n}}{n} \right]_{-\Omega_c}^{\Omega_c}$$

$$= \frac{1}{2\pi n} \quad e^{j\Omega_c n} - e^{-j\Omega_c n} = \frac{\sin \Omega_c n}{\pi n}$$

$$= \frac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n)$$

$h(n) \rightarrow$ IIR filter

7) **Modulation:**

$x(n) \longrightarrow$ message $\longleftrightarrow X(\lambda)$

$c(n) \longrightarrow$ carrier $\longleftrightarrow C(\Omega)$

$$Y(n) = x(n) \cdot c(n)$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n) \, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot c(n) \, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) \, e^{jn\lambda} \, d\lambda \right] \cdot c(n) \, e^{-jn\Omega}$$

$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) \left[\sum_{n=-\infty}^{\infty} c(n)\, e^{-jn(\Omega-\lambda)} \right] d\lambda$$

$$Y(\Omega) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) \cdot C(\Omega-\lambda)\, d\lambda$$

$$Y(\Omega) = X(\Omega) * C(\Omega)$$

8) FT of a periodic sequence :

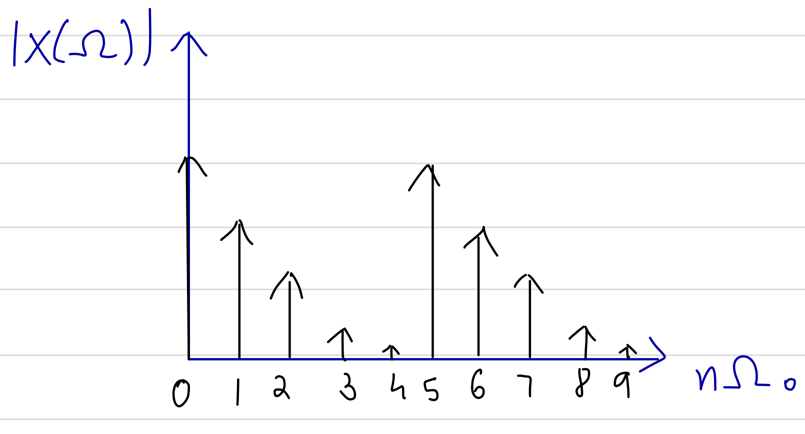
If $\left. x(n) \right|_N = \sum_{k=0}^{N-1} a_k\, e^{jk\Omega_0 n}$

But $e^{kj\Omega_0 n} \longleftrightarrow 2\pi\delta(\Omega - k\Omega_0)$

$$\therefore X(\Omega) = \sum_{k=0}^{N-1} a_k\, 2\pi\delta(\Omega - k\Omega_0)$$

NOTE : Suppose $N = 5$, $|a_0| > |a_1| > |a_2| > |a_3| > |a_4|$

$\Rightarrow$



Bandwidth of AM = 10 kHz

" " FM = 200 kHz

" " TV channels = 6 MHz

Sampled data or hybrid system

Considered sampled signal: $x_s(t)$, sampled on $x_a(t)$ at nT.

$$\Rightarrow X_a(\omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\omega t}\, dt.$$

Also
$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\omega t}\, dt$$
$$= \int_{-\infty}^{\infty} x_a(nT)\, e^{-j\omega nT}\, dt$$

But
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\Omega}\, d\Omega$$

and
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{jn\Omega}\, d\Omega \quad \rightarrow (1)$$

$$x_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_s(\omega)\, e^{j\omega t}\, d\omega \quad \rightarrow (2)$$

$$x_s(t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \int_{\frac{(2m-1)\pi}{T}}^{\frac{(2m+1)\pi}{T}} X_s(\omega)\, e^{j\omega t}\, d\omega \quad \rightarrow (3)$$

let $\omega = \omega + \frac{2m\pi}{T}$ and $t = nT$

$$x_s(\omega) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} X_s\left(\omega + \frac{2m\pi}{T}\right) e^{j\left[\omega + \frac{2m\pi}{T}\right]nT}\, d\omega$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} X_s\left(\omega + \frac{2m\pi}{T}\right) e^{j\omega Tn} \quad \rightarrow (4)$$

Sampled data systems

$x_a(t) \xrightarrow{\text{sampling}}$ @ $t = nT$ $\Rightarrow x_a(nT) = x(n)$

$\updownarrow$

$X_a(\omega)$ $\qquad x_s(t) \longleftrightarrow X_s(t)$

$$\Rightarrow x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)\, e^{j\omega t}\, dt \quad -(1)$$

$$x_a(nT) = x(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)\, e^{j\omega nT}\, d\omega \rightarrow (2)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{jn\Omega}\, d\Omega \rightarrow 3 \qquad (\Omega = \omega T)$$

$$2 \Rightarrow x(n) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{(2m-1)\frac{\pi}{T}}^{(2m+1)\frac{\pi}{T}} X_a\left(\omega + \frac{2m\pi}{T}\right) e^{jnT\left(\omega + \frac{2m\pi}{T}\right)} d\omega \rightarrow 4$$

$$\omega = \omega + \frac{2m\pi}{T}$$

$$x(n) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\pi/T} X_a\left(\omega + \frac{2m\pi}{T}\right) \cdot e^{jnT\left(\omega + \frac{2m\pi}{T}\right)} d\omega \rightarrow 5$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\pi/T} X_a\left(\omega + \frac{2m\pi}{T}\right) e^{jnT\omega}\, d\omega$$

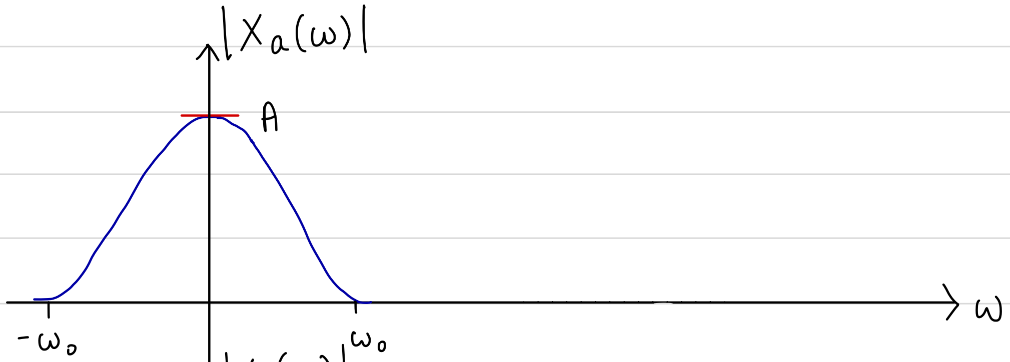
$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} X_a\left(\frac{\Omega}{T} + \frac{2m\pi}{T}\right) e^{jn\Omega}\, \frac{d\Omega}{T}$$

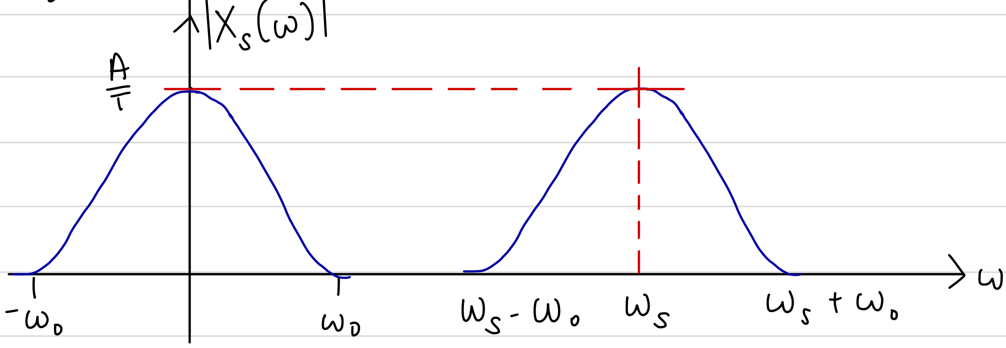
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{m=-\infty}^{\infty} X_a\left(\frac{\Omega}{T} + \frac{2m\pi}{T}\right)\right] e^{jn\Omega}\, d\Omega$$

from equation ③,

$$X(\Omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_a\left(\frac{\Omega}{T} + \frac{2m\pi}{T}\right) \rightarrow ⑦$$

$$X_s(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_a(\omega + m\omega_n) \quad \text{where} \quad \omega_s = \frac{2\pi}{T}$$





Discrete Fourier transform (DFT)

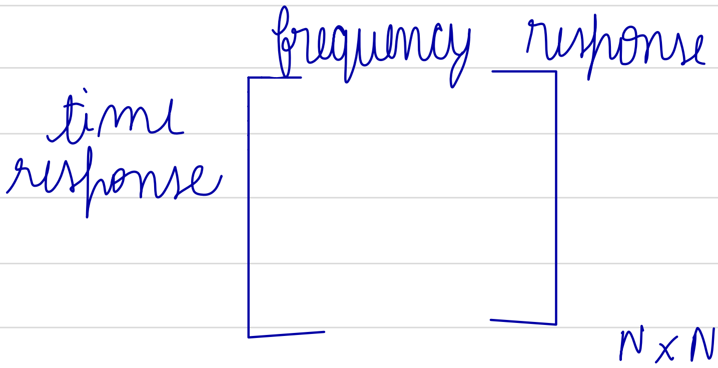
Truncation: $x_a(t) = \alpha^t u(t)$ where $\alpha < 1$

$x_a(t) \xrightarrow{\text{Sampling}} x_a(nT) = x_a(n)$

$x_a(n) \times w(n) = x(n)$

where $w(n) = \begin{cases} 1 & 1 \leq n \leq N-1 \\ 0 & n > N-1 \end{cases}$

DFT matrix:



The elements of this matrix are complex numbers

$$X(\Omega) = \sum_{n=0}^{N-1} x(n)\, e^{-jn\Omega_0} \rightarrow (1)$$

$$X(\Omega_k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_k n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-jn\frac{2\pi . K}{N}} \quad \text{for } K = 0, 1, \ldots N-1$$

NOTE: $x_a(nT) \rightarrow x(n)$

$X(\Omega_k) \rightarrow X(k)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} \rightarrow (2)$$

Also $$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n(k+N)}{N}}$$

$$= X(k)$$

NOTE $x(n) \longrightarrow \boxed{\text{DFT}} \longrightarrow X(k) \longrightarrow \boxed{\text{IDFT}} \longrightarrow x(n)$

$\uparrow$ Aperiodic

$\downarrow$ Periodic with period N

Inverse DFT (IDFT)

From DFT we have

$$X(k) = \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi}{N} lk}$$

Multiply both sides $e^{j\frac{2\pi}{N}kn}$

$$\Rightarrow X(k)\, e^{j\frac{2\pi nk}{N}} = \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi lk}{N}} \cdot e^{+j\frac{2\pi nk}{N}}$$

$$\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N} lk} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi k}{N}(l-n)}$$

$$= \sum_{l=0}^{N-1} x(l) \sum_{k=0}^{N-1} \alpha^k$$

where $\alpha = e^{-j\frac{2\pi}{N}(l-n)}$

$$\Rightarrow \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} = \sum_{l=0}^{N-1} x(l) \cdot \left(\frac{1-\alpha^N}{1-\alpha}\right)$$

But $$\frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{-j\frac{2\pi k(l-n)}{N}\cdot N}}{1-e^{-j\frac{2\pi(l-n)k}{N}}}$$

$$\Rightarrow \text{sum} = \begin{cases} N, & l=n \\ 0, & l\neq n \end{cases} = N\,\delta(l-n)$$

$$\therefore \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} = \sum_{l=0}^{N-1} x(l)\, N\,\delta(l-n)$$

$$= N \sum_{l=0}^{N-1} x(l)\cdot\delta(l-n)$$

$$= N\,x(n)$$

$$\therefore \boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}}}$$

Also $x(n+mN) = x(n)$

Properties of DFT

1> **Linearity:** $x_1(n) \longleftrightarrow X_1(k)$

$x_2(n) \longleftrightarrow X_2(k)$

If $\quad y(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\Rightarrow \quad Y(n) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] \, e^{-j\frac{2\pi n}{N}k}$$

$$Y(n) = a_1 X_1(n) + a_2 X_2(n)$$

2> **Time shift:** If $\quad y(n) = x(n-m)$

$$Y(k) = \sum_{n=0}^{N-1} x(n-m) \, e^{-\frac{2\pi n k}{N}}$$

Let $\quad n-m = l$

$$\Rightarrow \quad Y(k) = \sum_{l=m}^{m+N-1} x(l) \, e^{-j\frac{2\pi(m+l)k}{N}}$$

$$\Rightarrow \quad Y(k) = \left[\sum_{l=0}^{N-1} x(l) \, e^{-\frac{2\pi l k}{N}} \right] e^{-j\frac{2m\pi k}{N}}$$

$$\therefore \; Y(k) = e^{-j\frac{2\pi m k}{N}} \, X(k)$$

Also $|Y(k)| = |X(k)|$

3> **Convolution in time:**

$$Y(k) = H(k) \cdot X(k)$$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \cdot X(k)\, e^{j\frac{2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} h(m)\, e^{-j\frac{2\pi mk}{N}} \right] X(k)\, e^{j\frac{2\pi nk}{N}}$$

$$= \sum_{m=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi (n-m)k}{N}} \right] h(m)$$

$$y(n) = \sum_{m=0}^{N-1} h(m) \cdot x(n-m)$$

$\Rightarrow y(n) = h(n) \circledast x(n)$ → circular convolution

Eg: $x(n) = \{1, 2, 0, -1\}$ $N=4$, $h(n) = \{1, 3, -1, 2\}$ $N=4$
Estimate $y(n)$ using DFT properties.

Ans

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi nk}{4}} = \sum_{n=0}^{3} x(n)\,(-j)^{nk}$$

$$X(0) = \sum_{n=0}^{3} x(n) = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)\,(-j)^n = 1 - 2j - j = 1 - 3j$$

$$X(2) = \sum_{n=0}^{3} x(n)\,(-1)^n = 1 - 2 + 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)\,(j)^n = 1 + 2j + j = 1 + 3j$$

$$H(k) = \sum_{n=0}^{3} h(n)(-j)^{nk}$$

$$H(0) = \sum_{n=0}^{3} h(n) = 5$$

$$H(1) = \sum_{n=0}^{3} h(n)(-j)^{n} = 1-3j+1+2j = 2-j$$

$$H(2) = \sum_{n=0}^{3} h(n)(-1)^{n} = 1-3-1-2 = -5$$

$$H(3) = 2+j$$

$$\therefore X(k) = \{2, 1-3j, 0, 1+3j\}$$

$$\therefore H(k) = \{5, 2-j, -5, 2+j\}$$

$$\therefore Y(k) = X(k)\cdot H(k)$$

$$= \{10, -1-7j, 0, -1+7j\}$$

$$\therefore y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{j\frac{2\pi nk}{4}}$$

$$= \frac{1}{4}\sum_{k=0}^{3} Y(k)(j)^{nk}$$

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k) = 2 /\!/$$

$$y(1) = \frac{1}{4}\sum_{k=0}^{3} Y(k)(j)^{n} = \frac{10-j+7+j+7}{4} = 6 /\!/$$

$$y(2) = \frac{1}{4}\sum_{k=0}^{3} Y(k)(-1)^{n} = \frac{10+1+7j+1-7j}{4} = 3 /\!/$$

$$y(3) = \frac{1}{4}\sum_{k=0}^{3} Y(k)(-j)^{n} = \frac{10+1j-7-j-7}{4} = -1 /\!/$$

$$\therefore y(n) = [2, 6, 3, -1] /\!/$$

DFT estimate in matrix form

$x(n) \rightarrow$ finite sample : N i.e $n = 0, 1, \dots N-1$

$x(n) \longrightarrow \vec{x}_{N\times 1}$

$\Updownarrow$

$X(k) \longrightarrow \vec{X}_{N\times 1}$: $k = 0, 1, 2, \dots, N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} \rightarrow (1)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k n}{N}} \rightarrow (2)$$

Let $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n) \left(W_N\right)^{kn}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} = \begin{array}{c|c} & n \rightarrow 0 \qquad\qquad\qquad\qquad N-1 \\ \hline \begin{array}{c} k \downarrow 0 \\ \\ \\ N-1 \end{array} & \begin{bmatrix} W_N^0 & W_N^0 & \dots & W_N^0 \\ W_N^0 & W_N^{1,1} & \dots & W_N^{1,N-1} \\ \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1,1} & \dots & W_N^{N-1,N-1} \end{bmatrix}_{N\times N} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

$$\Rightarrow \vec{X} = W \vec{x}$$

$$\therefore \vec{x} = W^{-1} \vec{X}$$

Also $W^T = W$

From equation (2)

$$\vec{x} = W^{-1}\vec{X} = \frac{1}{N} W^{*}\vec{X}$$

$$\vec{x} = \frac{1}{N} W^{T*}\vec{X}$$

But $\vec{x} = W^{-1}\vec{X}$

$$\Rightarrow \frac{W^{T*}}{N} = W^{-1} \Rightarrow W^{T*} W = N I_N$$

NOTE: If $A^{*T} A = I$ then A is unitary matrix.

We have
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi nk}{N}}$$

$$= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left[\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}\right] e^{j\frac{2\pi nk}{N}}$$

$$= \sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)\, \frac{e^{-j\frac{2\pi nk}{N}}}{\sqrt{N}}\right] \frac{e^{j\frac{2\pi nk}{N}}}{\sqrt{N}}$$

Now let $W_N = \dfrac{e^{-j\frac{2\pi nk}{N}}}{\sqrt{N}}$

Now $W^{*T} \cdot W = I_N$

Also $N = 2^{r}$

NOTE: $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$ $\Rightarrow$ $\underbrace{\begin{array}{l} N \text{ complex multiplication} \\ N-1 \text{ complex addition} \end{array}}_{\text{for } N \text{ values of } k}$

DFT & FFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$ requires for 1 loop :

1> N complex multiplication
2> N-1 complex addition.

- If N is very large, $N-1 \approx N$, then for N loops, it requires
  N^2 complex multiplication and
  N^2 complex addition

- If N is very large, computation takes lot of time.

