ANALOG AND DIGITAL COMMUNICATION

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EC202

Module: 1

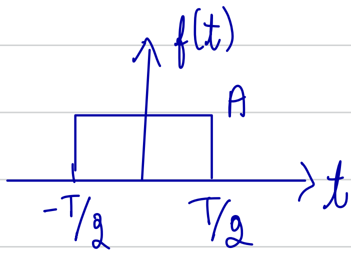
FOURIER TRANSFORM

$$\boxed{\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F(\omega)}$$

where $\mathcal{F}$ is the fourier operator.

- Also $$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega}$$

Example: compute fourier transform;



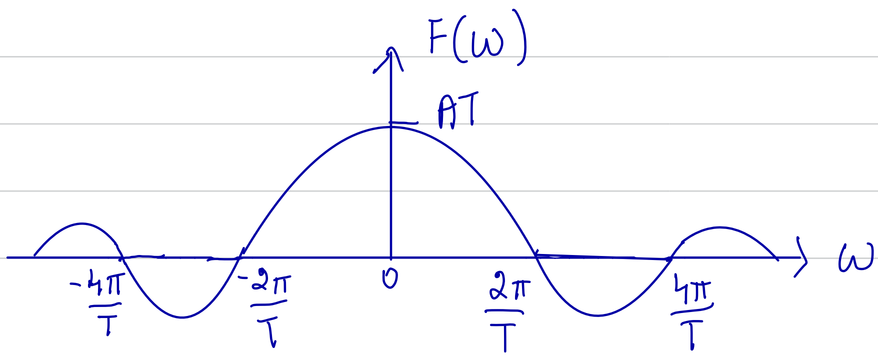
$\rightarrow$ $$F(\omega) = \int_{-T/2}^{T/2} A\, e^{-j\omega t}\, dt$$

$$= -\frac{A}{j\omega}\, e^{-j\omega t}\Big]_{-T/2}^{T/2}$$

$$= \frac{A}{j\omega}\left[e^{j\omega T/2} - e^{-j\omega T/2}\right] = \frac{2A}{\omega}\sin\left(\frac{\omega T}{2}\right)$$

$$= \frac{2A}{\omega T/2} \times \frac{T}{2}\sin\left(\frac{\omega T}{2}\right) = AT\, Sa\left(\frac{\omega T}{2}\right)$$

$F(\omega) =$



Ex 2: If $f(t) \longleftrightarrow F(\omega)$, express fourier transform of $f(t-t_0)$

Ans $\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-j\omega t}\, dt$

let $t - t_0 = x$ then $dt = dx$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-j\omega(x+t_0)}\, dx$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx = e^{-j\omega t_0} F(\omega) //$$

Ex 3: If $f(t) \longleftrightarrow F(\omega)$, express fourier transform of $e^{j\omega_0 t} f(t)$

Ans $\mathcal{F}[f(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)\, e^{-jt(\omega-\omega_0)}\, dt$

$$= F(\omega - \omega_0) //$$

Ex 4: If $f(t) \longleftrightarrow F(\omega)$, express fourier transform of $f(t)\cos(\omega_0 t)$, $f(t)\sin\omega_0 t$

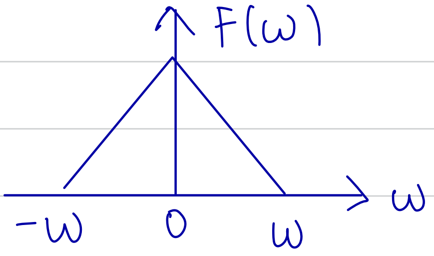
Ans $\mathcal{F}[f(t)\cos(\omega_0 t)] = \int_{-\infty}^{\infty} f(t)\cos(\omega_0 t)\, e^{-j\omega t}\, dt$

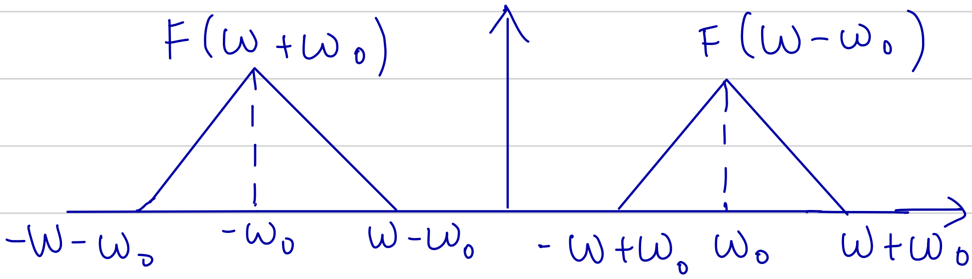
$$= \int_{-\infty}^{\infty} f(t) \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{f(t)}{2} \left(e^{-jt(\omega-\omega_0)} + e^{-jt(\omega+\omega_0)} \right) dt$$

$$= \frac{F(\omega+\omega_0) + F(\omega-\omega_0)}{2} //$$

$F(\omega) =$





$$\mathcal{F}\left(f(t)\sin\omega_0 t\right) = \int_{-\infty}^{\infty} f(t)\sin\omega_0 t \; dt \; e^{-j\omega t}$$

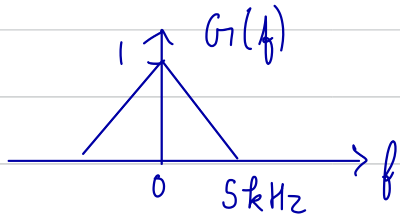
$$= \int_{-\infty}^{\infty} f(t)\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\frac{e^{-j(\omega-\omega_0)t} - e^{-j(\omega+\omega_0)t}}{2j}\right) dt$$

$$= \frac{F(\omega-\omega_0) - F(\omega+\omega_0)}{2j}$$

//

Ex5: Draw the fourier transform of $g(t)\cos 2\pi(50k)t$ if $G(f)$ is

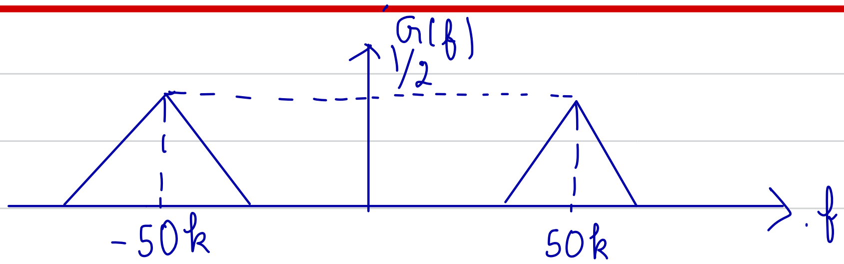


Ans $\mathcal{F}\left(g(t)\cos 2\pi(50k)t\right)$

$$= \frac{G(\omega+\omega_0) + G(\omega-\omega_0)}{2}$$

$$= \frac{G(2\pi(50k+5k)) + G(2\pi(50k-5k))}{2}$$

$$= \frac{G(2\pi(55k)) + G(2\pi(45k))}{2}$$



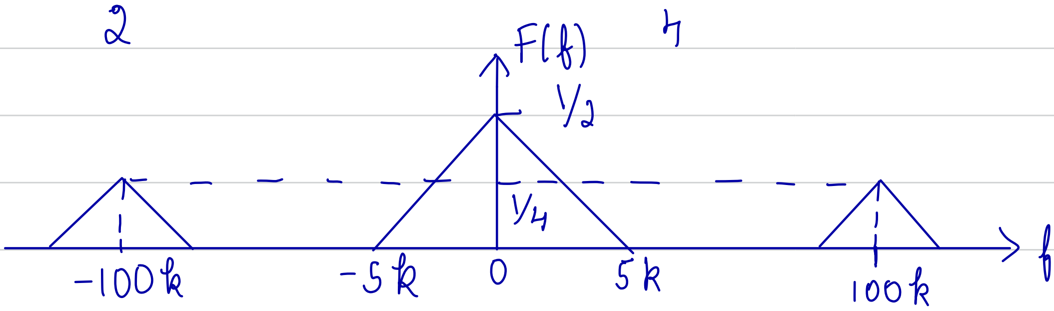
$g(t)\cos^2\omega_0 t$

$$\mathcal{F}(g(t)\cos^2\omega_0 t) = \int_{-\infty}^{\infty} g(t)\left(\frac{1+\cos 2\omega_0 t}{2}\right)e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}(g(t) + g(t)\cos 2\omega_0 t)\,e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\left[F(\omega) + \frac{F(\omega+2\omega_0)+F(\omega-2\omega_0)}{2}\right]$$

$$= \frac{F(\omega)}{2} + \frac{F(\omega+2\omega_0)+F(\omega-2\omega_0)}{4}$$



Properties of Fourier transform

1) Time differentiation : If $f(t) \longleftrightarrow F(\omega)$

then $\frac{d}{dt}(f(t)) \longleftrightarrow (j\omega)\,F(\omega)$

Proof: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$

$$\frac{d(f(t))}{dt} = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

$$\Rightarrow \frac{d(f(t))}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}\left(F(\omega)\,e^{j\omega t}\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, F(\omega)\, e^{j\omega t}\, d\omega$$

$$= j\omega \times \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

2) Frequency differentiation

If $f(t) \longleftrightarrow F(\omega)$ so $-jt\, f(t) \longleftrightarrow \frac{dF}{d\omega}$

Proof: $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

$$\frac{dF}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\omega}\left(f(t)\, e^{-j\omega t}\, dt\right)$$

$$= -jt\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

3) Time integration If $f(t) \longleftrightarrow F(\omega)$

$$\int_{-\infty}^{\tau} f(t)\, dt \longleftrightarrow \frac{1}{j\omega} F(\omega)$$

Let $\int_{-\infty}^{\tau} f(t)\, dt = \phi(t)$, $\Rightarrow \mathcal{F}\left(f(t)\right) = \mathcal{F}\left(\frac{d}{dt}\phi(t)\right)$

$F(\omega) = j\omega\, \phi(\omega)$

$\phi(\omega) = \frac{F(\omega)}{j\omega}$ $\Rightarrow \int_{-\infty}^{\tau} f(t)\, dt = \frac{F(\omega)}{j\omega}$

4) <u>Convolution:</u> $f_1(t) * f_2(t) \longleftrightarrow F_1(\omega) \cdot F_2(\omega)$

if $f_1(t) \longleftrightarrow F_1(\omega)$ & $f_2(t) \longleftrightarrow F_2(\omega)$.

Energy density spectrum

Energy of a signal is defined as energy dissipated by a voltage drop $f(t)$ applied across a $1\,k\Omega$ resistor. (Normalised energy)

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt$$

- The signal for which energy E is finite are known as energy signals.
- If the interval is infinite, it can we can compute average energy of the signal (average power). Such signals are known as power signals.

We have $E = \int_{-\infty}^{\infty} f^2(t)\, dt$

$$= \int_{-\infty}^{\infty} f(t) \cdot f(t)\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \right\} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left\{ \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot F(-\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |F(\omega)|^2 \, df$$

We have $F(\omega) \cdot F(-\omega) = |F(\omega)|^2$

We have $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

$$F^*(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt = F(-\omega)$$

$$\therefore \; F(\omega) \cdot F(-\omega) = F(\omega) \cdot F^*(\omega) = |F(\omega)|^2 \,//$$

Q1) Find $\mathcal{F}(\delta(t))$

Ans $f(t) = \delta(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega(0)} = 1 \,//$$

Q2) Find the inverse fourier i.e $\mathcal{F}^{-1}(2\pi\, \delta(\omega-\omega_0))$

Ans $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\, \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$

$$= \int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$= e^{j\omega_0 t} \,//$$

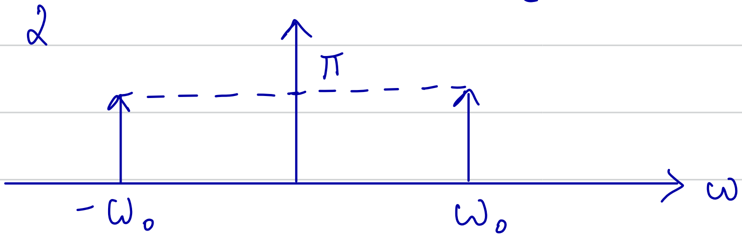
Q3) Find $\mathcal{F}(\cos\omega_0 t)$

Ans $F(\omega) = \int_{-\infty}^{\infty} \cos\omega_0 t\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\mathcal{F}(e^{j\omega_0 t}) + \frac{1}{2}\mathcal{F}(e^{-j\omega_0 t}) \,//$$

$$= \frac{2\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))}{2} = \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$$



Properties of Fourier transform

1) if $f(t) \longrightarrow F(\omega)$

then $f(-t) \longrightarrow 2\pi f(-\omega)$

Proof: We have $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$

let $x = \omega$, then $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)\, e^{jxt}\, dx$

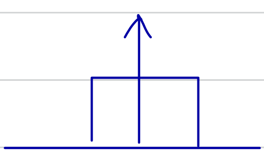
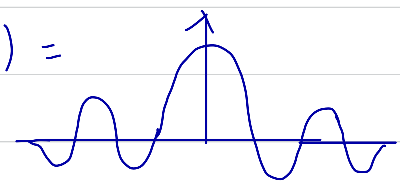
put $t = \omega$

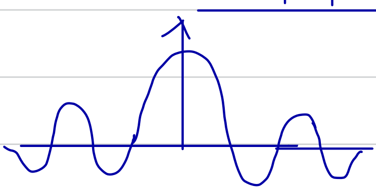
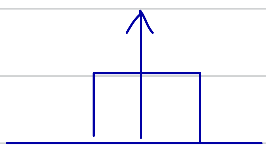
$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)\, e^{jx\omega}\, dx$$

Put $x = t$,

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{j\omega t}\, dt$$

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt$$

NOTE: If $f(t) =$  then $F(\omega) =$ 

If $f(t) =$  then $F(\omega) =$ 

2>

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \right\} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \left(\int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-j\omega t}\, dt \right) d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \cdot F_2(\omega)\, e^{-j\omega\tau}\, d\tau$$

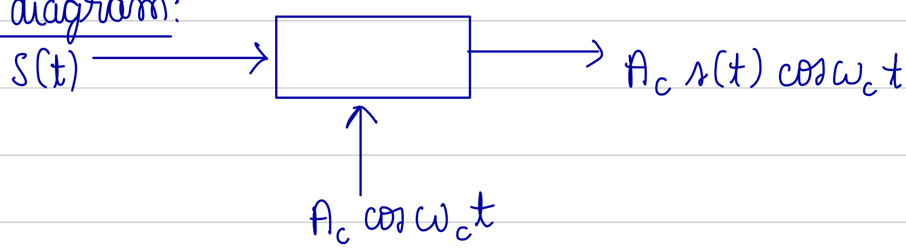
$$= F_2(\omega) \cdot \int_{-\infty}^{\infty} f_1(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= F_2(\omega) \cdot F_1(\omega)$$

AMPLITUDE MODULATION

$$x_{DSB}(t) = s(t) \cdot c(t)$$
$$= s(t) \cdot A_c \cos \omega_c t = A_c s(t) \cos \omega_c t$$

Block diagram:



