PH110

MANVITH PRABHU
211EC228

GAUSS LAW AND ELECTRIC POTENTIAL

Q1) Check whether $\vec{E} = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$ be an electrostatic field.

Ans
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ xy & 2yz & 3xz \end{vmatrix}$$
$$= -2y\hat{x} - 3z\hat{y} + x\hat{z} \neq 0$$

$\because \vec{\nabla} \times \vec{E} \neq \vec{0}$, $\vec{E}$ cannot be electrostatic field.

Q2) Check whether $\vec{E} = k[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}]$ is an electrostatic field

Ans
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

$$= \vec{0} \,//$$

$\therefore \vec{E}$ is an electrostatic field.

Potential due to a point charge

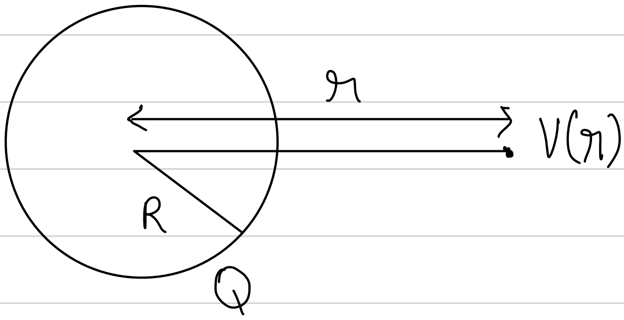
$$V(r) = -\int_{-\infty}^{r} \vec{E} \cdot d\vec{l}$$

$$= -\int_{-\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} \hat{r}' (\hat{r}' dr + r' d\theta\,\hat{\theta} + r' \sin\theta\, d\phi\,\hat{\phi})$$

$$= -\int_{\infty}^{r} \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

Potential due to a spherical shell

$$\vec{E}(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2} & r \geq R \\ 0 & r < R \end{cases}$$



$$V(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r} & r \geq R \\ \dfrac{Q}{4\pi\varepsilon_0 R} & r < R \end{cases}$$

- $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$

$\vec{E} = -\vec{\nabla} V \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{\nabla} V = -\rho/\varepsilon_0$

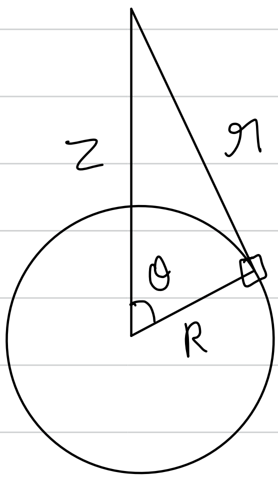
$\Rightarrow \boxed{\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}} \rightarrow$ Poisson's equation

NOTE 1) $\vec{\nabla} \cdot \vec{\nabla} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$. This is called Laplacian

2) For harmonic functions $\nabla^2 V = 0$

3) $V(r) = \dfrac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \dfrac{q_i}{r_i} = \dfrac{1}{4\pi\varepsilon_0} \int \dfrac{dq}{r}$

Potential of a uniformly charged spherical shell



$da = R^2 \sin\theta \, d\theta \, d\phi$

$$r = \sqrt{z^2 + R^2 - 2zR\cos\theta}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \, da}{r}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{\pi}\int_0^{2\pi} \frac{R^2 \sin\theta \, d\phi \, d\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= \frac{\sigma}{2\varepsilon_0} \int_0^{\pi} \frac{R^2 \sin\theta \, d\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

let $z^2 + R^2 - 2zR\cos\theta = u^2$

$$\cancel{2} zR \sin\theta \, d\theta = \cancel{2} u \, du$$

$$R^2 \sin\theta \, d\theta = \frac{Ru}{z} du$$

$$= \frac{\sigma}{2\varepsilon_0} \int_{\sqrt{z^2+R^2-2zR}}^{\sqrt{z^2+R^2+2zR}} \frac{R\cancel{u}}{z} \frac{du}{\cancel{u}}$$

$$= \frac{\sigma}{2\varepsilon_0} \frac{R}{z} \left(\sqrt{(R+z)^2} - \sqrt{(z-R)^2} \right)$$

$$= \frac{\sigma}{2\varepsilon_0} \frac{R}{z} \left(R+z - z+R \right) = \frac{\sigma R^2}{\varepsilon_0 z}$$

$$\therefore \quad V(z) = \begin{cases} \dfrac{\sigma R^2}{\varepsilon_0 z} & \text{for } z > R \\ \dfrac{\sigma R}{\varepsilon_0} & \text{for } z \leq R \end{cases}$$

Magnetostatics

- Magnetic field generated by steady currents.
- Static current do not change with time i.e $\frac{dI}{dt} = 0$

Force acting on a charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$ – This is known as Lorentz force.

When $\vec{E} = \vec{0}$, $\vec{F} = q(\vec{v} \times \vec{B})$

NOTE
- $1\,G$ (gauss) $= 10^{-4}\,T$ (Tesla)
- B_{earth} (average) $= 0.4\,G$

Current distribution

- If a line charge of density λ is moving with a velocity $\vec{v}$

$$\vec{I} = \lambda\vec{v}$$

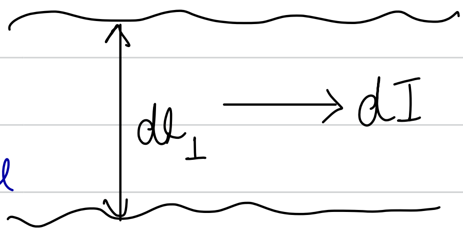
$$\lambda\vec{v} = \lambda\frac{dl}{dt} = \frac{dq}{dt} = I$$

- Force on a segment of current carrying wire,

$$\vec{F} = \int dq\,(\vec{v} \times \vec{B})$$
$$= \int \lambda\, dl\,(\vec{v} \times \vec{B})$$
$$= \int dl\,(\vec{I} \times \vec{B}) = \int I\,(d\vec{l} \times \vec{B})$$

Surface current density

If σ is the surface charge density which is flowing with velocity $\vec{v}$, then charge flowing through the strip in time dt is



$$dq = \sigma\,(\underbrace{\vec{v}\,dt}_{l})\,\underbrace{dl_\perp}_{b} \qquad \text{so } (\vec{v}\,dt)\,dl_\perp = \text{Area}$$

$$dI = \frac{dq}{dt} = \sigma \vec{v}\, dl_\perp$$

Surface current density is defined as,

$$\vec{K} = \frac{dI}{dl_\perp} = \sigma\vec{v}$$

- It is the current flowing through the surface per unit perpendicular distance.

Force on a current carrying strip,

$$\vec{F} = \int dq(\vec{v}\times\vec{B}) = \int \sigma\, da\,(\vec{v}\times\vec{B})$$
$$= \int (\vec{K}\times\vec{B})\, da$$

Volume current density: $\vec{J} = \dfrac{dI}{da_\perp} = \dfrac{dq}{dt\, da_\perp} = \dfrac{\rho\,\vec{v}\, d\vec{a}_\perp}{da_\perp} = \rho\vec{v}$

where ρ is volume charge density.

$$\therefore \vec{F} = \int dq\,(\vec{v}\times\vec{B}) = \int \rho\, d\tau\,(\vec{v}\times\vec{B})$$
$$= \int d\tau\,(\vec{J}\times\vec{B}) \;//$$

Net current flowing out of closed surface,

$$= \oint_S \vec{J} \cdot d\vec{a} = -\frac{dQ}{dt} = -\frac{d}{dt}\int \rho \, d\tau$$

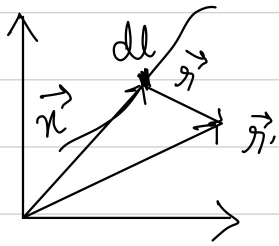
Applying divergence theorem,

$$\oint \vec{J} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{J} \, d\tau = -\frac{\partial}{\partial t}\int \rho \, d\tau$$

$$\boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0} \rightarrow \text{Continuity equation.}$$

Biot - Savart's law

$$\vec{B} = \frac{\mu_0}{4\pi}\int I \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



from surface current:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{k(x) \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, da$$

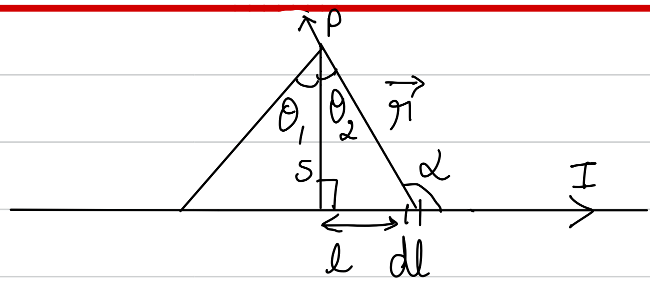
$$\mu_0 = 4\pi \times 10^{-7} \; N/A^2$$

from volume current:

$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(x) \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau$$

Q1) Find magnetic field due to a long straight wire carrying steady current I.