Fast Fourier transform (FFT)

→ It is a radix 2 algorithm.
→ 2^n samples requires n stages of computation.

1) Decimation in time (DIT) algorithm:

$$\underbrace{x(n)}_{N} = \underbrace{x_e(n)}_{\frac{N}{2}} \ \& \ \underbrace{x_o(n)}_{\frac{N}{2}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \quad \rightarrow (1)$$

$$= \sum_{n=\text{even}} x(n)\, W_N^{Nk} + \sum_{n=\text{odd}} x(n)\, W_N^{nk} \quad \rightarrow (2)$$

In equation (2) let
$$\left.\begin{aligned} n &= 2l \\ n &= 2l+1 \end{aligned}\right\} \ l = 0, 1, \cdots \frac{N}{2} - 1$$

$$\therefore X(k) = \sum_{l=0}^{N/2-1} x(2l)\, W_N^{2lk} + \sum_{l=0}^{N/2-1} x(2l+1)\, W_N^{(2l+1)} \quad \rightarrow (3)$$

NOTE: $x(n)\big|_N$ = N samples,

Normal DFT ⇒ N^2 - CM & N^2 CA

Now

N

$N/2$ $\quad$ $N/2 = M$

$x_1(n)$ $\quad$ $x_2(n)$

$\updownarrow$ $\quad$ $\updownarrow$

$X_1(k)$ $\quad$ $X_2(k)$

$$\Rightarrow \left\{ \begin{aligned} CM &= M^2 = N^2/4 \\ CA &= M^2 = N^2/4 \end{aligned} \right\} \text{2 times}$$

For complex adder:

Without decomposition $\rightarrow N^2$

with one decomposition $\rightarrow \frac{N^2}{4} \times 2 + N$

Now if $N = 10^5$ without decomposition $= 10^{10}$

with one decomposition $= \frac{10^{10}}{2} + 10^5 \approx 5 \times 10^9$

i.e 50% time saving

equation (3) $\Rightarrow$

$$X(k) = \sum_{l=0}^{N/2-1} x(2l)\, W_N^{2lk} + \sum_{l=0}^{N/2-1} x(2l+1)\, W_N^{(2l+1)k}$$

let $x(2l) = g(l)$ & $x(2l+1) = h(l)$

then,

$$X(k) = \underbrace{\sum_{l=0}^{N/2-1} g(l)\, W_N^{2lk}}_{\mathrm{I}} + W_N^k \underbrace{\sum_{l=0}^{N/2-1} h(l)\, W_N^{2lk}}_{\mathrm{II}} \rightarrow (4)$$

here $W_N^K = e^{-j\frac{2\pi}{N}}$

If $y(n)$ has $N/2$ samples,

$$Y(k) = \sum_{n=0}^{N/2-1} y(n)\, W_{N/2}^{nk} = \sum_{n=0}^{N/2-1} y(n)\, W_N^{2nk}$$

i.e $W_{N/2}^{nk} = e^{-j\frac{2\pi}{N/2}} = e^{-j\frac{2\pi \times 2}{N}} = W_N^{2nk}$

From equation (4)

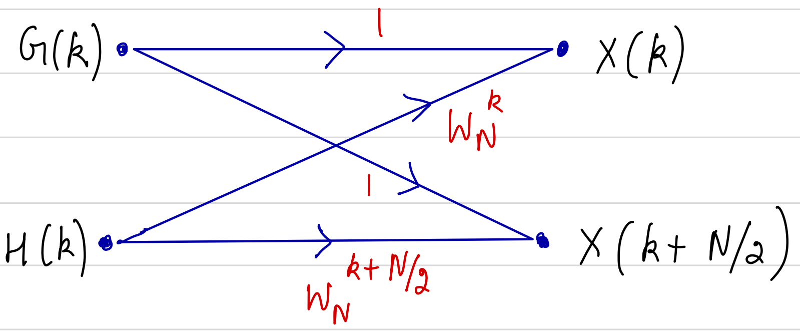
$$X(k) = G(k) + W_N^k H(k) \rightarrow (5)$$

(5) $\Rightarrow$ $X(k) = G(k) + W_N^k H(k) \rightarrow$ (6a)

$$X(k+N/2) = G(k) + W_N^{k+N/2} H(k) \rightarrow (6b)$$

$G(k+N/2) = G(k)$ & $H(k+N/2) = H(k)$ since both are periodic with period $N/2$

From 6a & 6b, we get following signal flow graph



From equation 5, requirement for this decomposition:

CM: $\frac{N^2}{4} \times 2 + N = \frac{N^2}{2} + N$

CA: $\frac{N^2}{4} \times 2 + N = \frac{N^2}{2} + N$

Now, for an 8 point DFT, DTT can be applied as:

$x(n) = \{x(0), x(1), \ldots, x(7)\}_{8\times 1}$

$g(l) = \{x(0), x(2), x(4), x(6)\}_{4\times 1}$

$h(l) = \{x(1), x(3), x(5), x(7)\}_{4\times 1}$

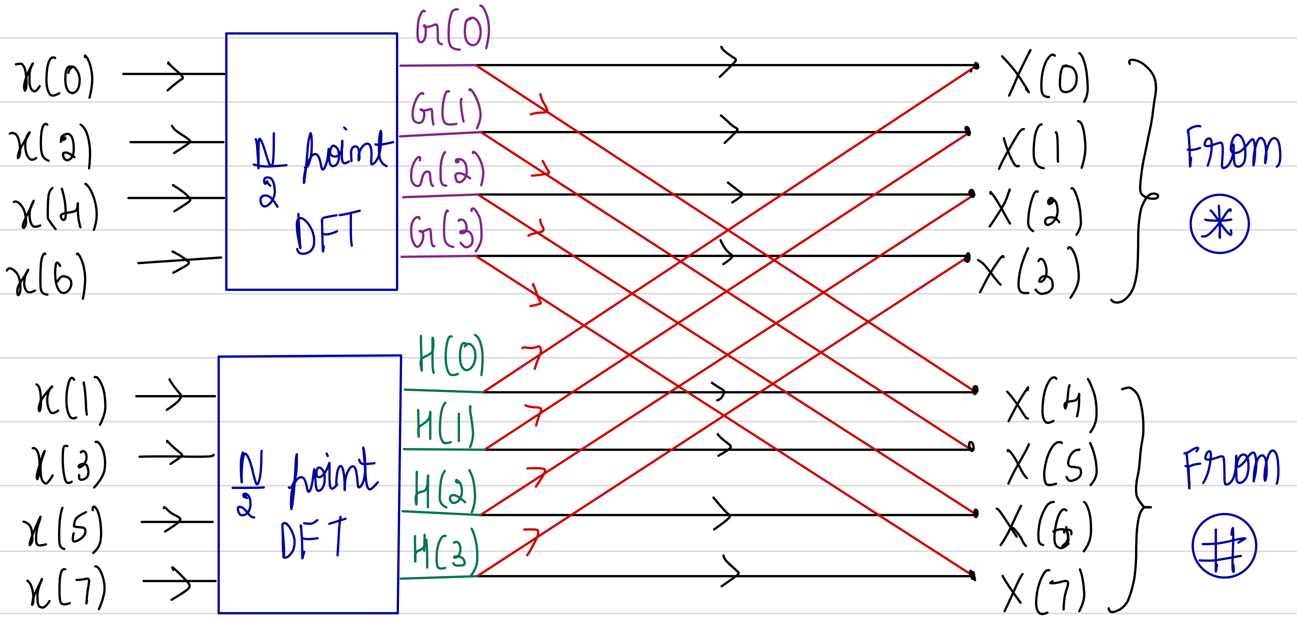
From 6a & 6b :

For $k = 0, 1, 2, 3$

$X(k) = G(k) + W_N^k H(k) \rightarrow$ *

$X(k+4) = G(k) + W_N^{k+4} H(k) \rightarrow$ #

This can be visualised as :



If we again decompose:

$$g(l) \rightarrow \begin{cases} g_1(l) = \{x(0), x(4)\} \longleftrightarrow G_1(k) \\ h_1(l) = \{x(2), x(6)\} \longleftrightarrow H_1(k) \end{cases}$$

$$h(l) \rightarrow \begin{cases} \tilde{g}(l) = \{x(1), x(5)\} \longleftrightarrow \tilde{G}(k) \\ \tilde{h}(l) = \{x(3), x(7)\} \longleftrightarrow \tilde{H}(k) \end{cases}$$

For 2nd decomposition $k = 0, 1$

$$G(k) = G_1(k) + W_{N/2}^{k} H_1(k) \rightarrow a$$
$$G(k+2) = G_1(k) + W_{N/2}^{k+2} H_1(k) \rightarrow b$$

$$H(k) = \tilde{G}(k) + W_{N/2}^{k} \tilde{H}(k) \rightarrow c$$
$$H(k+2) = \tilde{G}(k) + W_{N/2}^{k+2} \tilde{H}(k) \rightarrow d$$

$a \Rightarrow \quad G(k) = G_1(k) + W_N^{2k} H_1(k)$

$b \Rightarrow \quad G(k+2) = G_1(k) + W_N^{2(k+2)} H_1(k)$

$c \Rightarrow \quad H(k) = \tilde{G}(k) + W_N^{2k} \tilde{H}(k)$

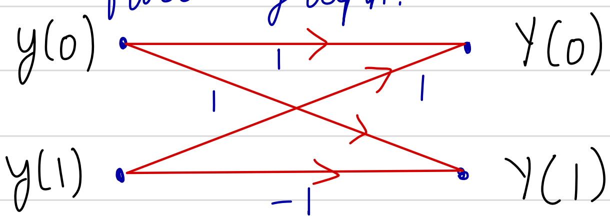
$d \Rightarrow \quad H(k+2) = \tilde{G}(k) + W_N^{2k+4} \tilde{H}(k)$

NOTE: If $y(n)$ has 2 points, $W_N \triangleq e^{-\frac{2\pi}{N}j} = e^{-2\pi/2j}$
$= e^{-j\pi} = -1$

$$\Rightarrow Y(k) = \sum_{n=0}^{1} y(n)\,(-1)^{nk}$$

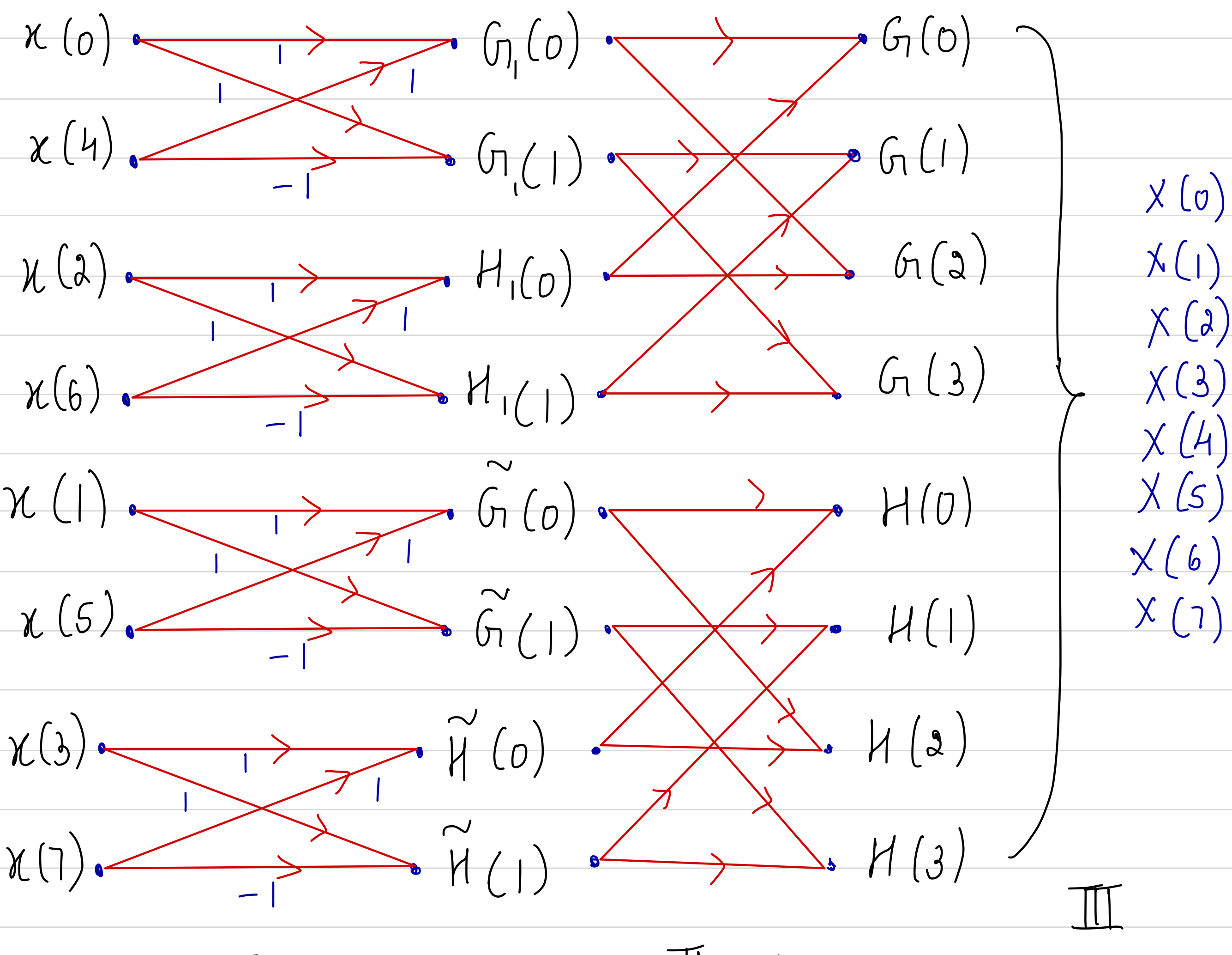
for $k = 0, 1$; $Y(0) = y(0) + y(1)$
$Y(1) = y(0) - y(1)$

Signal flow graph:



← Butterfly structure

For 8 point DFT:



Number of stages = $\log_2(n)$

$\therefore$ Total time complexity $O(n) = n \log_2 n$

Bit reversal :

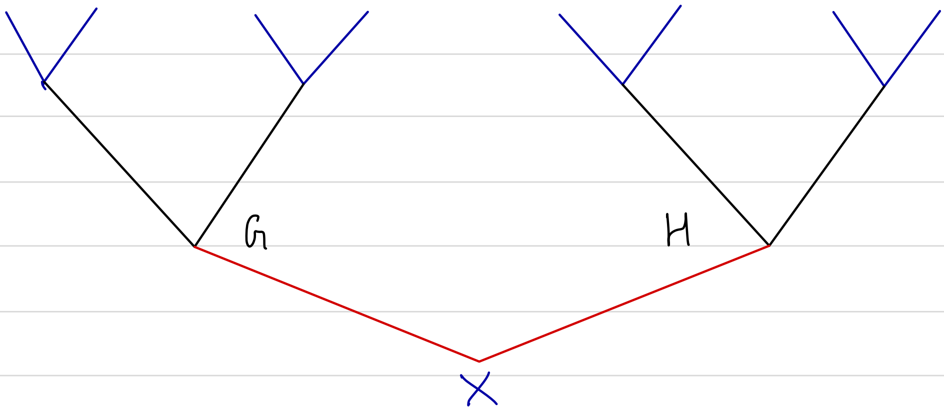
Eg: For 8 point sample :

$x \rightarrow$ 0 1 2 3 4 5 6 7

000 , 001 , 010 , 011 , 100 , 101 , 110 , 111

reverse: 000 , 100 , 010 , 110 , 001 , 101 , 011 , 111

0 4 2 6 1 5 3 7



DIF Algorithm (Decimation in Frequency)

We have $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$

$$X(k) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{nk} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{nk} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{nk}\, W_N^{N/2\, k} \rightarrow (1)$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{nk} + (-1)^k \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{nk} \rightarrow (2)$$

Here index $= k$.