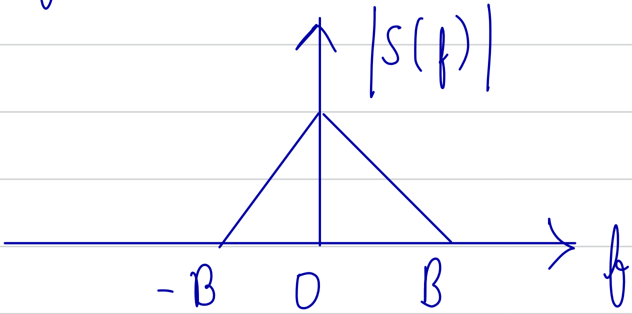
DSB = Double side band

Spectrum: $s(t) c(t) \longleftrightarrow S(f) * C(f)$

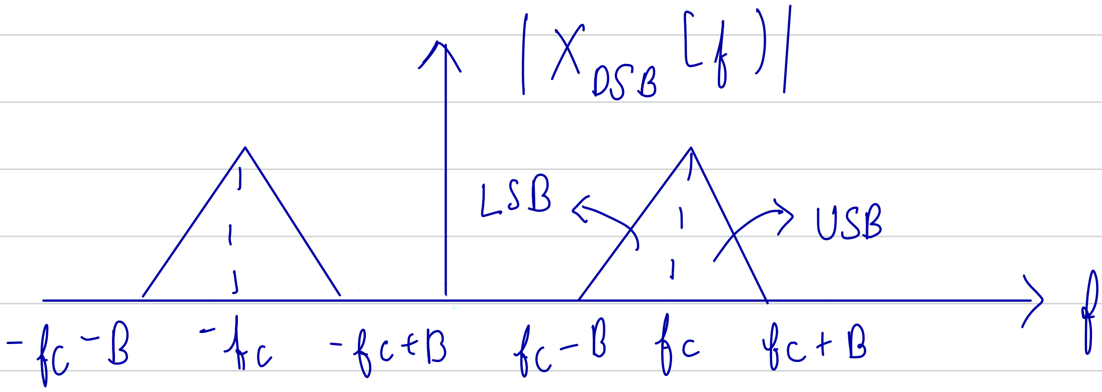
$$S(\omega) * C(\omega)$$
$$\longleftrightarrow S(\omega) * [\pi \{\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\}]$$

$$\therefore s(t) c(t) = \pi [S(\omega - \omega_c) + S(\omega + \omega_c)]$$

Q1> $S(f)$ is given. Find $S(f) * C(f)$ i.e $|X_{DSB}(f)|$



Ans



Demodulation

If we ignore noise and attenuation

$$r(t) = x_{DSB}(t) = A_c\, s(t) \cos\omega_c t$$

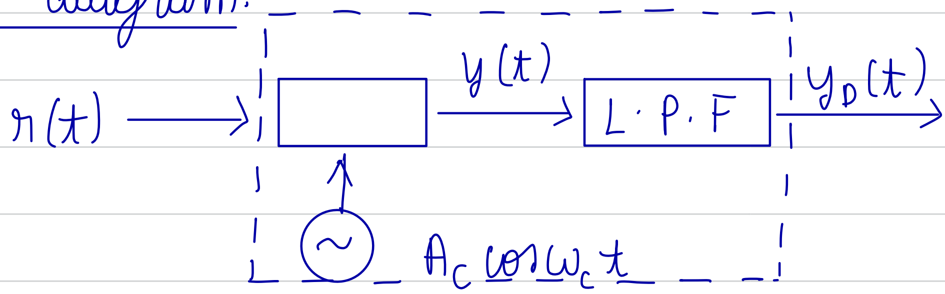
$$y(t) = r(t)\cdot c(t) = [A_c\, s(t) \cos\omega_c t]\, A_c \cos\omega_c t$$

$$= A_c^2\, s(t) \cos^2\omega_c t$$

$$= \frac{A_c^2}{2} s(t) + \frac{A_c^2}{2} s(t) \cos 2\omega_c t$$

The 2nd term can be filtered out by passing signal through low pass filter.

Block diagram:



$$\frac{A_c^2}{2} s(t)$$

L·P·F = Low pass filter.

If the carrier at the reciever has a frequency and phase deviation then

$$y(t) = r(t)c(t) = [A_c\, s(t) \cos\omega_c t]\, A_c \cos[(\omega_c + \Delta\omega)t + \phi]$$

$$= A_c^2\, s(t) \cos\omega_c t \cos[(\omega_c + \Delta\omega)t + \phi]$$

$$= \frac{A_c^2}{2} s(t) \left[\cos\{(2\omega_c + \Delta\omega)t + \phi\} + \cos\{\Delta\omega t + \phi\}\right]$$

The term with $2\omega_c$ will be eliminated by the filter. So output of demodulator will be

$$y(t) = \frac{A_c^2}{2} s(t) \cos(\Delta\omega t + \phi)$$

This is called coherent demodulation.

Coherent demodulation: A carrier which is frequency and phase synchronised at local oscillator at transmitter then frequency difference is zero.

Power, $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2_{DSB}(t)\, dt$

$= \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [A_c\, s(t) \cos \omega_c t]^2\, dt$

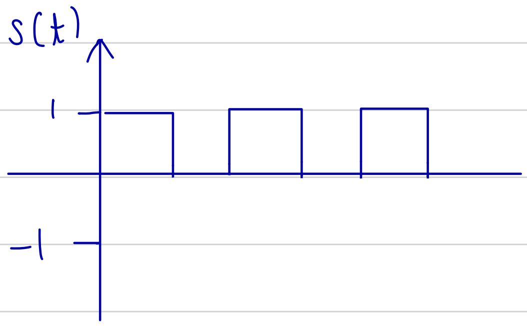
$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A_c^2\, s^2(t) \cos^2 \omega_c t\, dt$

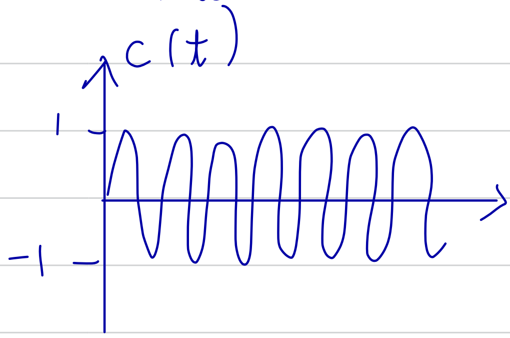
$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A_c^2\, s^2(t) \left[\frac{1 + \cos 2\omega_c t}{2}\right] dt$

$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A_c^2\, s^2(t)}{2}\, dt$ //

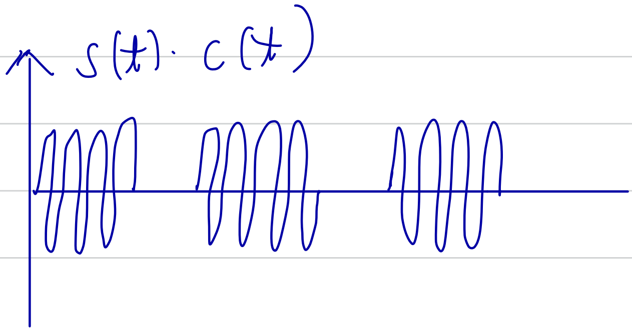
Q> $s(t)$ & $c(t)$ are shown. Draw $s(t) \cdot c(t)$

is





Ans



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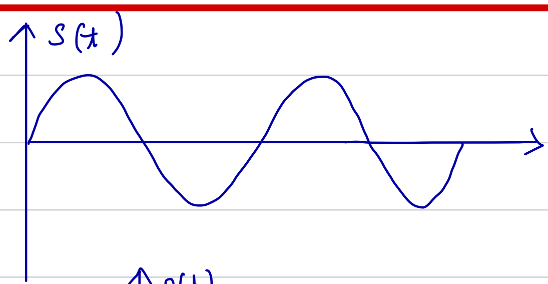
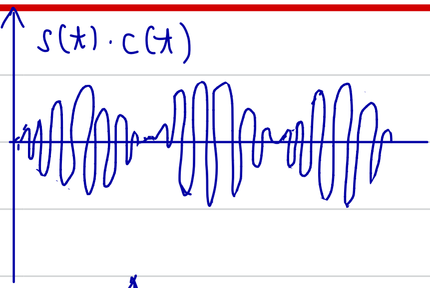


Ans:



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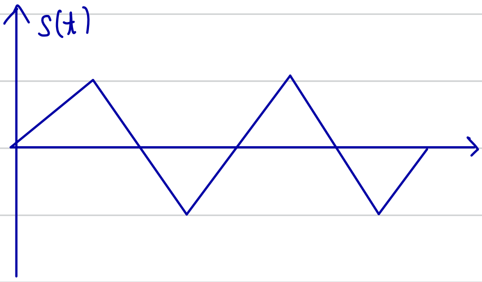
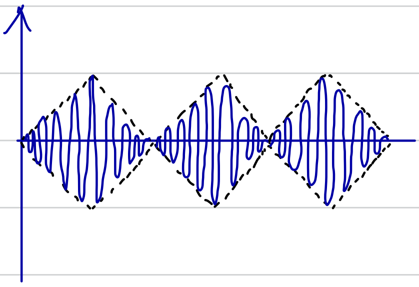
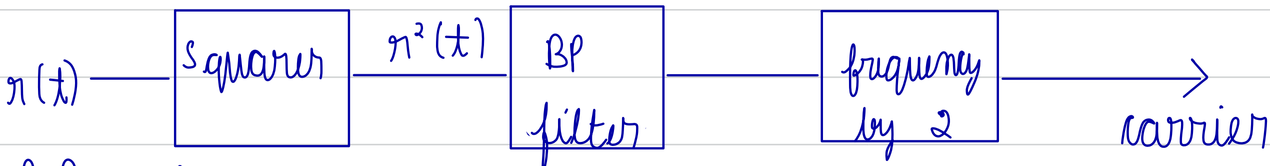


Ans:



Recovery of carrier



B.P → Band pass filter

Envelope function:

$$x(t) = A\cos\omega_c t + s(t)\cos\omega_c t$$

$$= A\left[1 + \frac{|\min s(t)|}{A_c} \cdot \frac{s(t)}{|\min s(t)|}\right]\cos\omega_c t$$

let $\dfrac{s(t)}{\min s(t)} = s_n(t)$ $\quad\quad s_n(t) \geq -1$

$\dfrac{|\min s(t)|}{A_c} = m_a$ $\quad\quad x_{am}(t) = A_c\left[1 + m_a s_n(t)\right]\cos\omega_c t$

Q If $s(t) = A_m\cos\omega_m t$ $\quad s_n(t) = ?$ $\quad m_a = ?$

Ans $s_n(t) = \dfrac{s(t)}{|\min s(t)|} = \dfrac{A_m\cos\omega_m t}{+A_m} = \cos\omega_m t$

$m_a = \dfrac{|\min s(t)|}{A_c} = \dfrac{A_m}{A_c}$

Q> Find min and max value of envelop.

Ans $x(t) = A_c[1 + m_a s_n(t)]$

$x(t)_{min} = A_c[1 - m_a]$

$x(t)_{max} = A_c[1 + m_a \cdot max(s_n(t))]$ //

NOTE: $m_a = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$

Q> If the modulated signal is $s(t) = A_m \cos\omega_m t$. what is $s_n(t)$, m_a and write expression $x_{AM}(t)$

Ans $s_n(t) = \cos\omega_m t$ $\quad m_a = \frac{A_m}{A_c}$

$x_{am}(t) = A_c[1 + m_a \cos\omega_m t] \cos\omega_c t$

Q> Find A_{max}, A_{min}:

Ans 1> $m_a = 1 = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$

$A_{max} + A_{min} = A_{max} - A_{min}$

$\Rightarrow A_{min} = 0$

Amplitude of $x_{am} = A_c[1 + m_a \cos\omega_m t]$

$A_{max} = 2A_c$ //

2> $m_a = 0.5$

$A_{max} = A_c[1 + 0.5] = 1.5A_c$ //

$0.5 = \frac{1.5A_c - A_{min}}{1.5A_c + A_{min}}$

$0.75A_c + 0.5A_{min} = 1.5A_c - A_{min}$

$A_{min} = 0.5A_c$ //

3) $m_a = 0.25$

$A_{max} = A_c[1+0.25] = 1.25 A_c$

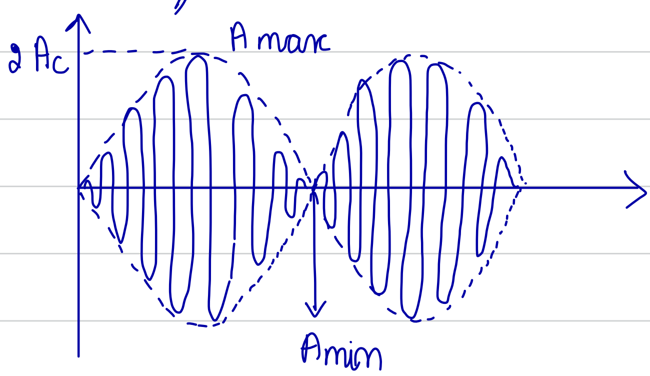
$0.25 = \dfrac{1.25 A_c - A_{min}}{1.25 A_c + A_{min}}$

$0.3125 A_c + 0.25 A_{min} = 1.25 A_c - A_{min}$

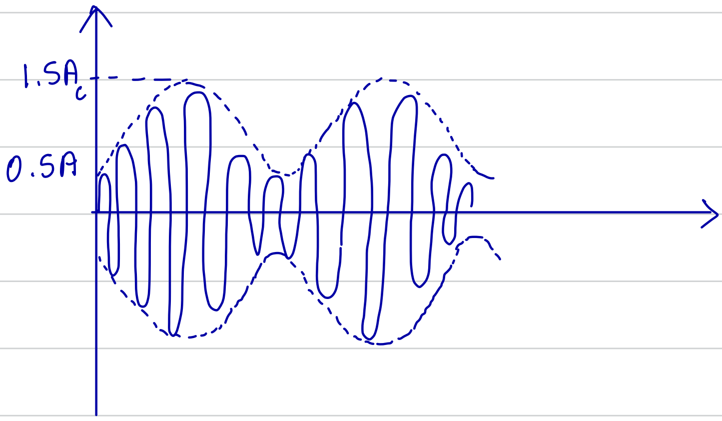
$1.25 A_{min} = 0.9375 A_c$

$\therefore A_{min} = 0.75 A_c$

Graphs: 1)



2)



Shape of the envelop will follow the shape of message signal as long as $A_{min} \geq 0$

i.e $A_{min} \geq 0$ $\quad A_c[1 - m_a] \geq 0$

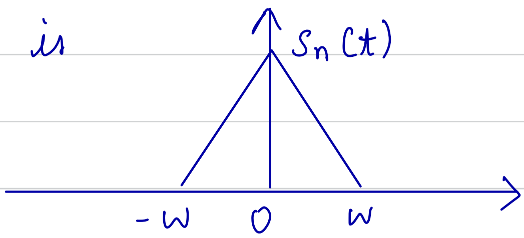
Find pictorial representation of x_{am}

$x_{am}(t) = A_c[1 + m_a s_n(t)] \cos \omega_c t$

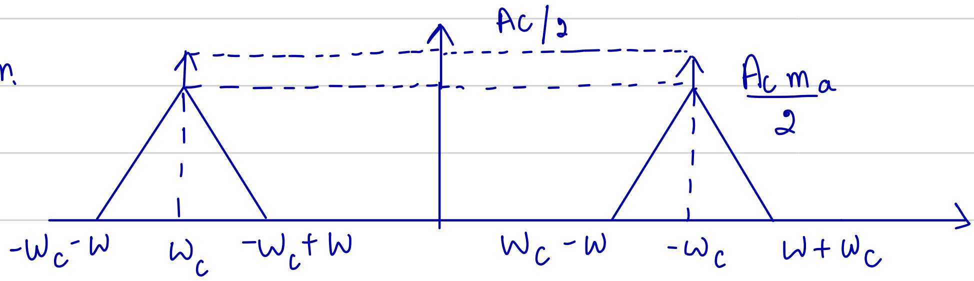
$= A_c \cos \omega_c t + A_c m_a s_n(t) \cos \omega_c t$

