LINEAR ALGEBRA AND PROBABILITY THEORY

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LINEAR ALGEBRA

Equation space: A system of equations represented as a form of $Ax + By + Cz + \dots = K$

Matrix - vector space: A system of equations represented in the form of multiplication of matrices.

3 kinds of system:

1> System with unique solution: (for 2 variable equation, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$)

2> System with no solution: (for 2 variable equation, $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$)

3> System with infinite solutions: (for 2 variable equation, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$)

Consistent and inconsistent systems

Consistent: Has atleast one solution (unique / infinite)

Inconsistent: Has no solution.

→ In Matrix - vector space for $ax + by = c$ & $px + qy = r$, i.e

$$\underbrace{\begin{bmatrix} a & b \\ p & q \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ r \end{bmatrix}$$

Augumented matrix: It is of the form:

$$\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$$

A system having m equations and n variables, then
i> $m=n$ ii> $m<n$ iii> $m>n$

- Now if $m \geq n$, then system can have no, unique or infinite solutions.
- If $m<n$ then system can have no solution or infinite solutions

Equation space $\rightarrow$ Matrix - vector space

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad\qquad & \;\;\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n &= b_n
\end{aligned}$$

$A_{m \times n} \cdot x_{n \times 1} = b_{m \times 1} \rightarrow$ Matrix - vector space.

Upper triangular matrix: Matrix having all entries below main diagonal are zeroes.

Lower triangular matrix: Matrix having all entries above main diagonal are zeroes.

Row transformation

1) Rows can be swapped
2) Rows can be multiplied by non zero number.
3) Rows can be added by multiples of other row.

Ex:
$$\begin{bmatrix} 3 & 1 & 0 \\ 5 & 3 & 6 \\ 7 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 5 & 3 & 6 \\ 14 & 4 & 8 \end{bmatrix}$$

Ex 1: Solve

$$x_1 - 2x_2 + x_3 = 0 \quad \to (1)$$
$$2x_2 - 8x_3 = 8 \quad \to (2)$$
$$-4x_1 + 5x_2 + 9x_3 = -9 \quad \to (3)$$

Ans Equation space: $(1) + (2) \Rightarrow x_1 - 7x_3 = 8 \Rightarrow x_1 = 8 + 7x_3 \to (4)$

Substitute (4) in (3)

$$-32 - 28x_3 + 5x_2 + 9x_3 = -9$$
$$5x_2 - 19x_3 = 23 \to (5)$$

$(2) = x_2 - 4x_3 = 4$

$\therefore \; 20 + 20x_3 - 19x_3 = 23$

$$x_3 = 3$$

$\therefore \; x_1 = 29 \qquad x_2 = 16 \qquad x_3 = 3$

Matrix Space: Augumented matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Now $R_3 \to 4R_1 + R_3$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$R_3 \to \frac{3}{2}R_2 + R_3$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2/2 \Rightarrow$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to 2R_2 + R_1 \to$

$$\sim \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 + 7R_3$ & $R_2 \to R_2 + 4R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore x_1 = 29$ // $\quad x_2 = 16$ // $\quad x_3 = 3$ //

Consistency validation

Q2) $x_2 - 4x_3 = 8 \qquad 2x_1 - 3x_2 + 2x_3 = 1 \qquad 5x_1 - 8x_2 + 7x_3 = 1$

Ans

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 8R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & 0 & -25 & 65 \end{bmatrix}$$

$R_1 \to R_1 + 3R_2$

$$\sim \begin{bmatrix} 2 & 0 & -10 & 25 \\ 0 & 1 & -4 & 8 \\ 5 & 0 & -25 & 65 \end{bmatrix}$$

$R_3 \to R_3 - (5/2)R_1$

$$\begin{bmatrix} 2 & 0 & -10 & 25 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}$$ //

$\Rightarrow$ Inconsistent system //

Echelon form: conditions:

1> All the zero rows should be at bottom of the matrix.
2> Leading non zero element should be atleast one position right to the leading non zero element in the row above.
3> All the elements below pivot positions should be zero.

Pivot columns: Columns containing leading non zero elements
Pivot positions: Positions of leading non zero elements.

Reduced row echelon form: Conditions:

1> All the zero rows should be at bottom of the matrix
2> Leading non zero element should be atleast one position right to the leading non zero element in the row above.
3> All the elements below pivot positions should be zero.
4> Leading non zero entries should be equal to 1.
5> Each leading 1 is the only non zero entry in that column.

Echelon form → Guass elimination
Reduced row echelon form → Guass - Jordan elimination
Number of pivot positions = Number of dependent variables.
Rest are free variables.

• Reduced echelon form is unique but echelon form is not unique.

Q1) Obtain solution of: $[\vec{A}|\vec{b}] = \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

Ans x_1, x_3, x_5 are dependent variables, 2 free variables x_2, x_4.

$R_1 \to R_1 - R_2 \sim \begin{bmatrix} 1 & 6 & 0 & 3 & -1 & -7 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

$R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3 \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

$R_2 \to R_2/2 \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

$\Rightarrow x_5 = 7$ $\quad x_3 - 4x_4 = 5$ $\quad x_1 + 6x_2 + 3x_4 = 0$

let $x_4 = \alpha$ & $x_2 = \beta$

then $x_1 = -(6\beta + 3\alpha)$ $\quad x_2 = \alpha$ $\quad x_3 = 5 + 4\alpha$ $\quad x_4 = \beta$ $\quad x_5 = 7$

Vector arithmetic, geometric representation & degree of freedom

Degree of freedom exists for a linear system is consistent.

Degree of freedom is the ability to change the value of a variable.

i.e Number of variables that are free to change its value.

Properties of vector arithmetic

1> Commutative : $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2> Associative : $(\vec{u} + \vec{v}) + \vec{w} = (\vec{u} + \vec{w}) + \vec{v}$
3> Distributive : $\alpha_1 \vec{u}_1 + \alpha_1 \vec{v} = \alpha_1 (\vec{u} + \vec{v})$

Vector Matrix: Array of m×1

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ can also be denoted as:

$$x \triangleq (x_1, x_2, x_3, \dots x_n)$$
or
$$x \triangleq [x_1 \quad x_2 \quad x_3 \dots \quad x_n]^T$$

Vector arithmetic

1> Scalar multiplication
2> Algebraic addition

Eg:
$$3x_1 - 5x_2 + x_3 = 11$$
$$2x_1 + 4x_2 - 3x_3 = -13$$
$$4x_1 - x_2 + 5x_3 = 4$$

This can be represented as:

$$x_1 \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -13 \\ 4 \end{bmatrix}$$

Eg2: Given $u = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ & $V = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find $3u + 4V \triangleq W$

Ans $W = 3\begin{bmatrix} 5 \\ 7 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 33 \end{bmatrix}$

Elementary vectors: vectors of unit magnitude whose linear combination gives any other vector.

Eg: $e_1 \equiv (1, 0)$ i.e $\hat{i}$ so $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$

$e_2 \equiv (0, 1)$ i.e $\hat{j}$ so $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}$

$\therefore$ $R^2 = \alpha e_1 + \beta e_2$ where $\alpha, \beta \in R^1$

$|||^{ly}$ $\mathbb{R}^3 = \alpha e_1 + \beta e_2 + \gamma e_3$ where $\alpha, \beta \in \mathbb{R}^1$

Q> Is every point of $\mathbb{R}^2$ a linear combination of $u = (5, 2)$ & $v = (7, 3)$

Ans Yes, $x = 3w_1 - 2w_2$ $y = 5w_2 - 7w_1$

Matrix Multiplication:

Let x', y' be such that $x \triangleq x' + h$, $y \triangleq y' + k$

Translation of axes: Here Origin $(0, 0)$ is shifted to (h, k)

Rotation of axes: $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

Generic equations: $x = a_{11}x' + a_{12}y'$
$y = a_{21}x' + a_{22}y'$

where x' & y' are old co-ordinate system

For rotation: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrix algebra:

1> $AB = AC$ does not mean $B = C$

Eg: $AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix}$

$\Rightarrow AB = AC$ but $B \neq C$.

2> $AB \neq BA$

3> $(AB)C = A(BC)$

4> $A \cdot (B + C) = AB + AC$

Diagnol matrix: Matrix in which only diagnol elements are non-zero.

Scalar matrix: Diagnol matrix with equal diagnol elements is called a scalar matrix.

Symmetric matrix: A matrix is said to be symmetric if $[a_{ij}]_{m\times n} = [a_{ji}]_{n\times m}$ i.e $A = A^T$. So $m=n$.

Complex conjugate transpose: $A \to \bar{A} \to (\bar{A})^T \to A^*$

$A \to A^T \to \overline{(A^T)} \to A^*$

Hermitian matrix: A matrix is said to be hermitian if $A = A^*$

- Symmetric matrices are special case of Hermitian matrix.

Span: Consider a set of vector $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$. The entire space obtained by all linear combinations of these vectors is called as span.

Eg: $\vec{u} \triangleq \{4, 2, 7\}$, $\vec{v} \triangleq \{3, 1, 4\}$, $\alpha\vec{u} + \beta\vec{v} = \vec{p}$

$$\alpha\begin{bmatrix}4\\2\\7\end{bmatrix} + \beta\begin{bmatrix}3\\1\\4\end{bmatrix} = \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}$$

$$\left[\begin{array}{cc:c}4 & 3 & p_1\\2 & 1 & p_2\\7 & 4 & p_3\end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \sim \left[\begin{array}{cc:c}2 & 1 & p_2\\4 & 3 & p_1\\7 & 4 & p_3\end{array}\right] \quad \begin{array}{l}R_3 \to R_3 - 2R_2\\R_2 \to R_2 - 2R_1\end{array}$$

$$\sim\begin{bmatrix}2 & 1 & p_2\\0 & 1 & p_1 - 2p_2\\-1 & -2 & p_3 - 2p_1\end{bmatrix} \quad R_1 \to R_3 + R_1 \quad \sim\begin{bmatrix}1 & -1 & p_2 + p_3 - 2p_1\\0 & 1 & p_1 - 2p_2\\-1 & -2 & p_3 - 2p_1\end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -1 & p_2+p_3-2p_1 \\ 0 & 1 & p_1-2p_2 \\ 0 & -3 & p_2+2p_3-4p_1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1+R_2 \\ R_3 \to R_3+3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & p_3-p_2-p_1 \\ 0 & 1 & p_1-2p_2 \\ 0 & 0 & 2p_3-5p_2-p_1 \end{bmatrix}$$

For system to be consistent $2p_3 - 5p_2 - p_1 = 0$

$\therefore$ Span of vector is satisfied by $p_1 + 5p_2 = 2p_3$ //

• <u>Note</u>: $A_{m\times n} = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$ where $\vec{a}_n = (a_n)_{m\times 1}$

$Ax = b \Rightarrow x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \dots x_n\vec{a}_n = \vec{b}_{m\times 1}$

Kernel or null space

• Kernel or Null is defined as

$Ker(A) = Null(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$ //

• If $\vec{x} = 0$ then it is known as trivial set.

Properties :

1) If vector $\vec{x}$ & $\vec{y}$ are $Ker(A)$ then $\vec{x}+\vec{y}$, $\alpha\vec{x}$, $\beta\vec{y}$ are also $Ker(A)$, where α & β are real valued coefficients.

2) If $A \sim B$ (equivalent) then $Ker(A) = Ker(B)$

Ex: Given: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \end{bmatrix}_{2 \times 3}$ $\vec{u} = \begin{bmatrix} 7 \\ 6 \\ -5 \end{bmatrix}$. Is vector $\vec{u}$ in $Ker(A)$?

Ans $7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ Yes, $\vec{u}$ is in $Ker(A)$

Rank of a Matrix

- The number of non zero rows in a reduced row echelon form of a matrix is called rank of a matrix.

Ex: Given: $A = \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & -2 & 16 & 1 & 7 \end{bmatrix}_{3 \times 6.}$ Find rank of this matrix.

Ans $R_3 \to R_3 - 4R_1$ $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & -8 & 16 & -7 & -53 \\ 0 & 0 & -10 & 20 & -7 & -61 \end{bmatrix}$$

$R_3 \to R_3 - 10/8\, R_2$, $R_3 \to 4R_3$

$$\sim \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & -8 & 16 & -7 & 53 \\ 0 & 0 & 0 & 0 & 7 & 21 \end{bmatrix}$$

Since we won't get any zero rows, rank = 3

NOTE: • Rank of a matrix = number of pivot positions.
• Rank of a matrix $\leq$ min (m, n) for $A_{m\times n}$

Properties:

1> Consider $A_{m\times n} \cdot \vec{x}_{n\times 1} = 0$ and if $n > m$ then solution is non trivial.

Ex: Estimate Pivot positions & Rank (A).

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 4 & 6 & 3 \\ 3 & 3 & 7 & 5 \end{bmatrix}$$

Ans $R_2 \leftrightarrow R_1$ $\sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 7 & 5 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$ $\sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & -9 & -11 & -4 \end{bmatrix}$

$R_3 \to 4R_2 + R_3$ $\sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, $R_2 \to R_2 + R_3$

$\sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, $R_3 \to R_3 + R_2$, $R_1 \to R_1 - 4R_2$ $\begin{bmatrix} 1 & 0 & -10 & -1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 5 & 1 \end{bmatrix}$

$R_1 \to R_1 + 2R_3$ $\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 5 & 1 \end{bmatrix}$ $R_2 \to R_2 - (4/5)R_3$, $R_3 \to R_3/5$ $\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix}$

$\therefore$ Rank = 3 and Pivot positions = 3

Linear dependence & Independence

Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p$ be linearly combined.
If $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \dots + \alpha_p \vec{v}_p = \vec{0}$ i.e $\sum_{i=1}^{p} \alpha_i \vec{v}_i = \vec{0}$

is true only for trivial set of α_i (i.e $\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_p = 0$) then these vectors are linearly independent.

Ex: $V_1 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ $V_2 = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}$ $V_3 = \begin{bmatrix} 0 \\ 17 \\ 11 \end{bmatrix}$. Linearly dependent?

Ans
$$\left[\begin{array}{ccc:c} 3 & -4 & 0 & 0 \\ 7 & 2 & 17 & 0 \\ 4 & 2 & 11 & 0 \end{array}\right]$$

$R_1 \to R_3 - R_1$
$R_2 \to R_2 - R_3$
$$\sim \begin{bmatrix} 1 & 6 & 11 & 0 \\ 3 & 0 & 6 & 0 \\ 4 & 2 & 11 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$
$$\sim \begin{bmatrix} 1 & 6 & 11 & 0 \\ 0 & -18 & -27 & 0 \\ 0 & -22 & -33 & 0 \end{bmatrix}$$

$R_2 \to -R_2/9$,
$R_3 \to R_3 + 11R_2$
$$\sim \begin{bmatrix} 1 & 6 & 11 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2 ($\because$ Pivot positions = 2)
Linearly dependent.