Ans $\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl} \times \hat{r}}{r^2}$



$\vec{dl} \times \hat{r} = dl \sin(90+\theta_2)$
$= dl \cos\theta$

Also $\cos\theta = \frac{s}{r}$ $\Rightarrow$ $r = \frac{s}{\cos\theta}$

$\therefore \vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{dl \cos\theta}{s^2/\cos^2\theta}$

$\tan\theta = \frac{l}{s}$ $dl = s \sec^2\theta \, d\theta$

$\therefore B = \frac{\mu_0 I}{4\pi} \int \frac{s \, d\theta \cos^3\theta}{\cos^2\theta \, s^2}$

$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$

$= \frac{\mu_0 I}{4\pi s} [\sin\theta_2 - \sin\theta_1]$

If wire is infinitely long
$\theta_1 = -90°$ $\theta_2 = +90°$

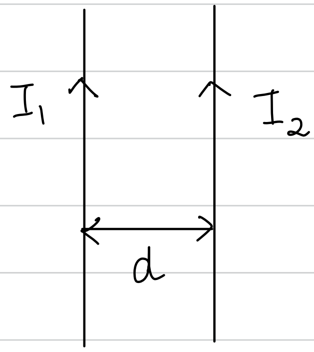
$\therefore B(P) = \frac{\mu_0}{4\pi} \frac{I}{s} (1-(-1))$

$\therefore B(P) = \frac{\mu_0 I}{2\pi s}$ //

Q2) Find the force / length exerted by I_1 on I_2 where I_1 & I_2 are current in 2 parallel infinite wires.

Ans $\vec{F} = I_2 \int d\vec{l} \times \vec{B}$

$$= I_2 \int_0^L dl \, \frac{\mu_0 I_1}{2\pi d} \sin 90^\circ$$



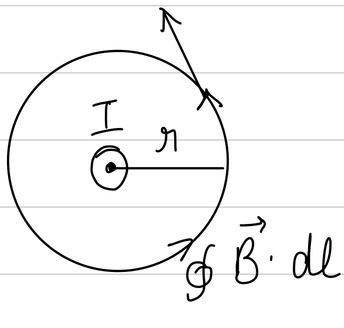
$$F = \frac{I_2 I_1 L \mu_0}{2\pi d}$$

$$\therefore \frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \quad \text{(attractive)}$$

Divergence & curl of Magnetic field

Consider $\oint \vec{B} \cdot dl = \oint B \, dl \cos\theta$

$$= \oint \frac{\mu_0 I}{2\pi r} dl$$



$$= \frac{\mu_0 I}{2\pi r} \oint dl$$

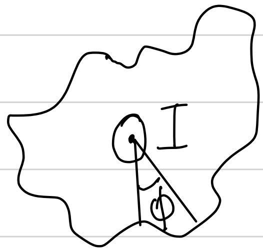
$$= \frac{\mu_0 I}{2\pi r} \times 2\pi r$$

$$= \mu_0 I$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Also $d\vec{l} = ds \, \hat{s} + s \, d\phi \, \hat{\phi} + dz \, \hat{z}$

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$



$$\therefore \vec{B} \cdot d\vec{l} = \frac{\mu_0 I \cancel{s} \, d\phi}{2\pi \cancel{s}} = \frac{\mu_0 I}{2\pi} d\phi$$

$$\therefore \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \int d\phi = \mu_0 I$$

$\therefore \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

But $I_{enc} = \int \vec{J}\cdot d\vec{a}$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

But $\oint \vec{B}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{B})\, d\vec{a}$ (from Stokes theorem)

$\Rightarrow \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$

$$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$$

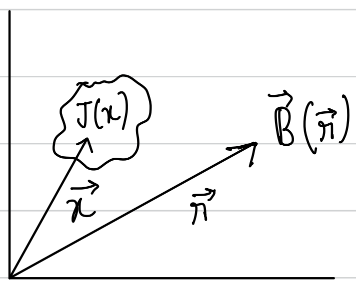
<u>Divergence of $\vec{B}$:</u>

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x})\times\hat{r}}{r^2}\, d\tau$$

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}(x)\times\frac{\hat{r}}{r^2}\right) d\tau$$

$$\vec{\nabla}\cdot\left(\vec{J}(x)\times\frac{\hat{r}}{r^2}\right) = \frac{\hat{r}}{r^2}\cdot\left(\vec{\nabla}\times\vec{J}(x)\right) - \vec{J}(x)\cdot\left(\vec{\nabla}\times\frac{\hat{r}}{r^2}\right)$$

$$= 0$$



$$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

Electrostatics:

1> $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ (Gauss' law)

2> $\vec{\nabla} \times \vec{E} = \vec{0}$

Magnetostatics

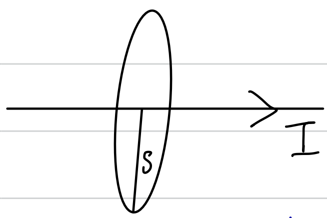
3> $\vec{\nabla} \cdot \vec{B} = 0$

4> $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's law)

Application of Ampere's law

i> line current:

$$\oint \vec{B} \cdot d\vec{l} = |\vec{B}| \int dl = B(2\pi s)$$

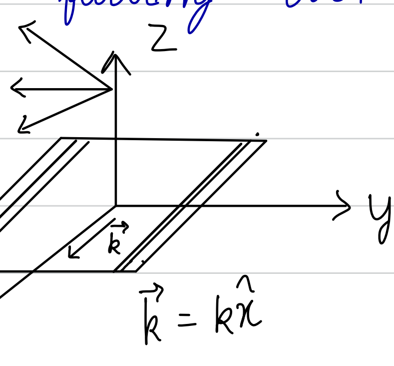


since $|\vec{B}|$ is constant on the amperean loop.

$$B \, 2\pi s = \mu_0 I$$

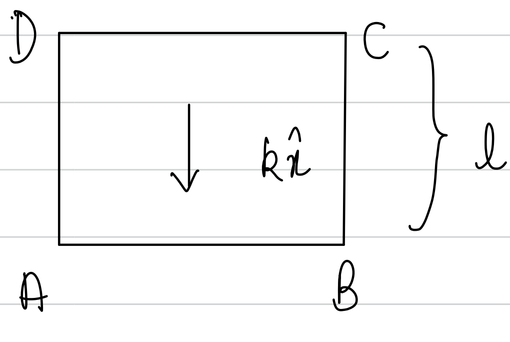
$$B = \frac{\mu_0 I}{2\pi s}$$

ii> Magnetic field of an infinite uniform surface current $\vec{k} = k\hat{x}$ flowing over xy plane.



Ans $\vec{B} = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{k}(\vec{x}) \times \hat{r}}{r^2}$

$$\oint_A^D \vec{B}\cdot\vec{dl} = \int_A^B \vec{B}\cdot\vec{dl} + \int_B^C \vec{B}\cdot\vec{dl} + \int_C^D \vec{B}\cdot\vec{dl} + \int_D^A \vec{B}\cdot\vec{dl}$$



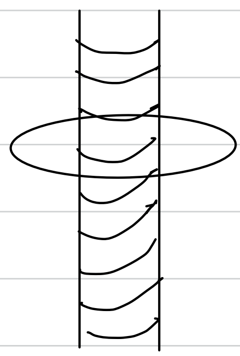
$$= 0 + Bl + 0 + Bl = 2Bl = \mu_0 I_{enc}$$

$$2Bl = \mu_0 k l$$

$$\therefore B = \frac{\mu_0 k}{2}$$

iii) Magnetic field due to a very long solenoid with density of turns n.

In plane of solenoid,



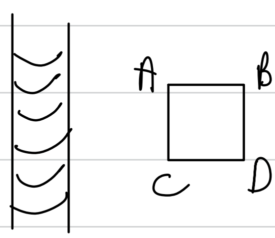
$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$$

$I_{enc} = 0 \quad \therefore B = 0$

$I_{enc} = 0$ because there is no current in plane of solenoid.

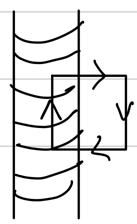
No current in a loop outside,

$\therefore B = 0$



Inside solenoid:

$$\oint \vec{B}\cdot\vec{dl} = B_{out}\, l + B_{in}\, l$$
$$= 0 + B_{in}\, l$$

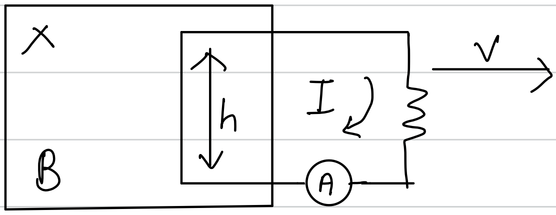


$$B_{in}\, l = \mu_0 I n l$$

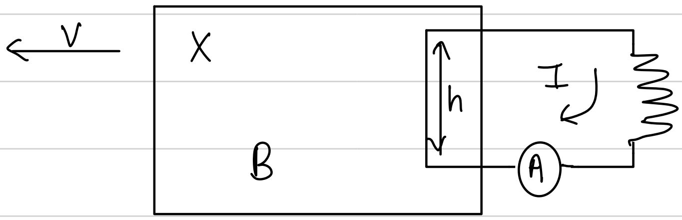
$$\therefore \vec{B} = \mu_0 n I \hat{z}$$

Faraday's laws

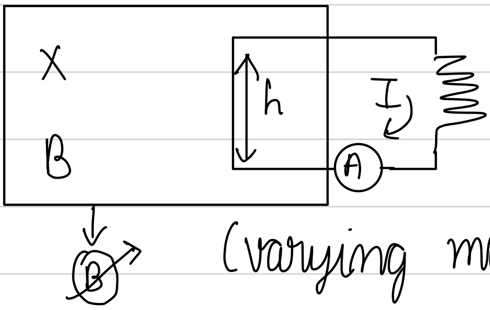
1) <u>Observation</u>: Current flows through wire



2) <u>Observation</u>: Current flows through the loop



3) <u>Observation</u>: Current flows through the loop.