$x_{am}(\omega) = \dfrac{(\delta(\omega - \omega_c) + \delta(\omega + \omega_c))A_c}{2} + \dfrac{A_c m_a}{2}(S_n(\omega - \omega_c) + S_n(\omega + \omega_c))$

If S_n is



then



Power content of AM signal

We have $x_{AM}(t) = A_c[1+m_a s_n(t)]\cos\omega_c t$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2_{AM}(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A_c^2[1+m_a s_n(t)]^2\cos^2\omega_c t\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A_c^2[1+m_a s_n(t)]^2\left(\frac{1+\cos 2\omega_c t}{2}\right)dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A_c^2[1+m_a s_n(t)]^2\,\frac{\cos 2\omega_c t}{2}\,dt = 0$

$$\therefore P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A_c^2}{2}[1+m_a s_n(t)]^2\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A_c^2}{2}[1+2m_a s_n(t) + m_a^2 s_n^2(t)]\,dt$$

But $\int_{-T/2}^{T/2} 2m_a s_n(t) = 0$ (Assuming $s_n(t)$ is sinusoidal)

$\therefore P_x = \frac{A_c^2}{2} + \frac{A_c^2 m_a^2 P_s}{2}$ where $P_s = \lim_{T\to\infty}\int_{-T/2}^{T/2} s_n^2(t)\,dt$

Power / Modulation / transmission efficiency

$$\eta = \frac{\text{Sideband power}}{\text{Total power}} = \frac{\left(A_c^2/2\right) m_a^2 P_s}{\left(\frac{A_c^2}{2}\right)\left(1 + m_a^2 P_s\right)} = \frac{m_a^2 P_s}{1 + m_a^2 P_s}$$

If $S_n(t)$ is sinusoidal $P_{S_n} = 1/2$

and if $m_a = 1$

$$\eta = \frac{1/2}{3/2} = \frac{1}{3}$$

If $S_n(t)$ is a square wave, $P_{S_n} = 1$

& if $m_a = 1$, $\eta = \frac{1}{2}$

Q) For an AM waveform $A_{max} = 3$, & $A_{min} = 1$. Find the modulation index & power efficiency assuming a sine wave.

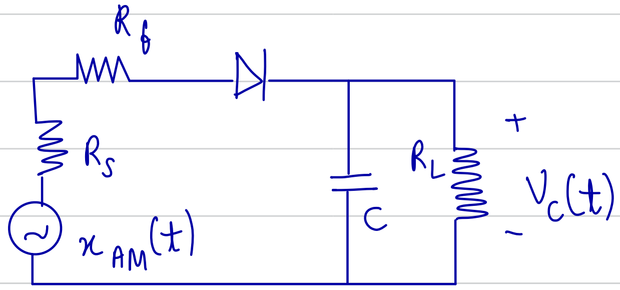
Ans $m_a = \frac{A_{max} - A_{min}}{A_{max} + A_{min}} = \frac{2}{4} = 0.5$

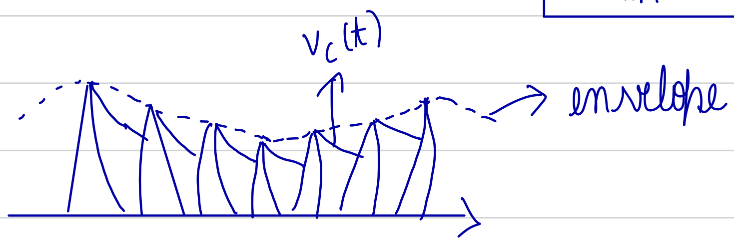
$$\eta = \frac{m_a^2 P_s}{1 + m_a^2 P_s} = \frac{(1/4)(1/2)}{1 + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)} = \frac{1}{9}$$

Q) Show that conventional AM can be demodulated coherently

Ans $x_{AM}(t) \cdot \cos\omega_c t$ → This can be passed through L.P.F directly to demodulate.

Envelope detector





- During the positive half cycle of the modulated carrier, the diode is forward biased and the capacitor charges.
- When the input voltage falls below capacitor voltage the diode is cutoff and the capacitor discharges through the resistor

- Inorder to make capacitor not to discharge rapidly,

1) $T_c <<< R_L C$ where T_c is time period of carrier wave.

2) Capacitor voltage should also follow the envelope, so $R_L C << T_B$ where T_B is the time period of highest frequency component of baseband signal.

$$\Rightarrow \quad \frac{1}{f_c} << R_L C << \frac{1}{f_B}$$

- Time constant should be small such that capacitor is able to follow changes in envelope.
- In ideal case charging time constant is zero.
- In actual case, $(R_s + R_f) \cdot C$

- Whenever input voltage exceeds capacitor voltage, diode conducts. Whenever input voltage falls below capacitor voltage, the diode does not conduct.
- Output of envelope detector has a triangular waveform superimposed, which can be eliminated using a low pass filter.

$$x_{AM}(t) \longrightarrow \boxed{\text{Envelope detector}} \xrightarrow{V_c(t)} \boxed{\text{L.P.F}} \longrightarrow V_D(t)$$

$V_D(t) \to$ detector output.

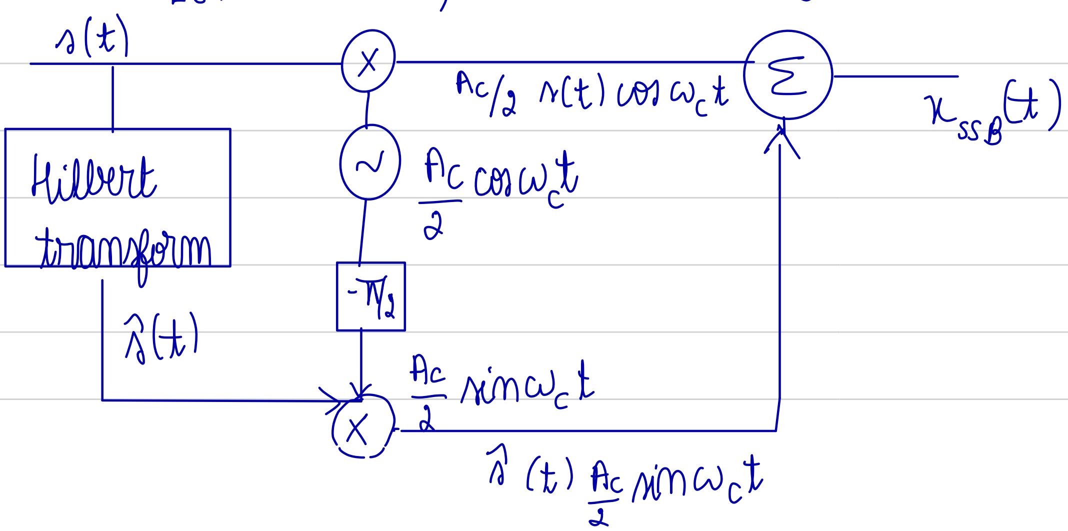
SSB (Single side band) Amplitude modulation

Methods to generate SSB

1> filtering method → convenient if baseband signal does not have spectrum content near zero frequency.
2> Phasing method.

$$x_{SSB}(t) = \frac{A_c}{2}\left[s(t)\cos\omega_c t - \hat{s}(t)\sin\omega_c t\right]$$

$\hat{s}(t) \to$ Hilbert transform of $s(t)$
Hilbert transform → Phase shift of $s(t)$ by $\pm 90°$.
$+90° \to$ LSB , $-90° \to$ USB.



Power content of SSB - AM

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2_{SSB}(t)\, dt$$

$$x^2_{SSB}(t) = \frac{A_c^2}{4}\left[s^2(t)\cos^2\omega_c t + \hat{s}^2(t)\sin^2\omega_c t + 2\, s(t)\,\hat{s}(t)\cos\omega_c t \sin\omega_c t \right]$$

$$\Rightarrow x^2_{SSB}(t) = \frac{A_c^2}{4}\left[\left(\frac{1+\cos 2\omega_c t}{2}\right)s^2(t) + \hat{s}^2(t)\left(\frac{1-\cos 2\omega_c t}{2}\right) + s(t)\,\hat{s}(t)\sin 2\omega_c t\right]$$

$$\therefore P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A_c^2}{4}\left[\frac{s^2(t)}{2} + \frac{\hat{s}^2(t)}{2} + 0\right]^2$$

$$= \frac{A_c^2}{8}\left[P_s + P_{\hat{s}}\right]$$

$$P_x = \frac{A_c^2}{4} P_s$$

Coherent demodulation of SSB signal

$$y(t) = x_{SSB}(t)\, 2\cos(\omega_c t + \phi)$$

$$= \frac{A_c}{2}\left[s(t)\cos\omega_c t - \hat{s}(t)\sin\omega_c t\right] 2\cos\omega_c t$$

$$= A_c\left[s(t)\cos\omega_c t\cos(\omega_c t+\phi) - \hat{s}(t)\sin\omega_c t\cos(\omega_c t+\phi)\right]$$

$$= \frac{A_c}{2}\left[s(t)\{\underbrace{\cos(2\omega_c t+\phi)}_{\times\ \text{LPF}} + (\cos\phi)\} - \frac{\hat{s}(t)}{2}\{\underbrace{\sin(2\omega_c t+\phi)}_{\times\ \text{LPF}} + \sin\phi\}\right]$$

$$= \frac{A_c}{2}\left[s(t)\cos\phi + \hat{s}(t)\sin\phi\right]$$

attenuation distortion

Envelope detection

$$x_c(t) = x_{SSB}(t) + k\cos\omega_c t \qquad [k = \text{carrier amplitude}]$$

$$= \frac{A_c}{2}\left[s(t)\cos\omega_c t + \hat{s}(t)\sin\omega_c t\right] + k\cos\omega_c t$$

$$= \left[\frac{A_c}{2}s(t) + k\right]\cos\omega_c t + \frac{A_c}{2}\hat{s}(t)\sin\omega_c t$$

$$= e(t)\cos(\omega_c t + \psi)$$

where

$$e(t) = \sqrt{\left(\frac{A_c}{2}s(t) + k\right)^2 + \left(\frac{A_c\hat{s}(t)}{2}\right)^2}$$

$$\psi = \tan^{-1}\left(\frac{(A_c/2)\,\hat{s}(t)}{\left(\frac{A_c}{2}\right)s(t) + k}\right)$$

$$e(t) = \sqrt{\frac{A_c^2}{2}s^2(t) + 2\frac{A_c}{2}s(t)k + k^2 + \frac{A_c^2}{4}\hat{s}^2(t)}$$

if $k \gg \frac{A_c}{2}s(t)$ $\qquad k \gg \frac{A_c}{2}\hat{s}(t)$

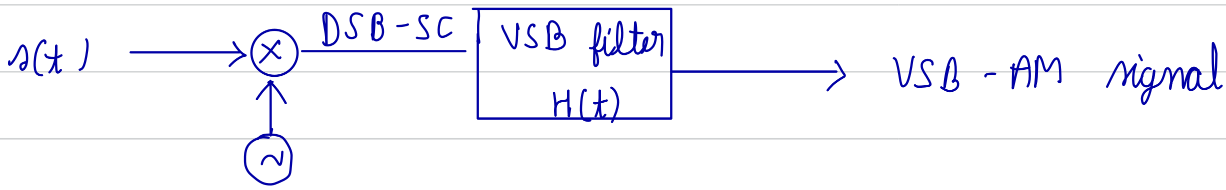
$$e(t) = \sqrt{A_c s(t) k + k^2}$$

$$= k\sqrt{1 + \frac{A_c s(t)}{k}}$$

$$= k\left(1 + \frac{1}{2}A_c\frac{s(t)}{k}\right)$$

$$= k + \frac{A_c}{2}s(t)$$

Vestigial sideband AM (VSB-AM)



$$C(t) = A_c \cos \omega_c t$$

$$x_{VSB} = x_{DSB}(t) \times h(t) \qquad (h(t) = \text{impulse response of filter})$$
$$= A_c\, s(t) \cos \omega_c t \otimes h(t)$$

In coherent demodulation output of the multiplier is

$$x_{VSB}(t) = x_{DSB}(f) \cdot H(f)$$

$$= \frac{A_c}{2}\left[S(f+f_c) + S(f-f_c)\right] H(f)$$

$$= \frac{A_c}{2}\left[\left(S(f) + S(f-2f_c)\right) H(f-f_c)\right] + \frac{A_c}{2}\left[\left(S(f+2f_c) + S(f)\right) H(f+f_c)\right]$$

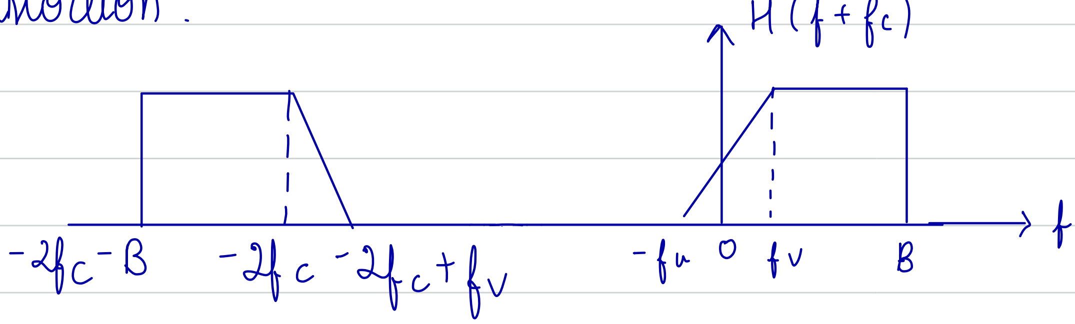
$$= \frac{A_c}{2} S(f)\left[H(f-f_c) + H(f+f_c)\right] + \frac{A_c}{2} S(f+2f_c)\left[H(f-f_c) + H(f+f_c)\right]$$

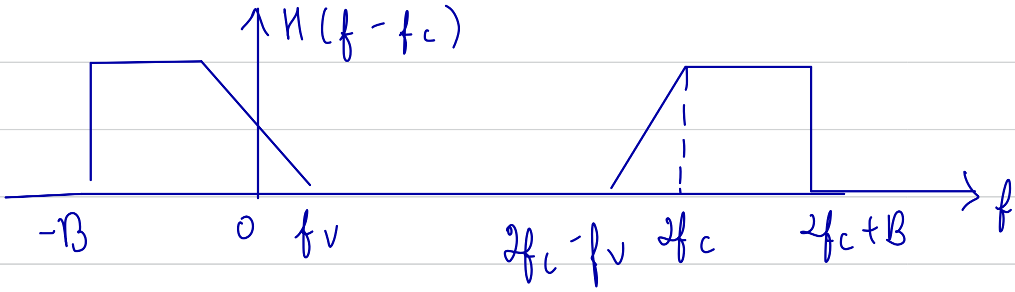
$\times$ (LPF)

$\therefore$ output of LPF $= \frac{A_c}{2} S(f)\left[H(f-f_c) + H(f+f_c)\right]$

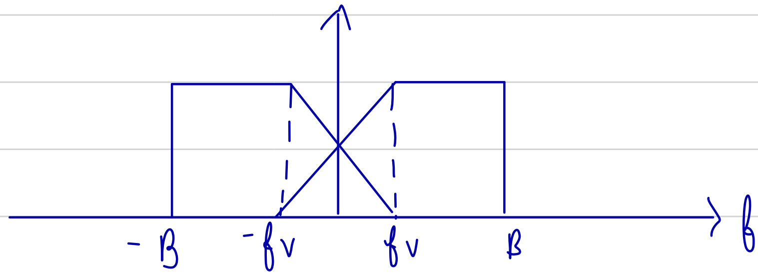
If $\left[H(f-f_c) + H(f+f_c)\right] = C \qquad f \leq$ Bandwidth

then message can be recovered without distortion.