Elementary matrices: There is a change in row or change in one element.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow E_1 \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\rightarrow E_2 = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$

$\rightarrow E_4 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \alpha I_2$

$\rightarrow E_5 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$

$\rightarrow E_6 = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$

Properties of Linearly dependent (LD) & Linearly independent (LI) vectors

1> K vectors R^n, one more vector $0_{n\times 1}$

$\rightarrow$ Set $S =$ total $(k+1)$ vectors

$\alpha_i \vec{v}_i \quad \forall i = 1$ to k

$(k+1)^{th} \rightarrow \vec{v}_{k+1} \triangleq \vec{0}_{n\times 1}$

$\rightarrow \sum_{i=1}^{k+1} \alpha_i \vec{v}_i = \vec{0}_{n\times 1}$ $\qquad \sum_{i=1}^{k} \alpha_i \vec{v}_i + (\alpha_{k+1} = 1)\, \vec{v}_{k+1} = 0$

• Single zero vector can change a set k independent vectors into a set of dependent vectors.

2> Let set $A = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\} \in \mathbb{R}^n$
a particular vector $\vec{v}_j$

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_j v_j + \dots + a_k u_k = \vec{0}_{n\times 1}$$

$$\Rightarrow \vec{v}_1 = -a_1\vec{v}_1 - a_2\vec{v}_2 - \dots - a_k\vec{v}_k$$

Basis / Bases

Set of vectors which
1> are linear independent
2> can be spanned throughout

Q> Given: $S_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$. Can these form a basis?

$$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$$

Ans For spanning, $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 1 & 1 & 0 & b_2 \\ 1 & 1 & 1 & b_3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{array} \sim \left[\begin{array}{cccc} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & b_3 - b_2 \end{array}\right]$$

$\alpha_1 = b_1 \qquad \alpha_2 = b_2 - b_1 \qquad \alpha_3 = b_3 - b_2$

$\Rightarrow$ This satisfies spanning.

For LI: $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ //

If $b_1 = b_2 = b_3 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$. $\therefore$ It is LI.

$\therefore$ It forms a basis.

Co-ordinates of a vector with respect basis.

If Set S forms a basis and $S \triangleq \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_m\}$ where $\vec{v}_i \in \mathbb{R}^n$ then any other vector $\vec{w} \in \mathbb{R}^n$ can be represented as:

$$\vec{w} = \sum_{i=1}^{m} a_i \vec{v}_i$$

Subspace

Let $S \triangleq \{\vec{v}_1, \vec{v}_2\}$ $\mathbb{R}^n$ and consider $\vec{u}$ & $\vec{w} \in \mathbb{R}^n$ and if

$\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$ $\qquad$ $\vec{w} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2$

then $p\vec{u} + q\vec{w} \in \mathbb{R}^n$

Determinants

Cofactor & minor:

Ex: $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ $\qquad$ Find M_{11}, A_{11}, M_{12}, A_{12}

Ans $M_{11} = 3 \times 4 - 1 \times 2 = 10$ $\qquad$ $M_{12} = 16 - 1 = 15$

$A_{11} = M_{11} = 10$ $\qquad$ $A_{12} = -15$

Properties: 1) Determinant of a triangle matrix is product of elements in principle diagnol.

2) $\det(A) = \det(A^T)$

3> If row $r_i \to r_i + \alpha r_j$ then det (A) remains the same

4> If row $r_i \leftrightarrow r_j$ then $\det(A') = -\det(A)$

5> If row $r_i \to \alpha r_i$ then $\det(A') = \alpha \det(A)$

6> If row $r_j = \alpha r_i$ then $\det(A) = 0$ //

If a_{ij} represents element of A in i^{th} row j^{th} column & A_{ij} represents cofactor of a_{ij} then

7> $\det A = \sum_{j=1}^{n} a_{kj} A_{kj}$ (k^{th} row)

$\det A = \sum_{i=1}^{n} a_{ik} A_{ik}$ (k^{th} column)

8> $\sum_{j=1}^{n} a_{kj} A_{rj} = 0$ where $r \in \{1, n\} - \{k\}$

$\sum_{i=1}^{n} a_{ik} A_{ir} = 0$ where $r \in \{1, n\} - \{k\}$

9> Determinant of diagonal matrix,

$\det(A) = a_{11} a_{22} a_{33} a_{44} \cdots a_{nn}$

10> $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} (M_{ij})$

11> $\det(AB) = \det(A) \cdot \det(B)$

Ex: $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 1 & 5 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 1 & 3 \\ 5 & 1 & 4 \end{bmatrix}$

verify if $\det(AB) = \det(A) \cdot \det(B)$

$$\det(A) = 1(-1-20) + 2(15+1)$$
$$= 32 - 21 = 11$$
$$\det(B) = 2(1) - 2(-19) + 3(-6)$$
$$= -18 + 2(20) = 40 - 18 = 22$$

$$AB = \begin{bmatrix} 12 & 4 & 11 \\ 27 & 9 & 22 \\ 2 & 8 & 22 \end{bmatrix}$$

$$\det(AB) = 11 \begin{vmatrix} 12 & 4 & 1 \\ 27 & 9 & 2 \\ 2 & 8 & 2 \end{vmatrix} = 22 \begin{vmatrix} 12 & 4 & 1 \\ 27 & 9 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$
$$= 22 \begin{vmatrix} 11 & 0 & 0 \\ 27 & 9 & 2 \\ 1 & 4 & 1 \end{vmatrix} = 22 \times 11 = \det(A) \cdot \det(B)$$

Cramer's rule

$$x_j = \frac{\det(A_j(b))}{\det(A)} \qquad 1 \le j \le n$$

Proof :=

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n]$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad I_j(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & x_j \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

$\det(A \cdot I_j(x))$

$A \cdot I_j(x) = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_j x \quad \dots \quad \vec{a}_n]$

$\det(A \cdot I_j(x)) = x \cdot \det(A)$

$$\therefore x_j = \frac{\det(A_j(x))}{\det(A)}$$

Eg: $\begin{bmatrix} 1 & 4 & 2 \\ 3 & -3 & 6 \\ 2 & 0 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$ Find x_2

Ans $\det(A) = 2(30) + 5(-15)$

$= 60 - 75 = -15$

$\det(A_2) = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 5 & 6 \\ 2 & -4 & 5 \end{vmatrix} = 2(8) + 4(0) + 5(-4)$

$= 16 - 20$

$= -4$

$$\therefore x_2 = \frac{4}{15}$$

Matrix inverse

- A^{-1} exists if and only if A is non singular.
- Inverse of A if it exists is unique.

Inverse of a standard 2x2 matrix

Let $A \triangleq \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad M \triangleq \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$AM = MA = I_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$ax + bz = 1$ $\qquad$ $cx + dz = 0$

$ay + bw = 0$ $\qquad$ $cy + dw = 0$

$$\Rightarrow \begin{bmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Augumented matrix:

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & \frac{ad-bc}{a} & 0 & -\frac{c}{a} \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 1 \end{array}\right]$$

$\left(\frac{ad-bc}{a}\right) w = 1$ $\qquad \Rightarrow w = \frac{a}{ad-bc}$

$\Rightarrow z = \frac{-c}{ad-bc}$

$ay + \frac{ba}{ad-bc} = 0$ $\qquad \Rightarrow y = \frac{-b}{ad-bc}$

$ax = 1 - bz$

$\therefore x = \frac{1}{a}\left(1 + \frac{bc}{ad-bc}\right) = \frac{d}{ad-bc}$

$$\therefore M = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{\operatorname{adj}(A)}{|A|}$$

Properties of Inverse:

1) $(A^{-1})^{-1} = A$

Proof: Let $(B^{-1}) = A$

then $A^{-1}B = BA^{-1} = I_n$

$\Rightarrow (A^{-1})^{-1} = B^{-1}$

$\Rightarrow (A^{-1})^{-1} = A$ //

2) $(KA)^{-1} = K^{-1}A^{-1} = \frac{1}{k}A^{-1}$

3) $(AB)^{-1} = B^{-1}A^{-1}$

Proof: Consider $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A I_n A^{-1} = AA^{-1} = I_n$ //

$\Rightarrow (B^{-1}A^{-1}) = (AB)^{-1}$ //

4) $(A^T)^{-1} = (A^{-1})^T$

Proof: consider $A^T \cdot (A^{-1})^T = (A^{-1}A)^T$ $\quad (\because (AB)^T = B^TA^T)$

$= I_n^T = I_n$

$\because A^T \cdot (A^{-1})^T = I_n \quad \Rightarrow (A^T)^{-1} = (A^{-1})^T$ //

NOTE: Any matrix that can be obtained from a identity matrix using single row operation is an elementary matrix.

Inverse of a matrix $A_{n\times n}$

$A_{n\times n}$ is invertible iff Columns $(\mathbb{R}^n)$ are Linearly independent.

Let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]_{n\times n}$

Set of scalars $C = c_1, c_2, \dots, c_n$

$c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0}$

i.e $\sum_{i=0}^{n} c_i \vec{a}_i = \vec{0}$. For linear independence, $c_i = 0$ i.e $\vec{c} = 0$

then

$$A^{-1} A \vec{c} = A^{-1} \vec{0}_{n\times 1} \rightarrow \text{has trivial solution.}$$

NOTE: $A_{n\times n}$ matrix is invertible iff the rank of A is n.

Properties of inverse

5) If $E_k(E_{k-1} \dots (E_3(E_2(E_1(A))))) = I_n$ where E_n represents n^{th} elementary row operation applied on A, then

$E_k(E_{k-1} \dots (E_3(E_2(E_1(I_n))))) = A^{-1}$

Q Find A^{-1}. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ using elementary row operations.

Ans
$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ 4 & -3 & -8 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \rightarrow R_3 - 4R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \quad \begin{array}{l} R_3 \rightarrow R_3 + 3R_2 \\ R_3 \rightarrow -\frac{R_3}{2} \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\begin{array}{l} R_1 \rightarrow R_1 - 3R_3 \\ R_2 \rightarrow R_2\ 2R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

LU factorization / LU decomposition

$A_{n\times n}\, x_{n\times n} = b_{n\times 1} \rightarrow (1)$

$A = L_{n\times n}\, U_{n\times n} \rightarrow (2)$

$L(Ux) = b \rightarrow (3)$; $Ux = y_{n\times 1} \rightarrow (4)$

$Ly = b_{n\times 1} \rightarrow (5)$

NOTE: if $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$

$A \sim U$

$E_m(E_{m-1} \cdots (E_2(E_1[A])))= U$

$A = (E_m(E_{m-1} \cdots (E_2(E_1\ )^{-1}[U]$

$A = \underbrace{E_1^{-1}(E_2^{-1}(\cdots E_m^{-1}}_{L}[U]))$ $\Rightarrow A = LU$

Q) Decompose A into L & U for

$A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$

Ans $R_2 \rightarrow R_2 - 2R_1$ & $R_3 \rightarrow R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{bmatrix}$ $R_3 \rightarrow R_3 - \frac{3}{2} R_2$

$\sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \rightarrow U$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 - 3R_1$ $\qquad$ $R_3 \to R_3 - \frac{3}{2} R_2$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3/2 & 1 \end{bmatrix} = L$$

$\therefore A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

LU decomposition consistency

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$$

$$= \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Inconsistent

Properties of LU decomposition

1) If $A_{n\times n}$ is invertible, then it's LU decomposition is unique.

Proof: Let A have 2 decompositions

i.e $A = L_1 U_1 = L_2 U_2$

$\Rightarrow \quad L_2^{-1} L_1 = U_2 U_1^{-1}$

$\rightarrow$ $U_2 U_1^{-1}$ is upper triangular matrix & $L_2^{-1} L_1$ is unit lower triangular matrix.

It is possible only when $L_2^{-1} L_1 = U_2 U_1^{-1} = I_n$

$\therefore \quad L_2^{-1} L_1 = I_n \quad \Rightarrow \quad L_2^{-1} = L_1^{-1} \quad \Rightarrow \quad L_1 = L_2$ //

$\therefore \quad U_2^{-1} U_1 = I_n \quad \Rightarrow \quad U_2^{-1} = U_1^{-1} \quad \Rightarrow \quad U_1 = U_2$ //

$\therefore$ By contradiction A has unique LU decomposition.

Linear transformation

case study: $\vec{x} = I_n \vec{x} = [\vec{e}_1 \ \vec{e}_2 \ \dots \ \vec{e}_n]_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_n$

$$\vec{x} = (x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n)$$

$$T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(e_2) + \dots + x_n T(e_n) \rightarrow ①$$

$T \rightarrow$ any transform

$A_{n\times n} \vec{x}_{n\times 1}$

Linear Transformation: Geometric aspects of LT : R^2

Reflection:

m_1: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ Reflection about x-axis

m_2: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ Reflection about y-axis

$m_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ Reflection about $y = x$

$m_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$ Reflection about $y = -x$

$m_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$ Reflection about origin

Companding:

m_1: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ x_2 \end{bmatrix}$

m_2: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_2 \end{bmatrix}$

Shearing:

m_1 : $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$

m_2 : $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix}$

Projection:

m_1 : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

m_2 : $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$.

Eigenvalues and Eigenvectors

$A\vec{x} = \lambda \vec{x}$ possible only for square matrix A.
(A may or may not be invertible)

Here λ is a scalar (real or complex)

$\lambda \Rightarrow$ eigenvalue

Any non zero vector $\vec{x}$ which satisfy $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is a eigenvector for A.

Eigenvalue $\rightarrow$ Latent root, characteristic value

Ex: $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find λ for $\vec{u}$ & $\vec{v}$

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$A\vec{u} \neq \lambda\vec{u}$ for any λ.

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\vec{v}$$

$\therefore A\vec{v} = 2\vec{v}$ $\therefore$ Eigenvalue $= 2$

Also $\vec{v}$ is a eigenvector for A.

Eg 2 $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find λ for $\vec{u}$ & $\vec{v}$

$$A\vec{u} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{u}$$

$A\vec{u} = 2\vec{u}$ $\therefore \lambda = 2$ for $\vec{u}$ & A

$$A\vec{v} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\vec{v}$$

$A\vec{v} = 3\vec{v}$ $\therefore \lambda = 3$ for $\vec{v}$ & A

Characteristic equation

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$A\vec{x} - \lambda I_n\vec{x} = \vec{0}$$
$$(A - \lambda I_n)\vec{x} = \vec{0}$$

But $\vec{x} \neq \vec{0}$

$B_{n\times n}\vec{x} = \vec{0}$ where $B = A - \lambda I_n$

If B is invertible

$$B^{-1} B\vec{x} = B^{-1}\vec{0}$$

$\Rightarrow$ $\vec{x} = \vec{0}$ which is a contradiction.

$\Rightarrow$ $\vec{B}$ is singular

$\therefore$ $\det(B) = 0$ $\Rightarrow$ $\det[A - \lambda I] = 0$

Ex: Given $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ is $\lambda = 7$ an eigenvalue?