Now put $k = 2r, 2r+1$

$$X(2r) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{2rn} + \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{2rk}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_{N/2}^{rn} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_{N/2}^{nr}$$

$$\Rightarrow X(2r) = \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2)\right] W_{N/2}^{nr} \rightarrow (3)$$

Let $g(n) = x(n) + x(n+N/2)$

$$\therefore X(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{nr} \; // \rightarrow (4)$$

Also $X(2r) = G(r)$

Now put $k = 2r+1$ in (2)

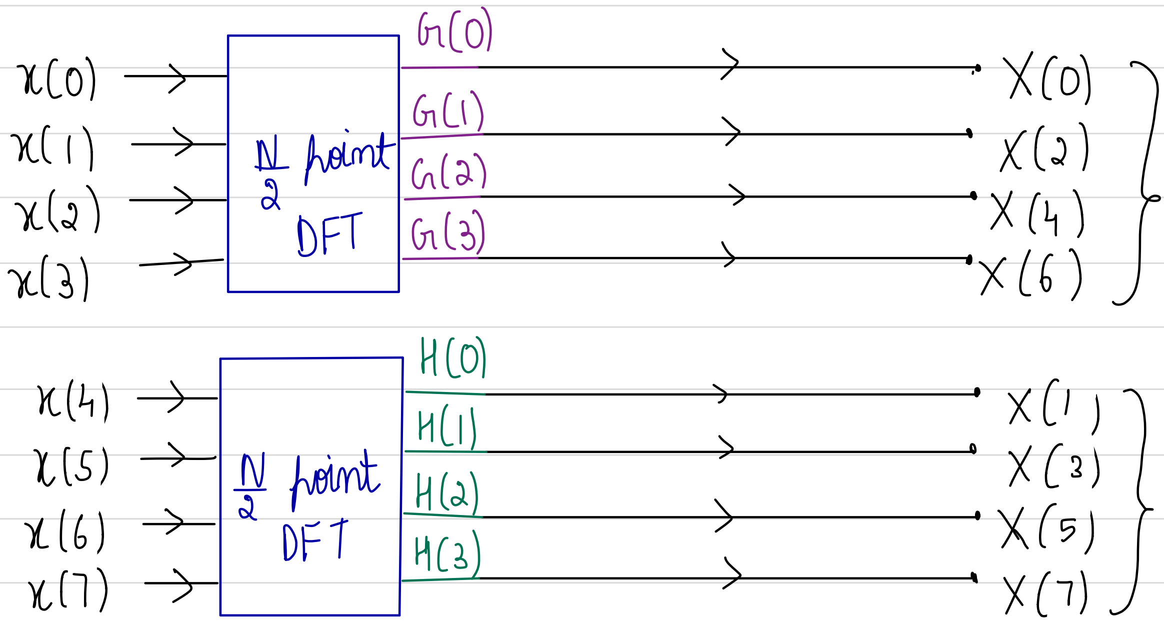
$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{(2r+1)n} - \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{n(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} W_{N/2}^{rn} \Big[\big[x(n) - x(n+N/2) \big] W_N^{n} \Big] \rightarrow 5$$

Let $h(n) = \big(x(n) - x(n+N/2)\big) W_N^{n} \Rightarrow X(2r+1) = H(r)$

$$\Rightarrow H(r) = \sum_{n=0}^{N/2-1} h(n)\, W_{N/2}^{rn}$$

Ex: 8 - point DFT using DIF



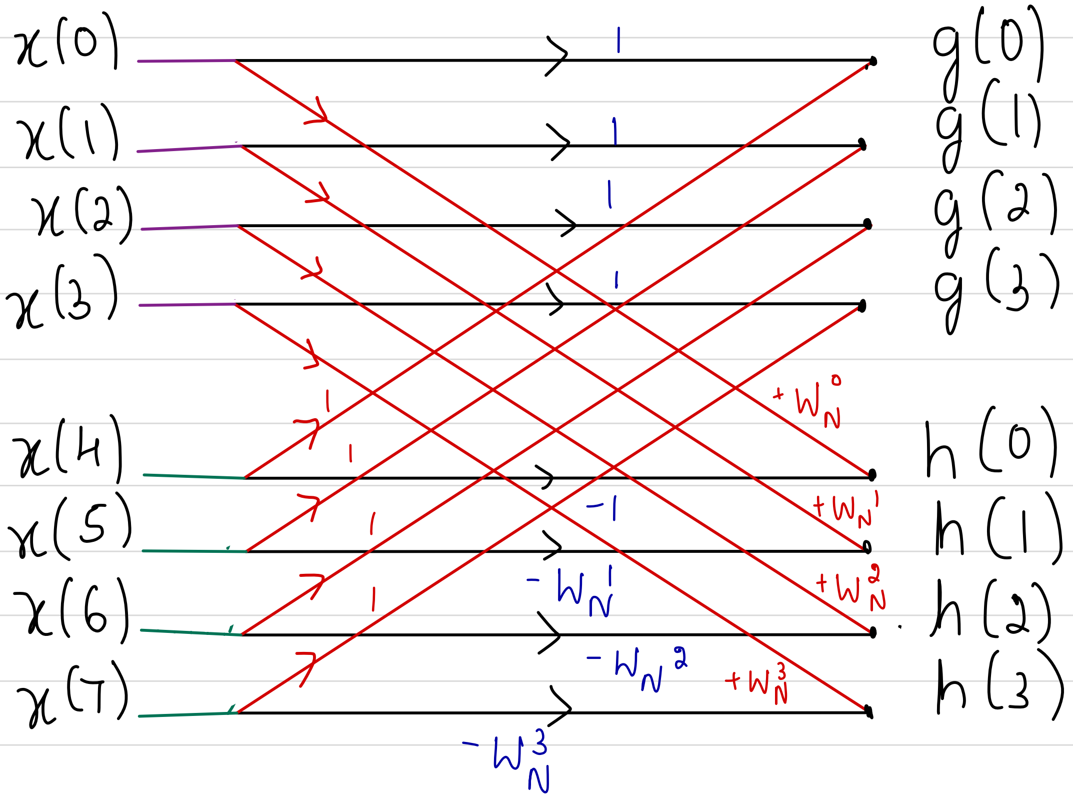
Splitting $g(n)$ & $h(n)$ again:

$$g_1(n) = g(n) + g(n+N/4)$$
$$g_2(n) = \left[g(n) - g\left(n+\frac{N}{4}\right) \right] W_{N/2}^{n}$$

$$h_1(n) = h(n) + h(n+N/4)$$
$$h_2(n) = \left[h(n) - h\left(n+\frac{N}{4}\right) \right] W_{N/2}^{n}$$

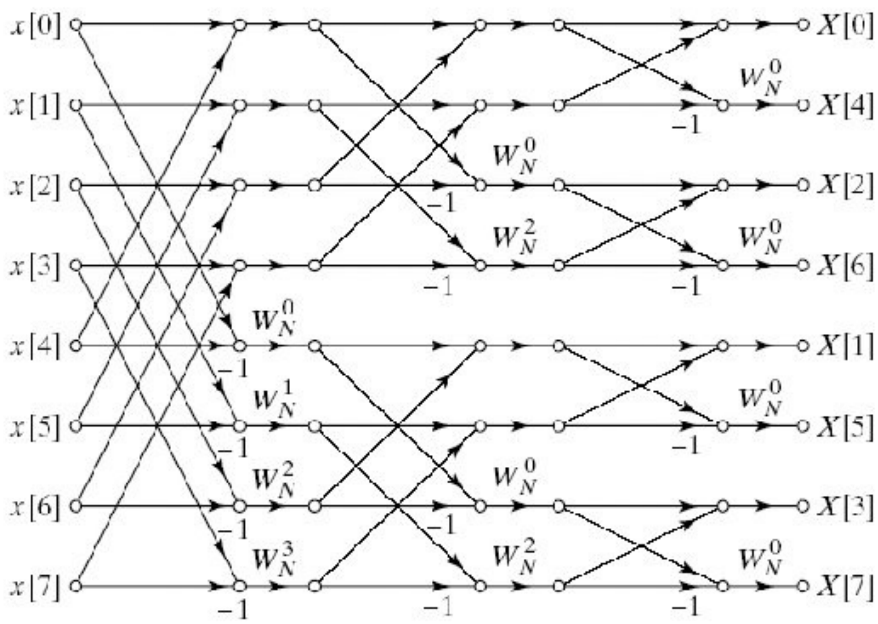
NOTE: In DIT: First stage maps time → Frequency
All other stages : Freq → Frequency

In DFT: Last stage maps time → Frequency
All other stages : time → time.



Decimation-In-Frequency FFT Algorithm

- Final flow graph for 8-point decimation in frequency



Total number of calculations = $2N \log_2 N$

Linear convolution using DFT

$h(n) = \{1, 2, 0, -1, 1\}$

$x(n) = \{1, 3, -1, -2\}$, $\Rightarrow x_p(n) = \{1, 3, -1, 2, 0\}$

$y_p(n) = h(n) \circledast x_p(n)$

$= \{5, 6, 3, -5, 6\}$

length of $y_l(n) = 4 + 5 - 1 = 8$

$h'(n) = \{1, 2, 0, -1, 1, 0, 0, 0\}$

$x_p'(n) = \{1, 3, -1, -2, 0, 0, 0, 0\}$

$y_l(n) = h'(n) \circledast x_p'(n)$

$y_l(n) = \{1, 5, 5, -5, 6, 4, 1, -2\}$

$y_p(n) = \{5, 6, 3, -5, 6\}$

In $y_l(n)$ 1st (L-K) samples & last (L-K) samples are lost. Addition gives L-K samples of periodic convolution.

Also $L - 2(L-K) = 2K - L$ samples have same value in periodic & linear convolution.

If you zeropad $h(n)$, $x(n)$ to 8 point sequence, then linear convolution = periodic convolution.

Z- transform

Let $x_s(t)$ be sampled version of $x_a(t)$, then

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(t) \cdot \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) \cdot \delta(t - nT_s)$$

$$\mathcal{L}(x_s(t)) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT_s)\, \delta(t - nT_s)\, e^{-st}\, dt$$

$$X_s(s) = \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-st}\, dt \rightarrow ①$$

$$X_s(s) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-nT_s s}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(e^{sT_s}\right)^{-n}$$

Now let $e^{sT_s} = z$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \rightarrow \text{bilateral z transform.}$$

If $x(n)$ is causal sequence,

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \rightarrow \text{unilateral z transform}$$

$$\therefore \boxed{X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}}$$

Eg: $$\delta(z) = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = 1$$

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

For $|z| < 1$

$$U(z) = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

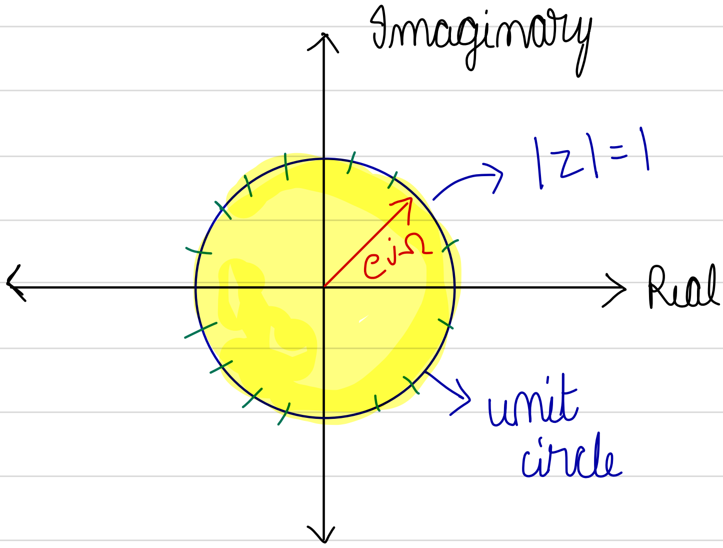
NOTE: z is complex $\Rightarrow z = \alpha + j\beta = re^{j\phi}$

if $z = e^{j\Omega}$

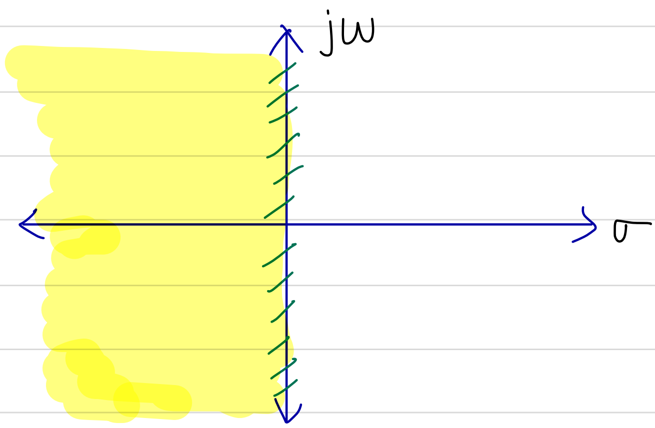
i.e $r = 1$ & $\phi = \Omega$

So $X(z)\Big|_{z=e^{j\Omega}} = X(\Omega)$

Z plane:



S-plane:



→ stability region

Bounding condition for $X(z)$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} x(n)\, r^{-n}\, e^{-jn\Omega}$$

For $X(z)$ to be bounded,

$$\left| x(n)\, r^{-n} \right| \leq L < \infty$$

Q) Find z transform & DTFT for

$x(n) = a^n u(n)$, case 1: $|a|<1$ & case 2: $|a|>1$

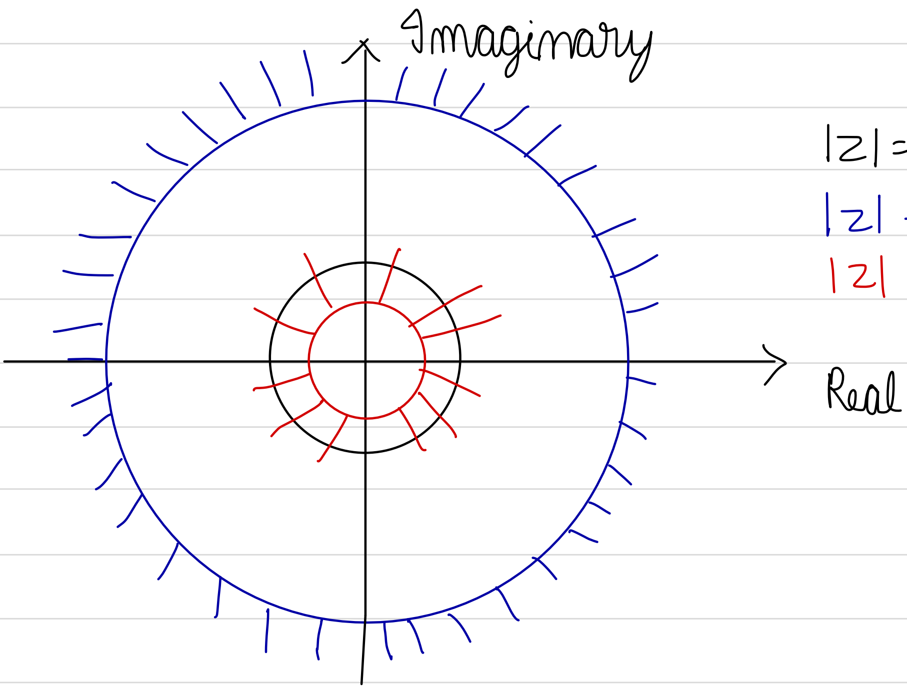
Ans

$$Z[x(n)] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a/z} = \frac{z}{z-a}$$

if $|z| > |a|$

When $|a| = 0.8$ & $|a| = 4$



$a = 0.8$ → stable → unit circle included

$a = 4$ → unstable → unit circle not included

$$X(\Omega) = \sum_{n=0}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

$$= \frac{1}{1 - ae^{-j\Omega}}$$

Q> Find z transform of $y(n) = a^n u(-n)$

Ans
$$Y(z) = \sum_{n=-\infty}^{\infty} a^n u(-n) z^{-n}$$

let $m = -n$

$$Y(z) = \sum_{m=-\infty}^{\infty} a^{-m} u(m) z^m$$

$$= \sum_{m=0}^{\infty} (a^{-1}z)^m = \frac{1}{1 - a^{-1}z} = \frac{a}{a-z}$$

$|a^{-1}z| < 1$

$\Rightarrow |z| < |a|$

Here $|a| = 0.8 \Rightarrow$ unstable

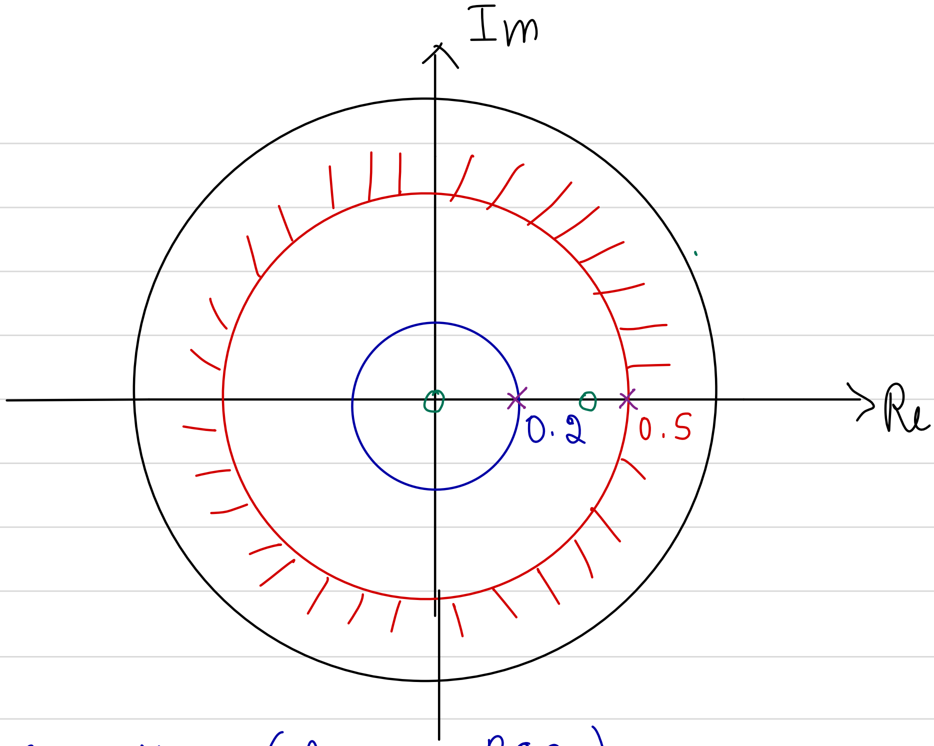
$|a| = 4 \Rightarrow$ stable.