(varying magnetic field)

Faraday arrived to the conclusion that EMF is induced due to change in magnetic flux.

$$\varepsilon = -\frac{d\phi_B}{dt}$$ where ε = EMF

This implies changing magnetic field induces electric field.

$$\text{EMF}, \ \varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt}$$

$$\boxed{\Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a}}$$ → Faraday's law

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \int -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Inconsistency with Ampere's law

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ But $\vec{\nabla}\cdot(\mu_0 J) \neq 0$.

From continuity equation, $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

But we have $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\rho = \vec{\nabla}\cdot(\varepsilon_0 \vec{E})$$

$$\therefore \quad \vec{\nabla}\cdot\left[\vec{J} + \frac{\partial(\varepsilon_0\vec{E})}{\partial t}\right] = 0$$

So Maxwell proposed that

$$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right]$$

So that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$

$$\Rightarrow \vec{\nabla}\cdot\mu_0\left[\vec{J} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right] = 0$$

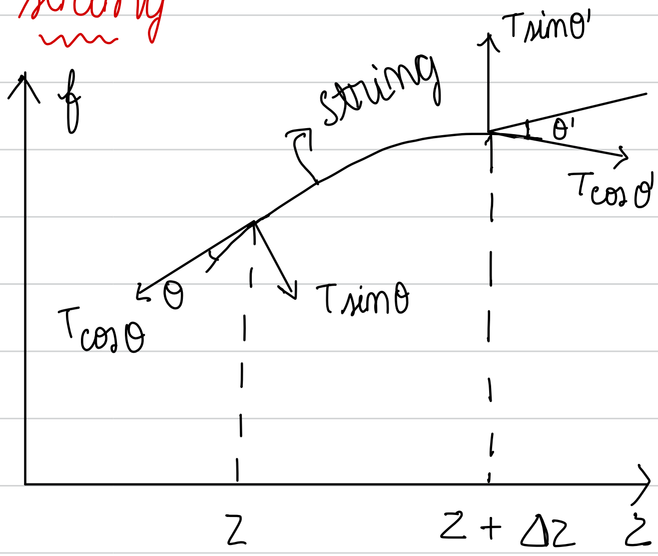
Displacement current

$$i_d = \varepsilon_0\frac{d\phi_e}{dt}$$

Maxwell's Equations

		(in vacuum) $\begin{pmatrix}\rho = 0 \\ J = 0\end{pmatrix}$
1)	$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ (Gauss' law)	$\vec{\nabla}\cdot\vec{E} = 0$
2)	$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
3)	$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)	$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
4)	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$	$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = 0$

Waves in a stretched string



$$F = T\sin\theta' - T\sin\theta$$

For small θ & θ'

$$F \approx T\tan\theta' - T\tan\theta$$

$$F = T\left[\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right]$$

$$F = T\Delta z\left[\frac{\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}}{\Delta z}\right]$$

$$= T\Delta z \frac{\partial^2 f}{\partial z^2}$$

Also $F = ma$ (μ = linear mass density of string)

$$= \mu\Delta z \frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow T\cancel{\Delta z}\frac{\partial^2 f}{\partial z^2} = \mu\cancel{\Delta z}\frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} - \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = 0}$$ → Equation of a wave.

Here $\sqrt{\frac{T}{\mu}}$ is the speed v of wave.

NOTE $f(z \pm vt)$ satisfies this equation.

- $g(z-vt)$ is a right moving wave
- $g(z+vt)$ is a left moving wave
- Wave equation satisfies superposition principle.
- Since wave equation is linear $f = af_1 + bf_2$ is also a solution. And this is what leads to interference.

Equation of a wave

- $f = A\cos(k(z-vt)+\delta)$

$A \to$ Amplitude $\qquad k \to$ wave number, $k = 2\pi/\lambda$

$k(z-vt)+\delta \to$ phase $\qquad \omega \to$ angular frequency, $\omega = 2\pi\nu$

$v \to$ velocity, $v = \dfrac{\lambda}{T} = \lambda\nu = \dfrac{\omega}{k}$

- Also $f = A\cos(kz - \omega t + \delta)$

$$f(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

Wave equation from Maxwell's equation

Consider $\vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0$

Now $\vec{\nabla}\times\left[\vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t}\right] = \vec{0}$

$$\Rightarrow \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E})}_{0} - (\vec{\nabla}\cdot\vec{\nabla})\vec{E} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = 0$$

$$\Rightarrow -\nabla^2\vec{E} + \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\Rightarrow -\frac{\partial^2\vec{E}}{\partial x^2} - \frac{\partial^2\vec{E}}{\partial y^2} - \frac{\partial^2\vec{E}}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad \text{where } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$\Rightarrow \vec{E}$ propogates like a wave with velocity, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$|||^{ly}$ $-\nabla^2 \vec{B} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$

Monochromatic plane waves

They have a single colour which means they have single wavelength.

$$\vec{E}(z,t) = \vec{\tilde{E}}_0 e^{i(kz - \omega t)}$$

In vacuum, $\vec{\nabla} \cdot \vec{E} = 0$,

$$\vec{\nabla} \cdot \left(\vec{\tilde{E}}_0 e^{i(kz - \omega t)} \right) = 0$$

$$\Rightarrow \underbrace{\partial_x \tilde{E}_{0x} e^{i(kz-\omega t)}}_{0} + \underbrace{\partial_y \tilde{E}_{0y} e^{i(kz-\omega t)}}_{0} + \partial_z \tilde{E}_{0z} e^{i(kz-\omega t)} = 0$$

$$\Rightarrow ik \tilde{E}_{0z} e^{i(kz-\omega t)} = 0 //$$

$\Rightarrow \tilde{E}_{0z} = 0 //$

$|||^{ly}$ $\vec{\nabla} \cdot \vec{B} = 0$, $\Rightarrow \tilde{B}_{0z} = 0$

$\Rightarrow \vec{E}$ & $\vec{B}$ are perpendicular to direction of propagation.

Also, $\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$

$$LHS = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \tilde{E}_{0x} e^{i(kz-\omega t)} & \tilde{E}_{0y} e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -ik\, \tilde{E}_{oy}\, e^{i(kz-\omega t)}\, \hat{x} + ik\, \tilde{E}_{ox}\, e^{i(kz-\omega t)}$$

$$RHS = -\frac{\partial \vec{B}}{\partial t} = i\omega\, \tilde{B}_{ox}\, e^{i(kz-\omega t)}\, \hat{x} + i\omega\, \tilde{B}_{oy}\, e^{i(kz-\omega t)}\, \hat{y}$$

$$\Rightarrow -ik\, \tilde{E}_{oy}\, e^{i(kz-\omega t)} = i\omega\, \tilde{B}_{ox}\, e^{i(kz-\omega t)}$$

$$\Rightarrow -k\tilde{E}_{oy} = \omega \tilde{B}_{ox}$$

Also $ik\tilde{E}_{ox}\, e^{i(kz-\omega t)} = i\omega\, \tilde{B}_{oy}\, e^{i(kz-\omega t)}$

$$\Rightarrow k\tilde{E}_{ox} = \omega \tilde{B}_{oy}$$

$$\therefore \tilde{B}_{ox} = \frac{-k}{\omega} \tilde{E}_{oy} \qquad \tilde{B}_{oy} = \frac{k}{\omega} \tilde{E}_{ox}$$

$$\Rightarrow \vec{\tilde{B}}_o = \frac{k}{\omega} (\hat{z} \times \vec{\tilde{E}}_o) \quad \Rightarrow \quad \vec{\tilde{B}}_o = \frac{1}{c} (\hat{z} \times \vec{\tilde{E}}_o)$$

$$\vec{E}(\vec{r}, t) = \vec{\tilde{E}}_o\, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

where $\vec{k} = k\hat{n} \rightarrow$ propagation vector

Light is an electromagnetic wave

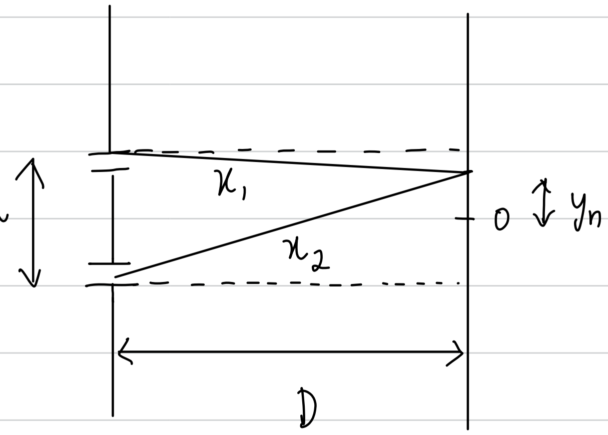
We have, $\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

$$\nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

- Since both of these are second order differential equations, superposition principle is applicable.
- This causes interference.

Young's double slit experiment

If y_1 and y_2 are solutions of an equation. Then $y = Ay_1 + By_2$ are also solutions by superposition principle



For constructive interference,

$|x_1 - x_2| = n\lambda$, $n \in \mathbb{Z}$

For destructive interference,

$|x_1 - x_2| = (2n+1)\frac{\lambda}{2}$, $n \in \mathbb{Z}$

Now, $x_1 = \sqrt{D^2 + \left(\frac{d}{2} - y_n\right)^2}$

$x_2 = \sqrt{D^2 + \left(\frac{d}{2} + y_n\right)^2}$

$x_1^2 - x_2^2 = 2dy_n$

$y_n = \frac{(x_1+x_2)(x_1-x_2)}{2d}$

$\therefore y_n = \frac{D(x_1 - x_2)}{d}$ $\quad (x_1 + x_2 \approx 2D)$

For constructive interference,

$y_n = \frac{n\lambda D}{d}$

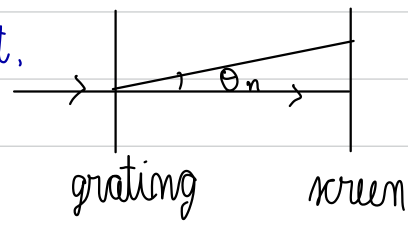
For destructive interference,

$y_n = \frac{(2n+1)\lambda D}{2d}$

Diffraction

Bending of a wave around an object.