Ans For λ to be eigenvalue $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} -6 & 6 \\ 5 & -5 \end{vmatrix} = 30 - 30 = 0$$

$\therefore$ λ can be eigenvalue

$$\therefore \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 6x_2 = 7x_1$ $\Rightarrow$ $x_1 = x_2$

$5x_1 + 2x_2 = 7x_2$

$\therefore$ $\lambda = 7$ is an eigenvalue for $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$, $a \in \mathbb{R}$ but $a \neq 0$

Ex 2: What are characteristic equation, eigenvalues, and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

Ans $|A - \lambda I| = 0$ $\Rightarrow$ $\begin{vmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = 0$

$$-2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$(\lambda - 3)(\lambda + 2) = 0$ $\lambda_1 = 3$ & $\lambda_2 = -2$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$2x_1 + x_2 = 3x_1 \quad \Rightarrow \quad x_1 = x_2$

$4x_1 - x_2 = 3x_2$

$\therefore$ Eigen vector $= \begin{bmatrix} a \\ a \end{bmatrix} \quad \forall\, a \in \mathbb{R} - \{0\}$

$(A - \lambda I)\vec{x} = 0$

$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \Rightarrow \quad 4x_1 + x_2 = 0 \quad \Rightarrow \quad x_2 = -4x_1$

$\vec{x} = \begin{bmatrix} a \\ -4a \end{bmatrix} \quad \forall\, a \in \mathbb{R} - \{0\}$

Eigenspace: It is a set of eigenvectors including zero vector

Q) $A = \begin{bmatrix} 3 & 0 & 0 & 13 \\ -25 & 7 & 11 & -6 \\ 18 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ Find all eigenvalues & atleast one eigenvector.

Ans $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 0 & 0 & 13 \\ -25 & 7-\lambda & 11 & -6 \\ 8 & 0 & 1-\lambda & 5 \\ 0 & 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)(3-\lambda)(7-\lambda)(1-\lambda) = 0$

$\lambda_1 = -2 \quad \lambda_2 = 3 \quad \lambda_3 = 7 \quad \lambda_4 = 1$

$(A - \ I)\vec{x} = 0$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 13 & 0 \\ -25 & 6 & 11 & -6 & 0 \\ 18 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right]$$

$x_4 = 0$ $\qquad\qquad$ $x_1 = 0$

$6x_2 = -11x_3$

Eigenvector for $\lambda = 1$ $= \begin{bmatrix} 0 \\ a \\ -11/6\, a \\ 0 \end{bmatrix}$ $\forall\, a \in \mathbb{R} - \{0\}$

<u>Trace of a matrix</u>: $(A) \triangleq a_{11} + a_{22} + \cdots + a_{nn}$

i.e Trace $(A) = \sum_{i=1}^{n} a_{ii}$

Characteristic polynomial : $C_A(\lambda) = 0$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$C_A(\lambda) = |A - \lambda I_2| = 0$

$= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$= (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow ad - \lambda(a+d) + \lambda^2 - bc = 0$

$\Rightarrow \lambda^2 - \lambda(a+d) + (ad-bc) = 0$

$\Rightarrow \lambda^2 - \lambda(\text{trace}(A)) + \det(A) = 0$

Q> $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ Find $C_A(\lambda)$.

Ans $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{vmatrix}$

$\Rightarrow (-1-\lambda)\left[(2-\lambda)^2 - 9\right] = 0$

$\Rightarrow -(1+\lambda)\left[4+\lambda^2-4\lambda-9\right] = 0$

$\Rightarrow -(1+\lambda)\left[\lambda^2-4\lambda-5\right] = 0$

NOTE So for $A_{n\times n}$:

$C_A(\lambda) = \lambda^n - \text{trace}(A)\,\lambda^{n-1} + \dots + c_2\lambda^2 + c_1\lambda + (-1)^n \det(A)$

Eigenvalues for matrix

If $A\vec{x} = \lambda\vec{x}$ have $\lambda = \lambda_1, \lambda_2$ & $\vec{x} = \vec{v}_1, \vec{v}_2$

then $A[\vec{v}_1 \ \vec{v}_2] = [\vec{v}_1 \ \vec{v}_2] \,\text{diag}(\lambda_1, \lambda_2)$

$\Rightarrow AP = PD \rightarrow (1)$ where $P = [\vec{v}_1, \vec{v}_2]$ $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$APP^{-1} = PDP^{-1} \Rightarrow A = PDP^{-1}$ (if P is invertible)

$P^{-1}AP = P^{-1}PD \Rightarrow D = P^{-1}AP$ (if P is invertible)

Q> Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Find λ.

Ans $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 1,$ $\Rightarrow u_1 = u_2$

$\Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ i.e $(A - \lambda I)\vec{x} = \vec{0}$

$\Rightarrow 0x_1 + x_2 = 0$ $\Rightarrow x_1 = \alpha \quad x_2 = 0$

Algebraic and geometric multiplicity

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1 \longleftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(\lambda - 1)^2 = 0$

$\Rightarrow$ For $\lambda = 1$, algebraic multiplicity is 2.

Geometric multiplicity $= 1$

($\because$ λ can be spanned only along one dimension)

Algebraic multiplicity = Number of repetitive roots for a particular eigenvalue.

Geometric multiplicity = Number of dimension in which eigenvalue can be spanned.

- Geometric multiplicity $= n - \text{rank}(A - \lambda I)$

Theorem:

Set of eigenvectors corresponding to distinct eigenvalues are linearly independent.

$A_{n\times n}$; $\quad Ax^{(i)} = \lambda_i x^{(i)} \quad 1 \leq i \leq k$

i.e Set $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is linearly independent.

Proof: Let $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is linearly dependent.

$$\Rightarrow \{\underbrace{(x^{(1)}, x^{(2)}, \ldots, x^{(j)})}_{\text{linearly independent}} \vdots\; x^{(j+1)} \ldots x^{(k)}\} \quad j < k$$

But $\quad x^{(j+1)} = \sum_{i=1}^{j} C_i x^{(i)}$

$$A\left[x^{(j+1)}\right] = A\left[\sum_{i=1}^{j} C_i x^{(i)}\right] \qquad A \rightarrow \text{linear transform}$$

$$\Rightarrow Ax^{(j+1)} = \sum_{i=1}^{j} C_i Ax^{(i)}$$

$$\Rightarrow \lambda_{j+1} x^{(j+1)} = \sum_{i=1}^{j} C_i \lambda_i x^{(i)}$$

$$\lambda_{j+1} \sum_{i=1}^{j} C_i x^{(i)} = \sum_{i=1}^{j} C_i \lambda_i x^{(i)}$$

$$\Rightarrow \sum_{i=1}^{j} C_i (\lambda_{j+1} - \lambda_i) x^{(i)} = 0$$

This can happen only if $\lambda_{j+1} = \lambda_i$ or all $C_i = 0$ or $x^{(i)} = 0$

But all eigenvalues are distinct $\Rightarrow \lambda_{j+1} \neq \lambda_i$

But $x^{(i)}$ is eigenvector which cannot be $\vec{0}$.

$\therefore$ all $C_i = 0 \Rightarrow x^{(j+1)}$ is independent.

$\therefore$ Set $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ is linearly independent.

Theorem: If $A_{n\times n}$ is having n distinct eigenvalues only then it can be represented as Product with diagnol matrix.
i.e $A = PDP^{-1}$

Ex: 1> $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ $\to \lambda_1 = 3 \to [1 \quad 1]^T$, $\to \lambda_2 = -2 \to [1 \quad -4]^T$ $\Rightarrow$ A can be expressed as PDP^{-1}

2> $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\to \lambda_1 = 1 \to [1 \quad 0]^T$, $\to \lambda_2 = 1 \to [1 \quad 0]^T$ $\Rightarrow$ A cannot be expressed as PDP^{-1}

Basis of eigen vectors:

$A_{n\times n} \longleftrightarrow \lambda_i \qquad i = 1 - n$

Eigen vector $\to$ $\begin{bmatrix} C_1 & C_2 & \cdots & C_n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

Ex: $A = \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix}$ $\longrightarrow \lambda_1 = 3 \to [1 \quad 0]^T$, $\longrightarrow \lambda_2 = 7 \to [5 \quad 4]^T$

$(A - 3I)x = 0$
$(A - 7I)x = 0$

Here $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ form basis.

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\xrightarrow{\lambda = 0}$, $\xrightarrow[\lambda = 0]{}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

But $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ do not form basis.

Theorem: Set of eigenvectors may or maynot form basis

Applications of eigenvalue and eigen vectors

1) Power of a square matrix $A_{n\times n}$

$A = PDP^{-1}$

$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$

$A^3 = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$

$\Rightarrow A^m = PD^mP^{-1}$

Eg: Given: $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ Find A^m if $\lambda_1 = 3$ $\lambda_2 = -2$

For $\lambda_1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 \rightarrow \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

Ans $P = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $D^m = \begin{bmatrix} 3^m & 0 \\ 0 & -2^m \end{bmatrix}$

$P^{-1} = \frac{-1}{5}\begin{bmatrix} -4 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{bmatrix}$

$$A^m = PD^mP^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3^m & 0 \\ 0 & -2^m \end{bmatrix}\begin{bmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4\cdot 3^m + (-2)^m}{5} & \frac{3^m - (-2)^m}{5} \\ \frac{4\cdot 3^m - 4(-2)^m}{5} & \frac{3^m + 4(-2)^m}{5} \end{bmatrix}$$

NOTE: Even if matrix is real, eigenvalues can be complex

eg Given: $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Find eigenvalues

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda)+2 = 0$

$3 - 4\lambda + \lambda^2 + 2 = 0$ $\Rightarrow$ $\lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm j$

$\lambda^2 - 4\lambda + 5 = 0$

$\therefore \lambda_1 = 2+j$ $\quad \lambda_2 = 2-j$

$(A - \lambda I)\vec{x} = 0$

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 2+j & 0 \\ 0 & 2+j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} -1-j & 1 \\ -2 & 1-j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Augmented matrix $= \left[\begin{array}{cc|c} -1-j & 1 & 0 \\ -2 & 1-j & 0 \end{array}\right]$

$R_2 \to R_1(-1+j) + R_2$ $\Rightarrow \left[\begin{array}{cc|c} -1-j & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow (-1-j)x_1 + x_2 = 0$ $\Rightarrow$ $x_2 = x_1(1+j)$ $\Rightarrow$ $V_1 = \begin{bmatrix} 1 \\ 1+j \end{bmatrix}$

let $x_1 = 1$ $\Rightarrow$ $x_2 = 1+j$

|||ly for λ_2

$$\left[\begin{array}{cc|c} -1+j & 1 & 0 \\ -2 & 1+j & 0 \end{array}\right]$$

$R_2 \to R_1(-1-j) + R_2$ $\to \begin{bmatrix} -1+j & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow (-1+j)x_1 + x_2 = 0$

$\Rightarrow x_2 = (1-j)x_1$

If $x_1 = 1$ $\quad x_2 = 1-j$ $\quad \therefore \vec{V}_2 = \begin{bmatrix} 1 \\ 1-j \end{bmatrix}$

$$\therefore P = \begin{bmatrix} 1 & 1 \\ 1+j & 1-j \end{bmatrix} \qquad D = \begin{bmatrix} 2+j & 0 \\ 0 & 2-j \end{bmatrix}$$

$$P^{-1} = \frac{-1}{2j}\begin{bmatrix} 1-j & -1 \\ -1-j & 1 \end{bmatrix} = \frac{j}{2}\begin{bmatrix} 1-j & -1 \\ -1-j & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} j+1 & -j \\ 1-j & j \end{bmatrix}$$

$$A = PDP^{-1}$$

Lab:

$$H(j\omega) = \mathcal{F}(h(t)) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$x(t) \longrightarrow \boxed{\text{Linear}} \longrightarrow y(t) = \alpha x(t - t_D)$$

$$x(t) \Updownarrow X(j\omega) \qquad y(t) \Updownarrow Y(j\omega) = \alpha e^{-j\omega t_d} X(j\omega)$$

Power of A:

If $A\vec{v} = \lambda \vec{v}$ then

$$A^2\vec{v} = A \cdot (A\vec{v})$$
$$= A \cdot \lambda \vec{v}$$
$$= \lambda \cdot A\vec{v} = \lambda^2 \vec{v},$$

III^{ly} $\quad A^3 \vec{v} = \lambda^3 \vec{v}$

$$\therefore A^k \vec{v} = \lambda^k \vec{v}$$

provided $A_{n\times n}$: λ_i exists for $i = 1 \dots n$

If $\vec{V}_i$ form basis then $\vec{x} = \sum_{j=1}^{n} c_j \vec{V}_j$

- $A^k \vec{x} = \underbrace{A \cdot A \cdots A}_{k \text{ times}} \cdot (A\vec{x})$

$= A^k \left[\sum_{j=1}^{n} c_j \vec{v}_j\right]$

$= \sum_{j=1}^{n} c_j A^k \vec{v}_j = \sum_{j=1}^{n} c_j \lambda_j^k \vec{v}_j$

Dynamical system $\triangleq$ $\vec{x}^{(k)} = A \vec{x}^{(k-1)}$

$\vec{x}^{(k-1)} = A \vec{x}^{(k-2)}$

$x^k = A(A x^{k-2}) = A^k x^{(0)}$

Eg: $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $\lambda_1 = a$

$\lambda_2 = b$

$\lambda_1 = a \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_1$ $\lambda_2 = b \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}_2$

$P \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ orthogonal (orthonormal)

$x \in \mathbb{R}^n$

$x = \begin{bmatrix} x1 \\ x2 \\ \vdots \\ x_n \end{bmatrix}$ Norm (length) of a vector $\triangleq ||\vec{x}||$

$= \sqrt{(\vec{x}) \cdot (\vec{x})}$

i.e $||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

$||\vec{v}_1|| = 1$ $||\vec{v}_2|| = 1$

let $x_n = \frac{1}{k}\begin{bmatrix} x1 \\ x2 \\ \vdots \\ xn \end{bmatrix} = \begin{bmatrix} x_1/k \\ x_2/k \\ \vdots \\ x_n/k \end{bmatrix}$

Hence $||x_n|| = 1$ so we can convert orthogonal to orthonormal.

Attraction, Repulsion, Saddle points are the 3 types of points.

$$x^{(0)} = \sum_{i=1}^{2} c_i \vec{V_i}$$

$$x^k = A^k x^{(0)}$$
$$= A^k \sum_{i=1}^{2} c_i \vec{V_i}$$
$$= \sum_{i=1}^{2} c_i A^k \vec{V_i} = \sum_{i=1}^{2} c_i \lambda^k \vec{V_i}$$
$$= c_1 \lambda_1^k \vec{V_1} + c_2 \lambda_2^k \vec{V_2}$$
$$= c_1 a^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 b^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence a & b are real values

$x^{(0)} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ given $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 01 \\ 01 \end{bmatrix}$

$\Rightarrow x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore c_1 = 1$ & $c_2 = 1$

$\therefore x^k = \frac{a^k}{b^m} [1, 1]$

case studies:

if i) $0 < a < 1$, $0 < b < 1$ } c converges to 1

ii) $0 < a < 1$, $b > 1$ repeller about y-axis $(0, b)$

iii) $0 < b < 1$, $a > 1$ repeller about x-axis

iv) $b > 1$, $a > 1$ } origin acts as repeller

Ill condition system:

$$\sum_{i=1}^{n} a_i x_i = b_j \qquad j = 1 \dots m$$

S1:	S2:	S3:
$x + y = 1$	$x + y = 1$	$x + y = 1$
$x + 1.01y = 1$	$x + 1.01y = 1.01$	$0.99x + y = 1.01$
$\Rightarrow (x, y) = (1, 0)$	$\Rightarrow (x, y) = (0, 1)$	$\Rightarrow (x, y) = (-1, 2)$

- So just small change in the coefficient of system, the solution changes drastically.
- Similarly, the error in component value gives rise to massive change in output.