Eg: $x(n) = (1/2)^n u(n) + (1/5)^n u(n)$

Ans
$x(n) = x_1(n) + x_2(n)$

$$X_1(z) = \frac{z}{z - 1/2} \qquad X_2(z) = \frac{z}{z - 1/5}$$

$$\therefore X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/5}$$



$$ROC = Max(ROC_1, ROC_2)$$

$$= Max\left(\frac{1}{2}, \frac{1}{5}\right) => 1/2$$

Also $$X(z) = \frac{z(2z-0.7)}{(z-0.5)(z-0.2)}$$

Eg 2: $x(n) = \cos \Omega_0 n \; u(n)$, $\Omega_0 = \pi/3$

Ans $$Z[x(n)] = \frac{1}{2}\sum_{n=0}^{\infty}\left[e^{j\Omega_0} + e^{-j\Omega_0 n}\right]z^{-n}$$

$$X(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(e^{j\Omega_0}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(e^{-j\Omega_0}z^{-1}\right)^n\right]$$

$$X(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\Omega_0}z^{-1}} + \frac{1}{1-e^{-j\Omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{j\Omega_0}} + \frac{z}{z-e^{-j\Omega_0}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-j\Omega_0} + z - e^{-j\Omega_0}}{(z-e^{j\Omega_0})(z-e^{-j\Omega_0})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (e^{j\Omega_0} + e^{-j\Omega_0})}{z^2 + 1 - z(e^{j\Omega_0} + e^{-j\Omega_0})}\right]$$

$$= z\left[\frac{z - \cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}\right]$$

Poles : $z = \dfrac{2\cos\Omega_0 \pm \sqrt{4\cos^2\Omega_0 - 4}}{2}$

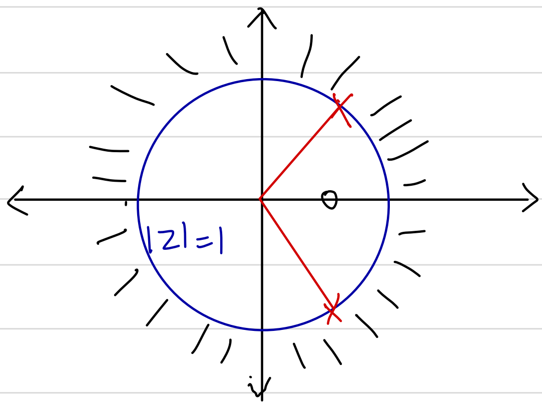
$= \cos\Omega_0 \pm j\sin\Omega_0$

$= (1/2) \pm j(\sqrt{3}/2)$

zeros : $0, \ \cos\Omega_0 \Rightarrow 0, \ 1/2$

To find ROC : $|e^{j\Omega} z^{-1}| < 1$

$\Rightarrow |z| > |e^{j\Omega_0}| \Rightarrow |z| > 1$



System is unstable as $|z|=1$ not included

Properties of z transform

1> Linearity : $x_1(n) \longleftrightarrow X_1(z)$

$x_2(n) \longleftrightarrow X_2(z)$

$y(n) = \alpha_1 x_1(n) \pm \alpha_2 x_2(n)$

$\downarrow$

$$Y(z) = \sum_{n=0}^{\infty} [\alpha_1 x_1(n) \pm \alpha_2 x_2(n)] z^{-n}$$

$$\therefore \; Y(z) = \alpha_1 X_1(z) \pm \alpha_2 X_2(z) \,//$$

2> <u>Time shift</u>: (delay n_0 unit) ;

$$x(n) \longrightarrow \boxed{D} \longrightarrow y(n) = x(n-n_0)$$

$$Y(z) = \sum_{n=0}^{\infty} x(n-n_0)\, z^{-n}$$

Let $n - n_0 = l$, when $n = 0$, $l = -n_0$

$n = \infty$, $l = \infty$

$$\therefore \; Y(z) = \sum_{l=-n_0}^{\infty} x(l)\, z^{-(l+n_0)}$$

$$= z^{-n_0} \left[\sum_{l=-n_0}^{-1} x(l)\, z^{-l} + \sum_{l=0}^{\infty} x(l)\, z^{-l} \right]$$

$$Y(z) = z^{-n_0} \left[\sum_{l=-n_0}^{-1} x(l)\, z^{-l} + X(z) \right] //$$

Q> Solve a Difference equation using z transform
$y(n) - \frac{1}{5} y(n-1) = \delta(n)$ with $y(-1) = 2$

and $x(n) = \delta(n)$

Ans Applying z transform:

$$Y(z) - \frac{1}{5}\left[\frac{Y(z)}{z} + y(-1)\right] = 1$$

$$Y(z)\left[1 - \frac{1}{5z}\right] = 1 + \frac{2}{5}$$

$$Y(z)\frac{(5z-1)}{z} = 7$$

$$\therefore Y(z) = \frac{7z}{5z-1}$$

$$y(n) = z^{-1}\left[\frac{7z}{5z-1}\right]$$

$$y(n) = \frac{7}{5}\left(\frac{1}{5}\right)^n u(n)$$

3> Frequency scaling: $x(n) \longleftrightarrow X(z)$
$y(n) = \alpha^n x(n) u(n)$

$$Y(z) = \sum_{n=0}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) (\alpha^{-1} z)^{-n}$$

$$= X(\alpha^{-1} z) = X\left(\frac{z}{\alpha}\right)$$

Eg: $x(n) = u(n)$ find $X_m(z)$ for $(0.3)^n u(n)$

Ans $X(z) = \frac{z}{z-1}$

$$X_m(z) = \frac{z/0.3}{\frac{z}{0.3} - 1} = \frac{z}{z-0.3}, \quad |z| > 0.3$$

Eg: Find $X_m(z)$ for $x_m(n) = \alpha^n \cos\Omega_0 n \, u(n)$

Ans For $x(n) = \cos\Omega_0 n \, u(n)$

$$X(z) = z\left[\frac{z - \cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}\right]$$

$$\therefore X_m(z) = \left(\frac{z}{\alpha}\right)\left[\frac{(z/\alpha) - \cos\Omega_0}{\left(\frac{z^2}{\alpha^2}\right) - 2\frac{z}{\alpha}\cos\Omega_0 + 1}\right]$$

$$= z\left[\frac{z - \alpha\cos\Omega_0}{z^2 - 2z\alpha\cos\Omega_0 + 1}\right]$$

4> Gradient in z plane: $x(n) \longleftrightarrow X(z)$

$$\frac{d}{dz}(X(z)) = \frac{d}{dz}\left[\sum_{n=0}^{\infty} x(n) z^{-n}\right]$$

$$= \sum_{n=0}^{\infty} x(n)(-n) z^{-n-1}$$

$$\frac{d}{dz}(X(z)) = \frac{-1}{z}\sum_{n=0}^{\infty} n\,x(n) z^{-n}$$

$$\therefore -z\frac{d}{dz}(X(z)) = \sum_{n=0}^{\infty} n\,x(n) z^{-n} \Rightarrow Z[n\,x(n)]$$

In general: $Z[n^k x(n)] = (-z)^k \left[\frac{d^k}{dz^k}(X(z))\right]$

5> <u>Initial value theorem</u> : $\lim_{n \to 0} x(n) = x(0)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots$$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots$$

$$\Rightarrow \lim_{z \to \infty} X(z) = x(0) //$$

$$\therefore \quad x(0) = \lim_{z \to \infty} X(z) //$$

Eg: Find $x(0)$ for $x(n) = u(n)$ & $\cos(\Omega_0 n)u(n)$ using $X(z)$

Ans $X(z) = \frac{z}{z-1}$ for $x(n) = u(n)$

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{z}{z-1} = \lim_{z \to \infty} \frac{z}{z(1-1/z)}$$

$$= \lim_{z \to \infty} \frac{1}{1-1/z} = 1 //$$

$$X(z) = z\left[\frac{z - \cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}\right]$$

$$\therefore \quad x(0) = \lim_{z \to \infty} z\left[\frac{z - \cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}\right]$$

$$= \lim_{z \to \infty} \frac{z^2}{z^2}\left[\frac{1 - (\cos\Omega_0/z)}{1 - (2\cos\Omega_0/z) + 1/z^2}\right] = 1 //$$

Eg: If $X(z) = \frac{z}{z-1} + \frac{z}{z-1/3} - \frac{(z-1/4)}{z^2 - \frac{z}{2} + 1}$

Verify IVT

Ans $x(0) = \lim_{z\to\infty} \frac{z}{z-1} + \lim_{z\to\infty} \frac{z}{z-1/3} - \lim_{z\to\infty} \frac{(z-1/4)}{z^2 - z/2 + 1}$

$\therefore x(0) = 1 + 1 - 0 = 2$ //

Also $x(n) = u(n) + (1/3)^n u(n) - z^{-1}\left[\frac{(z-1/4)}{(z-1/4)^2 + 15/16}\right]$

5> <u>Final value theorem</u> : $Z\left[x(n) - x(n-1)\right] = X(z) - \frac{X(z)}{z}$

$$\lim_{N\to\infty} \sum_{n=0}^{N} \left[x(n) - x(n-1)\right] z^{-n} = X(z)\left(1 - \frac{1}{z}\right)$$

But $\sum_{n=0}^{N} \left[x(n) - x(n-1)\right] z^{-n} = \left[x(N) - x(N-1)\right] z^{-N}$
$+ \left[x(N-1) - x(N-2)\right] z^{-N-1}$
$+ \ldots$

Now putting $\lim_{z\to 1}$ on both sides

$$\lim_{N\to\infty} \lim_{z\to 1} x(N) = \lim_{z\to 1} X(z)\left(1 - \frac{1}{z}\right)$$

$$\therefore x(\infty) = \lim_{z\to 1} X(z)\left(1 - \frac{1}{z}\right)$$ //

Eg: Find $x(\infty)$ for i> $x_1(n) = u(n)$ using FVT
ii> $x_2(n) = 2^n u(n)$

Ans i> $X_1(z) = \dfrac{z}{z-1}$

$$x(\infty) = \lim_{z\to 1} \left(\frac{z}{z-1}\right)\left(\frac{z-1}{z}\right) = 1 //$$

ii> $X_2(z) = \dfrac{z}{z-2}$

$$x(\infty) = \lim_{z\to 1} \left(\frac{z}{z-2}\right)\left(\frac{z-1}{z}\right)$$

$$= \lim_{z\to 1} \frac{z-1}{z-2} = 0 //$$

6> Convolution: $x_1(n) \longrightarrow X_1(z)$
$x_2(n) \longrightarrow X_2(z)$

$$x_3(n) = x_1(n) * x_2(n)$$

$$X_3(z) = \sum_{n=0}^{\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} x_1(n) \cdot x_2(n-k)\right] z^{-n}$$

Put $n-k = l$

$$\Rightarrow X_3(z) = \sum_{l=-k}^{\infty} \left[\sum_{k=0}^{\infty} x_1(k)\, x_2(l)\right] z^{-(k+l)}$$

$$= \left(\sum_{k=0}^{\infty} x_1(k) z^{-k}\right)\left(\sum_{l=0}^{\infty} x_2(l) z^{-l}\right)$$

$$\therefore X_3(z) = X_1(z) \cdot X_2(z)$$

Eg: Given $h(n) = \{\underset{\uparrow}{1}, 2, 0, -1, 1\}$ & $x(n) = \{\underset{\uparrow}{1}, 3, -1, -2\}$

$y_e(n) = \{1, 5, 5, -5, -6, 4, 1, -2\}$. Verify using Z transform.

Ans $H(z) = \sum_{n=0}^{4} h(n) z^{-n}$

$= 1 + 2z^{-1} - 1z^{-3} + z^{-4}$

$X(z) = \sum_{n=0}^{3} x(n) z^{-n} = 1 + 3z^{-1} - z^{-2} - 2z^{-3}$

$Y(z) = X(z) \cdot H(z)$

$= 1 + 3z^{-1} - z^{-2} - 2z^{-3} + 2z^{-1} + 6z^{-2} - 2z^{-3} - 4z^{-4}$
$- z^{-3} - 3z^{-4} + z^{-5} + 2z^{-6} + z^{-4} + 3z^{-5} - z^{-6} - 2z^{-7}$

$= 1 + 5z^{-1} + 5z^{-2} - 5z^{-3} - 6z^{-4} + 4z^{-5} + z^{-6} - 2z^{-7}$

$\therefore y(n) = \{1, 5, 5, -5, -6, 4, 1, -2\} = y_e(n)$

Verified.

Inverse z-transform

1) Circular integral method:

$x(n) \longleftrightarrow X(z)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

then
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \rightarrow ②$$

consider
$$\frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz = \frac{1}{2\pi j} \int_C \sum_{n=0}^{\infty} x(n) z^{k-n-1} dz$$

with circle c being in ROC.

Let $z = Re^{j\theta} \Rightarrow \frac{dz}{d\theta} = Re^{j\theta} j = jz$

$$\Rightarrow \frac{dz}{z} = j d\theta$$

$$\therefore \frac{1}{2\pi j} \oint_C \sum_{n=0}^{\infty} x(n) z^{k-n} \frac{dz}{z} = \sum_{n=0}^{\infty} \frac{x(n)}{2\pi j} \oint_C (R_1 e^{j\theta})^{k-n} j d\theta$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x(n) R_1^{k-n}}{2\pi} \int_0^{2\pi} e^{j(k-n)\theta} d\theta$$

now if $n = k$, $\int_0^{2\pi} e^{j(k-n)\theta} d\theta = 2\pi$

if $n \neq k$, integral $= 0$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x(n)}{2\pi} R_1^{(k-k)} \cdot 2\pi = \sum_{n=0}^{\infty} x(n) = x(n)$$

This integral is known as Cauchy Integral

2) <u>Power series / long division method</u>:

- If ROC $|z| > r_1$, then $H(z) = \frac{N(z)}{D(z)}$ should be arranged in such a way that both $N(z)$ & $D(z)$ should be in decreasing powers of z.

- If ROC $|z| < r_2$, then $H(z) = \frac{N(z)}{D(z)}$ should be arranged in such a way that both $N(z)$ & $D(z)$ should be in increasing powers of z.

Eg: Given: $\frac{z}{3z^2 - 4z + 1}$, case 1: $|z| > 1$
case 2: $|z| < 1/3$.

Find $x(n)$.

Ans

$$
\begin{array}{r|l}
 & 0z^0 + (1/3)z^{-1} + (4/9)z^{-2} + (13/27)z^{-3} + \dots \\
3z^2 - 4z + 1 & z \\
 & z - \frac{4}{3} + \frac{1}{3z} \\
\hline
 & + \frac{4}{3} - \frac{1}{3z} \\
 & + \frac{4}{3} - \frac{16}{9z} + \frac{4}{9z^2} \\
\hline
 & \frac{13}{9z} - \frac{4}{9z^2} \\
 & \frac{13}{9z} - \frac{52}{27z^2} + \frac{13}{27z^3} \\
\hline
 & \frac{10}{9z^2} - \frac{13}{27z^3}
\end{array}
$$

$$\therefore\ x(n) = \{\underset{\uparrow}{0},\ 1/3,\ 4/9,\ 13/27,\ \ldots\} \quad \text{if ROC}: |z|>1$$

if ROC : $|z| < 1/3$

$$X(z) = \frac{z}{1-4z+3z^2}$$

$$
\begin{array}{r|l}
 & 0z^0 + z + 4z^2 + 13z^3 + 30z^4 \\
\hline
1-4z+3z^2 & z \\
 & \underline{z - 4z^2 + 3z^3} \\
 & 4z^2 - 3z^3 \\
 & \underline{4z^3 - 16z^3 + 12z^4} \\
 & 13z^3 - 12z^4 \\
 & \underline{13z^3 - 52z^4 + 39z^5} \\
 & 30z^4 - 39z^5 \\
 & \vdots
\end{array}
$$

$$\therefore\ x(n) = \{\underset{\substack{\uparrow \\ n=-\infty}}{\ldots\ldots}\ 30,\ 13,\ 4,\ 1,\ \underset{\substack{\uparrow \\ n=0}}{0}\}$$

3> **Partial fraction expansion:**

Taking same example:

$$X(z) = \frac{z}{3z^2-4z+1}$$

$$\frac{X(z)}{z} = \frac{1}{(3z-1)(z-1)} = \frac{a}{3z-1} + \frac{b}{z-1}$$

$a = -3/2 \qquad b = 1/2$

$$\Rightarrow \frac{X(z)}{z} = \frac{-3}{2(3z-1)} + \frac{1}{2(z-1)}$$

$$X(z) = \frac{-3}{2} \frac{z}{3(z - 1/3)} + \frac{1}{2} \frac{z}{z-1}$$

$$\therefore x(z) = \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^n\right] u(n)$$

- System function for a causal DTS (also LTI)

N^{th} order system:

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k)$$

→ System is initially relaxed i.e all initial conditions are zero

$$\sum_{k=0}^{N} a_k \, z^{-k} \, Y(z) = \sum_{k=0}^{M} b_k \, z^{-k} \, X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{(z-z_0)(z-z_1)\ldots(z-z_n)}{(z-p_0)(z-p_1)\ldots(z-p_n)} = \frac{\prod_{i=0}^{M} (z-z_i)}{\prod_{i=0}^{N} (z-p_i)}$$

where z_i = zeroes & p_i = poles

- If we put $z = e^{j\Omega}$ then it becomes DTFT.

- $H(z)$ is known as system function

Eg: Given : $y(n) - 2y(n-1) + 2y(n-2) = x(n) + \frac{x(n-1)}{2}$

Find $H(z)$.

Ans $y(n) - 2y(n-1) + 2y(n-2) = x(n) + \frac{x(n-1)}{2}$

$$Y(z) - 2\frac{Y(z)}{z} + 2\frac{Y(z)}{z^2} = X(z) + \frac{X(z)}{2z}$$

$$\Rightarrow Y(z)\left[\frac{z^2 - 2z + 2}{z^2}\right] = X(z)\left[\frac{2z+1}{2z}\right]$$

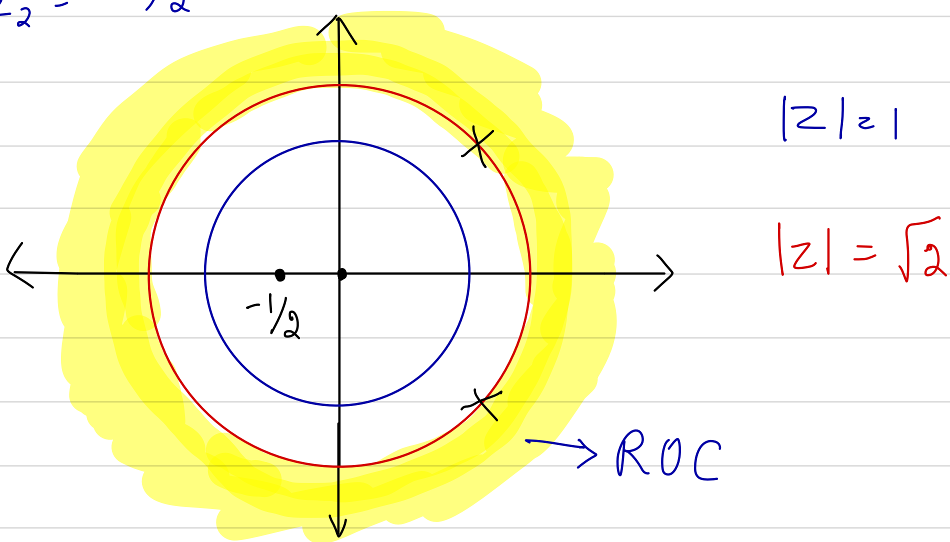
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(2z+1)}{2(z^2-2z+2)}$$

$$\Rightarrow P_1, P_2 = \frac{2 \pm \sqrt{4-8}}{2} \Rightarrow P_1, P_2 = 1 \pm j$$

$z_1 = 0 \qquad z_2 = -1/2$

X - Poles

• - zeroes



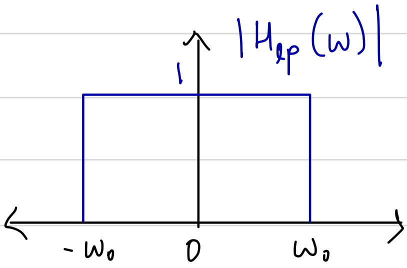
Filters:

Based on required sensitivity, active filters or passive filters can be chosen for a particular operation.