$d\sin\theta_n = n\lambda$



For red light, $\lambda = 600$ nm, $\sin\theta_n \approx 1/2$ let $n=1$,

$$d = \frac{\lambda}{0.5} = 1200 \text{ nm} = 1.2\ \mu m$$

For X ray, $\lambda = 0.1$ nm, $\Rightarrow d = \frac{1}{0.5} = 2$ nm

Bragg's law of X-ray diffraction

For constructive interference, $2d\sin\theta = n\lambda$

Q1) A single crystal of NaCl is irradiated with a beam of X rays of wavelength $\lambda = 0.25$ nm and Bragg reflection is observed at 26.3°. What is atomic spacing of crystal.

Ans $\theta = 26.3°$ $\quad \lambda = 0.25$ nm $\quad n = 1$

$$2d\sin\theta = n\lambda$$

$$d = \frac{0.25 \times 10^{-9}}{2 \times \sin(26.3°)} = 0.28 \times 10^{-9} = 0.28 \text{ nm}$$

Energy carried by EM waves

Intensity : It is given by Poynting vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}), \text{ units } J/sm^2 \text{ or } W/m^2$$

For plane waves,

$$\vec{E} = E_0 \sin(kz - \omega t)\,\hat{x} \qquad \vec{B} = B_0 \sin(kz - \omega t)\,\hat{y}$$

$$|\vec{S}| = \frac{E_0 B_0}{\mu_0}\sin^2(kz - \omega t) = \frac{E_0^2}{\mu_0 c}\sin^2(kz - \omega t)$$

$$\Rightarrow |\vec{S}| \propto E_0^2$$

If an em wave falls on a surface of area A.

• Then, net power recieved by surface $= \vec{S}\cdot\vec{A} \propto E_0^2$

NOTE **Cyclotron**

→ In cyclotron centripetal force is given by Lorentz force.

i.e $\frac{mv^2}{R} = qvB$

$\Rightarrow \quad R = \frac{mv}{qB} = \frac{p}{qB}$

$T = \frac{2\pi R}{v} = \frac{2\pi mv}{qB\cdot v} = \frac{2\pi m}{qB}$

$f = \frac{1}{T} = \frac{qB}{2\pi m}$

→ Maxwell's laws in integral form:

1) $\oiint \vec{E}\cdot d\vec{a} = \frac{q_{in}}{\varepsilon_0}$

2) $\iint \vec{B}\cdot d\vec{a} = 0$

3) $\oint \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt}$

4) $\int \vec{B}\cdot d\vec{l} = \mu_0 i + \mu_0\varepsilon_0 \frac{\partial\phi_E}{\partial t}$

→ $\vec{B} = \vec{\nabla}\times\vec{A}$

• Where $\vec{A}$ is the vector potential of magnetic field.

DUAL NATURE OF LIGHT

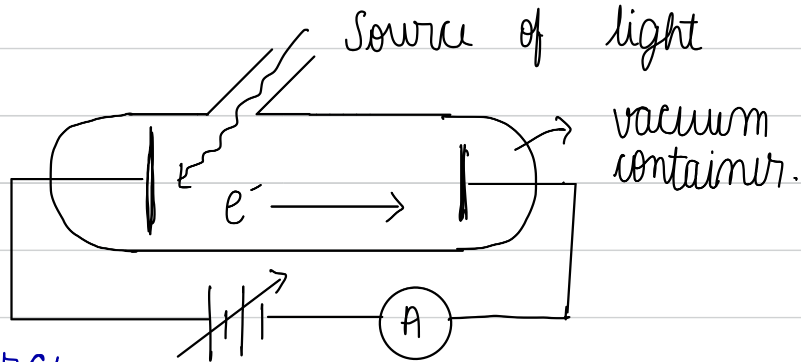
Light as a wave :

1> Interference
2> Diffraction
3> Intensity $\propto E^2$

Light as a particle

1> Photoelectric effect
2> Blackbody radiation
3> Compton effect

1> Photoelectric effect



Measurable quantities:
i> rate of e^- emmission by current
ii> Maximum Kinetic energy of electron by stopping potential i.e $K_{max} = -eV_s$

Prediction of wave theory:

1> Maximum Kinetic energy of electron, $K_{max} \propto E^2$
2> Photoelectric effect should be independent of frequency of incident light.
3> First electron should be emitted in a time of the order of seconds after radiation begins to fall on the metal.

Q1> To illustrate the 3rd point consider a laser beam which falls on sodium surface. Intensity of laser $I = 120\ W/m^2$. Energy required to remove an electron from Na surface is $\phi = 2.3\ eV$. Assume that electron is confined to an area of radius 0.1 nm. What is the time taken for the emission of first electron?

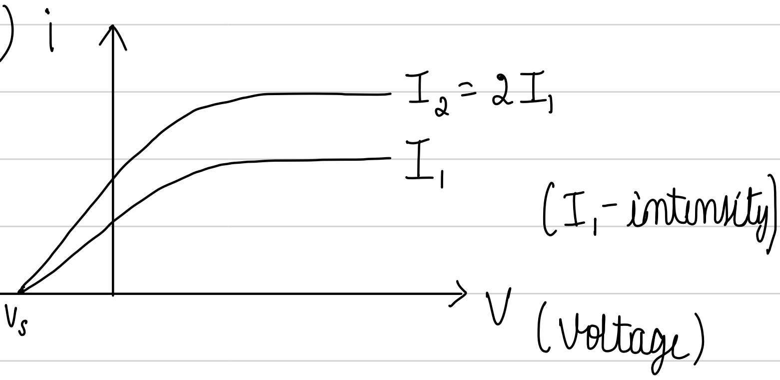
Ans
$$\frac{\phi}{\Delta t} = P_{av} = I \times A$$

$$\therefore \Delta t = \frac{\phi}{I \times A} = \frac{2.3 \times \cancel{10^{-19}}\ 10^{-1} \times 1.6}{120 \times \pi \times 0.01 \times \cancel{10^{-18}}}$$

$$= \frac{2.3 \times 10^{1}}{120 \times \pi} = 0.0976\ s \approx 0.1\ s$$

Actual observation of photoelectric experiment

1> For a light of fixed wavelength, K_{max} is independent of intensity.



2> Photoelectric effect does not take place for a light below a cutoff frequency.

3> First photoelectron is emitted almost instantaneously after turning on light source.

Quantum theory of photoelectric effect

1> Energy in an em wave is not distributed on the wavefront but it is concentrated on localized packets called quanta with energy $E = h\nu$ //

This implies that $K_{max} = h\nu - \phi$

2> There is a minimum frequency below which no photoelectric effect takes place and this frequency is called threshold frequency

Q2> What are the energy & momenta of red light of wavelength 650 nm. What wavelength of a photon of energy 2.4 eV?

Ans $E = \frac{hc}{\lambda} = \frac{1240}{\lambda}$ eV

$hc = 1240$ eV nm

$E = \frac{1240}{650} = 1.907$ eV //

$2.4 = \frac{1240}{\lambda}$

$\lambda = 516.66$ nm //

Q3> Work function of Tungsten is 4.52 eV. Find cutoff wavelength of Tungsten.

Ans $\lambda_c = \frac{hc}{\phi} = \frac{1240}{4.52} = 274.34$ nm //

ii> What is K_{max} if light of 198 nm is used?

Ans $K_{max} = \frac{hc}{\lambda} - \phi = \frac{1240}{198} - 4.52 = 1.74$ eV //

iii> What is stopping potential?

Ans $K_{max} = e V_{max} \Rightarrow V_{max} = \frac{1.74 \text{ eV}}{e} = 1.74$ V //

Thermal Radiation

It is the electromagnetic radiation emitted by all hot bodies.

Stefan's law: Total intensity radiated over all wavelengths, $I = \sigma e T^4$ where e is emissivity. and for blackbodies $e = 1$.

$\therefore I = \sigma T^4$

σ - Stefan - Boltzmann constant

$\sigma = 5.67 \times 10^{-8} \; W/m^2K^4$

Wein's displacement law: The wavelength λ_{max} at which the emitted intensity is maximum is proportional to inverse of temperature.

$T = \frac{b}{\lambda_{max}}$ where $b = 2.897 \times 10^{-3}$ mK

Q4) i) What is the wavelength at which an object at $20°C$ emit thermal radiation?

Ans $\lambda_{max} T = b$

$$\lambda_{max} = \frac{2.897 \times 10^{-3}}{293} = 9887 \text{ nm}$$

ii) To what temperature must we heat the object so that it's radiation peaks at $\lambda = 650$ nm

Ans $$T = \frac{2.897 \times 10^{-3}}{650 \times 10^{-9}} = 4457 \text{ K}$$

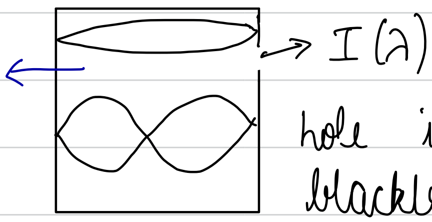
iii) How many times as much thermal radiation does it emit at higher temperature?