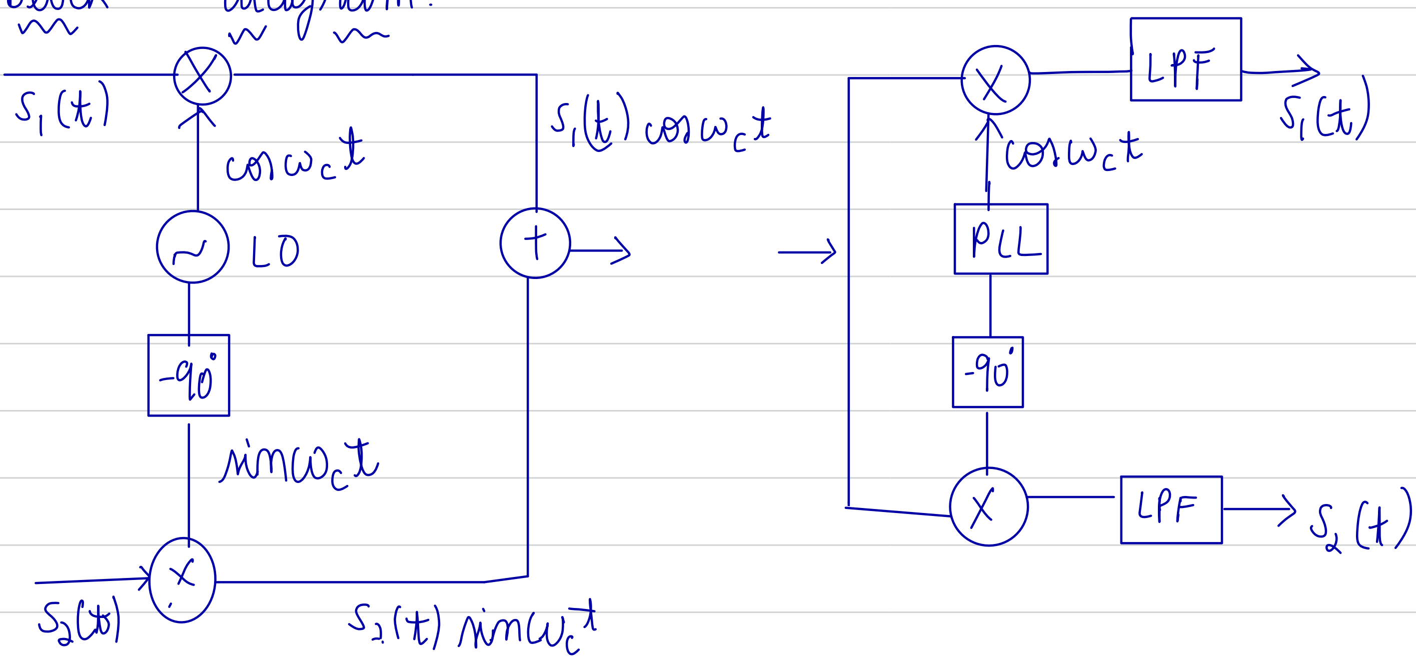
$H(f+f_c) + H(f-f_c)$



A carrier can be added to VSB signal so that demodulation can be using envelope detection.

Quadrature Multiplexing

Block diagram:



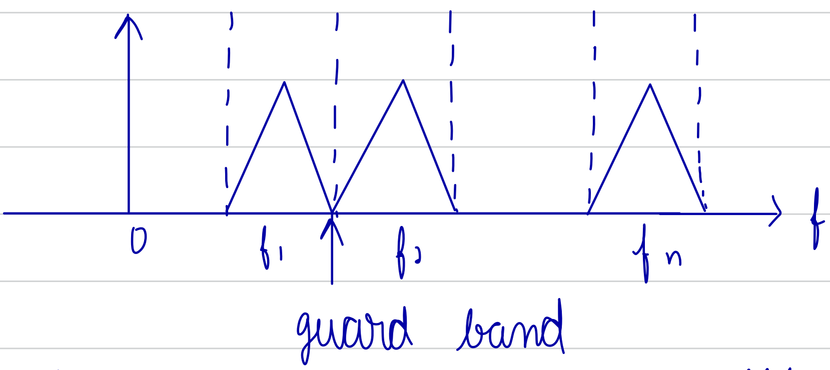
LO → Local oscillator.
PLL → Phase locked loop

$$x_{QAM}(t) = s_1(t)\cos\omega_c t + s_2(t)\sin\omega_c t$$

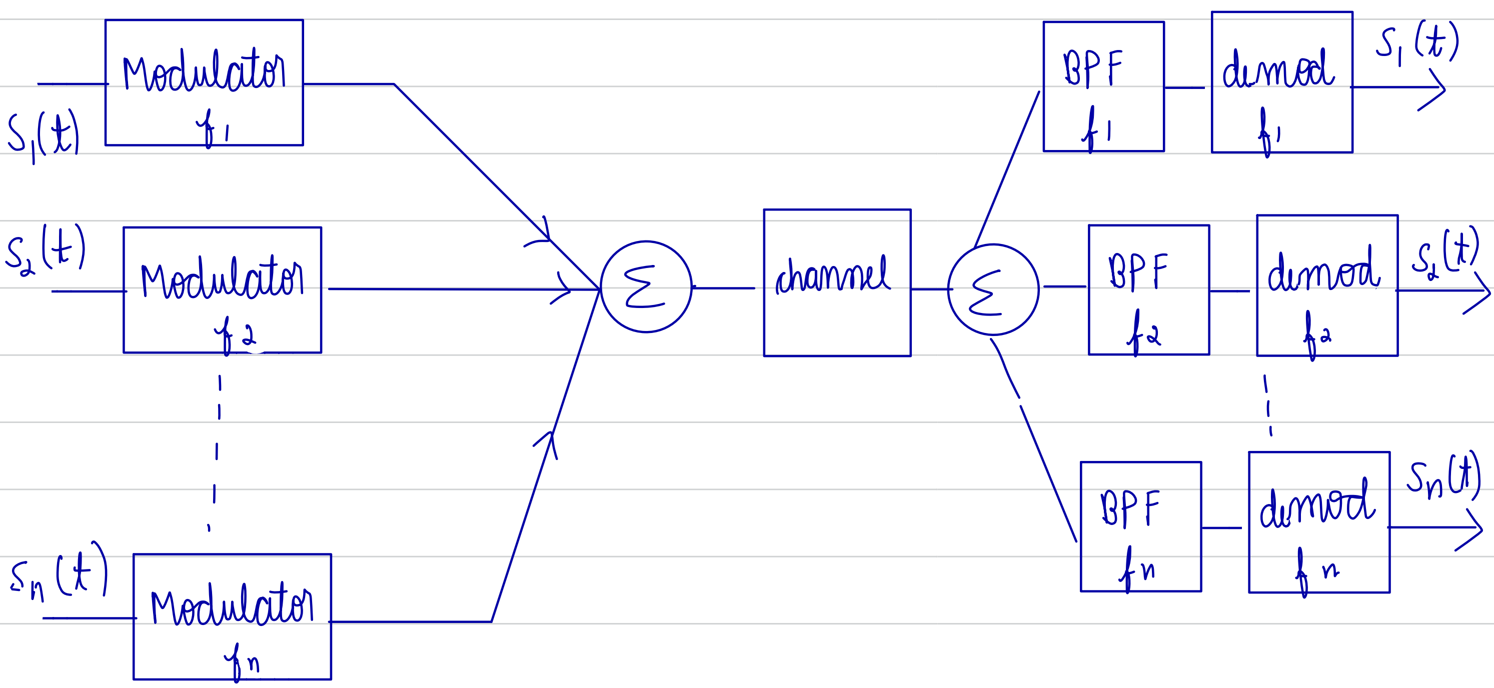
$$x_{QAM}(t)\cos\omega_c t = s_1(t)\cos^2\omega_c t + s_2(t)\sin\omega_c t\cos\omega_c t$$
$$= \frac{s_1(t)}{2}[1+\cos 2\omega_c t] + \frac{s_2(t)}{2}\sin 2\omega_c t$$

In case of QAM 2 DSB-SC signals share the same bandwidth - 2B. Thus it has the same bandwidth efficiency as that of SSB-SC.

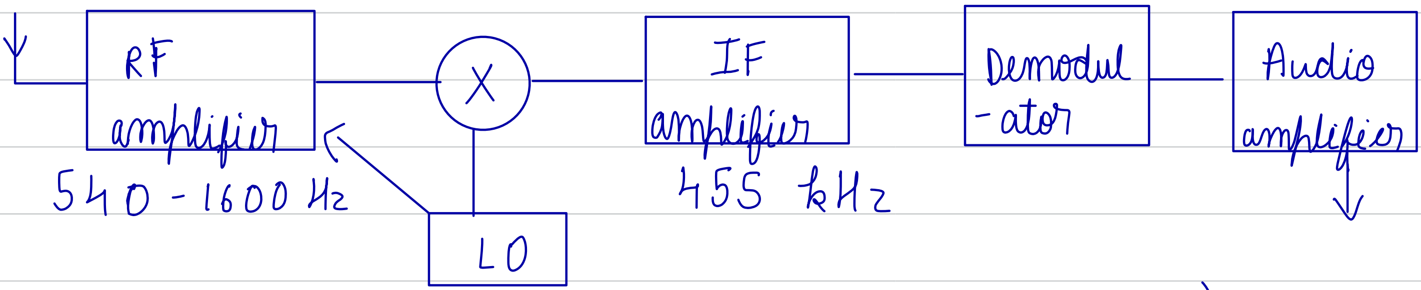
Multiplexing (FDM - frequency division multiplexing)



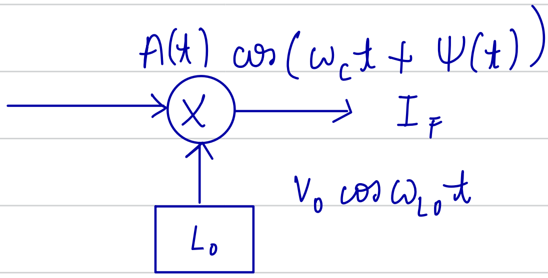
Spectrum of frequency multiplexed signal



Super heterodyne reciever



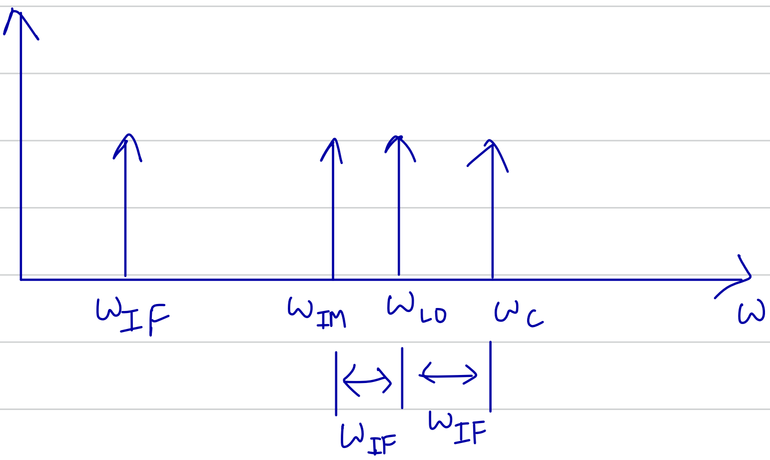
Down conversion mixer :



$$y(t) = x(t)\, V_0 \cos \omega_{L0} t$$

$$= V_0 A(t) \cos[\omega_c t + \psi(t)] \cdot \cos \omega_{L0} t$$

$$= \frac{V_0 A(t)}{2} \Big[\cos[(\omega_c + \omega_{L0})t + \psi(t)] + \cos[(\omega_c - \omega_L)t + \psi(t)] \Big]$$



$$\omega_{LO} = \omega_C - \omega_{IF}$$

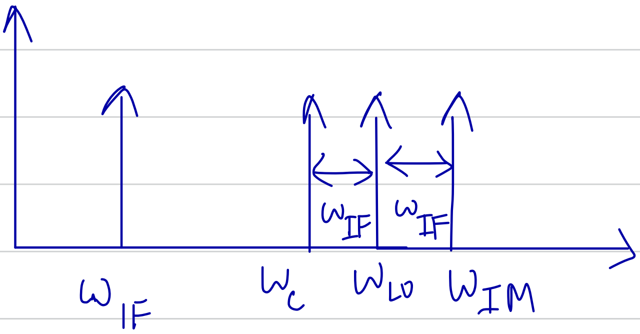
Low side injection

$$x_1(t) = A_1(t) \cos((\omega_c - 2\omega_{IF})t + \psi_1(t))$$

$$y_1(t) = V_0 A(t) \cos \omega_{L0} t \cdot \cos[(\omega_c - 2\omega_{IF})t + \psi_1(t)]$$

$$= V_0 A(t) \cos[(\omega_{L0} + \omega_c - 2\omega_{IF})t + \psi_1(t)] +$$

$$\cos[\{\omega_{L0} - (\omega_c - 2\omega_{IF})\}t + \psi_1(t)]$$



High side injection

$\omega_{LO} = \omega_c + \omega_{IF}$

The frequency band 535 kHz to 1605 kHz is allocated for AM broadcast.

The carrier frequency range for the broadcast station is 540 kHz to 1600 kHz

Q) Find the tuning range for local oscillator assuming a) High side injection
b) Low side injection.