Eg $6.5x + 7.2y = 10$ Here $x = -6.89818$

$5.1x + 3.7y = -7$ $y = 7.61641$

$6.477x + 7.203y = 10$ Here $x = -6.92054$

$5.072x + 3.692y = -7$ $y = 7.61132$.

PROBABILITY THEORY

Probability, Random variables & Random process

source → transmitter → channel → Reciever → Sink

Linear algebra, Random variables are used

AWGN → Additive white gaussian noise

Additive ↓ Superposition

white ↓ equally affects all frequencies

gaussian → model

Random process : A family of random variables changing with time or space generate a set of random processes.

Ex: Opamp based experiments with random variables such as resistance, bias current, offset voltages.

Sample space v/s Event space

Sample space - Set of all possible outcomes based on experiment defined

Event space: Required set of events in an experiment.

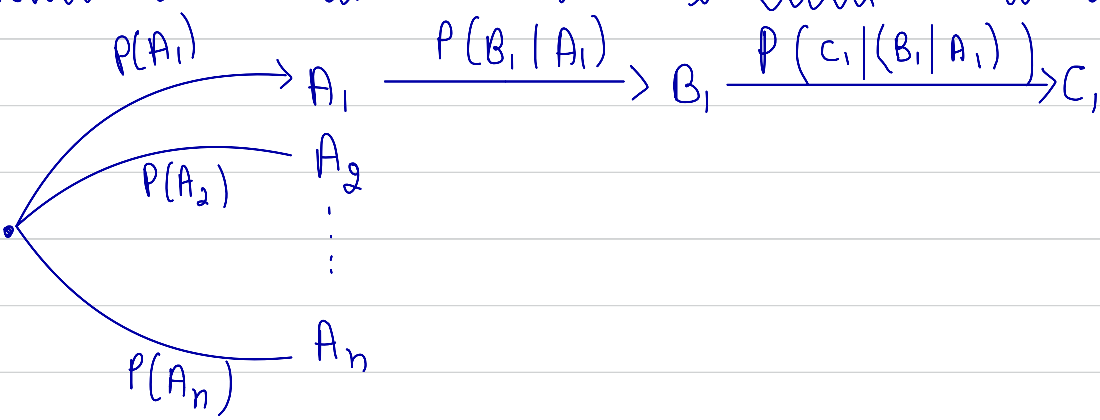
Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(A|B) \rightarrow$ Posteriori
$P(B) \rightarrow$ A priori

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Probability tree for sequential experiment



Bayes' theorem:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) \cdot P(A) = P(A \cap B)$$

$$\therefore \quad \boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

Random variable:

NOTE: Probability of a discrete random variable in continuous range is zero

2 types: 1> Continous random variable
2> Discrete random variable

Random variable is a mapping from event space to random variable space.

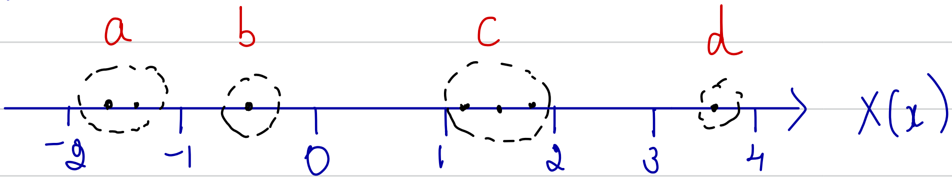
$P(s) = P(X(s))$
where s is an element of sample space S.

Probability mass function (PMF)

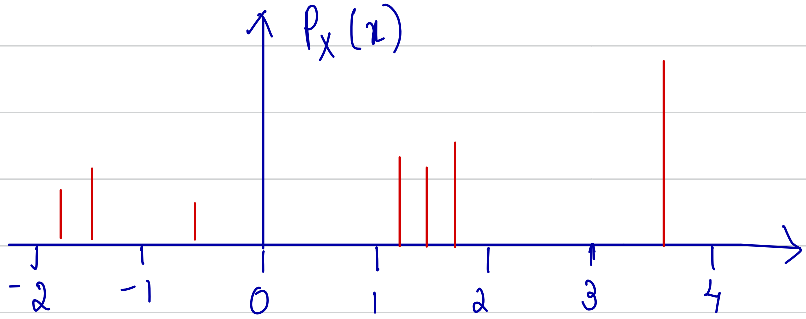
For $X(x)$, $P_X(x_0) \triangleq P(X = x_0)$

Graph for PMF

Eg:



Graph:



Note; $0 \leq P_X(x) \leq 1$

$$\sum_{x} P_X(x) = 1$$

$$F_X(x_0) \triangleq P\{X \leq x_0\}$$

Joint / compound PMF

$$\text{JPMF} \triangleq F_{X,Y}(x_0, y_0) = P\{X \leq x_0, Y \leq y_0\}$$

$$\sum_{x_0} \sum_{y_0} P_{X,Y}(x_0, y_0) = 1$$

Marginal PMF: $\sum_{\text{all } x_0} F_{X,Y}(x_0, y_0) = F_Y(y_0)$

Conditional PMF:

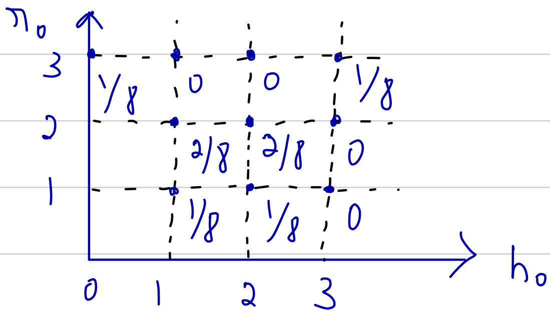
1> Condition is defined on some random variable.

$$F_{X|Y}(x_0 | y_0) \triangleq \frac{F_{X,Y}(x_0, y_0)}{F_Y(y_0)}$$

Eg: Flipping of coin

$H(h) = 0, 1, 2, 3, 4$
$R(n) = 1, 2, 3$

$$F_h(h_0) \triangleq P\{h \leq h_0\}$$



$$P_{r|h}(r_0|2) = \begin{cases} 1/3, & r_0 = 1 \\ 2/3, & r_0 = 2 \end{cases}$$

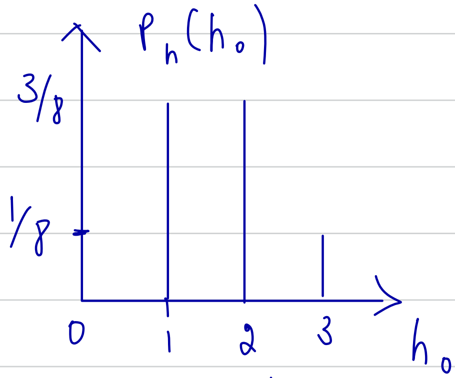
2> Condition is defined on some other event A.

$$F_{X,Y|A}((x_0,y_0)|A) = \frac{F_{X,Y}(x_0,y_0)}{P(A)} \quad \text{with } P(A) \neq 0$$

• Conditional PMF ⊂ Marginal PMF ⊂ Joint PMF

Eg: Marginal PMF of repeat count

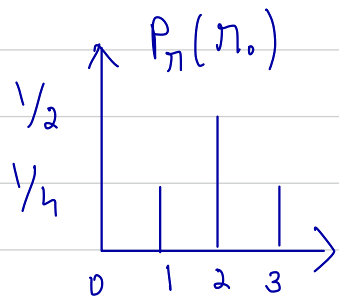
$\rightarrow$



i.e $\sum_{r_0=0}^{3} P_{r,h}(r_0, h_0) = P_h(h_0)$

Marginal PMF of head count

$\rightarrow$



• Joint PMF → Marginal PMF is irreversible process

Conditional Probability:

$$P_{x,y}(x_0, y_0 | A) \triangleq \begin{cases} \dfrac{P_{xy}(x_0, y_0)}{P(A)} & \text{if } (x_0, y_0) \text{ in } A \\ 0 & \text{if } (x_0, y_0) \text{ in } A' \end{cases}$$

Independence of conditional PMF

$$P_{x,y}(x_0 | y_0) = \frac{P_{x,y}(x_0, y_0)}{P_y(y_0)}$$

If $X(x_0)$ is independent of $Y(y_0)$ then
$P_{x,y}(x_0, y_0) = P_x(x_0) \cdot P_y(y_0)$

$\Rightarrow P_{x|y}(x_0 | y_0) = P_x(x_0)$

Probability distribution function : PDF

Let RV X : finite number of values
RV → random variable

$X: \; x_1 \quad x_2 \quad \ldots \quad x_n$

$f(x_1) \quad f(x_2) \quad \ldots \quad f(x_n)$

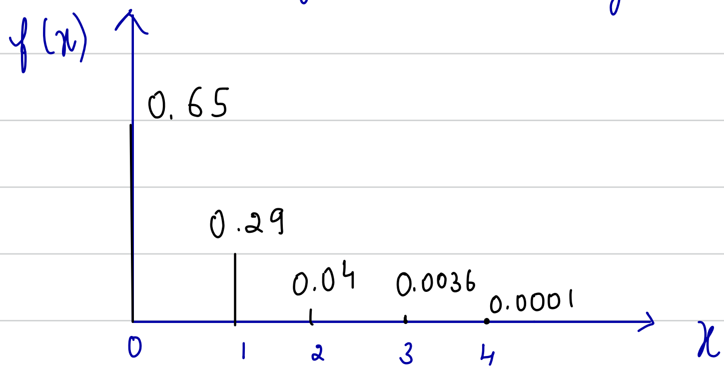
$$\sum_{i=1}^{n} f(x_i) = 1$$

Q> $X: \; x = 0, 1, 2, 3, 4$

$f(0) = 0.6561 \quad f(1) = 0.2916 \quad f(2) = 0.0486 \quad f(3) = 0.0036 \quad f(4) = 0.0001$

Draw the PDF for the given random variable.

Ans



Cumulative distribution function

→ It is an alternate way to define associated probability. (PMF)

$X : x_1 \quad x_2 \ldots \quad x_n$

CDF, $F_X(x_j) \triangleq P\{X \leq x_j\}$ i.e $F_X(x_j) = \sum_{i=1}^{j} f(x_i)$

Q> If $f(0) = 0.6561$ $\quad f(1) = 0.2916$ $\quad f(2) = 0.0486$

$f(3) = 0.0036$ $\quad f(4) = 0.0001$. Find $F_X(3)$

Ans $F_X(3) = P\{X \leq 3\} = \cup \{0, 1, 2, 3\}$

$= 1 - P\{X = 4\}$

$= 0.999$ //

NOTE:

Discrete RV	Continuous RV
1> PMF	Pdf (d- density)
2> CDF (PDF) ↓ distribution	PDF

Properties of CDF

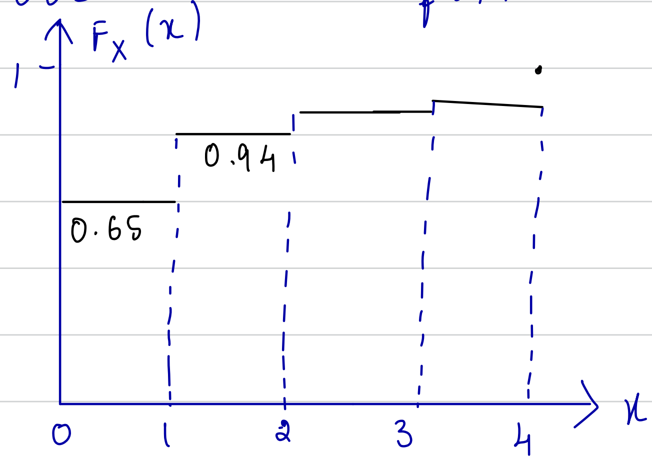
1> $0 \leq F_X(x_k) \leq 1$ for $k < n$

2> $F_X(-\infty) = 0$ 3> $F_X(\infty) = 1$

4> $F_X(x_2) > F_X(x_1)$ if $x_2 > x_1$

Q> If $f(0) = 0.6561$ $f(1) = 0.2916$ $f(2) = 0.0486$
$f(3) = 0.0036$ $f(4) = 0.0001$. Sketch $F_X(x)$

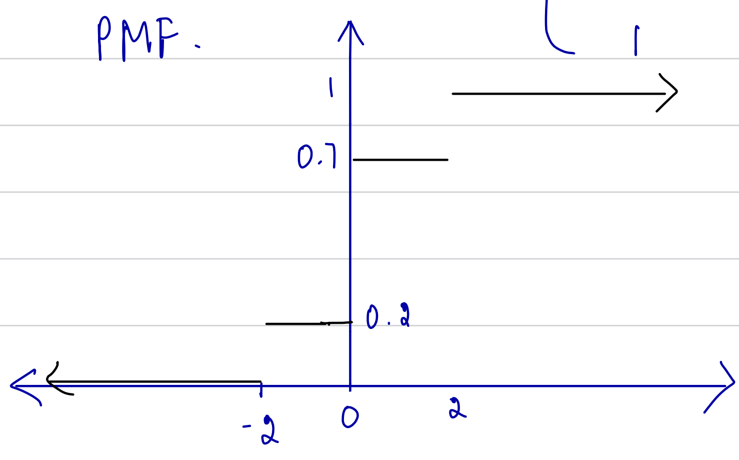
Ans



Q> Given CDF, $F_X(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \leq x < 0 \\ 0.7 & 0 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$

Find PMF.