Ans $\frac{I_2}{I_1} = \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{4457}{293}\right)^4 = 53,542.673$

Blackbody radiation

- Cosmic microwave background radiation is very close to blackbody.

$E(\lambda)\, d\lambda$



hole is a blackbody.

$I(\lambda)$ coming out of black body $\propto$ energy density inside the box.

- We will consider a collection of N harmonic oscillators inside the box of volume V kept at temperature T.

$$E(\lambda)\, d\lambda = \frac{N(\lambda)\, d\lambda\, E_{av}}{V}$$

- A collection of harmonic oscillators kept at temperature T have energies distributed by Maxwell - Boltzmann distribution:

$$N(E) = \frac{N}{kT} e^{-E/kT}$$

$$\int_0^\infty N(E)\, dE = N, \quad \text{(N - Total no. of molecules)}$$

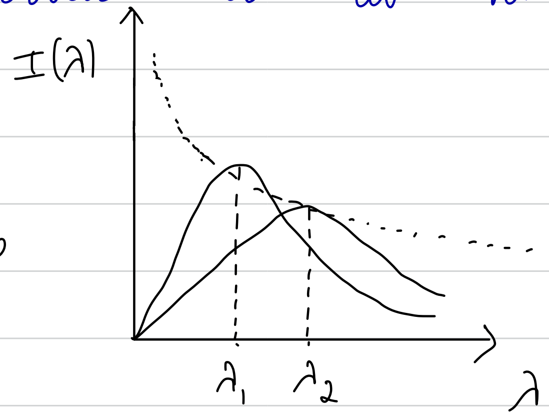
$$\int_0^\infty E\, N(E)\, dE \Big/ \int_0^\infty N(E)\, dE = E_{av} = kT$$

It can be shown that $N(\lambda)\, d\lambda = \frac{8\pi V}{\lambda^4} d\lambda$

$$\therefore U(\lambda)\,d\lambda = \frac{8\pi\nu}{\lambda^4}\,d\lambda \times \frac{kT}{\nu} = \frac{8\pi kT}{\lambda^4}\,d\lambda$$

where $E(\lambda)\,d\lambda$ is the energy density in the wavelength interval $d\lambda$ at some λ

$I(\lambda_1) > I(\lambda_2)$



As $\lambda \to \infty$, $I(\lambda) \to \infty$
Called UV catastrophe

- In quantum theory, $E_n = nh\nu$, $n = 1, 2, 3, \dots$

It can be shown that

$$N(n) = N_t\left(1 - e^{-h\nu/kT}\right) e^{-nh\nu/kT}$$

Also $\sum_{n=0}^{\infty} N(n) = N_t$

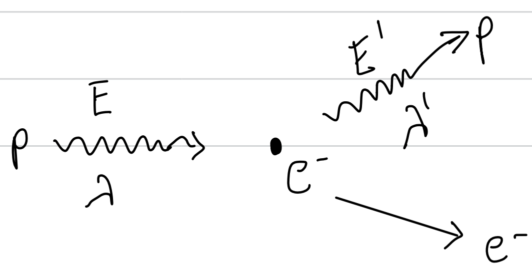
$$E_{av} = \sum_{n=0}^{\infty} N(n)\,E(n) \Big/ \sum_{n=0}^{\infty} N(n) = \frac{hc/\lambda}{e^{hc/kT\lambda} - 1}$$

$$\therefore U(\lambda)\,d\lambda = \frac{8\pi}{\lambda^4}\,\frac{hc/\lambda}{e^{hc/\lambda kT} - 1}$$

$$I(\lambda)\,d\lambda = \frac{c}{4}\,U(\lambda)\,d\lambda$$

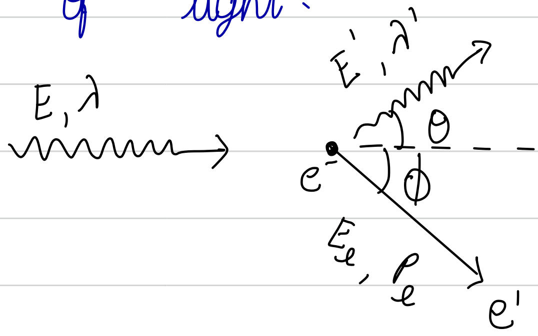
- This is called Planck's distribution.
- Using this distribution we can avoid UV catastrophe.

Compton effect



According to classic wave theory, $E' < E$,
But $\lambda = \lambda'$
But in reality $\lambda \neq \lambda'$ and it can be explained using quantum theory of light.

Now consider :



Conservation of energy gives : $E + m_e C^2 = E' + E_e \rightarrow (1)$
Conservation of momenta, P_x, initial $= P_x$, final
$P = P'\cos\theta + P_e \cos\phi \rightarrow (2)$
$P_{y,\,ini} = P_{y,\,final}$
$\Rightarrow 0 = P'\sin\theta - P_e \sin\phi$
$\Rightarrow P'\sin\theta = P_e \sin\phi \rightarrow (3)$
$(2) \Rightarrow P - P'\cos\theta = P_e \cos\phi \rightarrow (4)$
$(3)^2 + (4)^2$
$(P')^2 \sin^2\theta + P^2 + (P')^2\cos^2\theta - 2PP'\cos\theta = P_e^2$
$\Rightarrow (P')^2 + P^2 - 2PP'\cos\theta = P_e^2 \rightarrow (5)$

Also $E_e = E - E' + m_e C^2$

$\Rightarrow (E_e)^2 = (E - E' + m_e C^2)^2$ but $E_e^2 = m_e^2 C^4 + P_e^2 C^2$
From (5),
$P^2C^2 + P'C^2 - 2(PC)(P'C)\cos\theta + m_e^2 C^4 = LHS$
$RHS = E^2 + (E')^2 + m_e^2 C^4 - 2EE' + 2E m_e C^2 - 2E' m_e C^2$
Using $E = PC$ & $E' = P'C$
$LHS = E^2 + (E')^2 - 2EE'\cos\theta + m_e^2 C^4$
$\therefore E^2 + (E')^2 - 2EE'\cos\theta + m_e^2 C^4 = E^2 + (E')^2 + m_e^2 C^4 - 2EE'$
$+ 2E m_e C^2 - 2E' m_e C^2$

$$-2EE'\cos\theta = -2EE' + 2Em_eC^2 - 2E'm_eC^2$$

$$\Rightarrow EE'\cos\theta = EE' + m_eC^2(E'-E)$$

$$\Rightarrow EE'(1-\cos\theta) = m_eC^2(E-E')$$

Dividing by $EE'm_eC^2$

$$\Rightarrow \frac{1}{m_eC^2}(1-\cos\theta) = \frac{1}{E'} - \frac{1}{E}$$

we have $E = \frac{hc}{\lambda}$ & $E' = \frac{hc}{\lambda'}$

$$\Rightarrow \frac{(1-\cos\theta)}{m_eC^2} = \frac{\lambda'-\lambda}{hc}$$

$$\Rightarrow \lambda'-\lambda = \frac{h(1-\cos\theta)}{m_eC}$$

But $-1 \leq \cos\theta \leq 1$

$\therefore \lambda'-\lambda > 0 \qquad \therefore \lambda' \geq \lambda$

$$\frac{hc}{m_eC^2} = \frac{1240 \text{ eV}\cdot\text{nm}}{0.511 \text{ MeV}} = 2.426 \text{ pm (pico metres)}$$

- 2.426 pm is called compton wavelength.

Formula in compton effect

$$\lambda'-\lambda = \frac{h}{m_eC}(1-\cos\theta) = 2.426(1-\cos\theta) \text{ pm}$$

$$P_e\sin\phi = P'\sin\theta$$

$$P_e\cos\phi = P - P'\cos\theta$$

$$\tan\phi = \frac{p'\sin\theta}{p-p'\cos\theta}$$

$$K_e = E - E'$$

Eg1 X rays of wavelength 0.24 nm are Compton scattered. Scattered beam is observed at an angle of 60° relative to the incident beam.

Find :

a) Wavelength of scattered X ray

b) Energy of scattered X rays.

c) Kinetic energy of scattered e^-

d) Direction of scattered e^-

Ans a) $\lambda' - \lambda = 2.426 (1 - \cos\theta)$ $\quad \theta = 60°$

$= 2.426 (1 - \cos 60°)$

$= 1.213 \text{ pm}$

$\lambda' = 240 + 1.213 = 241.213 \text{ pm} = 0.241213 \text{ nm}$

b) $E' = \frac{hc}{\lambda'} = \frac{1240}{0.241213} = 5140.684 \text{ eV}$

c) $E_e = E - E' + m_e c^2$

$= \frac{1240}{0.24} - 5140.684 + m_e c^2$

$E_e - m_e c^2 = K_e = 5,166.66 - 5140.684$

$= 25.982 \text{ eV}$

d) $\tan\phi = \frac{p' \sin\theta}{p - p'\cos\theta} = \frac{E' \sin\theta}{E - E'\cos\theta}$

$= \frac{5140.684 \times 0.8660}{5166.66 - 5140.684 \times 0.5}$

$= \frac{4,451.8323}{2596.318} = 1.7146$

$\therefore \phi = \tan^{-1}(1.7146) = 59.75°$

Ex2 UV light of wavelength 350 nm & intensity 1 W/m^2 is directed at potassium surface ($\phi = 2.2$ eV)

a) Find K_{max} of photoelectron

b) If 0.5% of incident photons produce photo electrons, how many are emitted per second if potassium surface has an area of $1 cm^2$.