Assume IF = 455 kHz

Ans For High side injection, $\omega_c + \omega_{IF} = \omega_{LO}$

Range (455 + 540) to (455 + 1600)

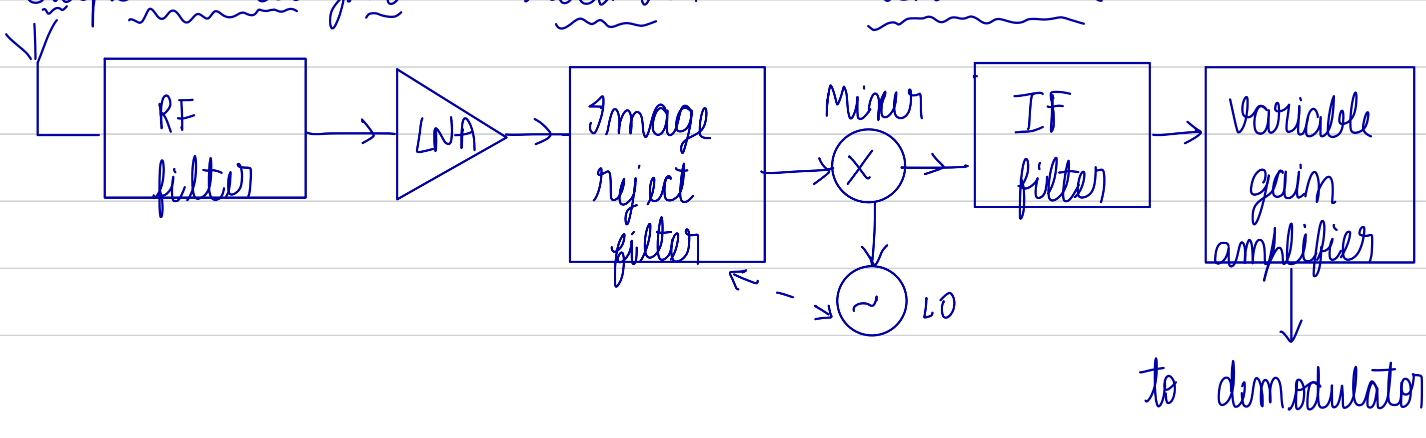
= 995 kHz to 2055 kHz

For low side injection $\omega_c - \omega_{IF} = \omega_{LO}$

Range (540 - 455) to (1600 - 455)

= 85 kHz to 1145 kHz

Superheterodyne reciever architecture



Amplitude - Phase representation

$$x(t) = A(t)\cos[\omega_c t + \Psi(t)] \rightarrow (1)$$

$$= A(t)\cos\omega_c t \cos\Psi(t) - A(t)\sin(\Psi(t))\sin(\omega_c t)$$

$$= \boxed{I(t)\cos\omega_c t - Q(t)\sin\omega_c t} \rightarrow \text{Quadrature representation} \rightarrow (2)$$

$$A(t) = \sqrt{I^2(t) + Q^2(t)}$$

$$\Psi(t) = -\tan^{-1}\left(\frac{Q(t)}{I(t)}\right)$$

Also $x(t)$ can be written as:

$$x(t) = Re\{A(t)\, e^{j(\omega t + \Psi(t))}\}$$

$$= Re\{A(t)\, e^{j\Psi(t)}\, e^{j\omega t}\}$$

$$= Re\{\tilde{x}(t)\, e^{j\omega t}\}$$

where $\tilde{x}(t) = A(t)e^{j\Psi(t)} = A(t)\cos\Psi(t) + jA(t)\sin\Psi(t)$

$$= I(t) + jQ(t)$$

let $x^+(t) = \tilde{x}(t)\, e^{j\omega t} \rightarrow (4)$

$$= Re\{\tilde{x}(t)\, e^{j\omega t}\} + j\{\tilde{x}(t)\, e^{j\omega t}\}$$

$x^+(t) = x(t) + j\hat{x}(t)$ $\quad$ ($\hat{x}(t) \rightarrow$ Hilbert transform of $x(t)$)

$x^+(t)$ is analytic signal

From (4)

$$x^+(t) = [I(t) + jQ(t)][\cos\omega_c t + j\sin\omega_c t]$$

$$x^+(t) = [I(t)\cos\omega_c t - Q(t)\sin\omega_c t]$$

$$+ j[I(t)\sin\omega_c t + Q(t)\cos\omega_c t]$$

$$I(t) = Re\{\tilde{x}(t)\}$$

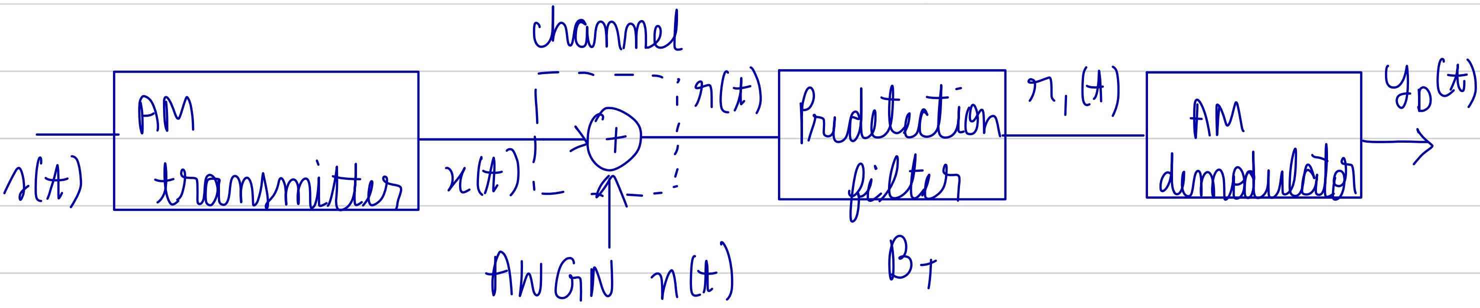
$$= Re\{x^+(t)\, e^{-j\omega t}\} \rightarrow \text{from } (4)$$

$$= Re\{[x(t) + j\,\hat{x}(t)][\cos\omega_c t - j\sin\omega_c t]\}$$

$$= x(t)\cos\omega_c t + \hat{x}(t)\sin\omega_c t$$

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

EFFECT OF NOISE ON THE PERFORMANCE OF AM SYSTEMS



AWGN → Added white gaussian noise
B_T → transmission bandwidth

Assumptions:

1) channel is distortionless
2) $x(t)$ & $n(t)$ are statistically independent
3) Message signal $s(t)$ is stationary. Zero mean, band limited to B Hz
4) Transmission bandwidth is B_T
5) The 2 sided power spectra density of noise is $G_{n_i}(f) = \frac{N_0}{2}$

Noise performance of DSB-SC

We have $x_{DSB}(t) = A_c\, s(t) \cos(\omega_c t + \phi)$

$$r(t) = \alpha\, x_{DSB}(t) + n_i(t)$$
$$= \alpha A_c\, s(t) \cos(\omega_c t + \phi) + n_i(t)$$

Signal power at the input of predetection filter

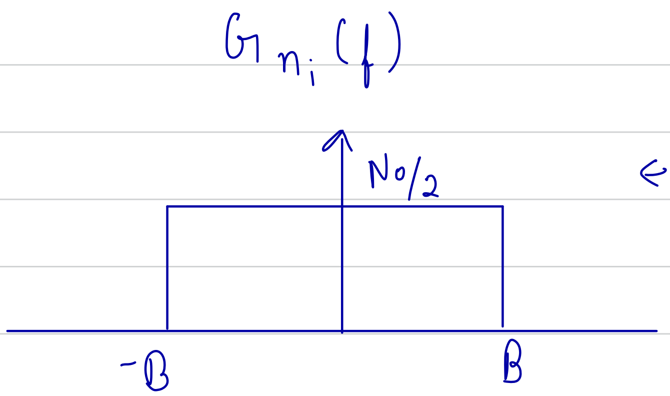
$$P_R = E\{(\alpha\, x_{DSB}(t))^2\}$$

$$= E\{\alpha^2 A_c^2 s^2(t) \cos^2(\omega_c t + \psi)\}$$

$$= \frac{1}{2}\alpha^2 A_c^2 \overline{s^2}(t) \quad \text{where} \quad \overline{s^2}(t) = E\{s^2(t)\}$$

Noise power measured in bandwidth of baseband

$$P_{n_i} = \frac{N_0}{2} \times 2B = N_0 B$$



← Power spectral density of n_i

Carrier power at the reciever input

Input noise power in the message signal bandwidth

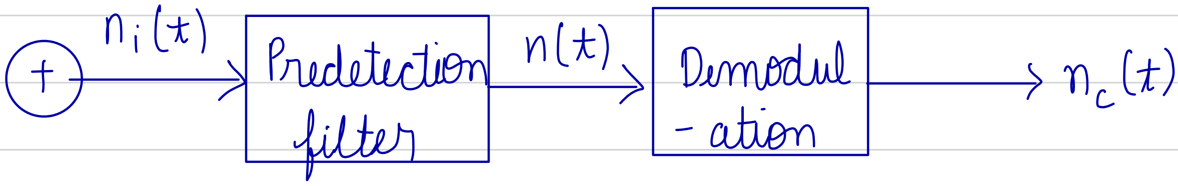
$$CNR_{in} = \frac{P_R}{P_{n_i}} = \frac{(1/2)\,\alpha^2 A^2 \overline{s^2(t)}}{BN_0}$$

CNR - Carrier noise ratio

At the output of predetection filter,

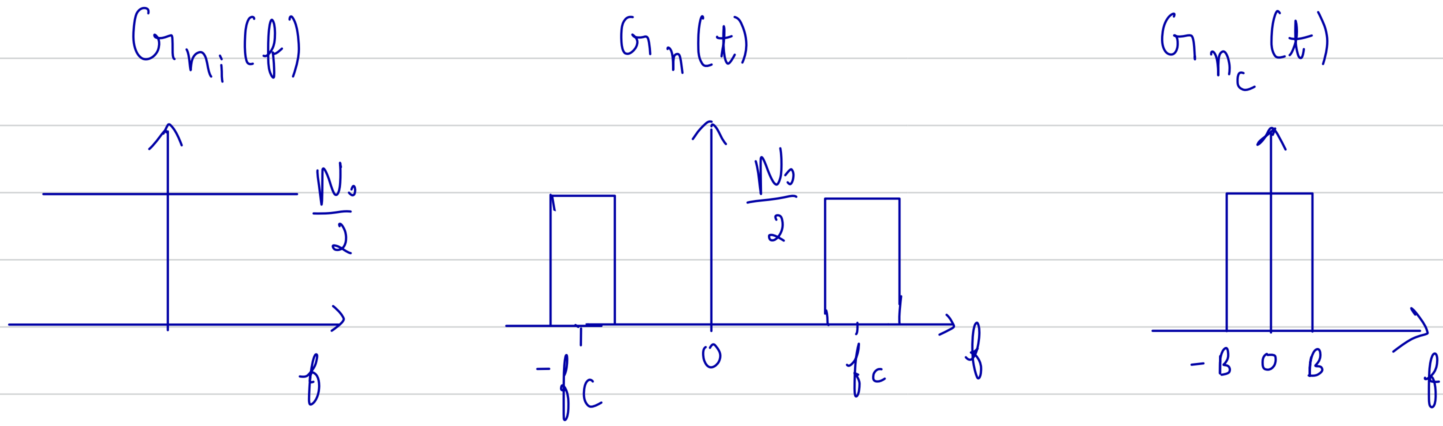
$$r_1(t) = \alpha A_c\, s(t) \cos(\omega_c t + \phi) + n(t)$$

$$= \underbrace{\alpha A_c\, s(t) \cos(\omega_c t + \phi)}_{2N_0 B} + n_c(t)\cos(\omega_c t + \phi) - n_i \sin(\omega_c t + \phi)$$

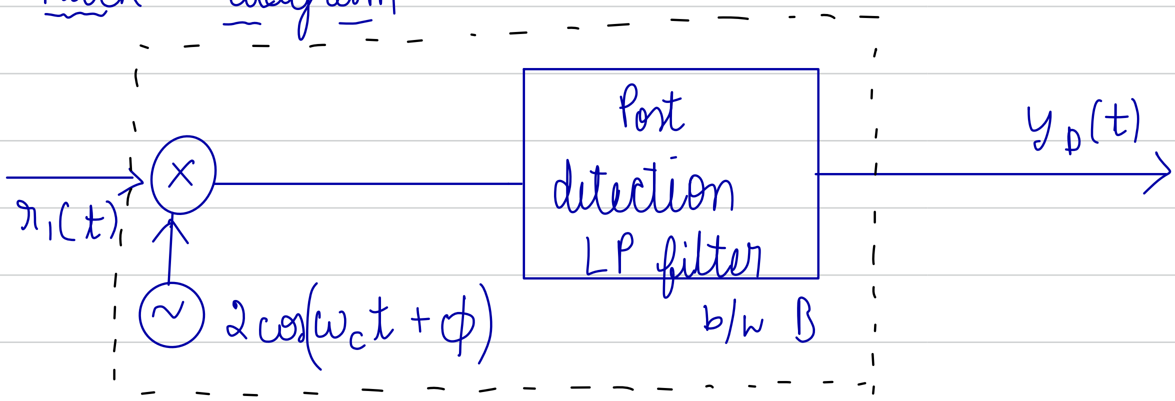


$n_i(t) \Rightarrow$ input noise

$n(t) \Rightarrow$ output noise



Block diagram



$$y(t) = r_1(t)\, 2\cos(\omega_c t + \phi)$$

$$= 2\alpha A_c\, s(t)\cos^2(\omega_c t + \phi) + 2n_c(t)\cos^2(\omega_c t + \phi) -$$

$$2n_s(t)\sin(\omega_c t + \phi)\cos(\omega_c t + \phi)$$

$$= \alpha A_c\, s(t)\left[1 + \cos(2\omega_c t + 2\phi)\right] + n_c(t)\left[1 + \cos(2\omega_c t + 2\phi)\right]$$

$$- n_s(t)\sin(2\omega_c t + 2\phi)$$

$$y_D(t) = \alpha A_c s(t) + n_c(t)$$

Signal modulator and demodulator output

$$P_o = E\left[\left(\alpha A_c s^2(t)\right)\right] = \alpha^2 A_c^2 \overline{s^2}$$

$$P_{n_c} = N_o \times 2B = 2N_o B$$

$$SNR = \frac{P_o}{P_{n_c}} = \frac{\alpha^2 A_c^2 \overline{s}^2}{2N_o B}$$

$$\frac{(SNR)_{DSB}}{CNR_{in}} = \frac{\alpha^2 A_c^2 \overline{s}^2}{2N_o B} \times \frac{N_o B}{\frac{1}{2}\alpha^2 A_c^2 \overline{s}^2} = 1$$

Noise performance of SSB

$$x_{SSB}(t) = \frac{A_c}{2}\left[s(t)\cos(\omega_c t + \phi) - \hat{s}(t)\sin(\omega_c t + \phi)\right]$$

$$r(t) = \alpha x_{SSB}(t) + n_i(t)$$

Pre detection filter input

$$P_R = E\left[\left(\alpha x_{SSB}(t)\right)^2\right]$$

$$= \frac{\alpha^2 A_c^2}{4} E\left[\left(s(t)\cos(\omega_c t+\phi) - \hat{s}(t)\sin(\omega_c t+\phi)\right)^2\right]$$

$$= \frac{\alpha^2 A_c^2}{4} E\left[s^2(t)\cos^2(\omega_c t+\phi) + \hat{s}^2(t)\sin^2(\omega_c t+\phi) - 2s(t)\hat{s}(t)\cos(\omega_c t+\phi)\sin(\omega_c(t)+\phi)\right]$$