→ Active filters have higher sensitivity.

Ideal v/s Practical filters:

LPF: Ideal : $H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{otherwise} \end{cases}$

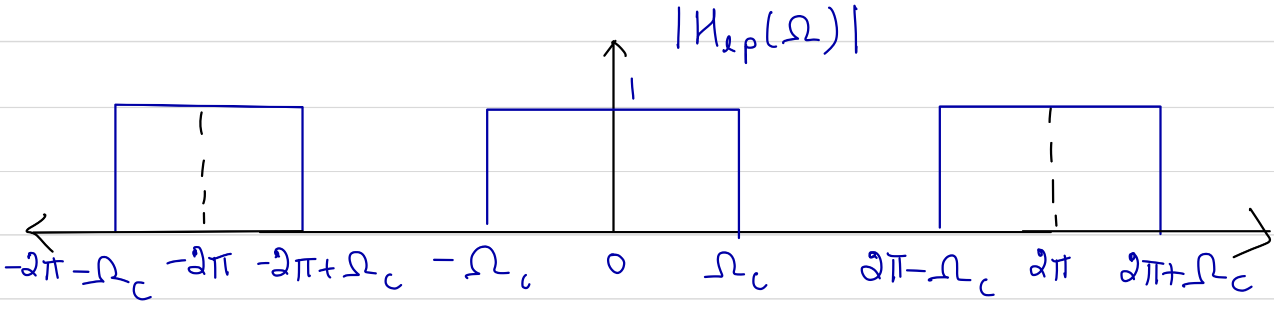


Inverse FT : $h(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \, e^{j\omega t} \, dt = \frac{1}{2\pi j} \left. \frac{e^{j\omega t}}{t} \right|_{-\omega_0}^{\omega_0}$

$$= \frac{1}{2\pi j t}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] = \frac{\sin \omega_0 t}{\pi t}$$

In discrete time domain:

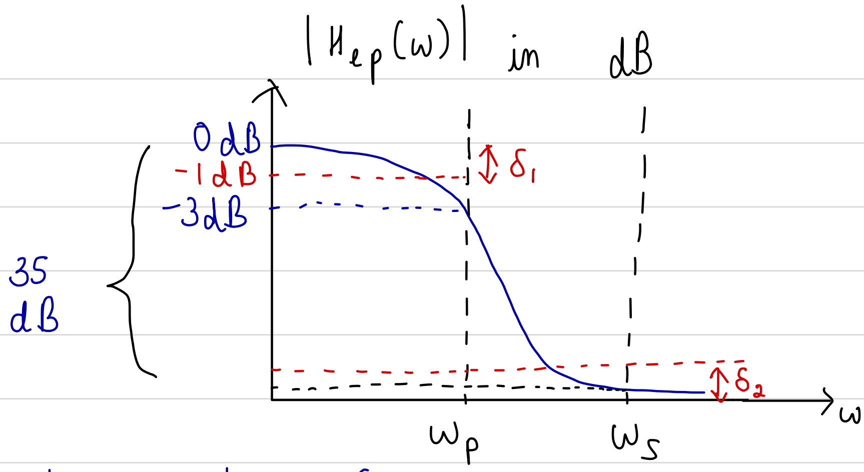
$H_{lp}(\Omega) = \begin{cases} 1, & |\Omega| \leq \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$



Practical LPF:

ω_p = passband

ω_s = stopband



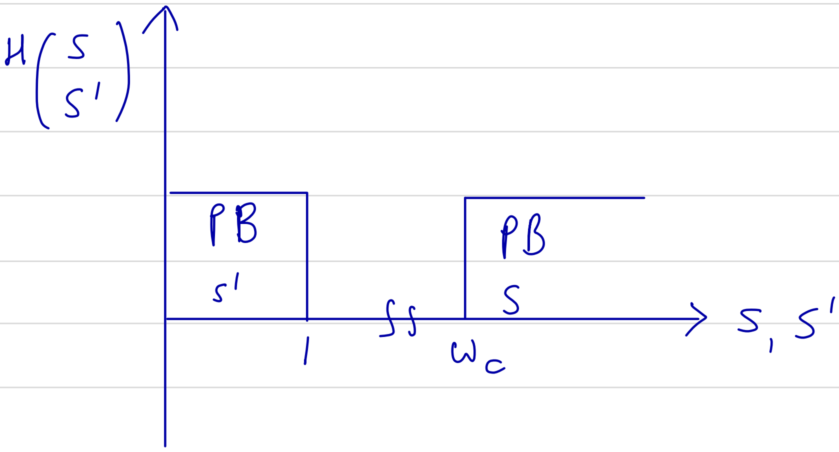
Transfer function: $$|H_{lp}(\omega)| = \begin{cases} 1 \pm \delta_1 , & |\omega| \leq \omega_p \\ \delta_2 , & |\omega| \geq \omega_s \end{cases}$$

Normalised LPF: Has cutoff frequency = 1 (unity)

Frequency transformation :

s' - plane $\rightarrow$ S - plane

$s' = \sigma' + j\omega' \rightarrow S = \sigma + j\omega$



Forward mapping $s' = f(s)$

Inverse mapping $s = g(s')$

1> <u>Normalise LPF $\rightarrow$ LPF with ω_c</u>

$\omega_c \Downarrow$ = unity

$s' = \sigma' + j\omega'$

Pass band $\rightarrow$ LPF ω

$0 \leq \omega' \leq 1 \rightarrow 0 \leq \omega \leq \omega_c$

$$\Rightarrow \frac{j\omega'}{1} = \frac{j\omega}{\omega_c}$$

$$\Rightarrow \omega = \omega'\omega_c$$

$$\therefore \boxed{\frac{s'}{1} = \frac{s}{\omega_c}}$$

Eg: If LPF is made L & C

then $X_L = Ls'$

$= L\dfrac{s}{\omega_c}$

$$\therefore L_{eq} = \frac{L}{\omega_c}$$

$$X_c = \frac{1}{s'C} = \frac{\omega_c}{sC} \qquad \therefore \; C_{eq} = \frac{C}{\omega_c}$$

2) LPF : ω_c' $\longrightarrow$ HPF : ω_c

Pass band : $0 \leq \omega' \leq \omega_c'$ $\longrightarrow$ $\omega_c \leq \omega \leq \infty$

$$\Rightarrow \frac{j\omega'}{\omega_c'} = \frac{\omega_c}{j\omega}$$

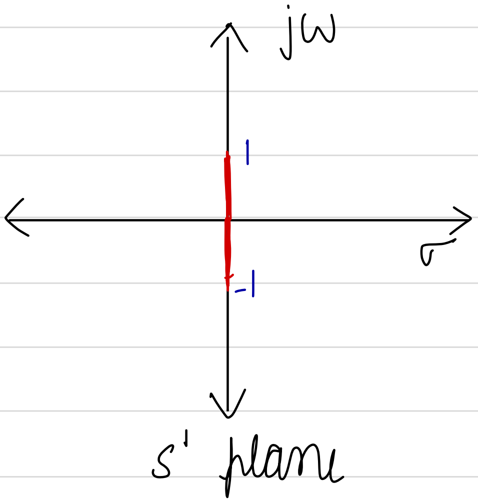
$$\Rightarrow \quad \omega = \left| -\frac{\omega_c'\omega_c}{\omega'} \right| \qquad \therefore \; \omega = \frac{\omega_c'\omega_c}{\omega'}$$

Also
$$\boxed{\frac{s'}{\omega_c'} = \frac{\omega_c}{s}}$$

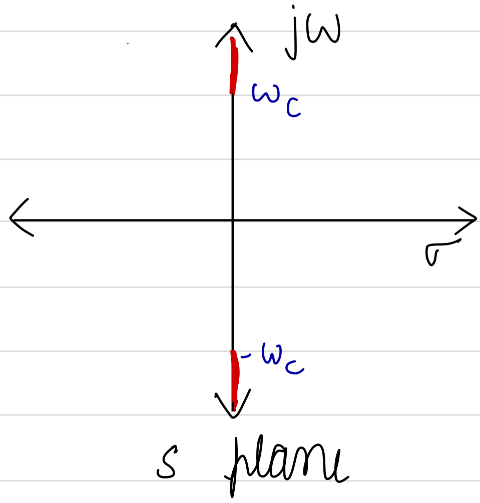
For normalised LPF : $s' = \dfrac{\omega_c}{s}$

So $$X_L = Ls' = \frac{L\omega_c}{s} = \frac{1}{C_{eq}s} \qquad \therefore \; C_{eq} = \frac{1}{L\omega_c}$$ //

$$X_c = \frac{1}{s'C} = \frac{s}{\omega_c C} = L_{eq}s \qquad \therefore \; L_{eq} = \frac{1}{C\omega_c}$$ //



→



3> $\underline{\text{LPF} \rightarrow \text{Band pass filter:}}$

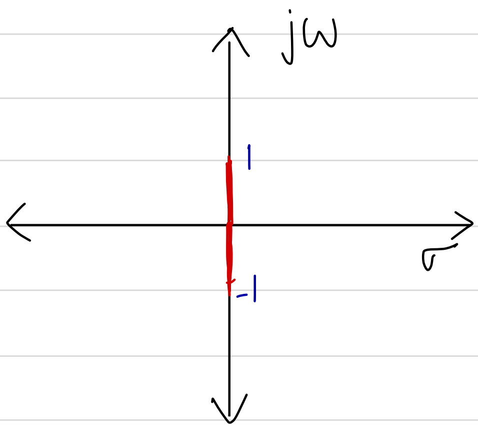
$s' \rightarrow s$

For BPF, let bandwidth $= B = \omega_2 - \omega_1$

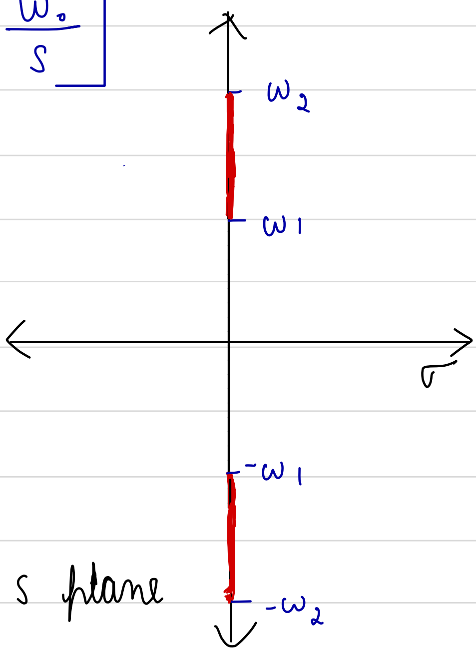
Mid band frequency $= \omega_0 = \sqrt{\omega_1 \omega_2}$

BPF $\rightarrow$ High pass filter cascaded by Low pass filter and not vice versa.

$$s' = \frac{\omega_0}{B}\left[\frac{s}{\omega_0} + \frac{\omega_0}{s}\right]$$



$\rightarrow$



$$s' = j1 = \frac{\omega_0}{B}\left[\frac{j\omega}{\omega_0} + \frac{\omega_0}{j\omega}\right]$$

$$Bj\,\omega_0 j\omega = \omega_0[-\omega^2 + \omega_0^2]$$

$$-B\omega = \omega_0^2 - \omega^2$$

$$\Rightarrow \omega^2 - B\omega - \omega_0^2 = 0$$

$$\therefore \omega = \frac{B}{2} \pm \sqrt{\frac{B^2}{4} + \omega_0^2}$$

$$= \frac{\omega_2 - \omega_1}{2} \pm \sqrt{\frac{\omega_2^2 + \omega_1^2 - 2\omega_1\omega_2 + 4\omega_1\omega_2}{4}}$$

$$\omega = \frac{\omega_2 - \omega_1}{2} \pm \frac{\omega_2 + \omega_1}{2}$$

$\omega = \omega_2$ or $-\omega_1$

Also $X_L = s'L = L\left[\frac{s}{B} + \frac{\omega_0^2}{sB}\right] = \frac{Ls}{B} + \frac{L\omega_0^2}{Bs}$

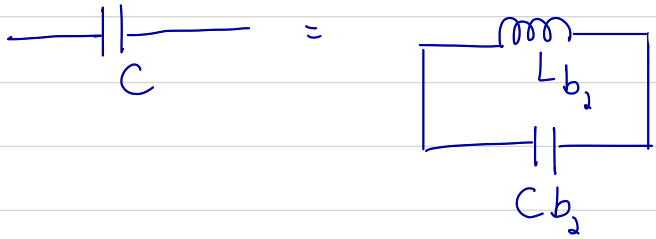
$$= L_{b_1}s + \frac{1}{sC_{b_1}}$$

$\therefore L_{b_1} = \frac{L}{B}$ $\qquad C_{b_1} = \frac{B}{L\omega_0^2}$

$$X_c = \frac{1}{s'C} = \frac{1}{\frac{Cs}{B} + \frac{C\omega_0^2}{B}\cdot\frac{1}{s}} = \frac{1}{C_{b_2}s + \frac{1}{L_{b_2}s}}$$

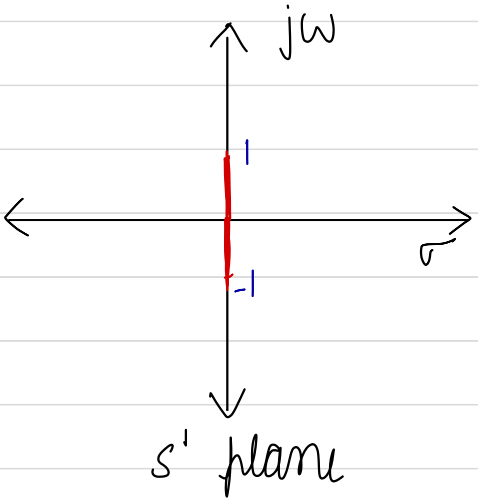
$\therefore C_{b_2} = \frac{C}{B}$ $\qquad L_{b_2} = \frac{B}{C\omega_0^2}$

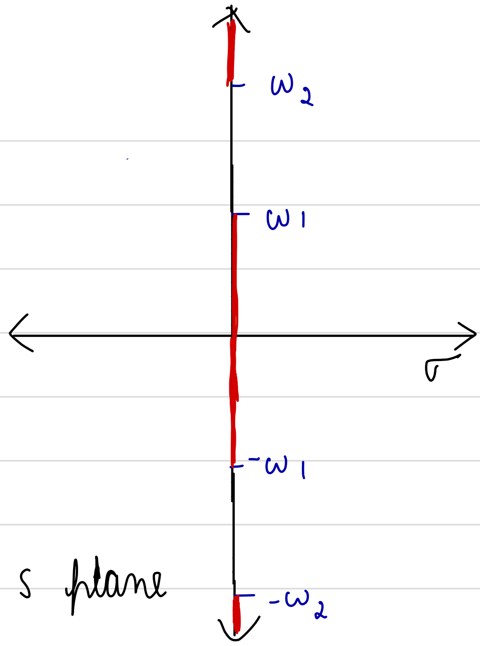
i.e



4) LPF to Bandstop filter:

PB: $0 \leq \omega' \leq \omega_c \rightarrow |\omega| \geq \omega_2$ & $|\omega| \leq \omega_1$





$$s' = \frac{1}{\frac{\omega_0}{B}\left[\frac{s}{\omega_0} + \frac{\omega_0}{s}\right]}$$

Chebyshev Polynomial for n^{th} order

$$C_n(\omega) \triangleq \cos(n \cos^{-1}\omega) \rightarrow ①$$

$$\cos^{-1}\omega = \theta \quad \Rightarrow \quad \omega = \cos\theta$$

$$C_n(\omega) = \cos(n\theta) \rightarrow ②$$

For ②, $\quad n = 0 \Rightarrow C_0(\omega) = 1$

$\quad n = 1 \Rightarrow C_n(\omega) = \cos\theta = \omega$

$$C_{n+1}(\omega) = \cos((n+1)\theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta \rightarrow ③$$

$$C_{n-1}(\omega) = \cos((n-1)\theta) = \cos n\theta \cos\theta + \sin n\theta \sin\theta \rightarrow ④$$

$$③ + ④ \Rightarrow \quad C_{n+1}(\omega) + C_{n-1}(\omega) = 2\cos n\theta \cos\theta$$

$$= 2\omega C_n(\omega) \rightarrow ⑤$$

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega) \rightarrow ⑥$$

$$C_2(\omega) = 2\omega C_1(\omega) - C_0(\omega)$$

$$= 2\omega \cdot \omega - 1$$

$$= 2\omega^2 - 1$$

$$C_3(\omega) = 2\omega(2\omega^2 - 1) - \omega$$

$$= 4\omega^3 - 2\omega - \omega$$

$$= 4\omega^3 - 3\omega$$

$\vdots$

$$C_{10}(\omega) = 512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$$

if $\omega \gg 1$, $\quad C_n(\omega) \cong 2^{n-1}\omega^n$

NOTE: MATLAB: C_{11}

```
>> syms w
>> C11 = cos(11 * acos(w));
>> C11 = expand(C11)
```