Ans a) $K_{max} = \frac{hc}{\lambda} - \phi_0 = \frac{1240}{350} - 2.2 = 1.34 \text{ eV}$

b) $N = \frac{P}{E}\eta = \frac{1 \times 1 \times 10^{-4} \times 0.5 \times 10^{-2}}{3.54 \times 1.6 \times 10^{-19}}$

$= 8.827 \times 10^{11} \, s^{-1}$

Eg 3 Find the shortest wavelength of X-ray coming from an X-ray machine whose accelerating potential is 50 kV

Ans $\frac{hc}{\lambda_{min}} = eV$

$\Rightarrow \lambda_{min} = \frac{1240}{50,000} = 0.0248 \text{ nm}$

Eg 4: X rays of wavelength 10 pm are scattered from a target. a) Find the wavelength of X rays are scattered at an angle of 45°

b) What is the maximum wavelength possible of scattered X rays?

c) Find max KE of recoil electrons.

Ans a) $\lambda' - \lambda = 2.426(1 - \cos\theta)$

$= 2.426\left(1 - 1/\sqrt{2}\right)$

$= 0.710 \text{ pm}$

$\therefore \lambda' = 10.710 \text{ pm}$

b) For max λ', $\cos\theta = -1$

$\Rightarrow \lambda' = 10 + 2.426 \times 2 = 14.852 \text{ pm}$

$K_x = E - E'$

$K_x = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{1240}{10 \times 10^{-3}} - \frac{1240}{14.852} \times 10^3$

$= (124 - 83.49) \times 10^3$

$= 40.51 \text{ keV}$

Eg5 A 1kW radio transmitter transmits 880 kHz. How many photons does it radiate in one second?

Ans $N = \frac{P}{E} = \frac{P}{h\nu} = \frac{1 \times 10^3}{6.626 \times 10^{-34} \times 880 \times 10^3}$

$= 1.715 \times 10^{-4} \times 10^{34}$

$= 1.715 \times 10^{30} \text{ s}^{-1}$

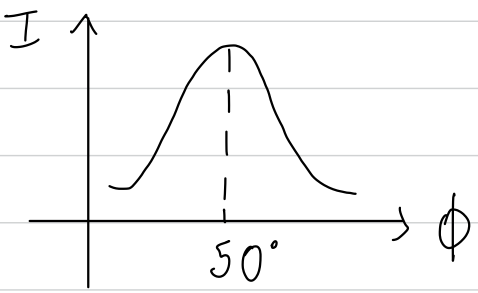
Wave nature of matter

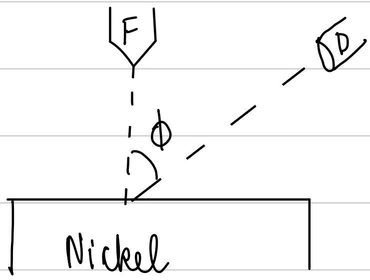
De Broglie proposed that particles also have a wave wave nature, with wavelength $\lambda = \frac{h}{p}$

This λ is known as De broglie wavelength

Davisson- Germer experiment:

F - Filament emitting electrons





If e^- had wavelength then they will cause diffraction

$\Rightarrow d \sin\phi = n\lambda$

For Ni, $d = 0.215$ nm $\quad \phi = 50^\circ \quad n = 1$

$$\lambda = \frac{d \sin\phi}{n} = 0.215 \sin 50^\circ = 0.165 \text{ nm}$$

From de-Broglie hypothesis,

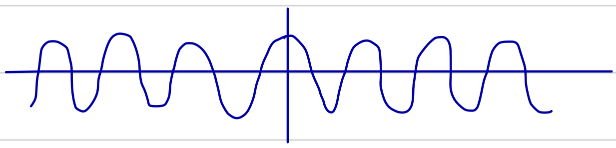
$$\lambda = \frac{h}{p} = \frac{hc}{pc}$$

$$p = \sqrt{2mK} \qquad pc = \sqrt{2mc^2K}$$

For e^- $\quad m_ec^2 = 0.511$ MeV $\qquad K = 54$ eV

$$\lambda = \frac{1240}{\sqrt{2 \times 0.511 \times 10^6 \times 54}} = \frac{1240}{7.42 \times 10^3} = 0.167 \text{ nm}$$

How can we represent particles by waves?

wave: 

Particle: 

⇒ In order to represent localised objects, we use wave packets.

Consider $\quad y_1 = A \cos(k_1 z - \omega_1 t)$

$\qquad\qquad y_2 = A \cos(k_2 z - \omega_2 t)$

$|\omega_1 - \omega_2|$ is a small number.

Now let $y = y_1 + y_2$

$$y = 2A \cos\left(\frac{\Delta k x}{2} - \frac{\Delta \omega t}{2}\right) \times \cos\left(\frac{(k_1+k_2)x}{2} - \frac{(\omega_1+\omega_2)t}{2}\right)$$

If $k_1 \cong k_2$

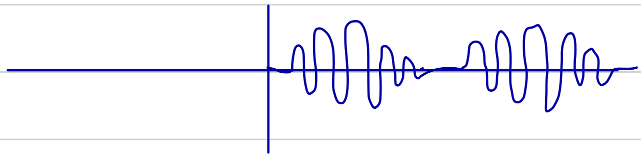
$$y = 2A \cos\left(\frac{\Delta k x}{2} - \frac{\Delta \omega t}{2}\right) \cos(kx - \omega t)$$

$$y = A' \cos(kx - \omega t) \quad \text{where } A' = 2A \cos\left(\frac{\Delta k x}{2} - \frac{\Delta \omega t}{2}\right)$$

Velocity of group wave is $v_g = \frac{d\omega}{dk}$

Phase velocity, $v_p = \frac{\omega}{k}$

$y \rightarrow$



Eg1 Ocean waves travel with a phase velocity $v_p = \sqrt{\frac{g\lambda}{2\pi}}$
What is group velocity?

Ans $v_p = \frac{\omega}{k} = \sqrt{\frac{g\lambda}{2\pi}} \Rightarrow \frac{\omega}{k} = \sqrt{\frac{g}{k}}$

$\Rightarrow \omega = \sqrt{gk}$

$v_g = \frac{d\omega}{dk} = \frac{d(\sqrt{gk})}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_p}{2}$ //

Eg 2 What is the phase velocity and group velocity of matter waves

Ans $v_p = \nu\lambda$

But $h\nu = \gamma mc^2$ where, $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\nu = \frac{\gamma mc^2}{h}$

$v_p = \frac{\gamma mc^2}{h} \frac{h}{p} = \frac{\gamma mc^2}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v}$ //

where v is the velocity of particle

$\Rightarrow v_p > c$

But particle moves at group velocity.

$\omega = 2\pi\nu = \frac{2\pi\gamma mc^2}{h}$

$k = \frac{2\pi}{\lambda} = \frac{2\pi\gamma mv}{h}$

$$V_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/v}$$

$$\frac{d\omega}{dv} = \frac{1}{h}\frac{d}{dv}\left(\frac{2\pi mc^2}{\sqrt{1-v^2/c^2}}\right) = \frac{2\pi mc^2}{h}\,\frac{1}{(1-v^2/c^2)^{3/2}} \times \left(\frac{-2v}{c}\right)\left(\frac{-1}{2}\right)$$

$$\frac{dk}{d\omega} = \frac{2\pi m}{h}\,\frac{d}{dv}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right) = \frac{2\pi m}{h}\left[\frac{1}{\sqrt{1-v^2/c^2}} - \frac{v(-2v)}{2c\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right]$$

$$\frac{dk}{d\omega} = \frac{2\pi m}{h}\,\frac{1}{(1-v^2/c^2)^{3/2}} = \frac{2\pi m\gamma^3}{h}$$

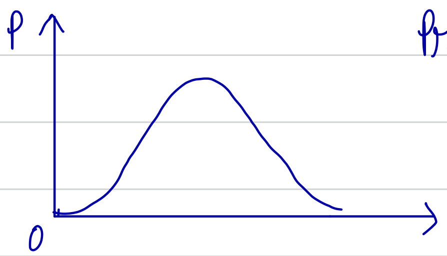
$$\frac{d\omega}{dv} = \frac{2\pi\gamma^3 mv}{h}$$

$$\therefore \frac{d\omega}{dk} = \frac{2\pi\gamma^3 mv/h}{2\pi\gamma^3 m/h} = v \quad\Rightarrow\quad V_g = v$$

∴ Matter wave moves with the speed of particle.

Max Born interpretation

He told that matter waves are probability waves. i.e



Probability of finding a particle $\propto |\psi|^2$ (i.e Probability density)

Heisenbergh's uncertainty principle

1) For a wave k & ω are exactly known.

But for a wave packet (matter wave) k and ω are not known perfectly.

2> For a wave position is not clearly known.
But for a particle position is known exactly

So $\Delta x \, \Delta p \geq \frac{\hbar}{2}$ where $\hbar = \frac{h}{2\pi}$

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For a single wave $y = A \sin kx$
⇒ Momentum or wave number is well known.
But $y(x)$ extends from $-\infty$ to ∞, so position is unknown.

For a wave packet $\Psi(x) = \sum_i A \sin k_i x$
⇒ Position is more clearly known.
But momentum is more uncertain.