Since ϕ is a uniformly distributed random variable in interval $[-\pi, \pi]$

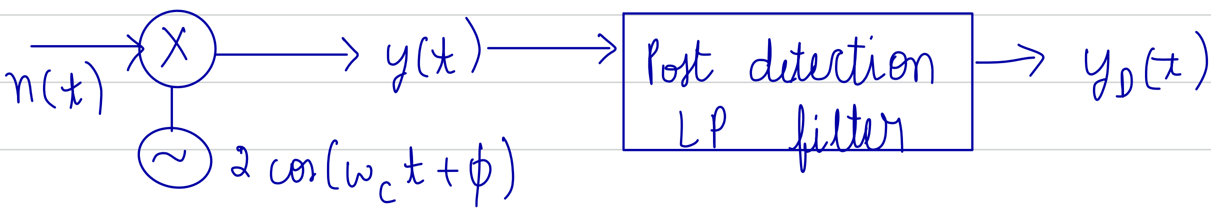
$$P_R = \frac{\alpha^2 A_c^2}{4}\left[E\left\{\frac{s^2(t)}{2}\right\} + E\left\{\frac{\hat{s}^2(t)}{2}\right\} - 0\right]$$

$$= \frac{\alpha^2 A_c^2}{4} \cancel{2}\frac{\overline{s^2}}{\cancel{2}} = \frac{\alpha^2 A_c^2 \overline{s^2}}{4}$$

Predetection filter output

$$r_1(t) = \alpha x_{SSB}(t) + n(t)$$

$n(t) \rightarrow$ narrow band white gaussian noise centred around $f_c + B/2$



Multiplier output: $y(t) = r_1(t) \cdot 2\cos(\omega_c t + \phi)$

$$= \frac{2\alpha A_c}{2}\left[\frac{s(t)}{2}\left[1 + \cos(2\omega_c t + 2\phi)\right] - \frac{\hat{s}(t)}{2}\cdot \sin(2\omega_c t + 2\phi)\right]$$

$$+ \frac{n_c}{2}\left[1 + \cos(2\omega_c t + 2\phi)\right] - \frac{n_c}{2}\sin(2\omega_c t + 2\phi)\Big]$$

Through L.P.F

$$y_D(t) = \frac{2A_c}{2}\left[s(t) + n_c(t)\right]$$

$$P_D = E\left\{\frac{\alpha^2 A_c^2}{4}\left[\frac{s^2(t)}{2} + \frac{\hat{s}^2(t)}{2}\right]\right\}$$

$$= \frac{\alpha^2 A_c^2}{4}\,\overline{s^2}(t)$$

$$P_{n_c} = \frac{N_0}{2} \times 2B$$

$$P_R = E\left\{\left(\frac{\alpha A_c\, s(t)}{2}\right)^2\right\}$$

$$= N_0 B$$

$$SNR_{SSB} = \frac{P_D}{P_{n_c}} = \frac{\alpha^2 A_c^2\, \overline{s}^2(t)}{4 N_0 B}$$

$$CNR_{in} = \frac{P_R}{P_{ni}} = \frac{\alpha^2 A_c^2\, \overline{s}^2(t)}{4 N_0 B}$$

$$\therefore \frac{SNR_{SSB}}{CNR_{in}} = 1$$

Noise performance of conventional AM

$x_{AM}(t) = A_c[1 + m_a s_n(t)] \cos(\omega_c t + \phi) \qquad |s_n(t)| \leq 1$

At predetection filter input:

$r(t) = \alpha x_{AM}(t) + n_i(t)$

where $n_i(t)$ is wideband white gaussian noise.

Recieved signal power $= E\{\alpha^2 x_{AM}^2(t)\}$

$P_R = E\{A_c^2[1 + m_a s_n(t)]^2 \cos^2(\omega t + \phi)\}$

$= E\{\alpha^2 A_c^2 [1 + m_a s_n(t)]^2\} \; E\{\cos^2(\omega t + \phi)\}$

$= E\{\alpha^2 A_c^2 [1 + m_a^2 s_n^2 t + 2 m_a s_n(t)]\} \times 1/2$

$= \frac{1}{2} \alpha^2 A_c^2 [1 + m_a^2 \overline{s_n^2}]$

$P_{n_i} = \frac{N_0}{2} \times 2B = N_0 B$

$CNR_{in} = \frac{P_R}{P_{n_i}} = \frac{\alpha^2 A_c^2 [1 + m_a^2 \overline{s_n^2}]}{2 N_0 B}$

Predetection filter output

$r_1(t) = \alpha x_{AM}(t) + n(t)$

where $n(t)$ is narrowband white gaussian noise.

$r_1(t) = \alpha A_c [1 + m_a s_n(t)] \cos(\omega_c t + \phi) + n_c(t) \cos(\omega t + \phi)$
$\qquad - n_s(t) \sin(\omega t + \phi)$

a) Coherent detection

Output of lowpass filter is :

$y_D(t) = 2\cos(\omega_c t + \phi)\, r_1(t)$

$= 2\left[\alpha A_c[1 + m_a s_n(t)]\cos^2(\omega_c t + \phi) + n_c(t)\cos^2(\omega_c t + \phi) - n_s(t)\sin(\omega_c t + \phi)\cos(\omega_c t + \phi)\right]$

On passing through LPF

$y_D(t) = \alpha A_c\left[1 + m_a s_n(t)\right] + n_c(t)$

- Since DC contain no information it is ignored in calculating signal to noise ratio.

Demodulated signal Power

$P_D = E\{\alpha^2 A_c^2 m_a^2 s_n^2(t)\}$

$= \alpha^2 A_c^2 m_a^2 \overline{s}_n^2(t)$

Noise power at output of post detection low pass filter.

$P_{n_i} = N_0(2B) = 2N_0B$

$SNR_{AM} = \frac{P_D}{P_n} = \frac{\alpha^2 A_c^2 m_a^2 \overline{s}_n^2(t)}{2N_0B}$

$\frac{SNR_{AM}}{CNR_{in}} = \frac{m_a^2 \overline{s}_n^2(t)}{1 + m_a^2 \overline{s}_n^2(t)} = \eta$ $\left\{ \begin{array}{l} \overline{s}_n^2(t) \text{ for square wave} = 1 \\ \text{for sine wave} = 1/2 \end{array} \right.$

$\therefore \eta \leq 0.5 \qquad \frac{SNR_{AM}}{CNR_{in}} \leq 0.5$

Envelop detection:

Since phase information is not used in envelop detection, for convenience we let $\phi = 0$. in the expression for AM signal

$$r_1(t) = \alpha A_c[1 + m_a s_n(t)] \cos\omega_c t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

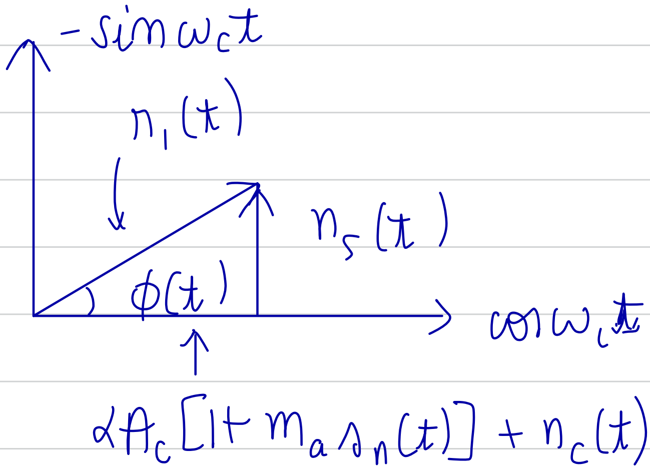
$$|r_1(t)| = \sqrt{(\alpha A_c[1+m_a s_n(t)] + n_c(t))^2 + n_s^2(t)}$$

case 1: Signal component much larger than noise component

$$|r_1(t)| = \alpha A_c[1 + m_a s_n(t)] + n_c(t)$$

eliminating DC, OP of LPF is

$$y_D(t) = \alpha A_c m_a s_n(t) + n_c(t)$$



$$\phi(t) = \tan^{-1}\frac{n_s(t)}{\alpha A_c(1+m_a s_n(t)) + n_c(t)}$$

case 2: Signal component much less than noise component

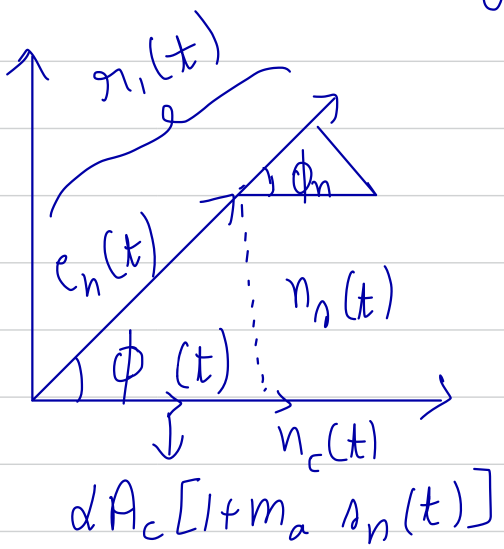
Input of demodulator :

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$
$$= e_n(t)\cos(\omega_c t + \phi_n(t))$$

where $e_n(t) = \sqrt{n_c^2(t) + n_s^2(t)}$, $\phi_n = \tan^{-1}\frac{n_s(t)}{n_c(t)}$

Phasor diagram:

$$\phi(t) = \tan^{-1}\left(\frac{n_s(t)}{\alpha A_c[1+m_a s_n(t)] + n_c(t)}\right)$$



where $\alpha A_c[1+m_a s_n(t)] << e_n(t)$

$$r_1(t) \cong e_n(t) + \alpha A_c[1+m_a s_n(t)]\cos\phi_n(t)$$

output of LPF eliminating DC

$$y_D(t) = e_n(t) + \alpha A_c m_a s_n(t)\cos\phi_n(t)$$

We observe that for high value of CNR the SNR varies linearly, when the CNR falls below a certain threshold the output SNR deteriorates more rapidly than the CNR.

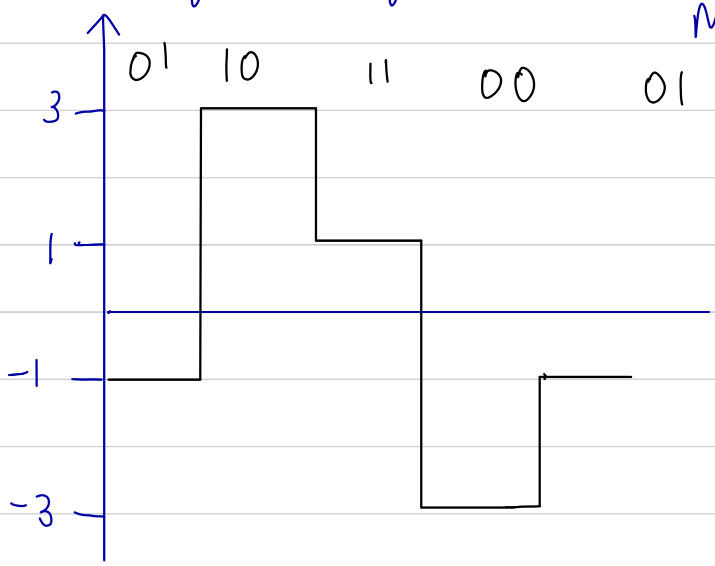
MODULE - 4

DIGITAL BASEBAND MODULATION

- Baseband modulation without carrier frequency

→ Timing information is needed (i.e clock)

M-ary pulses



00	-3
01	-1
11	+1
10	3

Line coding schemes

1> Unipolar:
2> Polar
3> Bipolar (AMI)
4> Manchester

Desirable features:

1> No DC or low frequency components near zero.
2> clock synchronization / recovery
3> small bandwidth

1) Unipolar:

1 → +ve pulse (A volts)
0 → no pulse (0 volts)

2) Polar:

1 → +ve pulse (A)
0 → -ve pulse (-A)

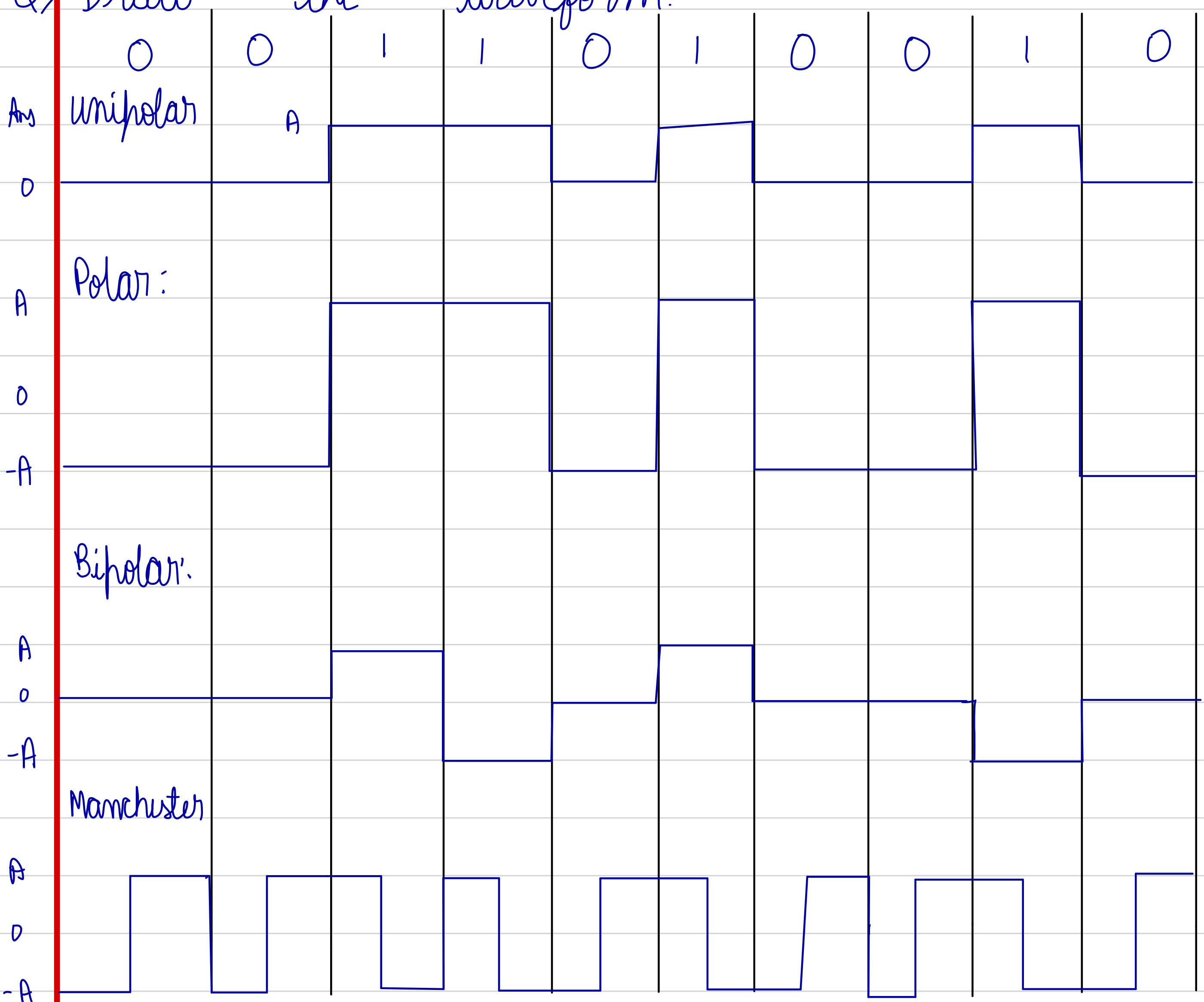
3) Bipolar (Alternate mark inversion)

1 → +ve & -ve pulse alternately (±A)
0 → no pulse (0)
(3 levels - pseudoternary)

4) Manchester:

1 → transition from +ve level to -ve level in the middle of the interval
0 → transition from -ve level to +ve level in the middle of the interval.