Analyse $C_n(\omega)$ for various ω

1> **Passband ($0 \leq \omega \leq 1$):**

$$C_n(\omega) = \cos(n\cos^{-1}\omega) \Rightarrow C_n(0) = \cos(n\pi/2)$$

If $n =$ even, $\quad C_n(0) = 1$
$\quad n =$ odd, $\quad C_n(0) = -1$

$$C_n(1) = \cos(n\cos^{-1}(1))$$
$$= \cos(n \cdot 2m\pi) = 1$$

2> **If $\omega > 1$:** $\quad C_n(\omega) = \cos(n\cos^{-1}\omega)$
$$= \cosh(n\cosh^{-1}(\omega))$$

3> **If $\omega \gg 1$:** $\quad C_n(\omega) \cong 2^{n-1}\omega^n$

Chebyshev LPF characteristic

$|C_n(\omega)| \leq 1 \quad$ for $\quad |\omega| \leq 1$

$$C_n(\omega) = \cos(n\cos^{-1}\omega)$$
$$C_n(-\omega) = \cos(n\cos^{-1}(-\omega))$$
$$= \cos[n(\pi + \cos^{-1}\omega)]$$
$$= \cos[n\pi + n\cos^{-1}\omega]$$
$$= \cos n\pi \cos(n\cos^{-1}\omega) - \sin n\pi \sin(n\cos^{-1}\omega)$$
$$= (-1)^n C_n(\omega)$$

For a small number ξ
$$F^2(\omega) = \xi^2 C_n^2(\omega)$$

$$|H(j\omega)|^2 = \frac{A_0}{1 + F^2(\omega)} = \frac{1}{1+\xi^2 C_n^2(\omega)}$$

For passband $(0 \leq \omega \leq 1)$:

$$|H(j\omega)^2| = \frac{1}{1+\xi^2}$$

$$|H(j\omega)|_{max} = 1$$

$$|H(j\omega)|_{min} = \frac{1}{\sqrt{1+\xi^2}}$$

$$\text{Attenuation (dB)} = 10 \log_{10}\left(\frac{1}{1+\xi^2}\right)$$

$$\Rightarrow \qquad \alpha_p = -10 \log_{10}(1+\xi^2)$$

- If $\xi^2 C_n^2(\omega) \ggg 1$, $|H(j\omega)|^2 \cong \frac{1}{\xi^2 C_n^2(\omega)}$

$$\therefore |H(j\omega)|^2 \approx \frac{1}{\xi^2\, 2^{2(n-1)}\, \omega^{2n}}$$

$$|H(j\omega)|^2 \text{ in dB} = -10 \log_{10}(\xi^2\, 2^{2(n-1)}\, \omega^{2n})$$

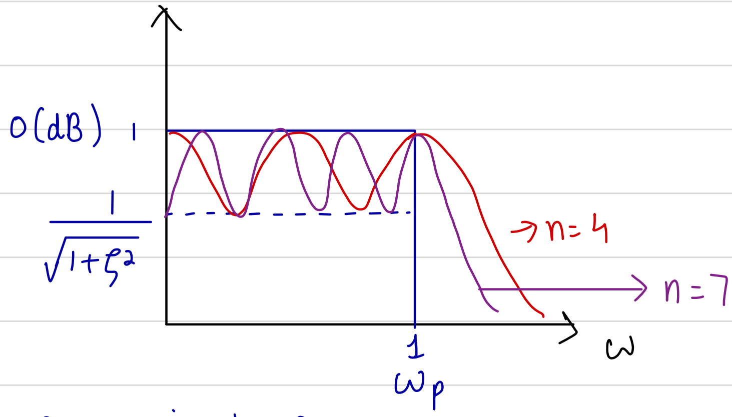
$$\Rightarrow \qquad \alpha_s(\omega) = -20 \log(\xi\, 2^{(n-1)}\, \omega^n)$$

$$= -20 \log_{10}\xi - 20(n-1)\log 2 - 20n \log\omega$$

$$\therefore \alpha_s(\omega) = -20 \log_{10}\xi - 6(n-1) - 20n\log\omega$$

NOTE: α_p = attenuation in passband

α_s = attenuation in stopband



Sum of maxima + minima = n

Eg: Estimate n. $\alpha_p = 1$ dB $\alpha_s = 25$ dB

$\frac{\omega_s}{\omega_p} = 1.5$ $|H(j\omega)|^2 = \frac{1}{1+\xi^2 C_n^2(\omega)}$

Ans

$$\alpha_p = 10 \log(1+\xi^2) = 1$$
$$\log(1+\xi^2) = 1/10$$
$$1+\xi^2 = 10^{0.1}$$
$$\xi^2 = 0.259$$

$$\alpha_s = 10 \log[1+\xi^2 C_n^2(\omega)]$$
$$25 = 10 \log[1+0.259\ C_n^2(1.5)]$$
$$C_n^2(1.5) = 1217.4$$
$$C_n(1.5) = \sqrt{1217.4}$$
$$C_n = 34.9$$

$\Rightarrow$ $C_n(\omega) = \cos(n \cos^{-1}\omega)$

But $\omega > 1 \Rightarrow$ $C_n(\omega) = \cosh(n \cosh^{-1}(\omega))$

$\Rightarrow$ $34.9 = \cosh(n \cosh^{-1}(1.5))$

$$n = \frac{\cosh^{-1}(34.9)}{\cosh^{-1}(1.5)} = 4.41 \cong 5^{th} \text{ order}$$

For butterworth, $|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$

$n = 9$, for butterworth filter.

Digital filter:

FIR filter: Finite impulse response
IIR filter: Infinite impulse response

LTI $\longrightarrow$ LSI
(Linear time invariant) (linear shift invariant
Analog Digital

$$y(n) = \sum_{k=0}^{q} a_k x(n-k) - \sum_{k=1}^{p} b_k y(n-k) \rightarrow (1)$$

$$y(n) + \sum_{k=0}^{p} b_k y(n-k) = \sum_{k=0}^{q} a_k x(n-k)$$

$$\downarrow$$

$$Y(z)\left[1 + \sum_{k=1}^{p} b_k z^{-k}\right] = \sum_{k=0}^{q} a_k z^{-k} X(z) \rightarrow (2)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{q} a_k z^{-k}}{1 + \sum_{k=1}^{p} b_k z^{-k}} \rightarrow (3)$$

$$H(z) = A \frac{\prod_{k=1}^{q} (1 - \alpha_k z^{-1})}{\prod_{k=1}^{p} (1 - \beta_k z^{-1})}$$

case 1: if $p > q$ $\Rightarrow$ proper system function

case 2: if $p < q$ $\Rightarrow$ improper system function

Impulse response for a rational system function:

$\alpha_i \neq \beta_j \Rightarrow$ No pole zero cancellation (1st order)

if proper rational function, $p > q$

PFE $\Rightarrow$ $H(z) = \sum_{k=1}^{p} \frac{A_k}{1 - \beta_k z^{-1}}$

IZT $\downarrow$

$$h(n) = \sum_{k=1}^{p} A_k (\beta_k)^n u(n) \quad , \quad |\beta_k| < 1 \Rightarrow \text{IIR}$$

When $q > p$

$$H(z) = \sum_{k=0}^{q-p} \alpha_k z^{-k} + \sum_{k=1}^{p} \frac{A_k}{1 - \beta_k z^{-1}}$$

$$\therefore h(n) = \underbrace{\sum_{k=0}^{q-p} \alpha_k \delta(n-k)}_{\text{FIR}} + \underbrace{\sum_{k=1}^{p} A_k \beta_k^n u(n)}_{\text{IIR}} = \text{IIR}$$

Suppose $p = 0$, $h(n) = \text{FIR}$

$p > 0$, $h(n) = \text{IIR}$

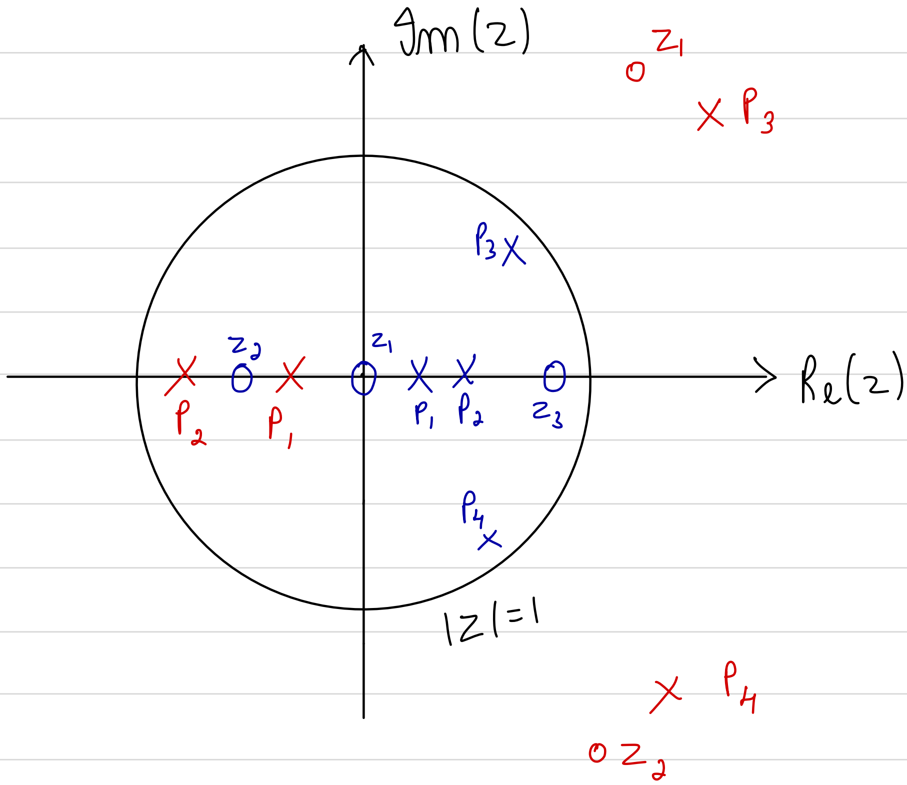
Inverse system

If $h_2(n)$ is inverse of $h_1(n)$ then,
$$h_1(n) * h_2(n) = \delta(n)$$

$$\Rightarrow \quad H_1(z) \cdot H_2(z) = 1$$
$$\therefore \quad H_2(z) = \frac{1}{H_1(z)}$$

Minimum phase system

Given a pole zero plot;



- For inverse system, zeros become poles and poles become zeros.
- For blue coloured zeros & poles, they remain within $|z| = 1$
- For red coloured zeros & poles, they can make inverse unstable.

Minimum phase response $\Rightarrow$ $|z_k| < 1$ & $|P_k| < 1$ in

$$H(z) = \frac{\prod_{k=1}^{q} (z - z_k)}{\prod_{k=1}^{p} (z - P_k)}$$

Eg: $|H(e^{j\omega})|^2 = \frac{17}{16} - \frac{1}{2}\cos\omega$. Check if system is minimum phase system.

Ans $|H(e^{j\omega})|^2 = \frac{17}{16} - \frac{1}{2}\frac{(e^{j\omega} + e^{-j\omega})}{2}$

$$|H(z)|^2 = \frac{17}{16} - \frac{1}{4}\left(z + \frac{1}{z}\right)$$

Frequency response of an LTI FIR & IIR system

$$h(n) = \begin{cases} 1 & , \quad N_1 \leq n \leq N_2 \quad , \quad N_2 > N_1 \\ 0 & \quad \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = \sum_{n=N_1}^{N_2} h(n)\, e^{-j\omega n}$$ Here $\omega = \Omega$

Difference equation:

$$\sum_{k=0}^{N} d_k\, y(n-k) = \sum_{k=0}^{M} p_k\, x(n-k)$$

$$\sum_{k=0}^{N} d_k\, e^{-jk\omega}\, Y(e^{j\omega}) = \sum_{p=0}^{M} p_k\, e^{-jk\omega}\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} p_k\, e^{-jk\omega}}{\sum_{k=1}^{N} d_k\, e^{-jk\omega}}$$

Eg: Frequency response of moving average filter

$$h(n) = \frac{1}{M}\{\underset{\substack{\uparrow\\0}}{1}, 1, 1, \ldots \underset{\substack{\uparrow\\M-2}}{1}, \underset{\substack{\uparrow\\M-1}}{1}, 0, 0\}$$

$$h(n) = \frac{1}{M}\{\underset{\substack{\uparrow\\0}}{\alpha_1}, \alpha_2, \alpha_3, \ldots \underset{\substack{\uparrow\\M-1}}{\alpha_M}, 0, 0\}$$

<u>DTFT:</u> $$H(e^{j\omega}) = \sum_{n=0}^{M-1} \frac{1}{M}\, e^{-j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1}\left(e^{-j\omega}\right)^n$$ → finite GP sum

$$= \frac{1}{M}\frac{1-e^{-j\omega M}}{1-e^{-j\omega}} =$$

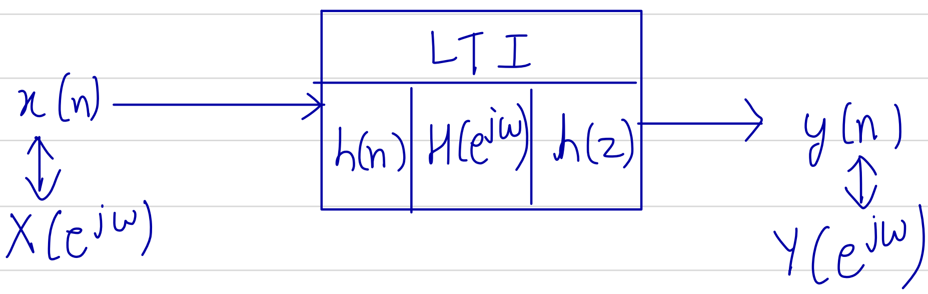
$$= \frac{1}{M}\frac{e^{-j\omega M/2}\left[e^{+j\omega M/2} - e^{-j\omega M/2}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{j\omega/2}\right]}$$

$$= \frac{1}{M}\, e^{j\omega(1-M)/2}\, \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

Magnitude response: $|H(e^{j\omega})| = \frac{1}{M}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$

This behaves like LPF

Phase & group delay:



$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$
$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

More spectral:

let $x(n) = A\cos(\omega_0 n + \phi)$

then

$$y(n) = A\,|H(e^{j\omega})| \cdot \cos\left[\omega_0 n + \phi + \theta(\omega)\right]$$

$$y(n) = A\left|H(e^{jw})\right| \cdot \cos\left[w_0\left(n + \frac{\theta(w)}{w_0}\right) + \phi\right]$$

$$= A\left|H(e^{jw})\right| \cdot \cos\left[w_0(n - \tau_p) + \phi\right]$$

where $\tau_p = -\dfrac{\theta(w)}{\omega_0}$

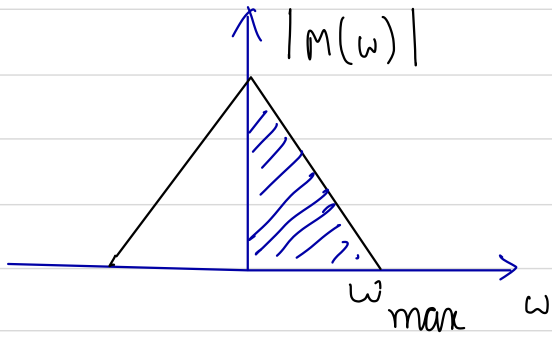
Continuous time counterpart of τ_p,

group delay, $\tau_g = -\dfrac{d(\theta(\omega))}{d\omega}$

Consider a system:

LTI

$m(t) \rightarrow$ Amplifier / LPF



• Also $\tau_g = -\dfrac{d(\theta(\omega))}{d\omega}$

If τ_g is fixed quantity, then it is fixed slop.