So $\Delta p \, \Delta x \geq \frac{h}{4\pi}$

Therefore there are no experiments that measure p and x simultaneously.

NOTE Standard deviation of a quantity M denoted by ΔM or $\sigma_M = \sqrt{\langle M^2 \rangle - \langle M \rangle^2}$

- In Heisenberg's uncertainty principle, Δx & Δp mean σ_x & σ_p respectively.
- σ_M can be thought as most probable value taken by M.
- Heisenberg's uncertainty principle can also be written as $\Delta E \, \Delta t \geq \frac{\hbar}{2}$

Ex 1 An electron moves along x-axis with a speed of 3×10^6 m/s. We can measure its speed with an uncertainty of 1%. With what precision can we simultaneously measure it's x co-ordinate?

Ans $\Delta p_x = m \Delta v_x$

$= 9.1 \times 10^{-31} \times 3 \times 10^6 \times \frac{1}{100}$

$\Delta p_x = 27.3 \times 10^{-27}$

$\Delta x = \frac{h}{4\pi \Delta p_x} = \frac{6.626 \times 10^{-34}}{4 \times 3.14 \times 27.3 \times 10^{-27}}$

$= 1.93 \times 10^{-9}$ m $= 1.93$ nm

Ex 2 In nuclear beta decay electrons are observed to be emitted from the atomic nucleus. Assume that the electrons are trapped in the nucleus and occasionally they escape. Take diameter of nucleus to be 10^{-14} m and estimate the range of kinetic energy that such an electron must have.

Ans $\Delta x = 10^{-14}$ m $= 10$ fm Also $\hbar c = 197$ MeV·fm

$\Delta p = \frac{\hbar}{2\Delta x} = \frac{\hbar c}{2 \Delta x c}$

$\Delta p = \frac{197 \text{ MeV}\cdot\text{fm}}{2 \times 10 \text{ fm} \times c} = 9.85$ MeV/c

$E = \sqrt{\Delta p^2 c^2 + (m_e c^2)^2}$

$= \sqrt{(9.85)^2 + (0.511)^2}$

$= \sqrt{97.2836} = 9.8632$ MeV

$K = E - m_e c^2$

$= 9.8632 - 0.511$

$= 9.35$ MeV

Eg3 A charged π meson has a rest mass of 140 MeV and a lifetime of 26ns. Find the energy uncertainty of π meson and also the uncertainty as a fraction of energy

Ans $E_\pi = 140$ MeV

lifetime, $\Delta t = 26$ ns

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{6.626 \times 10^{-34}}{4 \times \pi \times 26 \times 10^{-9}}$$

$$= 2.028 \times 10^{-25} \text{ J}$$

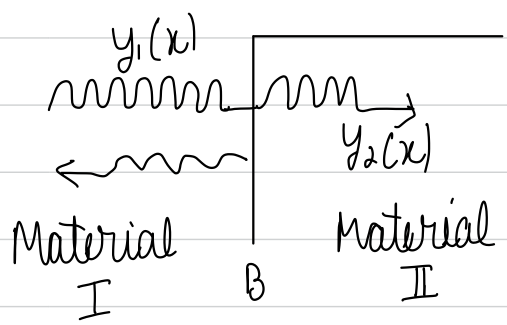
$$= \frac{2.028 \times 10^{-25}}{1.6 \times 10^{-19}} = 1.2675 \times 10^{-6} \text{ eV}$$

$$\therefore \frac{\Delta E}{E} = \frac{1.2675 \times 10^{-6}}{140 \times 10^{6}} = 9.05 \times 10^{-15}$$

Schrodinger's equation

Equation of motion governing quantum mechanics is called Schrodinger equation.

Waves at boundaries:



When a wave is incident at a boundary:

1> Part of the wave will be transmitted and another part will be reflected.

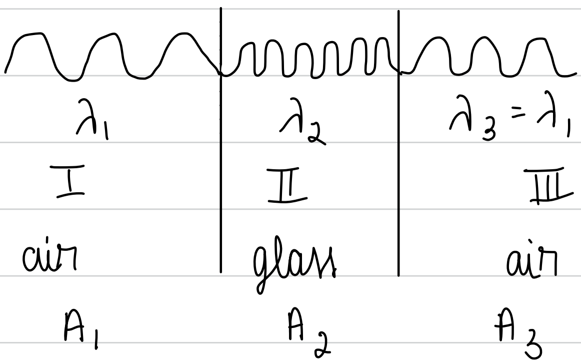
2> Wave and it's first derivative will be continuous across boundaries.

3> When a wave encounters boundary to a

forbidden region, the wave will penetrate by a few wavelength before it's reflected.

At the boundary:

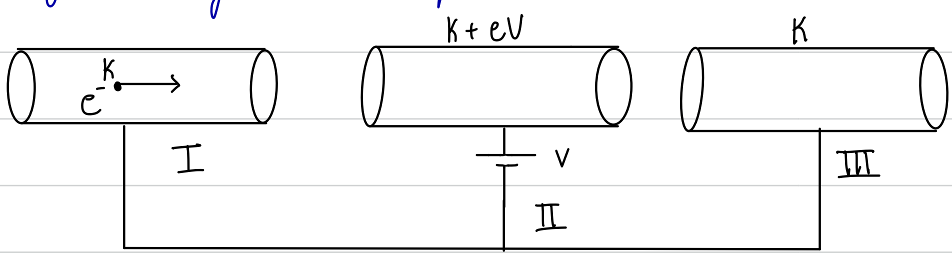
$$y_1(B) = y_2(B) \quad \& \quad \left.\frac{\partial y_1}{\partial x}\right|_B = \left.\frac{\partial y_2}{\partial x}\right|_B$$



$A_1 \neq A_2 \neq A_3$

$A \rightarrow$ amplitude

Consider following example:



$p = \frac{h}{\lambda} \Rightarrow \lambda_I = \frac{h}{p} = \frac{h}{\sqrt{2mK}}$ (not relativistic)

$\lambda_{II} = \frac{h}{\sqrt{2m(K+eV)}}$ $\qquad$ $\lambda_{III} = \frac{h}{\sqrt{2mK}}$

$y_1 = C_1 \sin\left(\frac{2\pi x}{\lambda_1} - \phi_1\right)$ | $y_2 = C_2 \sin\left(\frac{2\pi x}{\lambda_2} - \phi_2\right)$ | $y_3 = C_3 \sin\left(\frac{2\pi x}{\lambda_3} - \phi_3\right)$

Eg1 Boundary of first 2 mediums is at $x = 0$. Boundary between II and III is at $x = 20$ cm.

$C_1 = 11.5$ cm $\qquad \lambda_1 = 4.97$ cm $\qquad \phi_1 = -65.3°$ $\qquad \lambda_2 = 10.5$ cm

Find $y_2(x)$ & $y_3(x)$

Ans i> $y_1(x=0) = y_2(x=0)$

$$y_1(x) = 11.5 \sin\left(\frac{2\pi x}{4.97} + 65.3°\right)$$

$$y_1(0) = 11.5 \sin(65.3°) = C_2 \sin(-\phi_2)$$

$$10.447 = -C_2 \sin\phi_2 \rightarrow ①$$

ii> $y_1'(x=0) = y_2'(x=0)$

$$11.5 \times \frac{2\pi}{4.97} \cos\left(\frac{2\pi x}{4.97} + 65.3°\right)\Bigg|_0 = C_2 \frac{2\pi}{\lambda_2} \cos\left(\frac{2\pi x}{\lambda_2} - \phi_2\right)\Bigg|_0$$

$$\frac{11.5 \cos(65.3°)}{4.97} = C_2 \cos(-\phi_2) \times \frac{1}{10.5}$$

$$10.15 = C_2 \cos\phi_2 \rightarrow ②$$

① ÷ ②

$$\Rightarrow 1.029 = -\tan\phi_2$$

$$\Rightarrow \phi_2 = \tan^{-1}(-1.029) = -45.82°$$

$$\therefore C_2 = \frac{10.447}{-\sin(-45.82°)} = 14.57 \text{ cm}$$

iii> $y_2(20) = y_3(20) \qquad (\lambda_3 = \lambda_1)$

$$14.57 \sin\left(\frac{2\pi \times 20}{10.5} + 45.82°\right) = C_3 \sin\left(\frac{2\pi \times 20}{4.97} - \phi_3\right)$$

$$14.57 \sin(57.78) = C_3 \sin(25.28 - \phi_3)$$

$$12.32 = C_3 \sin(25.28 - \phi_3) \rightarrow ③$$

iv> $y_2'(20) = y_3'(20)$

$$14.57 \cos\left(\frac{2\pi x}{10.5} + 45.82°\right) \times \frac{2\pi}{10.5}\Bigg|_{20} = \frac{2\pi}{4.97} \times C_3 \cos\left(\frac{2\pi x}{4.97} - \phi_3\right)\Bigg|_{20}$$

$$\frac{14.57\cos(57.78^\circ)}{10.5} = \frac{C_3}{4.97}\cos(25.28-\phi_3)$$

$$3.677 = C_3 \cos(25.28-\phi_3) \rightarrow (4)$$

(3) ÷ (4)

$$\Rightarrow \tan(25.28-\phi_3) = 3.35$$

$$\phi_3 = -48.1^\circ$$

$$C_3 = 12.85 \text{ cm}$$

Confined particle :

If a particle is represented waves it can only have wavelength such that $\frac{n\lambda}{2} = L$

i.e $\lambda_n = \frac{2L}{n}$

$$P_n = h/\lambda_n = nh/2L$$

$$E_n = \frac{P_n^2}{2m} = \frac{n^2h^2}{8mL^2}$$

Does a confined particle violate uncertainty principle?