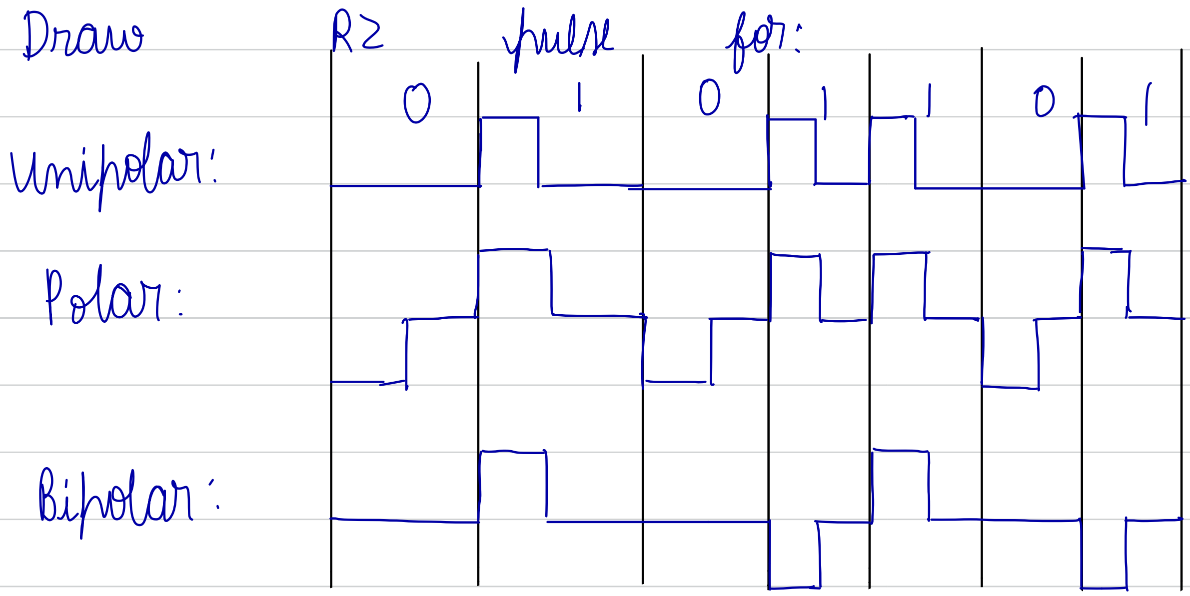
Q> Draw the waveform:



Two types of pulses:

1> Return to zero (RZ)
2> Non return to zero (NRZ)

Q> Draw RZ pulse for:



Transmitted signal:

It can be represented as

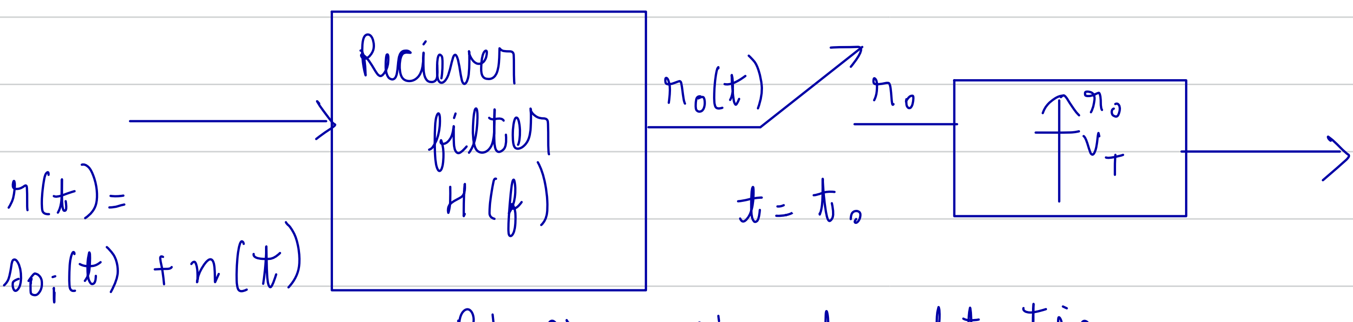
$$s(t) = \begin{cases} s_1(t) & 0 \leq t \leq T_b \\ s_2(t) & 0 \leq t \leq T_b \end{cases}$$

Recieved signal:

$$r(t) = s(t) + n(t), \quad 0 \leq t \leq T_b$$

with 2 sided power spectral density = $N_0/2$

The detector consists of TI filter with response $H(t)$



Binary signal detection

The detector pulse LTI followed by a

sampler and threshold comparator. The initial conditions of filter is set to zero prior to the arrival of each new pulse.

$$r_0(t) = s_{01}(t) \mid s_{02}(t) + n_0(t)$$

$$s_{01}(t) = h(t) * s_1(t)$$

$r_0(t)$ is sampled at time t_0 during the bit interval $0 \leq t \leq T_0$, the resulting sample $r_0(t)$ is a random variable. $i = 1$ or 2 where $s_{0i} = s_{01}(t)\Big|_{t=0}$

$$n_0 = n_0(t)\Big|_{t=t_0}$$

→ r_0 is a gaussian random variables with mean zero.

$$\sigma_0^2 = \overline{n_0^2} = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 \, df$$

(variance)

NOTE: $\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 \, p(x) \, dx$

The Matched filter

We need to find the optimum filter transfer function, that maximises the peak signal to noise power ratio.

$$s_{01} = s_{01}(t)\Big|_{t=t_0} = s_1(t) * h(t)\Big|_{t=t_0}$$

$$s_{o1}(t)\Big|_{t=t_0} = \mathcal{F}^{-1}\left\{ H(f)\cdot S(f) \right\}\Big|_{t=t_0} = \int_{-\infty}^{\infty} H(f)\,S(f)\,e^{j2\pi f t_0}\,df$$

$$\frac{s_{o1}^2}{\overline{n_o^2}} = \frac{\left| \int_{-\infty}^{\infty} S(f)\,H(f)\,e^{j2\pi f t_0}\,df \right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df}$$

Optimization seeks for an expression that maximises the expression on RHS. This can be accomplished by maximizing the numerator using cauchy - schwarz inequality.

<u>Cauchy - Schwarz inequality</u>:

$$\left| \int_{-\infty}^{\infty} X(t)\,Y(t)\,dt \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2\,df$$

where X & Y are output function of real variables x & y

Equality is obtained only when

$$X(f) = K\,Y^*(f) \quad \text{(complex conjugate)}$$

$$\left| \int_{-\infty}^{\infty} H(f)\,S(f)\,e^{j2\pi f_0}\,df \right|^2 = \int_{-\infty}^{\infty} |H(f)|^2\,df \int_{-\infty}^{\infty} |S_1(f)|^2\,df$$

$$\frac{S_{o1}^2}{\sigma_1^2} = \frac{\int_{-\infty}^{\infty} |H(f)|^2\,df \int_{-\infty}^{\infty} |S_1(f)|^2\,df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df}$$

$$\leq \frac{2}{N_0}\int_{-\infty}^{\infty} |s_1(f)|^2\,df$$

$$H_{opt}(f) = s_1^*(f)\, e^{j2\pi f t_0}$$

Taking inverse fourier transform on both sides.

$$H_{opt}(t) = s(t_0 - t)$$

Observations

1> $h(t)$ is independent of $N_0/2$

2> The SNR for the matched filter detector at the sampler o/p is

$$(SNR)_{MF} = (SNR)_{max} = \left| \frac{s_{01}^2}{\sigma_0^2} \right|$$

$$= \frac{1}{\frac{N_0}{2}} \int_{-\infty}^{\infty} |S_1(f)|^2 \, df$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} s_1^2(t)\, dt = \frac{2E}{N_0}$$

where E = energy of $s_1^2(t)$

SNR depends on energy of signal not the waveform.

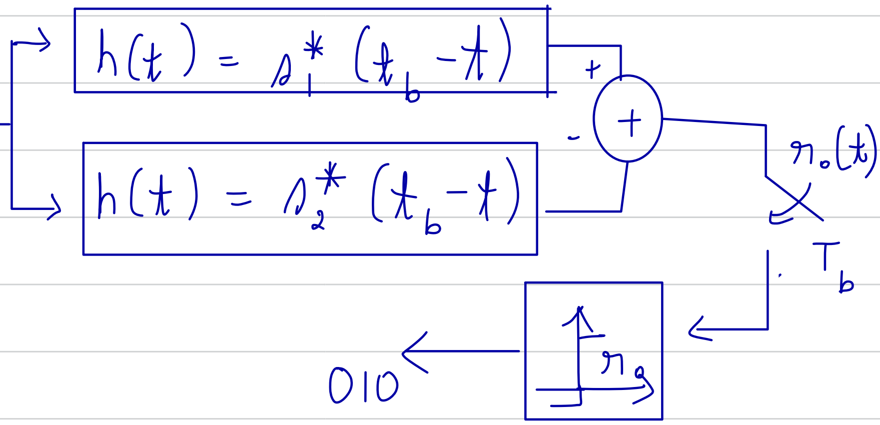
- E can be increased by increasing amplitude or duration.

- If $S_2(t) \neq 0$, by approach similar to above we can increase E.

$$h_{opt}(t) = s_1^*(t_0 - t) - s_2^*(t_0 - t)$$

$$r(t) = s_{0i}(t) + n_0(t)$$

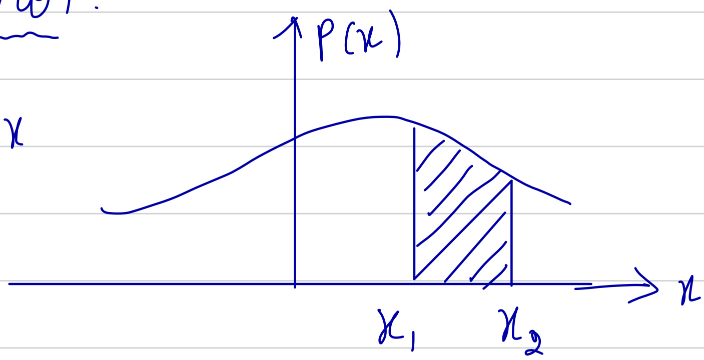
$i = 1, 2$



For $t < T_b$
The filter will have non zero impulse response for $t < 0$ ∴ It is unrealisable.
Hence we take $t_0 = T_b$

Probability of bit error:

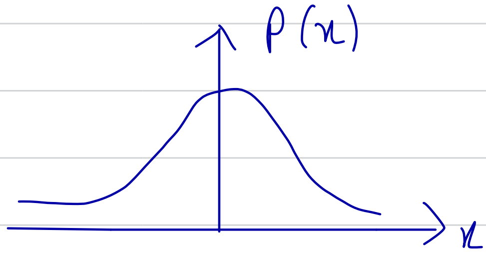
$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\,dx$$



Also $$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

For gaussian distribution:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



σ - variance $\quad$ m - mean

$$\int_{x_1}^{\infty} p(x)\,dx = Q(x_1)$$

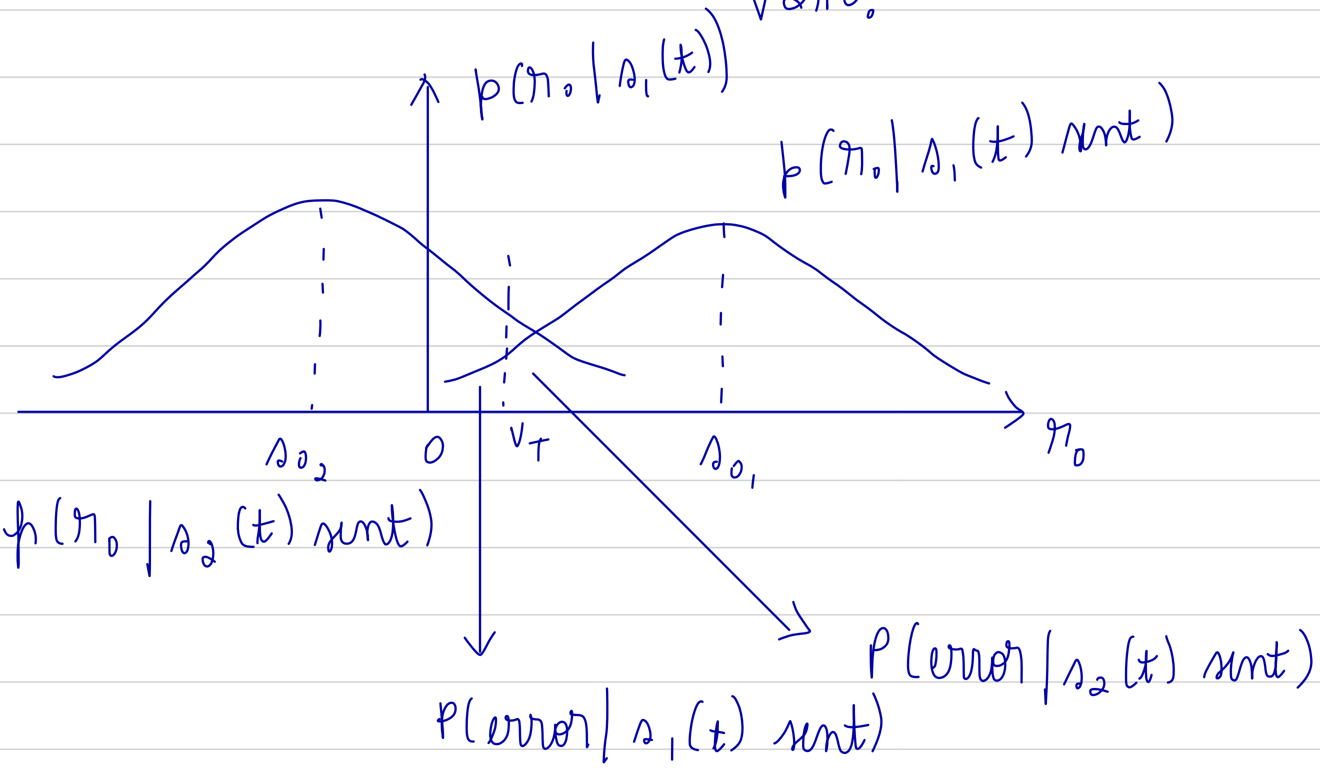
$$P(x)\Big|_{\substack{\sigma^2=1\\ m=0}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\therefore\ Q(x_1) = \int_{x_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$\phi(x_1) = P(-\infty < x < x_1)$$
$$Q(x_1) = P(x_1 \leq x < \infty)$$
$$\phi(x_1) + Q(x_1) = 1$$

Conditional Probability density

$$p(r_0 | s_i(t) \text{ transmitted}) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(r_0 - s_{0i})^2}{2\sigma_0^2}}$$



• The output of the sampler is compared with threshold voltage V_T and decision is made.