Amplitude and phase distortion:

$s_1(t) = V_1 \cos \omega t$ → Linear system / Frequency response → $V_1 |A(j\omega)| \cdot e^{j\phi(\omega)}$

$\uparrow$

$s_2(t) = V_2 \cos((\omega + \Delta\omega)t)$

$A(j\omega) = |A(j\omega)| \angle \phi(\omega)$

for s_1

$$V_1 |A(j\omega)| e^{j\phi(\omega)} = V_1 |A(j\omega)| \cos(\omega t + \phi(\omega))$$

$$= V_1 |A(j\omega)| \cos\left(\omega\left(t + \frac{\phi(\omega)}{\omega}\right)\right)$$

$$= V_1 |A(j\omega)| \cos[\omega(t - \tau_{p_1})]$$

where $\tau_{p_1} \triangleq \frac{-\phi(\omega)}{\omega} \rightarrow (1)$

for s_2

<u>Response:</u> $V_2 |A(j(\omega+\Delta\omega))| e^{j\phi(\omega+\Delta\omega)}$

$$= V_2 |A(j(\omega+\Delta\omega))| \cos[(\omega+\Delta\omega)t + \phi(\omega+\Delta\omega)]$$

$$= V_2 |A(j(\omega+\Delta\omega))| \cos[(\omega+\Delta\omega)(t - \tau_{p_2})] \rightarrow (2)$$

where $\tau_{p_2} = \frac{-\phi(\omega+\Delta\omega)}{\omega+\Delta\omega} \rightarrow (3)$

But,

$$\phi(\omega+\Delta\omega) = \phi(\omega) + \frac{d\phi}{d\omega}\Delta\omega$$

consider $\frac{1}{\omega+\Delta\omega} = (\omega+\Delta\omega)^{-1}$

$$= \omega^{-1}\left(1 + \frac{\Delta\omega}{\omega}\right)^{-1}$$

$$= \omega^{-1}\left(1 - \frac{\Delta\omega}{\omega} + \left(\frac{\Delta\omega}{\omega}\right)^2 + \dots\right)$$

Ignoring higher order terms,

$$\frac{1}{\omega+\Delta\omega} \simeq \frac{1}{\omega}\left(1 - \frac{\Delta\omega}{\omega}\right)$$

From equation (3), $\frac{\phi(\omega+\Delta\omega)}{\omega+\Delta\omega} = \left(\frac{\phi(\omega)}{\omega} + \frac{d\phi}{d\omega}\frac{\Delta\omega}{\omega}\right)\left(1 - \frac{\Delta\omega}{\omega}\right)$

$$\therefore \tau_{p_2} = \overset{\uparrow \tau_{p_1}}{-\frac{\phi(\omega)}{\omega}} + \frac{\phi(\omega)}{\omega}\frac{\Delta\omega}{\omega} - \frac{\Delta\omega}{\omega}\frac{d\phi}{d\omega}$$

To make $\tau_{p_1} = \tau_{p_2}$

$$\tau_{p_2} = \tau_{p_1} + \frac{\Delta\omega}{\omega}\left(\frac{\phi(\omega)}{\omega} - \frac{d\phi}{d\omega}\right)$$

$$\Rightarrow \frac{\Delta\omega}{\omega}\left(\frac{\phi(\omega)}{\omega} - \frac{d\phi}{d\omega}\right) = 0$$

$$\Rightarrow \int \frac{d\omega}{\omega} = \int \frac{d\phi}{\phi(\omega)}$$

$$\ln \phi(\omega) = \ln \omega + \ln K$$

$$\phi(\omega) = K\omega \,//$$

$\boxed{\phi(\omega) \propto \omega}$ $\rightarrow$ condition for zero phase distortion

Also $d\phi = K\,d\omega \Rightarrow \frac{d\phi}{d\omega} = K = -\tau_g$

$\therefore \tau_g = \text{constant} \,//$

Frequency domain equalization

Consider a practical channel; Frequency response $= C(\omega)$

Ideal situation $\Rightarrow$ Behave like response channel
FR: $C_{ref}(\omega) = e^{-j\omega D}$

where $D =$ delay

Practical channel to ideal channel is made by cascading an equalizer:

$$X(\omega) \longrightarrow \underset{C(\omega)}{\boxed{\text{LTI}}} \longrightarrow \underset{H(\omega)}{\boxed{\text{equalizer}}} \longrightarrow Y(\omega)$$

$$\Rightarrow Y(\omega) = C(\omega) \cdot H(\omega) \cdot X(\omega)$$

For ideal channel: $Y(\omega) = C_{ref}(\omega) \cdot X(\omega)$

$$\therefore \quad C_{ref}(\omega) = C(\omega) \cdot H(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{C_{ref}(\omega)}{C(\omega)} = \frac{e^{-j\omega D}}{C(\omega)}$$

$H(\omega) \rightarrow$ FIR filter
$\downarrow$ IDTFt
$h(l) =$ defined for $0 \leq l \leq n-1$

$$\Rightarrow H(\omega) = \sum_{l=0}^{L-1} h(l)\, e^{-j\omega l}$$

Discretizing ω to M samples,

$$H(m) = \sum_{l=0}^{L-1} h(l)\, e^{-j\frac{2\pi}{M} m l}$$

where $m = 0, 1, 2, \ldots, M-1$

Also $H(\omega) = \sum_{l=0}^{L-1} h(l)\, e^{-j\omega l} = a^T(\omega) \cdot h$

where $a(\omega) = [e^{-j0}, e^{-j1\omega}, e^{-j2\omega}, \ldots]$, $a^T(\omega)$ = Transpose of a

$h = [h(0), h(1), h(2), \ldots, h(L-1)]_{L\times 1}$

$\therefore H(m) = a_m^T \cdot h$

$$a_m = \left[1, e^{-j\frac{2\pi m}{M}}, e^{-j\frac{2\pi m \times 2}{M}}, e^{-j\frac{2\pi \times 3}{M}}, \ldots, e^{-j\frac{2\pi m}{M}(L-1)}\right]_{L\times 1}$$

<u>NOTE:</u> FIR filters are preferred over IIR filters because:

1) It is easier to maintain Linear phase response in FIR filter.

2) FIR filters are always stable. IIR filters may or maynot be stable.

Least Square approximation Algorithm

Strategy: Use of digital filter to approxima-te a system $H(\omega)$ or $H(m)$

Use FIR filter of length L, having frequency response $E(\omega)$

We try to approximate $E(\omega) \approx H(\omega)$ from ⊛. So that we can use a FIR filter instead of the previous block diagram system. Since $H(\omega)$ / $H(m)$ is just a abstract math model we use this approach to realize the model.

$$E(\omega) = \sum_{n=0}^{L-1} \underbrace{h(n)}_{\text{array of length 'L'}} \; e^{-j\omega n} \rightarrow a(\omega)$$

$\therefore \; E(\omega) = a(\omega) \cdot h$

where $h = \{h(0), \; h(1), \; \ldots, \; h(L-1)\}_{1\times L}$

$a(\omega) = \{1, \; e^{-j\omega}, \; e^{-2j\omega}, \; \ldots, \; e^{-j(L-1)\omega}\}_{1\times L}$

Taking T = unity $(T=1)$

$a(\omega) = \{1, \; e^{-j\omega T}, \; e^{-2j\omega T}, \; e^{-3j\omega T}, \; \ldots, \; e^{-j(L-1)\omega T}\}_{1\times L}$

$h \rightarrow$ Weighted coefficient array or filtered coefficient array

$a \rightarrow$ Phase shift array

$$E(\omega) = \vec{a}^T(\omega) \cdot h$$

Discretizing ω to M samples

$$E(m) = \sum_{n=0}^{L-1} h(n)\, e^{-j\frac{2\pi}{M} m n} \qquad m = 0, 1, 2, \ldots M-1$$

$$\text{Error} = e(m) = H(m) - E(m)$$

$$a(m) = \{ a(0),\ a(1), \ldots,\ a(M-1) \}_{1 \times M}$$

$E(m)$ can be written as:

$$E(m) = A^T \cdot h$$

where

$$A^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & \cdots & \cdots & e^{-j\frac{2\pi}{M}(M-1)(L-1)} \end{bmatrix}_{L \times M}$$

$$\text{Error} = \vec{e} = \vec{H} - A \cdot \vec{h}$$

To minimize error we vary $\vec{h}$.

<u>Optimization:</u> $\min |(e_m)^2| = \min\limits_{h} \| H - Ah \|^2$

Transfor function (frequency response) classificati-on based on phase characteristics

TF: $H(z) \rightarrow$ real valued coefficients $h(n)$

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

Define: $F(z) = H(z)\, H(z^{-1})$, $F(z)$ = zero phase TF

$$= H(e^{jw}) \cdot H(e^{-jw})$$

$$F(e^{jw}) = |H(e^{jw})|^2\, e^{j0}$$

At $z = v\ (P/z)$ (Pole/zero) for $H(z)$

then for $H(z^{-1})$, $z = \frac{1}{v\ (P/z)}$ $\Rightarrow z = v^{-1}$

FIR: $A(z) = \sum_{i=0}^{N} h(i)\, z^{-i}$

then zero phase transfer function:

$$B(z) = \sum_{i=0}^{N} h(i)\left[z^{i} + z^{-i}\right]$$

$$x(n) \longrightarrow \underset{\text{LTI}}{\boxed{H(z)}} \xrightarrow{v(n)} \underset{\text{LTI}}{\boxed{H(z^{-1})}} \longrightarrow w(n)$$

$\Downarrow$ or put $u(n) = v(-n)$

$$x(n) \longrightarrow \underset{\text{LTI}}{\boxed{H(z)}} \xrightarrow{v(n)} \quad \xrightarrow[u(n)]{} \underset{\text{LTI}}{\boxed{H(z)}} \longrightarrow w(n)$$

$$x(n) \longleftrightarrow X(z) \quad \Rightarrow \quad x(-n) \longleftrightarrow X(z^{-1})$$

Also $U(e^{jw}) = V(e^{-jw}) = V^*(e^{jw})$

$$y(n) = w(-n)$$

$V(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$
$U(e^{j\omega}) = X^*(e^{j\omega}) \cdot H^*(e^{j\omega})$
$W(e^{j\omega}) = U(e^{j\omega}) \cdot H(e^{j\omega}) = X^*(e^{j\omega}) \cdot H^*(e^{j\omega}) H(e^{j\omega})$
$Y(e^{j\omega}) = W^*(e^{j\omega})$

NOTE: Zero phase response is practically non realisable.

Linear phase transfer function

- We try to achieve linear phase only in passband.

$$x(n) \longrightarrow \boxed{D} \longrightarrow y(n) = x(n-D)$$

$\Rightarrow Y(e^{j\omega}) = e^{-j\omega D} X(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D} \;//$$

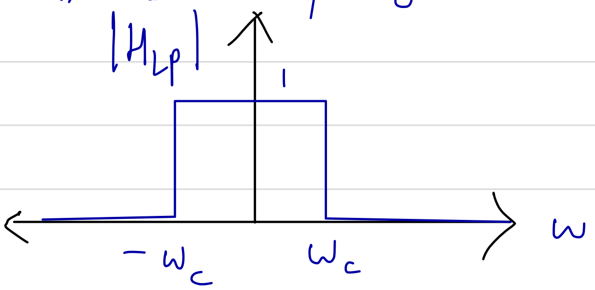
$\Rightarrow |H(e^{j\omega})| = 1 \qquad \Rightarrow \angle H(e^{j\omega}) = -\omega D \;//$

Eg: If $x(n) = Ae^{j\omega n}$ then for linear phase response $y(n) = Ae^{j\omega n} \cdot e^{-j\omega D}$
$\Rightarrow y(n) = Ae^{j\omega(n-D)} \;//$

Q) Given IR of ideal linear phase LPF (DT) Find linear phase TF.

Ans

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_0}, & |\omega| \leq \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$



$$h_{lp}(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega n_0} e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-n_0)} \, d\omega$$

$$h_{lp}(n) = \frac{1}{2\pi} \left. \frac{e^{j\omega(n-n_0)}}{j(n-n_0)} \right|_{-\omega_c}^{\omega_c}$$

$$= \frac{\sin \omega_c(n-n_0)}{\pi(n-n_0)} \qquad -\infty < n < \infty \Rightarrow \text{IIR}$$

if $n_0 = N/2$

$$h_{lp}(n) = \frac{\sin \omega_c(n-N/2)}{\pi(n-N/2)} \qquad 0 \le n \le N$$

$\Rightarrow$ FIR filter of length $N+1$

$$H_{Lp}(\omega) = \sum_{n=0}^{N} h_{lp}(n) e^{-j\omega n} = e^{-j\omega N/2} H_{Lp}^{U}(\omega)$$

$H_{Lp}^{U}(\omega) =$ zero phase TF.

Types of linear phase FIR filter

$h(n)$ = real valued.

N samples ; $n = 0, 1, 2, \ldots, N-1$

length of $h(n) = N+1$ for $0 \le n \le N$

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n) e^{-j\omega n}$$

Mathematically, $H(e^{j\omega}) = e^{j(c\omega+\beta)} \; H^{U}(\omega)$

$\downarrow$

Zero phase response (only magnitude)

$|H^{U}(e^{j\omega})| = |H^{U}(\omega)|$

Zero phase TF $H^{U}(\omega)$ can be even or odd.

$$H^{U}(-\omega) = H^{U}(\omega) \rightarrow \text{even}$$
$$H^{U}(-\omega) = -H^{U}(\omega) \rightarrow \text{odd}$$

$\rightarrow$ $H^{*}(e^{-j\omega}) = \sum_{n=0}^{N} h(n) e^{-j\omega n} = H(e^{j\omega}) \rightarrow$ ①

$\rightarrow$ $H(e^{j\omega}) = e^{j(c\omega+\beta)} \; H^{U}(\omega) \rightarrow$ 2a

$H(e^{j\omega}) = e^{jc\omega} \cdot e^{j\omega\beta} \; H^{U}(\omega) \rightarrow$ 2b

$\rightarrow$ $H^{U}(\omega) = e^{-j(c\omega+\beta)} \; H(e^{j\omega}) \rightarrow$ ③

if $H^{U}(\omega)$ is even function

$H^{U}(-\omega) = e^{-j(c(-\omega)+\beta)} \; H(e^{-j\omega}) \rightarrow$ ④

$H^{*}(e^{-j\omega}) = e^{-j(c(-\omega)+\beta)} \cdot H^{U}(-\omega)$

$$\therefore \quad H^*(e^{-j\omega}) = e^{j\omega c} \; e^{-j\beta} \; H^U(-\omega) \rightarrow (5)$$

<u>case study 1:</u> $H^U(\omega)$ is even function

$$H^U(\omega) = H^U(-\omega)$$

Compare equation 5 & 2b

$$e^{j\beta} = e^{-j\beta} \rightarrow (6)$$

True for $\beta = 0 \; , \pm\pi$

$$H(e^{j\omega}) = e^{j\omega}(\pm 1)\, H^U(\omega)$$

$H^U(\omega) = \pm\, e^{-jc\omega}$. $\quad H(e^{j\omega}) = \pm\, e^{-jc\omega} \sum_{n=0}^{N} h(n)\, e^{-j\omega n}$

$$H^U(\omega) = \pm \sum_{n=0}^{N} h(n)\, e^{-j(c+n)\omega} \rightarrow (7)$$

Put $\omega = -\omega$ in equation (7) & Put $l = n$

$$H^U(-\omega) = \pm \sum_{l=0}^{N} h(l)\, e^{j\omega(c+l)} \rightarrow (8)$$

Now put $l = N-n$ & $c = -N/2$

$$H^U(-\omega) = \pm \sum_{n=0}^{N} h(N-n)\, e^{j\omega(-N/2+N-n)}$$

$$= \pm \sum_{n=0}^{N} h(N-n)\, e^{-j\omega(n-N/2)} \rightarrow (9)$$

From (7) & (9),

$$H^U(-\omega) = H^U(\omega) \Rightarrow h(n) = h(N-n)$$

<u>case 2;</u> $H^U(\omega)$ is odd function

if $H^U(-\omega) = -H^U(\omega)$

$$H^U(\omega) = e^{-jc\omega}\, e^{-j\beta}\, H(e^{j\omega})$$

$$H^U(-\omega) = e^{jc\omega}\, e^{-j\beta}\, H(e^{-j\omega})$$

For odd $e^{j\beta} = -e^{-j\beta}$ if $\beta = \pm\pi/2$

$$H(e^{jw}) = e^{jcw} \quad j\, H^U(w)$$

$$H^U(w) = -j\, e^{-jcw} \sum_{n=0}^{N} h(n)\, e^{-jwn}$$

$$= -j \sum_{n=0}^{N} h(n)\, e^{-jw(c+n)} \rightarrow (10)$$

$$H^U(-w) = -j \sum_{\ell=0}^{N} h(\ell)\, e^{jw(c+\ell)} \rightarrow (11) \qquad (\text{Put } \ell = n)$$

Put $\ell = N-n$ & $c = -N/2$

$$H^U(-w) = -j \sum_{n=0}^{N} h(N-n)\, e^{+jw(N-n-N/2)}$$

$$= j \sum_{n=0}^{N} [-h(N-n)]\, e^{-jw(n-N/2)} \rightarrow (12)$$

$\therefore$ From (10) & (12),

$$H^U(-w) = -H^U(w) \Rightarrow h(n) = -h(N-n)$$

4 types of filter

i) $H^U(w)$ & N both are even
ii) $H^U(w)$ & N both are odd
iii) $H^U(w)$ is even & N is odd
iv) $H^U(w)$ is odd & N is even

- If $H^U(w) =$ even, $h(n)$ is symmetric as $h(n) = h(N-n)$

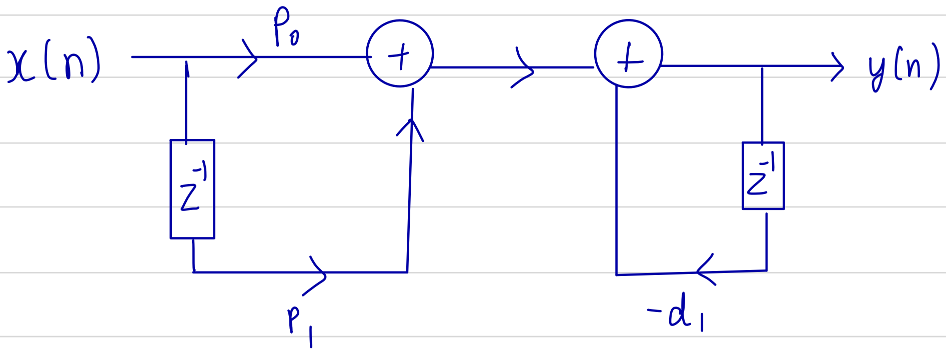
- If $H^U(w) =$ odd, $h(n)$ is anti-symmetric as $h(n) = -h(N-n)$

Digital filter structure

$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

Difference equation:
$$y(n) = -\sum_{k=1}^{N} d_k\, y(n-k) + \sum_{k=0}^{M} P_k\, x(n-k)$$

1st order IIR: if $N=1$ & $M=1$
$$y(n) = -d_1\, y(n-1) + P_0\, x(n) + P_1\, x(n-1)$$



→ If the number of delay units is greater than the order of system, it is known as Non canonical structure

→ If the number of delay units is equal to the order of system, it is known as canonical structure.