Ans



$\Rightarrow f(-2) = 0.2$

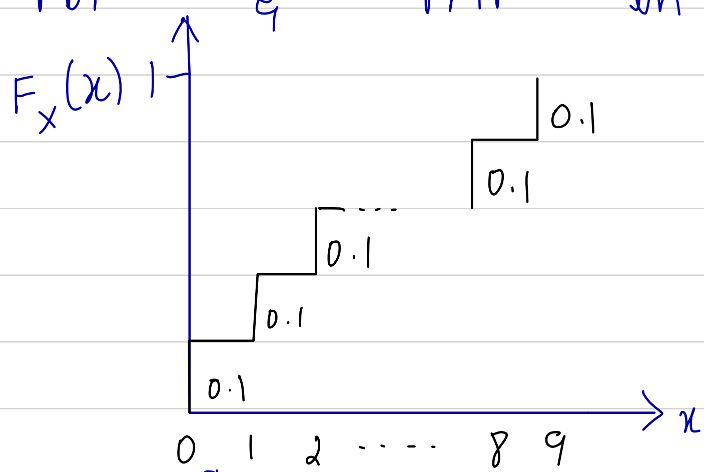
$f(0) = 0.7 - 0.2 = 0.5$

$f(2) = 1 - 0.7 = 0.3$

Unit step representation of PDF, $\sum_{i=1}^{m} \alpha_i u(n-i)$ where α_i is PMF.

i.e $\quad PDF = 0.2\, u(n+2) + 0.5\, u(n) + 0.3\, u(n-2)$

Qs Express PDF & PMF in mathematical form for:



Ans $\quad$ PDF, $F_X(x) = \sum_{i=1}^{9} 0.1\, u(x-i)$

PMF, $f_X(x) = \sum_{i=1}^{9} 0.1\, \delta(x-i)$

Mean & variance of a DRV

- Expectation of Mean $\triangleq E[X] = \sum_{i=1}^{n} x_i f(x_i) = \mu$

where $f(x_i) \triangleq P\{X = x_i\}$

- Variance $\triangleq \sigma^2 = V[x] = E\left[(X-\mu)^2\right] = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$

$$= E[X^2] - (E[X])^2$$

$$= E[x^2] - \mu^2$$

Q> If $X = 0, 1, 2, 3, 4$ with $f(x_i) = 0.6561, 0.2916, 0.0486, 0.0036, 0.0001$ respectively. Find σ^2 & μ

Ans $\mu = \sum_{i=0}^{4} x_i f(x_i)$

$= 0 \times 0.6561 + 1 \times 0.2916 + 2 \times 0.0486 + 3 \times 0.0036 + 4 \times 0.0001$

$\therefore \mu = 0.4$ //

$\sigma^2 = \sum_{i=0}^{4} (x_i - 0.4)^2 f(x_i)$

$= 0.16 \times 0.6561 + 0.36 \times 0.02916 + 2.56 \times 0.0486 + 6.76 \times (0.0036) + 12.76 \times 0.0001$

$= 0.265$ //

NOTE: 1) $E[h(x)] \triangleq \sum_{i=1}^{n} h(x_i) \cdot f(x_i)$

2) If $E[X] = \mu_x$ & $E[Y] = \mu_y$
and $Y \rightarrow aX + b$
then $\mu_y = a\mu_x + b$

3) $\sigma^2_y = a^2 \sigma_x^2$

Proof: $\sigma^2_y = E[(y - \mu_y)^2] = E[((ax+b) - (a\mu_x + b))^2]$

$= E[a^2 (x - \mu_x)^2] = a^2 E[(x - \mu_x)^2]$

$= a^2 \sigma_x^2$ //

Types of discrete random variables

1> Uniform DRV: In this all random variables have same PMF.

i.e $X = x_1, x_2, x_3, \dots, x_n$ then $f_X(x_i) = 1/n$

case study: $X: x \in \{a, a+1, a+2, \dots, b-1, b\}$

i.e $a \le x \le b$ No. of measurement $= b-a+1$

$$f(x_i) = \frac{1}{b-a+1}$$

$$E[X] = \sum_{i=a}^{b} x_i f(x_i) = \sum_{i=a}^{b} \frac{1}{b-a+1} i$$

$$= \frac{1}{b-a+1} \sum_{i=a}^{b} i = \frac{1}{b-a+1}\left(\sum_{i=0}^{b} i - \sum_{i=0}^{a-1} i\right)$$

$$= \frac{1}{b-a+1}\left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right)$$

$$= \frac{1}{b-a+1} \frac{b^2+b-a^2+a}{2} = \frac{1}{b-a+1} \frac{(b+a)(b-a+1)}{2}$$

$$\therefore E[X] = \frac{b+a}{2} = \mu$$

$$\sigma^2(x) = E[(X-\mu)^2]$$

$$= \frac{(b-a+1)^2-1}{12}$$

2> Binomial Random Variable: X is a Binomial RV if PMF is defined as follows:

$$P_X(x) = \begin{cases} {}^nC_x \, p^x (1-p)^{n-x} \end{cases} \qquad x = 0, 1, 2, \ldots n$$

where $0 < p < 1$

a> <u>Bernoulli Random variable</u>: Binomial random variable, with $x = 0$ or 1. , $n = 1$

$$P_X(0) = {}^1C_0 \, p^0 (1-p)^1 = 1-p$$

$$P_X(1) = {}^1C_1 \, p^1 (1-p)^0 = p$$

$$\therefore P_X(x) = \begin{cases} 1-p & , \quad x = 0 \\ p & , \quad x = 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Q> In a digital channel, packets are transmitted in a group of 4 bits with probability of error, $p = 0.1$ for each bit. Find the probability that there are 2 bits in error.

Ans $X = 0, 1, 2, 3, 4 \Rightarrow n = 4$

$$P_X(2) = {}^4C_2 \, p^2 (1-p^2)$$

$$= 6 \times (0.1)^2 (0.9)^2$$

$$= 10^{-4} \times 6 \times 81$$

$$= 486 \times 10^{-4} = 0.0486$$

NOTE: For Binomial distribution, $\mu = np$

$\sigma^2 = np(1-p)$

case study: $X \to$ Bernoulli RV

$$P_X(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \end{cases}$$

$$\mu = E[X] = \sum_{i=0}^{1} x_i f_x(x_i) = 0 \times (1-p) + p = p$$

$$\sigma_0^2 = E\left[(X-\mu)^2\right] = \sum_{i=0}^{1} (x-p)^2 f_x(x_i)$$

$$= (0-p)^2(1-p) + (1-p)^2 p$$

$$= p^2 - p^3 + (1+p^2-2p)p$$

$$= p^2 - p^3 + p + p^3 - 2p^2$$

$$= p - p^2 = p(1-p)$$

$\therefore \sigma_0^2 = p(1-p)$

b) Geometric distribution:

$X \to$ No of attempts

$\downarrow$

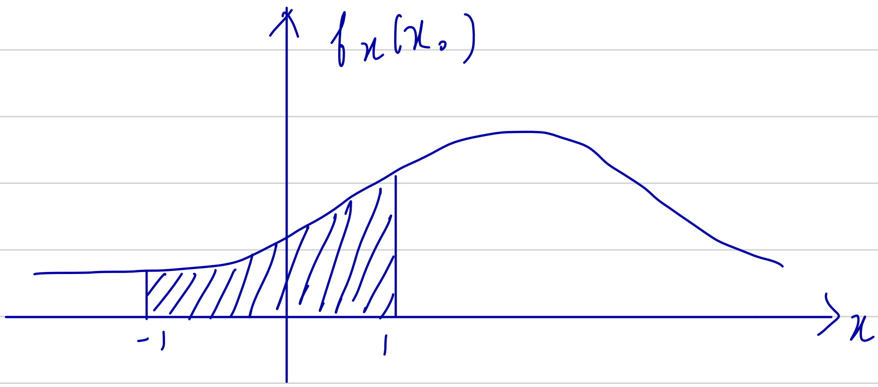
$x = 1, 2, 3, \ldots,$

$f_x(x) = (1-p)^{x-1} \, p \longrightarrow$ successful attempt

Continuous random variable:

Probability density function is used instead of PMF.

Probability density function = PdF



$$P\{|x| \leq 1\} = \int_{-1}^{1} f_x(x)\, dx$$

CDF for CRV: $P\{x \leq x_0\} = \int_{-\infty}^{x_0} f_x(x)\, dx$

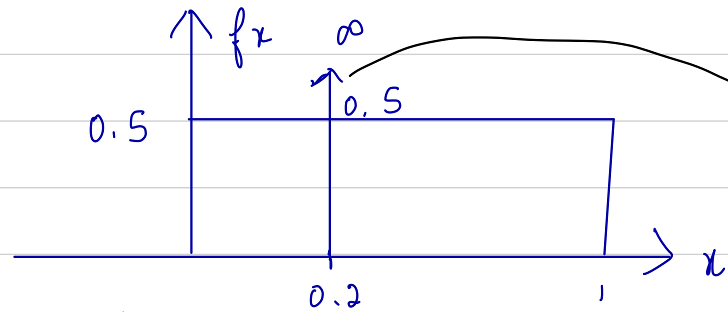
<u>Properties:</u>

1> $P_X(\infty) = 1$ 2> $P_X(-\infty) = 0$

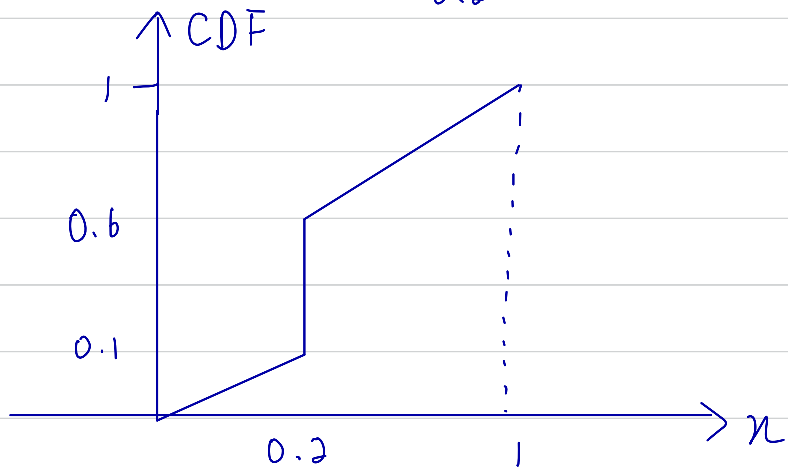
3> $$P\{a < x \leq b\} = \int_a^b f_x(x)\, dx$$
$$= P_X(x \leq b) - P_X(x \leq a)$$

Q> Experiment consists of flipping a coin once with equally likely head & tail outcome. If head comes out $x = 0.2$ if Tails comes out rotate a wheel with values between 0 & 1 with $x =$ value obtained. Sketch the PdF.

Ans



Ending with arrow means area under arrow is 0.5



Ending with dot (•) means it is amplitude

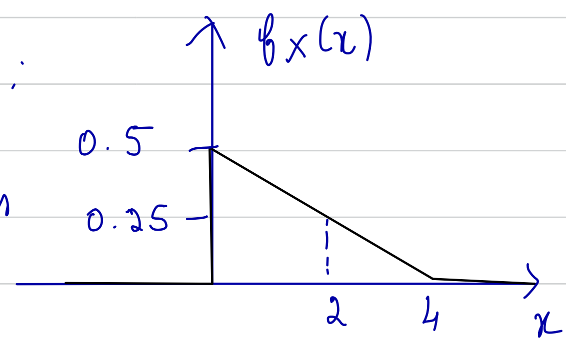
CDF in terms of PdF

$$P_X \leq (x) = \int_{-\infty}^{\infty} f_X(x)\,dx \rightarrow (1)$$

PdF in terms of CDF

$$f_X(x) = \frac{d}{dx} P_X \leq (x)$$

Eg PdF information is given:
It represents information of life of an IC where x is months.



Find:

1> Probability that IC fails in 2nd month of operation

2> Probability that IC fails in 2nd month provided it worked properly in 1st month.

Ans i> $P_X(1 \leq x \leq 2) = \int_1^2 f_X(x)\,dx$

$= \int_1^2 \frac{4-x}{8}\,dx$

$= \left.\frac{x}{2} - \frac{x^2}{16}\right]_1^2 = \frac{1}{2} - \frac{3}{16} = \frac{8-3}{16} = \frac{5}{16}$

ii> A = Fails in 2nd month

B = Fails in 1st month

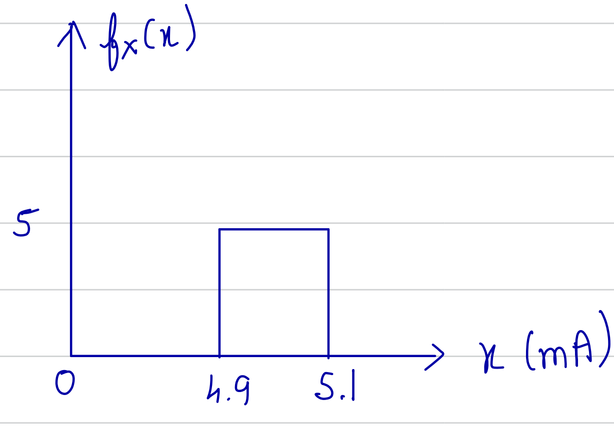
$\therefore P(A|B') = \frac{P(A \cap B')}{P(B')}$

$P(B) = \int_0^1 \frac{4-x}{8}\,dx$

$= \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$

$= \frac{5/16}{9/16} = \frac{5}{9}$

Eg Given PdF :



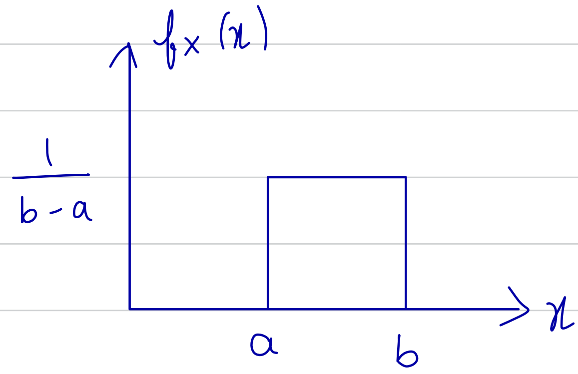
Find the probability that current varies between 4.95 & 5 mA

Ans $P_X(4.95 \leq x \leq 5) = \int_{4.95}^{5} 5\,dx = \left. 5x \right]_{4.95}^{5}$

$= 0.25$

Mean and Variance of PdF

1) Uniform PdF:



$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = E[X] = \int_a^b \frac{1}{b-a} x \, dx = \left.\frac{x^2}{2}\right]_a^b \frac{1}{b-a}$$

$$= \frac{(b+a)(b-a)}{2(b-a)} = \frac{b+a}{2}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_a^b (x-\mu)^2 f_X(x) \, dx$$

$$= \int_a^b (x-\mu)^2 \frac{1}{(b-a)} dx$$

$$= \left.\frac{(x-\mu)^3}{3}\right]_a^b \frac{1}{b-a}$$

$$= \left[\left(b - \frac{(b+a)}{2}\right)^3 - \left(a - \frac{(b+a)}{2}\right)^3\right] \frac{1}{3(b-a)}$$

$$= \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right] \frac{1}{3(b-a)}$$

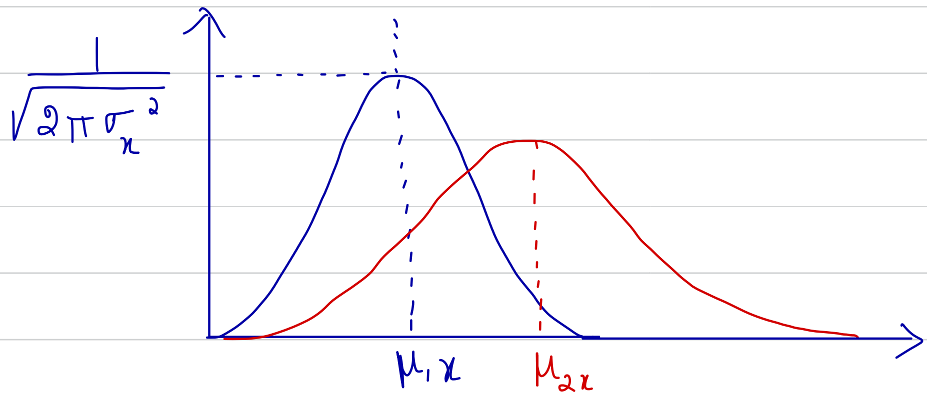
$$= \frac{(b-a)^3}{8} \times 2 \times \frac{1}{3(b-a)} = \frac{(b-a)^2}{12}$$

Central limit theorem : Gaussian random Variable, standard Normal Random Variable.