Here Δx or $\langle x \rangle = L$ & $\Delta p = \langle p \rangle = \frac{nh}{2L}$

$\therefore \Delta x \, \Delta p = \frac{nh}{2} \gg \frac{h}{4\pi}$

$\Rightarrow$ uncertainty principle is not violated

Schrodinger's equation

Time dependent equation: $-i\hbar \dfrac{\partial \Psi(x,t)}{\partial t} = \dfrac{-\hbar^2}{2m} \dfrac{\partial^2 \Psi(x,t)}{\partial x^2} + U\Psi(x,t)$

Motivation:

with constant energy: $\Psi(x,t) = A \sin(kx)\, e^{\frac{iEt}{\hbar}}$

$LHS = -i\hbar \dfrac{iE}{\hbar} \Psi(x,t) = E\Psi(x,t)$

$RHS = \left(\dfrac{\hbar^2 k^2}{2m} + U\right) \Psi(x,t)$

$\hbar k = \dfrac{h}{2\pi} \times \dfrac{2\pi}{\lambda} = \dfrac{h}{\lambda} = p \Rightarrow RHS = \left(\dfrac{p^2}{2m} + U\right) \Psi(x,t)$

$RHS = (K + U)\Psi(x,t)$

Time independent equation: $-\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \Psi(x)}{\partial x^2} + U\Psi(x) = E\Psi(x)$

Solutions to Schrodinger Equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U\Psi(x) = E\Psi(x)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{-2m}{\hbar^2} (E-U)\Psi(x)$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-U)\Psi(x) = 0$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial x^2} + k^2 \Psi(x) = 0$$

Case 1 : $U = 0$ $\Rightarrow$ $k^2 = \dfrac{2mE}{\hbar^2}$

General solution: $\Psi(x) = A \sin\left(\sqrt{\dfrac{2mE}{\hbar^2}}\, x\right) + B \cos\left(\sqrt{\dfrac{2mE}{\hbar^2}}\, x\right)$

How do we get A & B ?

1> Boundary conditions 2> Normalisation

1> Using boundary condition:

Suppose at $x = 0$, $\Psi(0) = 0$

$\Psi(0) = A \sin(0) + B \cos(0) \Rightarrow B = 0$

$\Psi(x) = A \sin kx$

2> Using normalisation condition

Probability of finding a particle between x & $x + dx$ is

$\Rightarrow P(x)\, dx = |\Psi(x)|^2\, dx$

$$\Rightarrow \int_{x_1}^{x_2} P(x)\, dx = \int_{x_1}^{x_2} |\Psi(x)|^2\, dx$$

But $\displaystyle\int_{-\infty}^{\infty} P(x)\, dx = 1$ $\Rightarrow$ $\displaystyle\int_{-\infty}^{\infty} |\Psi(x)|^2\, dx = 1$

Using this we can find A.

Infinite Potential Wall

We have,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x)}{\partial x^2} + U\Psi(x) = E\Psi(x)$$

I	II	III
$U = \infty$	$U = 0$	$U = \infty$
$\Psi_1(x)$	$\Psi_2(x)$	$\Psi_3(x)$

$x = 0$ $x = L$

$\Psi_1(x) = \Psi_3(x) = 0$ } In forbidden region

In region II $U=0$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$$

$$\Rightarrow \frac{\partial^2\psi(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

If we define $k^2 = \frac{2mE}{\hbar^2}$, $k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\Rightarrow \frac{\partial^2\psi(x)}{\partial x^2} + k^2\psi(x) = 0$$

This has solution $\sin(kx)$, $\cos(kx)$

i.e $\psi_{II}(x) = A\sin kx + B\cos kx$

Now we use boundary conditions

At $x=0$

$$\psi_I(x) = \psi_{II}(x) \quad \text{i.e} \quad \psi_I(0) = \psi_{II}(0)$$

$\Rightarrow B = 0$

At $x = L$, $\psi_{II}(L) = \psi_{III}(L)$

$\Rightarrow A\sin kL = 0$

$\Rightarrow kL = n\pi \quad \Rightarrow k = n\pi/L$

Thus from 2 boundary conditions

$$\psi_{II}(x) = A\sin\left(\frac{n\pi x}{L}\right), \quad \psi_1(x) = \psi_3(x) = 0$$

In order to find A we use normalisation

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$\Rightarrow \int_0^L |\psi_2(x)|^2 dx = 1$$

$$A^2 \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = 1$$

$$\frac{A^2}{2} \int_0^L \left(1 - \cos \frac{2n\pi x}{L}\right) dx = 1$$

$$\frac{A^2}{2} L = 1 \quad \Rightarrow \quad A = \pm\sqrt{\frac{2}{L}}$$

Thus, $\Psi(x)$ is

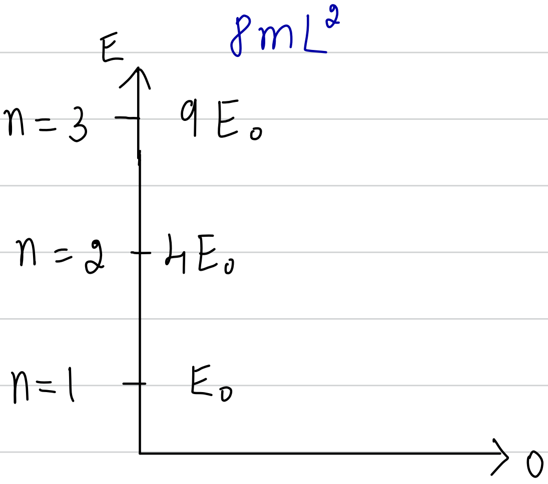
$$\Psi(x) = \begin{cases} 0 & \text{in I} \\ \pm\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & \text{in II} \\ 0 & \text{in III} \end{cases}$$

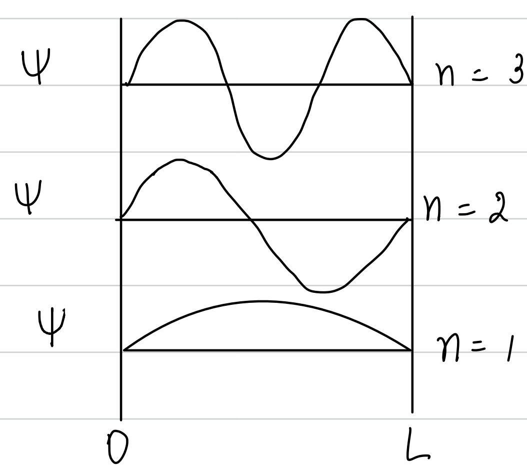
To find energy,

$$k = \frac{n\pi}{L} = \sqrt{\frac{2mE}{\hbar^2}}$$

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2\pi^2 h^2}{(4\pi^2)(2mL^2)}$$

$$= \frac{n^2 h^2}{8mL^2} = n^2 E_0$$





Average position of the particle is

$$\langle x \rangle = \int_{-\infty}^{\infty} P(x)\, x\, dx \quad \text{where} \quad P(x) = |\Psi(x)|^2$$

Average value of any quantity O is given by:

$$\langle O \rangle = \int_{-\infty}^{\infty} P(x) \cdot O(x)\, dx = \int_{-\infty}^{\infty} |\Psi(x)|^2 \, O(x)\, dx$$

$\therefore$ Average position in infinite potential well:

$$\langle x \rangle = \int_{-\infty}^{0} |\Psi_1(x)|^2 x\, dx + \int_{0}^{L} |\Psi_2(x)|^2 dx + \int_{L}^{\infty} |\Psi_3(x)|^2 x\, dx$$

$$= \int_0^L \left| \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right|^2 x\, dx$$

$$= \frac{2}{L} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) x\, dx$$

$$= \frac{1}{L} \int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) x\, dx$$

$$= \frac{1}{L} \int_0^L x\, dx - \frac{1}{L} \int_0^L x \cos \frac{2n\pi x}{L}\, dx$$

$$= \frac{1}{L} \left. \frac{x^2}{2} \right]_0^L - 0$$

$$= L/2$$

Case 2: When $U \neq 0$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U\Psi(x) = E\Psi(x)$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial x^2} + (E-U) \frac{2m}{\hbar^2} \Psi(x) = 0$$

subcase a: $E > U$. Let, $(E-U)\frac{2m}{\hbar^2} = k^2$, $k^2 > 0$

$$\frac{\partial^2 \Psi}{\partial x^2} + k^2 \Psi(x) = 0$$

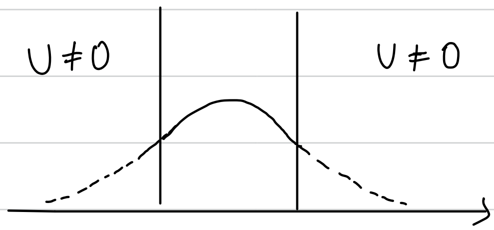
$\Psi(x) = A\sin kx + B\cos kx$

subcase b: $U > E$ then let $k^2 = \frac{2m}{\hbar^2}(U-E)$

$\Rightarrow \frac{\partial^2 \Psi(x)}{\partial x^2} - k^2 \Psi(x) = 0$

$\Psi(x) = Ae^{kx} + Be^{-kx}$

For this subcase:



NOTE: De Broglie wavelength of a particle excited through a potential of V volts:

$$\lambda = \frac{hc}{\sqrt{qV(qV + 2mc^2)}}$$