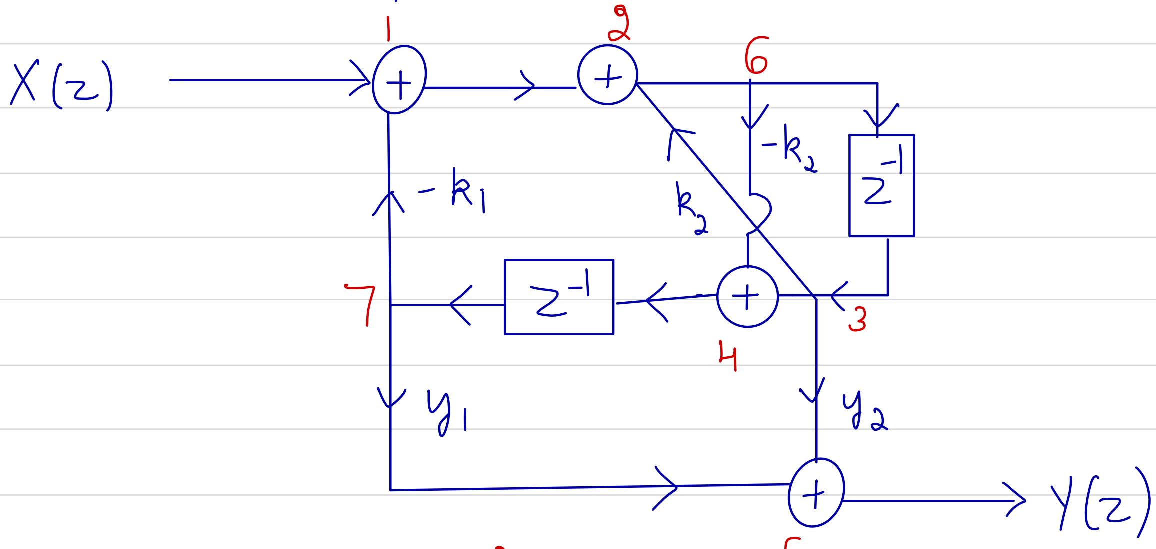
Equivalent structure & Transpose operation

Transpose : i> Replace adder by take off point and vice versa

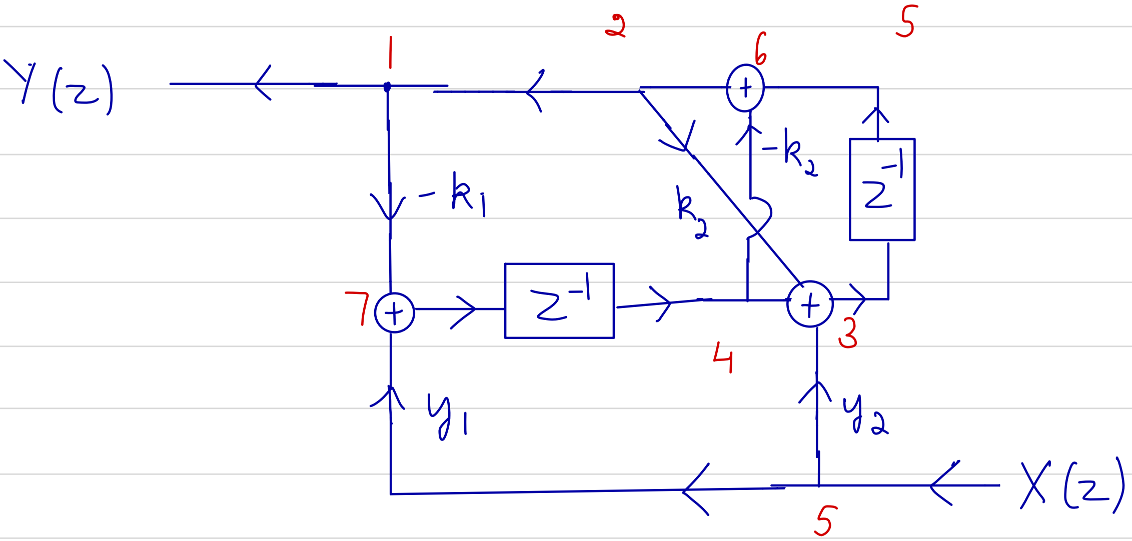
ii> Reverse the direction of arrows

iii> change the port : $x(n) \leftrightarrow y(n)$

Eg 1> Find the transpose of:



Ans

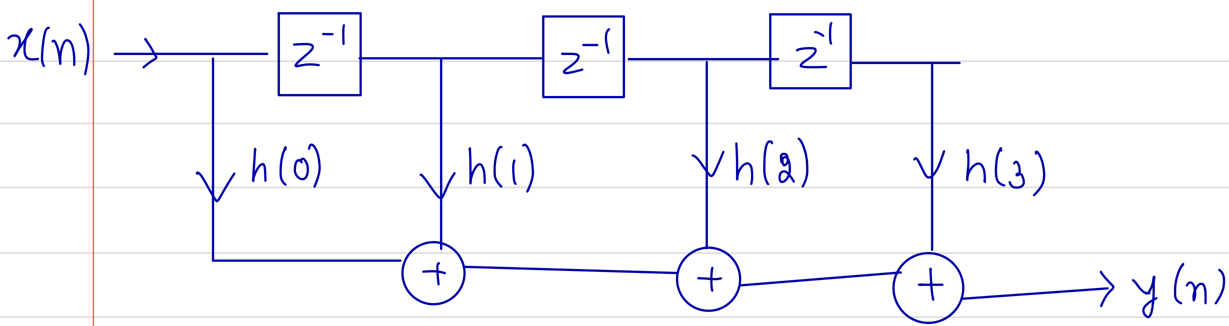


This is the transpose of system

FIR filter structure

1> Direct → order N, (N+1) → filter coefficient
$h(0) \ldots h(N)$

Eg: $N=3$
$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3)$$



Taking transpose and inverting

Adder → N
Multiplier → N+1 delay = order
Suppose we had,
$$y(n) = h(0)x(n) + h_1 x(n-1) + h(2)x(n-2) + h(3)x(n-3) + \alpha_1 y(n-1) + \alpha_2 y(n-2) + \alpha_3 y(n-3)$$

if one of $\alpha_i \neq 0$ then feedback exists & it is IIR filter

2> <u>Cascade form:</u> (FIR) → order N

Taking a higher N,

N = even ⇒ $k = N/2$

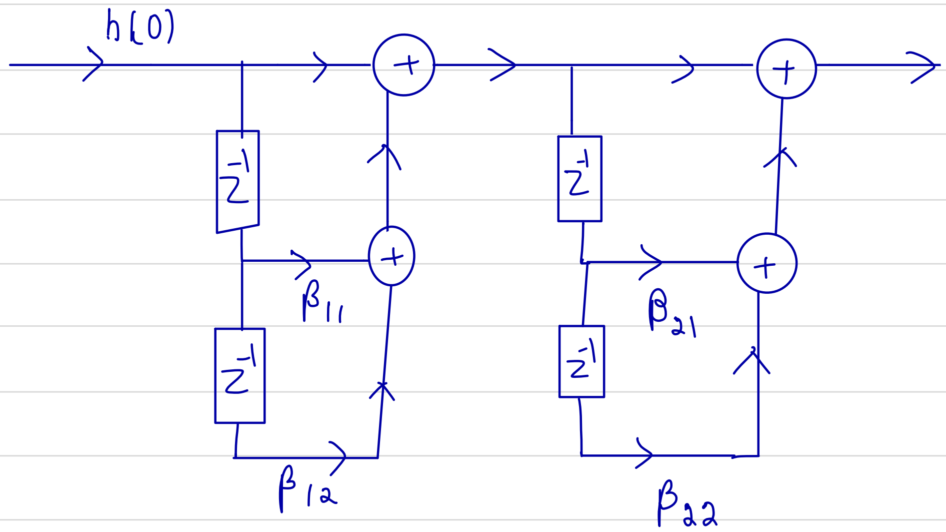
N = odd ⇒ $k = \frac{N+1}{2}$ with $\beta_{2i} = 0$

TF: factored $H(z) = \sum_{k=0}^{N} h(k) z^{-k}$

$$= h(0) \prod_{i=1}^{k} (1 + \beta_{1i} z^{-1} + \beta_{2i} z^{-2})$$

if $N = 4$, $k = 2$

$$H(z) = h(0) \prod_{i=1}^{2} (1 + \beta_{1i} z^{-1} + \beta_{2i} z^{-2})$$



3> <u>Polyphase realisation:</u> FIR of order N = 8

$$H(z) = \sum_{k=0}^{8} h(k) z^{-k}$$

$$= h(0) + h(1) z^{-1} + \dots + h(8) z^{-8}$$

$$= h(0) + h(2) z^{-2} + h(4) z^{-4} + h(6) z^{-6} + h(8) z^{-8}$$
$$+ z^{-1} [h(1) + h(3) z^{-2} + h(5) z^{-4} + h(7) z^{-6}]$$

→ 1

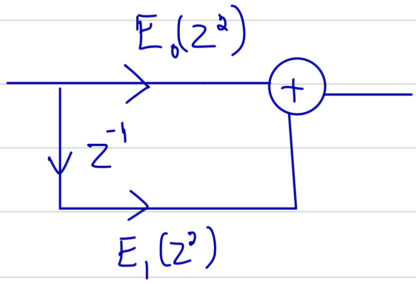
2nd level decomposition:

let $E_0(z) = h(0) + h(2)z^{-1} + h(4)z^{-2} + h(6)z^{-3} + h(8)z^{-4}$

$E_1(z) = h(1) + h(3)z^{-1} + h(5)z^{-2} + h(7)z^{-3}$

$H(z) = f[E_0(z) \,\&\, E_1(z)]$

$\therefore H(z) = E_0(z^2) + z^{-1}E_1(z^2)$



3rd level decomposition:

$E_0(z) = h(0) + h(3)z^{-1} + h(6)z^{-2}$

$E_1(z) = h(1) + h(4)z^{-1} + h(7)z^{-2}$

$E_2(z) = h(2) + h(5)z^{-1} + h(8)z^{-2}$

then $H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$

- Now consider a generic FIR filter of order N & size $S = N+1$

then $$H(z) = \sum_{m=0}^{L-1} z^{-m} E_m(z^L)$$

and $$E_m(z) = \sum_{n=0}^{\left\lfloor \frac{N+1}{L} \right\rfloor} h(Ln+m)\, z^{-n} \qquad 0 \leq m \leq L-1$$

where $\lfloor x \rfloor$ is floor function

Multirate system

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow x_u(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

upsampler

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow x_d(n) = \begin{cases} x(nM), & n = \pm 1/M, \pm 2/M, \\ 0, & \text{otherwise} \end{cases}$$

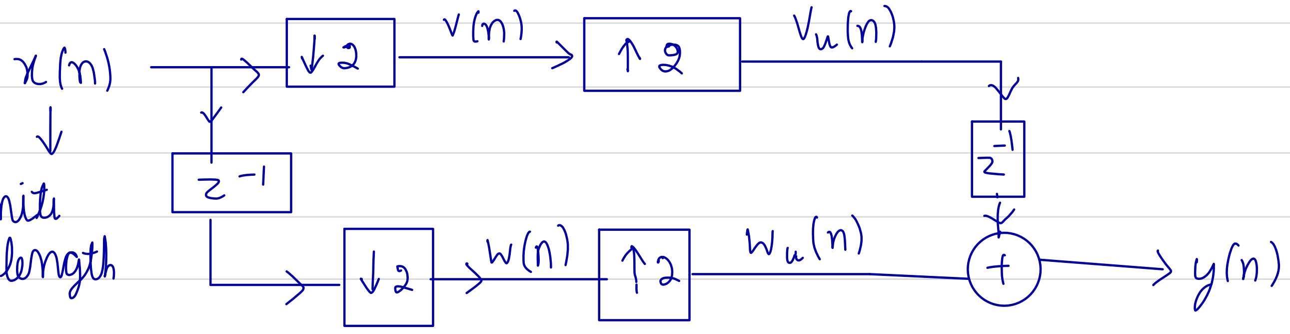
downsampler

$f_{s_i} = 1/T$ (sampling frequency)

for upsampler $f_{s_o} = L f_{s_i} = L/T$

for downsampler $f_{s_o} = \frac{f_{s_i}}{M} = \frac{1}{MT}$

Q>



Ans

n	0	1	2	3	4	5	6	7	8
$x(n)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	$x(8)$
$v(n)$	$x(0)$	$x(2)$	$x(4)$	.....				.....	$x(16)$
$w(n)$	$x(-1)$	$x(1)$	$x(3)$	.....				....	$x(15)$
$V_u(n)$	$x(0)$	0	$x(2)$	0	$x(4)$	......			$x(8)$
$W_u(n)$	0	$x(1)$	0	$x(3)$	0	.....		$x(7)$	0

$$y(n) = W_u(n) + V_u(n-1)$$

$\therefore$ $y(n)$ | $x(-1)$ $x(0)$ $x(1)$ .... $x(7)$

IIR filter

Based on analog filter design techniques:

I: Impulse invariant

II: Bilinear transformation

Digital LPF:

$$H(j\omega) \longrightarrow H(s) \xrightarrow[\text{Laplace}]{\text{Inverse}} h(t) \xrightarrow{\text{sampling}} h(nT) = h(n)$$

$$h(n) \xrightarrow{z \text{ transform}} H(z)$$

I **Impulse invariant:** $h(nT) \longrightarrow h(n)$

II **Bilinear transformation:-**

$$s \text{ plane} \longleftrightarrow z\text{-plane}$$

$$s = \frac{2}{T} \frac{z-1}{z+1} \quad \text{or} \quad z = \frac{1 + sT/2}{1 - sT/2}$$

let $z = r^{-1} e^{j\Omega}$

then $$s = \frac{2}{T} \frac{1 - re^{-j\Omega}}{1 + re^{-j\Omega}} \times \left(\frac{1 + re^{-j\Omega}}{1 + re^{-j\Omega}} \right)$$

$$s = \frac{2}{T} \left[\frac{1 - r^2 + j(2r \sin\Omega)}{1 + r^2 + 2r\cos\Omega} \right]$$

$$\Rightarrow \quad s = \frac{2}{T} \left[\frac{1 - r^2}{1 + r^2 + 2r\cos\Omega} + j \frac{2r\sin\Omega}{1 + r^2 + 2r\cos\Omega} \right]$$

$$s = \sigma + j\omega$$

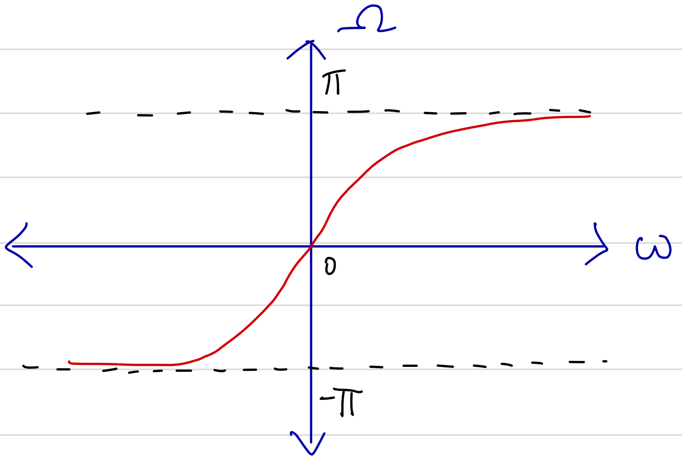
if $r < 1 \Rightarrow \sigma > 0$

if $r < 1 \Rightarrow \sigma < 0$

if $r = 1 \Rightarrow \sigma = 0$ & $\omega = \frac{2}{T}\tan\frac{\Omega}{2}$

$\therefore \Omega = 2\tan^{-1}\left(\frac{\omega T}{2}\right)$ for $r = 1$

For $r = 1$



Q> Butterworth digital filter should be designed using bilinear transformation ($T = 2$).
Specifications: Pass band (Permissible attenuation 2 dB, 0.2π)
Stop band (Attenuation more than 10 dB, frequency: $0.4\pi \leq \Omega \leq \pi$)

Ans Given: $\Omega_p = 0.2\pi$

$\Rightarrow \omega_p = \frac{2}{2}\tan\left(\frac{0.2\pi}{2}\right) = 0.3249$

$\Omega_s = 0.4\pi \Rightarrow \omega_s = \tan\left(\frac{0.4\pi}{2}\right) = 0.7265$

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}$$ for butterworth filter

Taking $\omega_c = 1$ (since it is not given)

Attenuation: $|-2\text{ dB}|^2 = \frac{1}{1 + \omega^{2N}}$ at $\omega = \omega_p$

$\Rightarrow N = 1.695 \simeq 2$

Now $|-2|^2 = \dfrac{1}{1+\left(\dfrac{\omega_p}{\omega_c}\right)^{2\times 2}}$

$\Rightarrow \quad \omega_c = 0.4195$

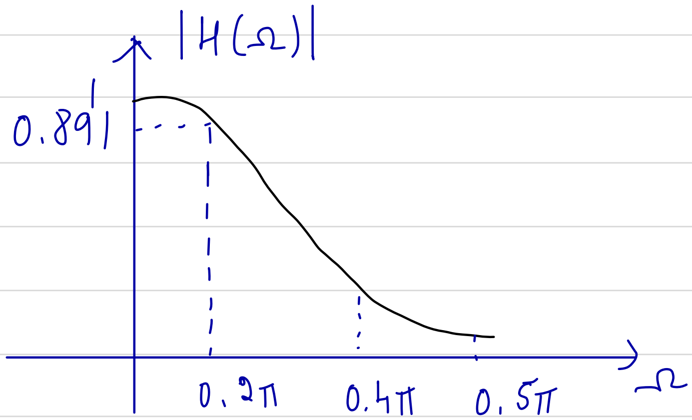
$$H(s) = \frac{1}{\left(\dfrac{s}{\omega_c}\right)^2 + \sqrt{2}\,\dfrac{s}{\omega_c} + 1}$$

$$H(s) = \frac{0.176}{s^2 + 0.5935s + 0.176}$$

$$H(z) = \frac{0.176}{\left(\dfrac{z-1}{z+1}\right)^2 + 0.5935\left(\dfrac{z-1}{z+1}\right) + 0.176}$$

$$\therefore \quad H(z) = \frac{0.135\,(z+1)^2}{z^2 - 2.1742z + 1.716}$$

Put $z = e^{j\Omega}$



FIR filter using window

$$h_{lp}(n) = \frac{\sin\left[w_c(n - N/2)\right]}{\pi(n - N/2)} \qquad -\infty < n < \infty$$

1) **Rectangular window**

$$\hat{h}_{lp}(n) = h_{lp}(n) \cdot w_R(n)$$

where $$W_R(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

2) **Hamming window:**

$$W_{hamming}(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

3) **Blackman window**

$$W_{BM}(n) = \begin{cases} 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right) & \text{for } 0 \leq n \leq N-1 \\ 0 \quad , \quad \text{otherwise} \end{cases}$$