- If there is a large number of random variable, then summation of all random variable form a gaussian random variable. (i.e PdF follows gaussian distribution).

$$PdF = f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_X^2}} \quad -\infty < x < \infty$$

μ_x = mean $\qquad \sigma_x^2$ = variance



If $\mu_{2x} > \mu_{1x}$, $\sigma_{1x}^2 > \sigma_{2x}^2$

$$P(X \geq a) = \int_a^\infty f_X(x)\, dx$$

NOTE: 1) $P\{\mu_x - \sigma_x \leq x \leq \mu_x + \sigma_x\} \approx 0.68$

2) $P\{\mu_x - 2\sigma_x \leq x \leq \mu_x + 2\sigma_x\} = 0.94$

3) $P\{\mu_x - 3\sigma_x \leq x \leq \mu_x + 3\sigma_x\} = 0.997$

- Gaussian width = $6\sigma_x$

<u>Change of variable:</u>

let $\dfrac{x-\mu_x}{\sigma_x} \triangleq z$

$$\frac{dx}{\sigma_x} = dz$$

$$P\{X \geq a\} = \int_{\frac{a-\mu_x}{\sigma_x}}^{\infty} \frac{1}{\sigma_x^2\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$\Rightarrow$ If $\sigma_x = 1$ or $\mu_x = 0$

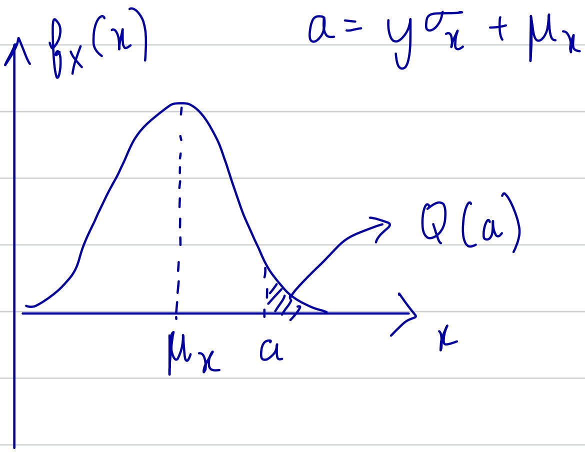
$$P\{X \geq a\} = \int_{a}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

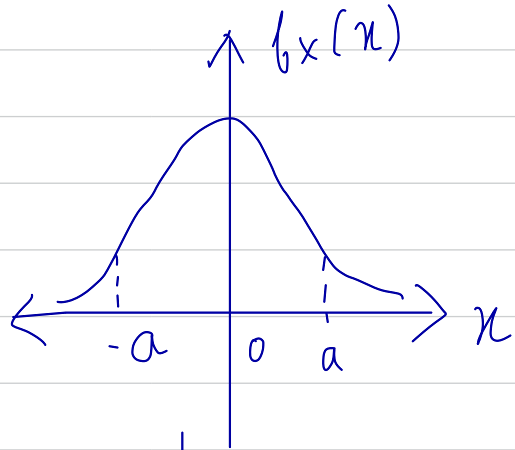
This PdF is called standard or Normal gaussian distribution.

<u>Q-function:</u> $Q(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{y}^{\infty} e^{-z^2/2} \, dz \quad , y > 0$

where $y = \dfrac{a-\mu_x}{\sigma_x}$ $\Rightarrow a = y\sigma_x + \mu_x$

$$P\{X \geq a\} = Q\left[\frac{a-\mu_x}{\sigma_x}\right]$$



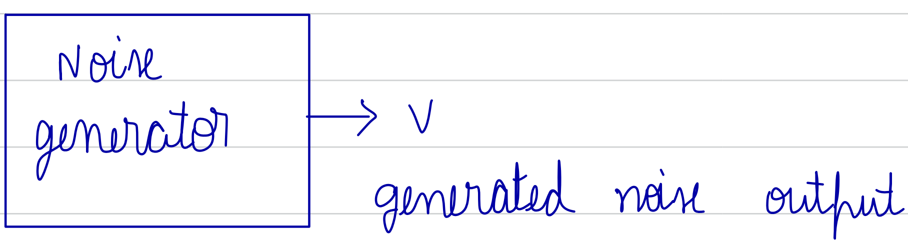


$$P\{X \leq 0\} = P\{X \geq 0\} = 1/2$$

$$P\{-a \leq x \leq a\} = 2P\{-a \leq x \leq 0\} = 2P\{0 \leq x \leq a\}$$

$$= 1 - 2Q(a)$$

Eg: An experiment is conducted to validate SNR estimation. If V (generated noise output) is modelled using standard normal Random variable, find $P\{V > 2.3\}$ & $P\{1 \leq V \leq 2.3\}$



Ans For standard normal distribution, $\mu_x = 0$ & $\sigma_x^2 = 1$

$$P\{V > 2.3\} = \int_{2.3}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \Rightarrow Q(2.3)$$

$$\therefore P(V > 2.3) = Q(2.3) = 0.11 \text{ (from table)}$$

$$P\{1 \leq V \leq 2.3\} = 1 - Q(2.3) - [1 - Q(1)]$$
$$= Q(1) - Q(2.3)$$
$$= 0.148$$

Exponent Density function:

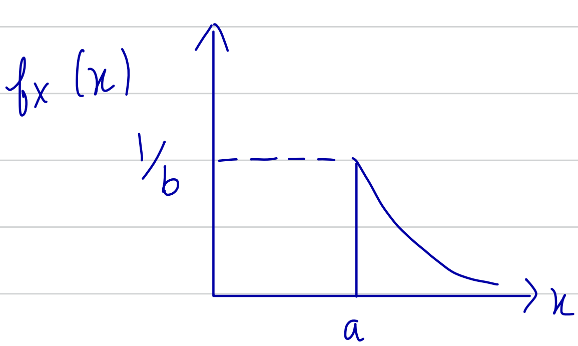
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}}, & x > a \\ 0, & x < a \end{cases}$$

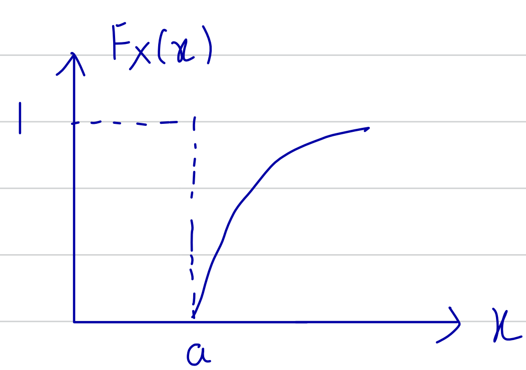
$$\Rightarrow CDF = F_X(x) = \int_{-\infty}^{x} f_X(\lambda)\, d\lambda = \int_{-\infty}^{x} \frac{1}{b} e^{-\frac{(\lambda-a)}{b}}\, d\lambda$$

$$= \frac{1}{b} e^{-\frac{(x-a)}{b}} \quad x > a$$

$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & x > a \\ 0 & x < a \end{cases}$$

Used in estimating strength of recieved signal in Radar system.



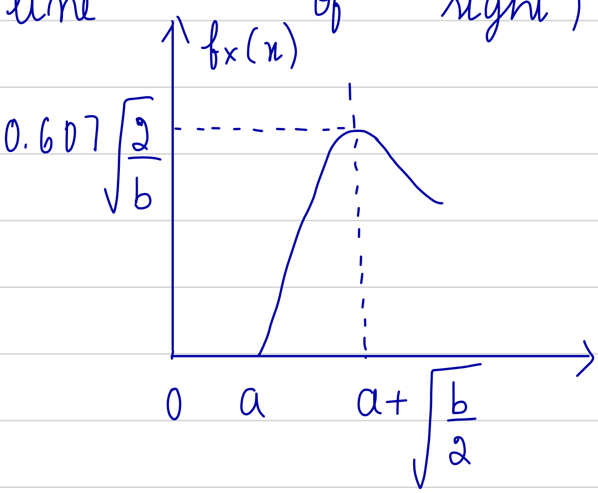


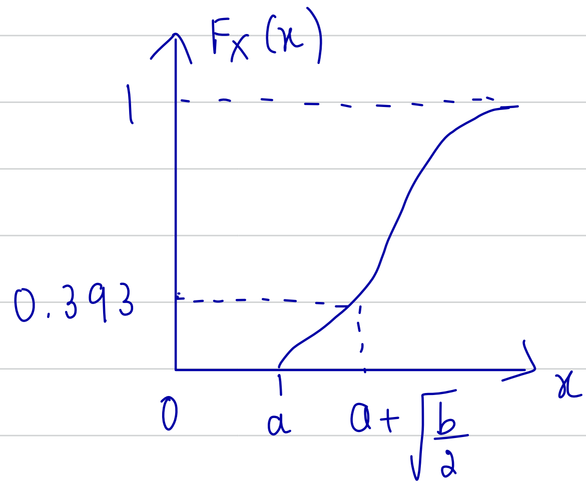
Rayleigh density function

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}, & x \geq a \\ 0, & x < a \end{cases}$$

$$\Rightarrow F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)^2}{b}}, & x \geq a \\ 0, & x < a \end{cases}$$

used in wireless communication (for non line of sight)





- For both line of sight & non line of sight:
  - Rician density function is used.

Two random variables: Joint, Marginal & Conditional density function.

$\left.\begin{array}{l} X(x) \\ Y(y) \end{array}\right\} f_{X,Y}(x, y) \rightarrow$ Joint density function

- Joint CDF, $F_{X,Y}(x, y) \triangleq P\{X \leq x, Y \leq y\}$
- If X, Y are independent then
  $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$

  CDF $= F_X(x, y) = P\{X \leq x\} \cdot P\{Y \leq y\}$
- $f_{X,Y}(x, y) = \dfrac{\partial^2 (F_{X,Y}(x, y))}{\partial x\, \partial y}$
- $F_{X,Y}(x_0, y_0) = \displaystyle\int_{-\infty}^{y_0} \int_{-\infty}^{x_0} f_{X,Y}(x, y)\, dx\, dy$
- Joint PdF $= \displaystyle\lim_{dx, dy \to (0,0)} P\left[\{x \leq X \leq x + dx\} \cap \{y \leq Y \leq y + dy\}\right]$

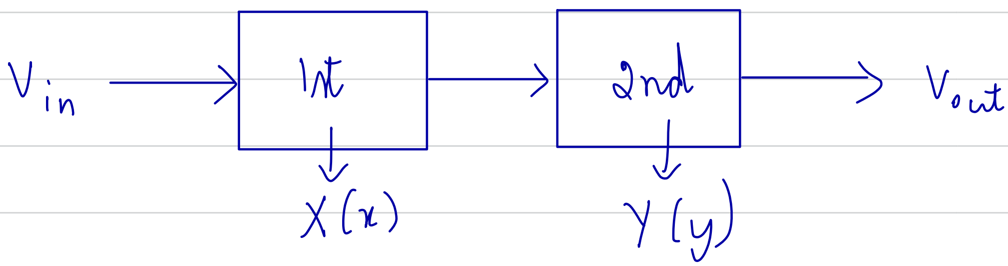
  $= f_{X,Y}(x, y)\, dx\, dy$
- Marginal PdF $=$ $f_X(x)$ $= \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$ | $f_Y(y) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$

Conditional PdF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Eg: Given a 2 stage system:



if $f_{X,Y}(x,y) = \begin{cases} axy & 1 \le x \le 3 \\ & 2 \le y \le 4 \\ 0 & \text{otherwise.} \end{cases}$

Find a, Marginal PdF and $F_Y(y)$

Ans we know that, $\int_2^4 \int_1^3 axy \, dx \, dy = 1$

i) $\Rightarrow a \left[\frac{x^2}{2}\right]_1^3 \left[\frac{y^2}{2}\right]_2^4 = 1$

$\frac{a}{\cancel{4}} \times \cancel{8}^{2} \times 12 = 1 \quad \Rightarrow \quad a = \frac{1}{24}$

ii) $f_X(x) = \int_2^4 f_{X,Y}(x,y) \, dy = \int_2^4 \frac{xy}{24} dy = \frac{x}{24} \times \left.\frac{y^2}{2}\right]_2^4$

$$\therefore f_X(x) = \begin{cases} \dfrac{x}{4} & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_1^3 f_{X,Y}(x,y)\,dx = \int_1^3 \frac{xy}{24}\,dx = \frac{y}{24} \left. \frac{x^2}{2} \right]_1^3 = \frac{y}{6}$$

$$\therefore f_Y(y) = \begin{cases} \dfrac{y}{6} & 2 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$F_Y(y) = \int_2^y f_Y(y)\,dy = \int_2^y \frac{y}{6}\,dy = \frac{y^2-4}{12}$$

$$\therefore F_Y(y) = \begin{cases} 0 , & y < 2 \text{ or } y > 4 \\ \dfrac{y^2-4}{12} , & 2 \leq y \leq 4 \end{cases}$$

Function of 2 random variables:

$$\mu_{x,y} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

$$\sigma_{x,y}^2 = E\left[(x-\mu_x)^2 (y-\mu_x)^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_y)^2 (x-\mu_x)^2 f_{X,Y}(x,y)\,dx\,dy$$

Covariance, $\sigma_{XY} = E[(x-\mu_x)(y-\mu_y)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y)\, f_{x,y}(x,y)\, dx\, dy$$

Co-relation ratio $\triangleq \rho = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

NOTE: $-1 \leq \rho \leq 1$

Conditional expected value

$$E[g(X,Y)\,|_{Y=y}] = \int_{-\infty}^{\infty} g(x,y) \frac{f_{x,y}(x,y)}{f_Y(y)}\, dx$$

If X & Y are Statistically independent (SI)

$X \rightarrow g(x)$ $Y \rightarrow h(y)$

then $E[g(x)h(y)] = E[g(x)] \cdot E[h(y)]$

Characteristic functions & Moment generating functions

$$\Psi_X(\omega) = E\left[e^{jx\omega}\right]$$

$$= \int_{-\infty}^{\infty} e^{jx\omega} f_X(x)\, dx = \int_{-\infty}^{\infty} \left[f_X(x)\, e^{-j\omega x}\right]^* dx$$

$\Rightarrow \quad \Psi_X(\omega) =$ complex conjugate of Fourier tranform

We have $\quad f(t) \longleftrightarrow F(j\omega)$

IIIly $\quad f_X(x) \overset{*}{\longleftrightarrow} \Psi_X(\omega)$

or $\quad f_X(x) \longleftrightarrow \Psi_X^*(\omega)$

Properties of characteristic function:

1> Paring: $\quad f_X(x) \longleftrightarrow \Psi_X^*(\omega) = \Psi_X(-\omega)$

2> $\Psi_X(\omega)\Big|_{\omega=0} = 1 \qquad$ area of f_X

3> $\displaystyle\int_{-\infty}^{\infty} \left| f_X(x) \cdot e^{j\omega x} \right| dx \le 1 \qquad$ (Here order goes upto ∞)

We have $\quad \Psi_X(\omega) = E\left[e^{jx\omega}\right]$

$$\triangleq \int_{-\infty}^{\infty} e^{jx\omega} \cdot f_X(x)\, dx \quad \Rightarrow (1)$$

Also $\quad \Psi_X(\omega)\Big|_0 = \Psi_X(0) = 1 \quad \Rightarrow (2)$

$$\frac{d}{d\omega}\left(\Psi_X(\omega)\right) = \int_{-\infty}^{\infty} f_X(x)\, e^{jx\omega}\, jx\, dx$$

$$\frac{d^2}{d\omega^2}\left(\Psi_X(\omega)\right) = j^2 \int_{-\infty}^{\infty} f_X(x)\, e^{jx\omega}\, x^2\, dx$$

at $\omega = 0$

$$\frac{d^2}{d\omega^2}\left(\Psi_X(\omega)\right) = j^2 \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= j^2 E[X^2] \quad \text{at } \omega = 0$$

$$\boxed{\therefore E[X^k] = \frac{1}{j^k} \frac{d^k}{d\omega^k}\left(\Psi_X(\omega=0)\right)}$$

2 Random variable space:

Space: 2 RVs: $X_1 \downarrow \omega_1$ $X_2 \downarrow \omega_2$

$$\Psi_{X_1,X_2}(\omega_1,\omega_2) = E\left[e^{j\omega_1 x_1} \cdot e^{j\omega_2 x_2}\right] = E\left[e^{(j\omega_1 x_1 + j\omega_2 x_2)}\right]$$

$$\Psi_{X_1,X_2}(0,0) = 1$$

$$E(X_1^m X_2^n) = (-j)^{m+n} \frac{\partial^m \Psi_{X_1}(\omega_1)}{\partial \omega_1^m} \cdot \frac{\partial^n \Psi_{X_2}(\omega_2)}{\partial \omega_2^n}$$

Moment generating function:

$$M_X(t) \longrightarrow \Psi_X(t/j)$$

$$\Rightarrow M_X(t) = \Psi_X(\omega)\Big|_{t/j}$$

$$= E\left[e^{(jx\, t/j)}\right] = \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx$$

where $x \in [x_1, x_2]$ $t \in [t_1, t_2]$. These are boundary conditions for region of convergence.

Bivariant Gaussian PdF

$$\underset{(x)}{\overset{RV}{X}} \longrightarrow \boxed{\begin{array}{c}\text{Linear system} \\ \hline \text{Amplifier} \,/\, \int \,/\, \frac{d}{dt} \,/\, \text{filter}\end{array}} \longrightarrow \underset{(y)}{\overset{RV}{Y}}$$

Linear system preserves the shape of PdF.

	X		Y
uniform	(a_X, b_X)	$\longrightarrow$	(a_Y, b_Y)
gaussian	(μ_X, σ_X)	$\longrightarrow$	(μ_Y, σ_Y)
Exponential	(a_X, b_X)	$\longrightarrow$	(a_Y, b_Y)
Rayleigh	(a_X, b_X)	$\longrightarrow$	(a_Y, b_Y)

Bivariant gaussian function is given by:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

where $\rho = \dfrac{\sigma_{xy}}{\sigma_x \sigma_y} = \dfrac{E[(x-\mu_x)(y-\mu_y)]}{\sigma_x \sigma_y}$

Marginal PdF, $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

Complex random variable : Let RVs be $X(x)$ & $Y(y)$ defined on real axis.

$$Z(z) = X + jY$$

$$= \sqrt{X^2 + Y^2}\, e^{j\phi} \quad , \quad \phi = \tan^{-1}(Y/X)$$

$$E[g(Z)] = \int_{-\infty}^{\infty} g(z)\, f_Z(z)\, dz$$

$$\mu_Z = \int_{-\infty}^{\infty} z\, f_Z(z)\, dz$$

$$\sigma_Z^2 = E[|z - \mu_Z|^2] \longrightarrow \text{(Similar to power calculation)}$$

$$\text{Covariance} = E[(X - \mu_x)^* (Y - \mu_y)]$$

or

$$= E[(X - \mu_x)(Y - \mu_y)^*]$$

Multiple Random variables :

$$X_1 \quad X_2 \quad \cdots \quad X_m$$

$$x_1 \quad x_2 \quad \cdots \quad x_m$$

Joint Distribution : $F_{X_1, X_2 \cdots X_m}(x_1, x_2 \ldots x_m)$

$$= P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_m \leq x_m)$$

Joint PdF $f_{X_1 X_2 \cdots X_m}(x_1, x_2 \cdots x_m)$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{Joint PdF } dx_2\, dx_3 \cdots dx_m$$
$$x_2 \quad x_3 \quad\quad x_m$$

$$f_{X_1 X_2}(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{Joint PdF } dx_3\, dx_4 \cdots dx_m$$
$$x_3 \quad x_4 \quad\quad x_m$$

Conditional probability

$$f_{X_1, X_2 | X_3}(x_1, x_2 | x_3) = \frac{f_{X_1 X_2 X_3}(x_1, x_2, x_3)}{f_{X_3}(x_3)} \; //$$

$$f_{X_2 | X_1 X_3}(x_2 | x_1 x_3) = \frac{f_{X_1 X_2 X_3}(x_1, x_2, x_3)}{f_{X_1 X_3}(x_1, x_3)} \; //$$

Expected value & conditional PdF involving function of several random variables

$$g(X_1 X_2 X_3 X_4) = \sum_{i=1}^{4} \alpha_i X_i = g(X_{4\times 1}) = Y$$

$$E[g(X_1 X_2 X_3 X_4)] =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1 x_2 x_3 x_4)\, f(x_1 x_2 x_3 x_4)\, dx_1\, dx_2\, dx_3\, dx_4$$

Important parameters of multiple RVs
experiment space

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}_{m\times 1} \to \text{RV}$$

Set of RV

$$E[X_i] = \mu_{X_i}$$

Covariance $\to$ $\sigma_{X_i X_j} = E[X_i X_j] - \mu_{X_i}\mu_{X_j}$

If $X_i = X_j$, $\sigma_{X_i X_j} = \sigma_{X_i}^2 = E[X_i^2] - \mu_{X_i}^2$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} = (x_1 \ x_2 \ \dots \ x_m)^T$$

$x_m \to$ Arbitrary value of RV

$$E[\vec{X}] = \mu_{\vec{X}} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_m] \end{bmatrix} \longleftarrow \text{Average vector}$$

Covariance matrix set $[\vec{X}]$

$$\Sigma_X = \begin{bmatrix} \sigma_{X_1 X_1} & \sigma_{X_1 X_2} & \cdots & \sigma_{X_1 X_m} \\ \sigma_{X_2 X_1} & \sigma_{X_2 X_2} & \cdots & \sigma_{X_2 X_m} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_m X_1} & \sigma_{X_m X_2} & \cdots & \sigma_{X_m X_m} \end{bmatrix}_{m\times m}$$

$\therefore \Sigma_X = E[X \cdot X^T] - \mu_X \mu_X^T$

- Also $\sigma_{x_i x_j} = 0$ for $i \neq j$ then X_i & X_j are uncorrelated.

- If m random variables are independent then,

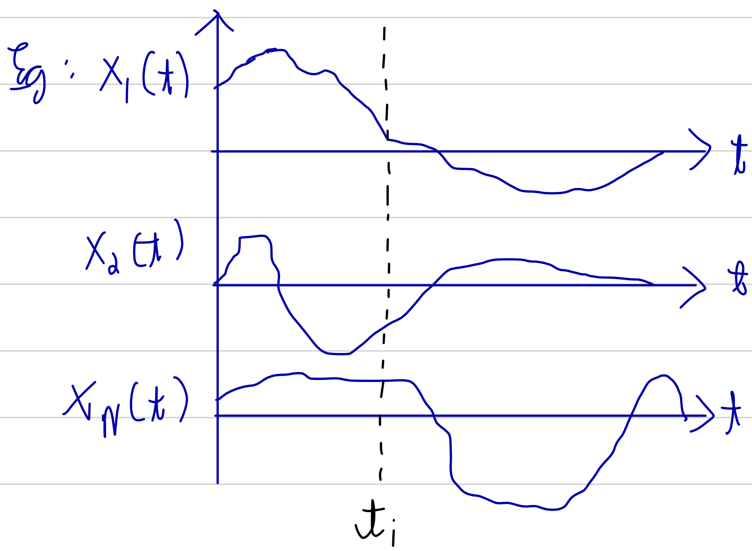
$$f_{X_1 X_2 \dots X_m}(x_1, x_2, \dots x_m) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_m}(x_m)$$

The multivariate gaussian PdF having m RVs:

$$f_X(x) = \frac{1}{(2\pi)^{m/2} |\Sigma_X|^{1/2}} e^{\left[\frac{-(X-\mu_X)^T}{2} (\Sigma_X)^{-1} (X-\mu_X)\right]}$$

Random process

Ensemble member or ensemble : Collection of random signal



→ continuous state continuous time Random Process

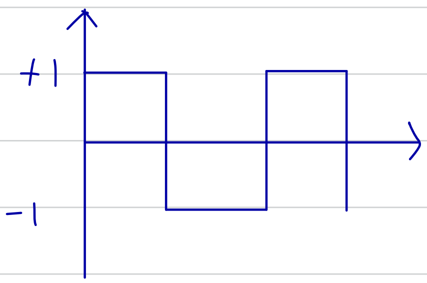
$X = \{X_1(t), X_2(t), \ldots, X_N(t)\}$

$X(S, t) \longrightarrow X(S, t_i) \rightarrow$ RV vector of size $N \times 1$

$X(S, t) \longrightarrow X(S_j, t) \rightarrow$ Ensemble member

$X(S, t) \longrightarrow x(S_j, t_i) \rightarrow$ Scalar random value.

Discrete state continuous time random Process



- If time is discrete it is called random sequence instead of random process.

Notion of mapping:

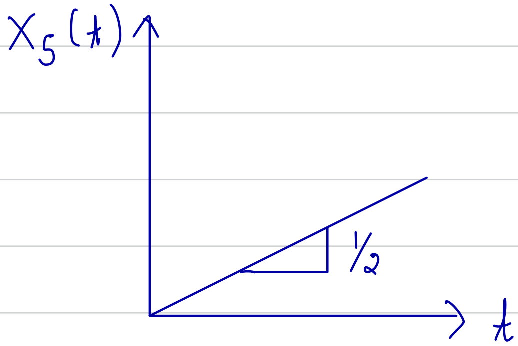
Eg: A die is thrown, $S = \{1, 2, 3, 4, 5, 6\}$

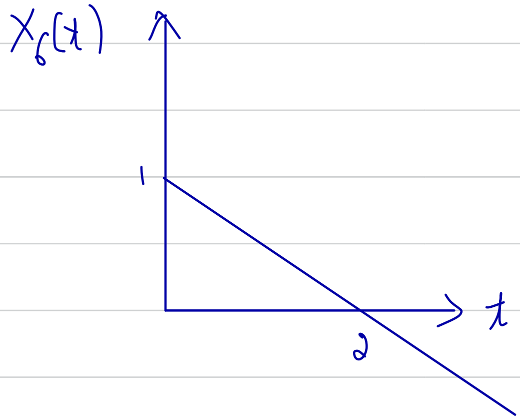
$1 \downarrow X_1(t)$, $6 \downarrow X_6(t)$

For $t \geq 0$

$X_1(t) = -4\,V$ $\quad X_2(t) = -2\,V$ $\quad X_3(t) = 2\,V$ $\quad X_4(t) = 4\,V$

$X_5(t) = t/2$ $\quad X_6(t) = 1 - t/2$





- For a ensemble if average of one slot is same as average of other one, then random process is said to be stationary.

- Random process can be real or complex.

Eg: $X(t) = A(t) \cos[2\pi f_c t + \phi(t)]$

$$= \text{Real part of } \left[A(t)\, e^{j(2\pi f_c t + \phi(t))}\right]$$

$$= \text{Real part of } \left[A(t)\, e^{j\phi(t)} \cdot e^{j2\pi f_c t}\right]$$

$$X(t) = \text{Real}\left[W(t) \cdot e^{j2\pi f_c t}\right]$$

Let $W(t) = A(t)\, e^{j\phi(t)}$

$$W(t) = A(t)\cos\phi(t) + j\,A(t)\sin\phi(t)$$
$$W(t) = P(t) + j\,Q(t)$$

Joint PDF

$$F_X(x_1, x_2, \ldots x_N ; t_1, t_2 \ldots t_N) \triangleq P[X(t_1) \le x_1, X(t_2) \le x_2 \ldots X(t_N) \le x_n]$$

$\underbrace{x_1, x_2, \ldots x_N ; t_1, t_2 \ldots t_N}_{\text{ensembles}}$

$f_X(x_1, t_1) \longrightarrow$ density function $\quad f_X(x_1, t_1) \triangleq \dfrac{d}{dx}(F_X(x_1, t_1))$

|||ly $\quad f_X(x_1, x_2 \ldots x_N ; t_1, t_2 \ldots t_N) \triangleq \dfrac{\partial^n (F_X(x_1, x_2 \ldots x_N ; t_1, t_2 \ldots t_N))}{\partial x_1 \, \partial x_2 \, \partial x_3 \ldots \partial x_N}$

Random Process: Analytical expression

$x(t) = A(t) \cos(2\pi \times 10^8 t + \theta(t))$ is transmitted to n users.

Signal recieved by users:

$U_1 = a_1 A \cos(2\pi \times 10^8 t + \theta_1)$

$U_2 = a_2 A \cos(2\pi \times 10^8 t + \theta_2)$

$\vdots$

$U_N = a_N A \cos(2\pi \times 10^8 t + \theta_N)$

S_θ

deterministic signal $\longrightarrow$ $\boxed{\text{channel}}$ $\longrightarrow$ Random signal

Multiple random process

If $X(t)$ & $Y(t)$ are 2 random processes,

$X(t) : x_1 \; x_2 \dots x_N ; \; t_1 \; t_2 \dots t_N$

$Y(t) : y_1 \; y_2 \dots y_M ; \; t'_1 \; t'_2 \dots t'_M$

Joint distribution function

$$F_{X,Y}(x_1\, x_2 \dots x_N ; t_1\, t_2 \dots t_N ; \; y_1\, y_2 \dots y_M ; \; t'_1\, t'_2 \dots t'_M)$$

$$\triangleq P[X(t_1) \le x_1 , X(t_2) \le x_2 \dots X(t_N) \le x_N , Y(t'_1) \le y_1 , Y(t'_2) \le y_2 , \dots Y_M(t'_M) \le y_m]$$

If X & Y are statistically independent, then,

$$f_{X,Y}(x_1\, x_2 \dots x_N ; t_1\, t_2 \dots t_N ; \; y_1\, y_2 \dots y_M ; \; t'_1\, t'_2 \dots t'_M)$$

$$= f_X(x_1\, x_2 \dots x_N ; t_1\, t_2 \dots t_N) \cdot f_Y(y_1\, y_2 \dots y_M ; t'_1\, t'_2 \dots t_M)$$

1st order Stationary random process:

$E[X(t)] =$ constant

Proof:

$$F_X(x_1\, t_1) = F_X(x_1 , t_1 + \Delta t) \Rightarrow f_X(x_1\, t_1) = f_X(x_1 , t_1 + \tau)$$

$$E[X(t_1)] = E[x_1] = \int_{-\infty}^{\infty} x_1 f_X(x_1 , t_1) \, dx_1$$

$\circledast$

$$E[x_2] = \int_{-\infty}^{\infty} x_2 \, f_X(x_2 \; t_2) \, dx_2$$

$$= \int_{-\infty}^{\infty} x_1 \, f_X(x_1, t_2) \, dx_1 \qquad (\because x_2 \text{ is dummy integration variable})$$

$$= \int_{-\infty}^{\infty} x_1 \, f_X(x_1 \; t_1 + \tau) \, dx_1 \qquad (\because t_2 = t_1 + \tau \text{ for some } \tau)$$

$$= \int_{-\infty}^{\infty} x_1 \, f_X(x_1, t_1) \, dx_1 \qquad (\text{from } *)$$

$$\therefore \; E[x_2] = E[x_1]$$

$$\Rightarrow \; E[X(t)] = \text{constant}$$

2nd order & wide-sense stationarity

$$f_{X_1 X_2}(x_1, x_2 ; t_1, t_2) = f_{X_1 X_2}(x_1, x_2 ; t_1 + \tau, t_2 + \tau)$$

Wide sense stationarity is observed when 2 conditions are satisfied:

1) if $E[X(t)] = \text{constant}$

2) if process of 2nd order is time invarient i.e

$$f_{X_1 X_2}(x_1, x_2 ; t_1, t_2) = f_{X_1 X_2}(x_1, x_2 ; t_1 + \tau, t_2 + \tau)$$

Autocorrelation:

$X(t)$ @ $t = t_1 \Rightarrow X_1(x_1)$

@ $t = t_2 \Rightarrow X_2(x_2)$

The 2^{nd} condition of wide sense stationarity can also be written as:

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] \quad (\text{for real valued RV})$$
$$R_{XX}(t_1, t_2) = E[X^*(t_1) \cdot X(t_2)] \quad (\text{for complex RV})$$

$R_{XX}(t_1, t_2 + \tau)$ is a function of τ only

<u>NOTE</u>:

- Autocovariance, $C_{XX} = R_{XX}(t_1, t_2) - \mu_X(t_1)\,\mu_X(t_2)$
- Autocorrelation coefficient (ratio)

$$r_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1) \cdot C_{XX}(t_2, t_2)}}$$

<u>Eg1)</u> A random process

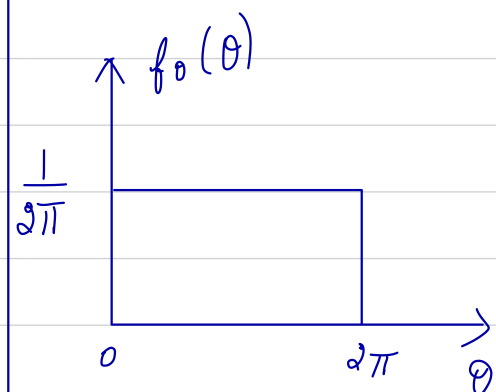
$$X(t) = A\cos(\omega_0 t + \theta) \rightarrow (1)$$

where A & ω_0 are constant & θ is a random variable.
Characterize θ using uniform PdF on $[0, 2\pi]$
Verify WSS (wide sense stationarity)

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Ans
- $R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$

$$= \int_0^{2\pi} \frac{A^2}{2\pi} \cos(\omega_0 t + \theta)\cos(\omega_0(t+\tau) + \theta)\, d\theta$$



$$= \frac{A^2}{4\pi} \int_0^{2\pi} [\cos(2\omega t + 2\theta + \omega\tau) + \cos(\omega\tau)] \, d\theta$$

$$= \frac{A^2}{4\pi} \left[\frac{\sin(2\omega t + 2\theta + \omega\tau)}{2} \Bigg|_0^{2\pi} + \cos(\omega\tau)\, \theta \Bigg|_0^{2\pi} \right]$$

$$= \frac{A^2}{4\pi} \times 2\pi \cos\omega\tau = \frac{A^2}{2} \cos\omega\tau \rightarrow \text{Function of } \tau$$

$\Rightarrow$ 2nd condition satisfied.

- $E[X(t_1)] = \frac{1}{2\pi} \int_0^{2\pi} A\cos(\omega t_1 + \theta)\, d\theta$

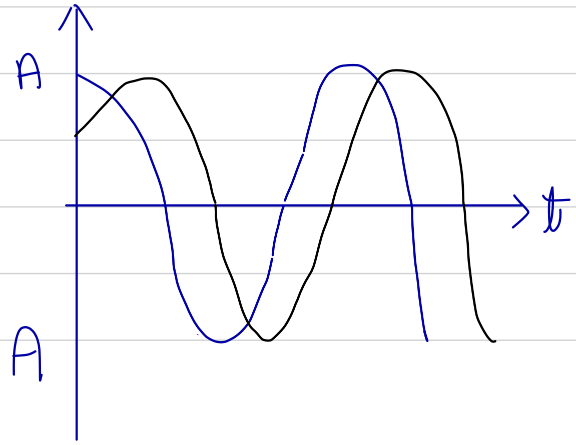
$$= \frac{A}{2\pi} \sin(\omega t_1 + \theta) \Big]_0^{2\pi} = 0$$

|||ly $E[X(t_2)] = 0$

$\Rightarrow E[X(t)] =$ constant. $\therefore$ 1st condition is verified.

$\therefore$ The given example satisfies WSS.

$x(t) = A\cos(\omega_0 t + \theta)$



— $A_0 \cos\omega_0 t$

— $x(t)$

2 Random process:

$X(t)$ & $Y(t)$

Crosscorrelation, $R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$

Joint WSS

$\Rightarrow R_{XY}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only

$= R_{XY}(\lambda)$

N^{th} order process & strict sense stationarity

$X(t)$ @ $t_1 \quad t_2 \quad \ldots \quad t_N$

$$\begin{array}{ccc} t_1 & t_2 & \ldots & t_N \\ \downarrow & \downarrow & & \downarrow \\ x_1 & x_2 & & x_N \end{array}$$

$$f_X(x_1, x_2 \ldots x_n;\, t_1\, t_2 \ldots t_N) = f_X(x_1\, x_2 \ldots x_N;\, t_1+\tau,\, t_2+\tau \ldots t_N+\tau)$$

Stationarity for $K \leq N \Rightarrow SSS$

Ergodic process

$$E[X(t) \cdot X(t+\tau)] \triangleq R_{XX}(\tau)$$

Average (A): $\quad A[x(t)] = \lim\limits_{T \to \infty} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} x(t)\, dt = \bar{x}$

Time average, $E[x(t)] = \bar{x}$

- Time average is calculated over one waveform

Time autocorrelation function $= R_{xx}(t, \tau + t)$

$= A[x(t), x(t+\tau)]$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_i(t)\, x_i(t+\tau)\, dt$$

Time average = Statistical average.
(when variance of $\bar{x} = 0$)

Erogodic process $\Longleftrightarrow$ (time average = statical average)

<u>NOTE</u>: Gaussian Random process satisfies strict sense stationarity and ergodicity.

Autocovariance

$$C_{xx}(t_1, t_2) = E[(x(t_1) - \underbrace{\mu_x(t_1)}_{K_1})(x(t_2) - \underbrace{\mu_x(t_2)}_{K_2})]$$

$= E[(x(t_1) \cdot x(t_2))] - K_2 E[x(t_1)] - K_1 E[x(t_2)] + K_1 K_2$

$= R_{xx}(t_1, t_2) - K_2 K_1 - K_1 K_2 + K_1 K_2$

$= R_{xx}(t_1 t_2) - \mu_x(t_1)\, \mu_x(t_2)$

If $X(t_1)$ & $X(t_2)$ are independent,
$C_{XX} = 0$
But if $C_{XX} = 0$ $X(t_1)$ & $X(t_2)$ need not be independent.

Crosscovariance, $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$

If $X(t_1)$ & $Y(t_2)$ are orthogonal, $R_{XY}(t_1, t_2) = 0$

Properties of autocorrelation function of real valued wide sense stationary random process

$$R_{XX}(t_1, t_2) = \text{function of } (t_2 - t_1)$$
$$\triangleq E[X(t_1)\,X(t_1+\tau)] = R_{XX}(\tau)$$

Properties

1> $R_{XX}(0) = E[X^2(t)] \geq 0$
2> $R_{XX}(-\tau) = R_{XX}(\tau)$ $\Rightarrow$ R_{XX} is even function
3> $|R_{XX}(\tau)| \leq R_{XX}(0)$

Proof: Consider $E[\{X(t+\tau) - X(t)\}^2] \geq 0$

$$= E[X^2(t+\tau)] + E[X^2(t)] - 2E[X(t+\tau)\cdot X(t)] \geq 0$$

$$= 2E[X^2(t)] - 2E[X(t+\tau)\,X(t)] \geq 0$$

$$\Rightarrow 2R_{XX}(0) - 2R_{XX}(\tau) \geq 0$$

$$\Rightarrow R_{XX}(0) \geq R_{XX}(\tau)$$

$$\therefore |R_{XX}(\tau)| \leq R_{XX}(0)$$

Properties of Crosscorrelation of WSS RP

$X(t)$ & $Y(t)$

$$\Rightarrow R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

let $t_1 = t$

$t_2 = t + \tau$

$$= E[X(t)\, Y(t+\tau)]$$

$=$ function of τ

1> $R_{XY}(\tau) = R_{XY}(-\tau)$

2> $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)\, R_{YY}(0)} \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$

Eg: Given: $X(t) = K \cos \omega t$, $t \geq 0$, $\omega \rightarrow$ constant, $K \rightarrow$ uniformly distributed RV on $[0, 2]$. Find $E[X(t)]$ & Autocorrelation.

Ans: $E[X(t)] = \int_0^2 \frac{1}{2}\, k \cos \omega t \, dk = \frac{\cos \omega t}{2} \left.\frac{k^2}{2}\right]_0^2$

$= \cos \omega t$

$$R_{XX}(t, t+\tau) = E[X(t)\cdot X(t+\tau)]$$

$$= \int_0^2 \frac{1}{4} k^2 \cos(\omega t)\cdot\cos(\omega(t+\tau))\, dk$$

$$= \frac{1}{4}\frac{k^3}{3} \cos(\omega t)\cdot \cos(\omega(t+\tau))\Big]_0^2$$

$$= \frac{2}{3}\left[\cos(\omega t + \omega t + \omega\tau) + \cos(\omega t + \omega\tau - \omega\tau)\right] \times \frac{1}{2}$$

$$= \frac{1}{3}\left[\cos(2\omega t + \tau) + \cos(\omega\tau)\right]$$

Eg2> A set of $2n$ RVs A_i & B_i, $i = 1, 2, \ldots n$ uncorrelated & Joint Gaussian distribution having $E[A_i] = E[B_i] = 0$ & $E[A_i^2] = E[B_i^2] = \sigma^2$ and let $X(t) = \sum_{i=1}^{n} A_i \cos\omega_i t + B_i \sin\omega_i t$.

verify strict sense stationarity.

Ans $$E[X(t)] = \sum_{i=1}^{n}\{E[A_i]\cos\omega_i t + E[B_i]\sin\omega_i t\}$$

$$= 0$$

$$R_{XX}(t, t+\tau) =$$

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i\cos\omega_i t + B_i \sin\omega_i t)(A_j\cos\omega_j(t+\tau)) + B_j \sin\omega_j(t+\tau))\right]$$

Only $i=j$ $E \neq 0$

$$E\Big[\sum_{i=1}^{n} A_i^2 \cos(\omega_i t)\cos(\omega_i(t+\tau)) + \underbrace{A_i B_i \cos\omega_i t \sin\omega_i(t+\tau)}_{(0)} +$$

$$\underbrace{A_i B_i \cos(\omega_i(t+\tau))\sin\omega_i t}_{(0)} + B_i^2 \sin\omega_i t \sin\omega_i(t+\tau)\Big]$$

$\therefore \; R_{XX}(t, t+\tau) = \sum_{i=1}^{n} \sigma^2 \cos(\omega_i t - \omega_i t - \omega_i \tau)$

$$= \sum_{i=1}^{n} \sigma^2 \cos \omega_i \tau \;//$$

$$= R_{XX}(\tau)$$

$\therefore$ WSS is verified

This can be extended to $X_k(t)$. $\therefore$ SSS is verified.

Power spectral density (PSD) $(S_{XX}(f)$ or $S_{XX}(\omega))$

If $X(t)$ is WSSRP,

$$\mathcal{F}[R_{XX}(\tau)] = PSD \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$\therefore \; R_{XX}(\tau) = \mathcal{F}^{-1}(PSD) = \mathcal{F}^{-1}(S_{XX}(f))$$

$$= \int_{-\infty}^{\infty} S_{XX}(f)\, e^{j2\pi f\tau}\, df$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

Eg 1) Estimate $R_{XX}(\tau)$ for

$$S_{XX}(\omega) = \begin{cases} S_0 & |\omega| \leq \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Ans $R_{XX}(\tau) = \mathcal{F}^{-1}\left(S_{XX}(\omega)\right)$

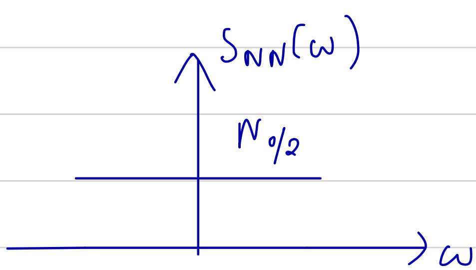
$$= \frac{1}{2\pi} \int_{-\omega_0}^{\omega} S_0 \, e^{j\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} S_0 \left. \frac{e^{j\omega\tau}}{j\tau} \right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{2\pi j\tau} S_0 \left(e^{j\omega_0\tau} - e^{-j\omega_0\tau}\right)$$

$$= \frac{S_0}{\pi\tau} \sin \omega_0 \tau \,//$$

Eg 2)



Find $R_{NN}(\tau)$

Ans $R_{NN}(\tau) = \frac{N_0}{2} \delta(\tau)$