The decision rule is;

$r_0 > V_T \rightarrow$ declare that $s_1(t)$ was sent

$r_0 < V_T \rightarrow$ declare that $s_2(t)$ was sent.

Average probability of error / Bit error rate

BER (Bit error rate) $= P(s_1(t) \text{ sent}) \cdot P(\text{error} \mid s_1(t) \text{ sent}) + P(s_2(t) \text{ sent}) \cdot P(\text{error} \mid s_2(t) \text{ sent})$

If $P(s_1(t) \text{ sent}) = p$, then

$$BER = p \cdot P(\text{error} \mid s_1(t) \text{ sent}) + (1-p)\, P(\text{error} \mid s_2(t) \text{ sent})$$

1) $$P(\text{error} \mid s_1(t) \text{ sent}) = P_r(r_0 < V_T \mid s_1(t) \text{ sent}) = \int_{-\infty}^{V_T} P(r_0 \mid s_1(t) \text{ sent})\, dr_0$$

$$= \int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{(r_0 - s_{01})^2}{2\sigma_0^2}}\, dr_0$$

let $\dfrac{r_0 - s_{01}}{\sigma_0} = y$ $\qquad dr_0 = \sigma_0\, dy$

when $r_0 = -\infty \rightarrow y = -\infty$

$r_0 = V_T \rightarrow y = \dfrac{V_T - s_{01}}{\sigma_0}$

$$\therefore P(\text{error} \mid s_1(t) \text{ sent}) = \int_{-\infty}^{\frac{V_T - s_{01}}{\sigma_0}} \frac{1}{\sqrt{2\pi}\,\sigma_0}\, e^{-y^2/2}\, \sigma_0\, dy$$

$$= \int_{-\infty}^{\frac{V_T - s_{01}}{\sigma_0}} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, \cancel{\sigma_0}\, dy$$

Put $x = -y$ $\qquad dx = -dy$

$$= \int_{\left(\frac{s_{01} - V_T}{2}\right)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = Q\left(\frac{s_{01} - V_T}{2}\right)$$

2) $P(\text{error} \mid s_2(t) \text{ sent}) = \int_{V_T}^{\infty} f(r_0 \mid s_2(t) \text{ sent}) \, dr_0$

$$= \int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi \sigma_0^2}} e^{-\frac{(r_0 - s_{02})^2}{2\sigma_0^2}} \, dr_0$$

$\frac{r_0 - s_{02}}{\sigma_0} = y$ $\qquad$ then $= \int_{\frac{V_T - s_{02}}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$

$$= Q\left(\frac{V_T - s_{02}}{\sigma_0}\right) //$$

$$BER = P_e = p\, Q\left(\frac{s_{01} - V_T}{\sigma_0}\right) + (1-p)\, Q\left(\frac{V_T - s_{02}}{\sigma_0}\right)$$

$$\max_{H(F)} \left(\frac{s_{01} - s_{02}}{\sigma}\right)$$

$$s_{01} = s_{01}(t)\Big|_{t = T_b}$$

$$= s_1(t) * h_{opt}(t)\Big|_{t = t_b}$$

$$= s_1(t) * [s_1^*(T_b - t) - s_2^*(T_b - t)]$$

$$= \int_{-\infty}^{\infty} |s_1(t)|^2 \, dt - \int_{-\infty}^{\infty} s_1(t)\, s_2^*(t)\, dt$$

$$s_{02} = s_{02}(t) \Big|_{t = T_b}$$

$$= s_2(t) * h_{opt}(t) \Big|_{t_0 = T_b}$$

$$= s_2(t) * \left[s_1^*(T_b - t) - s_2^*(T_b - t) \right]$$

$$= -\int_{-\infty}^{\infty} |s_2(t)|^2 \, dt + \int_{-\infty}^{\infty} s_1^*(t)\, s_2(t)\, dt$$

let $d^2 = s_{01} - s_{02}$

$$= \int_{-\infty}^{\infty} |s_1(t)|^2 \, dt - \int_{-\infty}^{\infty} s_1(t)\, s_2^*(t)\, dt + \int_{-\infty}^{\infty} |s_2(t)|^2 \, dt$$
$$- \int_{-\infty}^{\infty} s_1^*(t)\, s_2(t)\, dt$$

$$= \int_{-\infty}^{\infty} \Big(s_1(t) \big(s_1^*(t) - s_2^*(t) \big) - s_2(t) \big(s_1^*(t) - s_2^*(t) \big) \Big) dt$$

$$= \int_{-\infty}^{\infty} \big(s_1(t) - s_2(t) \big) \big(s_1(t) - s_2(t) \big)^* dt$$

$$= \int_{-\infty}^{\infty} | s_1(t) - s_2(t) |^2 \, dt$$

$$\therefore \; d^2 = \int_{-\infty}^{\infty} | s_1(t) - s_2(t) |^2 \, dt$$

$$d^2 = \int_{-\infty}^{\infty} \left| s_1(t-T_b) - s_2(t-T_b) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left| s_1(T_b-t) - s_2(T_b-t) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left| s_1^*(T_b-t) - s_2^*(T_b-t) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left| h_{opt}(t) \right|^2 dt \;//$$

$$= \int_{-\infty}^{\infty} \left| H_{opt}(f) \right|^2 df \;//$$

Noise power at output of matched filter can be written as ; $\sigma_0^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} \left| H_{opt}(f) \right|^2 df$

$$\Rightarrow \quad \sigma_0^2 = \frac{N_0 d^2}{2}$$

When there are 2 signal waveforms, probability of error,

$$= p\,Q\left(\frac{S_{01} - V_T}{\sigma_0}\right) + (1-p)\,Q\left(\frac{V_T - S_{02}}{\sigma_0}\right)$$

For $p = 1/2$

$$BER = \frac{1}{2}\left[Q\left(\frac{S_{01} - V_T}{\sigma_0}\right) + Q\left(\frac{V_T - S_{02}}{\sigma_0}\right)\right]$$

Bit error rate can be minimised by choosing optimum values for V_T & for filter transfer function.

$$BER = Q\left(\frac{s_{01} - s_{02}}{2\sigma_0}\right)$$

$$= Q\left(\frac{d^2}{2\sqrt{\frac{N_0}{2} d^2}}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$\boxed{\therefore \; BER = Q\left(\frac{d}{\sqrt{2N_0}}\right)}$$

Implementation of optimum filter

Correlation detector:

For the matched filter we have :

$$r_0(t) = r(t) * h_{opt}(t) \quad \text{(convolution)}$$

$$= \int_{-\infty}^{\infty} h_{opt}(t-\tau)\, r(\tau)\, d\tau$$

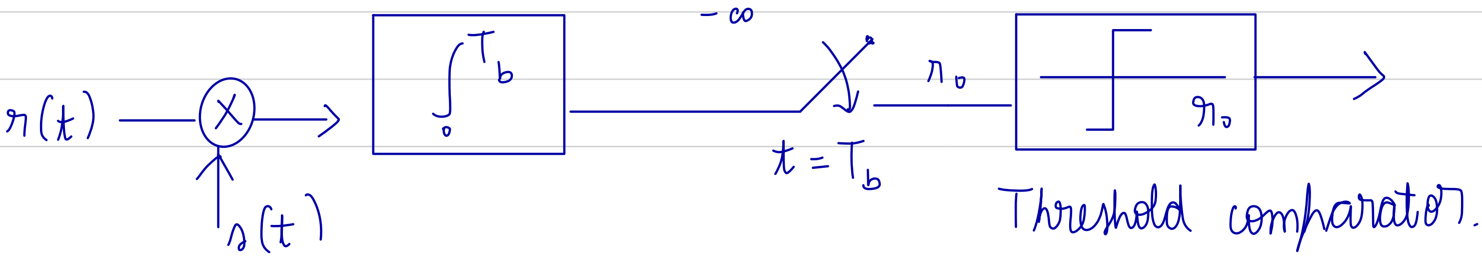
with $h_{opt}(t) = s^*(T_b - t)$, we have

$$r_0(t) = \int_{-\infty}^{\infty} s^*(T_b - t - \tau)\, r(\tau)\, d\tau$$

at $t = T_b$

$$r_0(T_b) = r_0(t)\Big|_{T_b} = \int_{-\infty}^{\infty} s^*(-\tau)\, r(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} s(\tau)\, r(\tau)\, d\tau$$



Since $s(t)$ is of finite duration (0 to T_b) we have :

$$r_0(T_b) = \int_{-\infty}^{\infty} s(\tau)\, r(\tau)\, d\tau = \int_0^{T_b} s(\tau)\, r(\tau)\, d\tau.$$

For a rectangular input pulse, $s(t) = 1$ from 0 to T_b

$$r_0(T_b) = \int_0^{T_b} r(\tau)\, d\tau$$

Q) Find expression for average error probability for the case when there are 2 input signals $s_1(t)$ & $s_2(t)$.

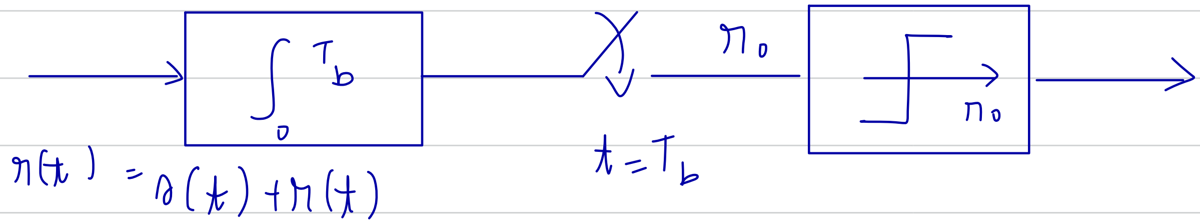
Q) Write expression for Bit error rate (BER) in terms of Q function with optimum threshold & Optimum filter

Q) Derive expression for impulse function for matched filter.

Q) What is a matched filter, and why is it named so?

Q) Starting from a matched filter detector obtain structure of correlation detector & integrate and dump detector.

Integrate & dump filter:



Performance of binary coding schemes

1) Unipolar NRZ - on / off signalling

$$BER = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

where $d^2 = \int_{-\infty}^{\infty} |s_1(t) - s_2(t)|^2 \, dt$

$$\left.\begin{array}{l} s_1(t) = A \\ s_2(t) = 0 \end{array}\right\} \quad 0 \le t \le T_b$$

$$d^2 = \int_0^{T_b} |A - 0|^2 \, dt = A^2 T_b$$

Average energy per bit $= \frac{1}{2}\int_0^{T_b} A^2 \, dt + \frac{1}{2}\int_0^{T_b} 0 \, dt$

$$= \frac{A^2 T_b}{2} = E_b$$

$$\therefore BER = Q\left(\sqrt{\frac{2E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

2) Polar NRZ or Antipodal NRZ

$$BER = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$\left.\begin{aligned} s_1(t) &= A \\ s_2(t) &= -A \end{aligned}\right\} \quad 0 \leq t < T_b$$

$$d^2 = \int_0^{T} |A-(-A)|^2 \, dt = 4A^2 T_b$$

$$\text{Average energy per bit} = \frac{1}{2}\int_0^{T_b} A^2 \, dt + \frac{1}{2}\int_0^{T_b} (-A)^2 \, dt$$

$$= A^2 T_b = E_b$$

$$\therefore BER = Q\left(\sqrt{\frac{4E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Polar > Unipolar

- For same BER, unipolar has more E_b/N_0 than polar $\Rightarrow$ Polar is more efficient.
- Unipolar needs 3 dB more SNR to get same SNR as polar.

Orthogonal signalling:

Two signals $s_1(t)$ & $s_2(t)$ are orthogonal over the interval $[0, T_b]$ if

$$\int_0^{T_b} s_1(t)\, s_2(t) = 0$$

Ex: $s_1(t) = \begin{cases} A & 0 \le t \le T_b/2 \\ 0 & \frac{T_b}{2} \le t \le T_b \end{cases}$

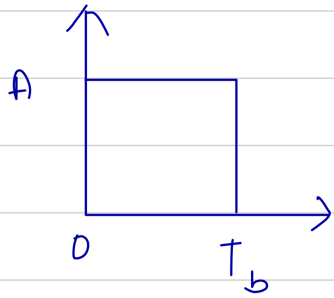
$s_2(t) = \begin{cases} 0 & 0 \le t \le T_b/2 \\ A & T_b/2 \le t \le T_b \end{cases}$

Q Find BER for unipolar RZ signalling & polar RZ signalling.

Bipolar NRZ signalling

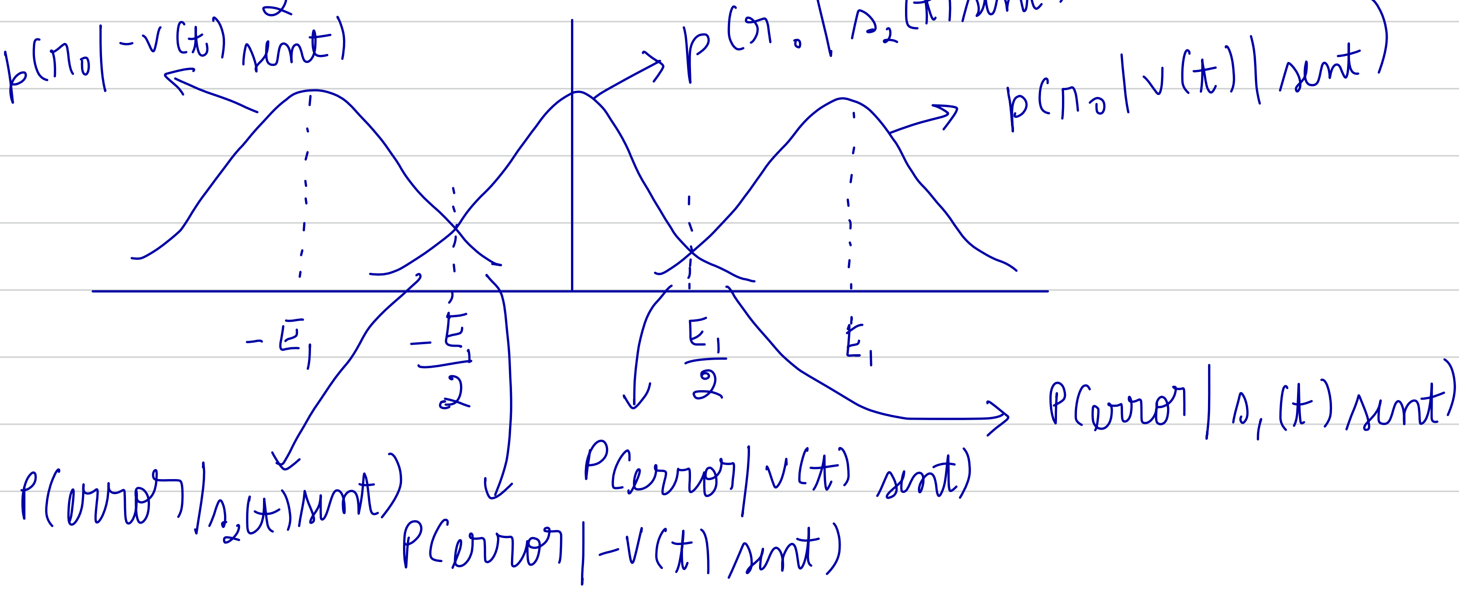
$s(t) = v(t) = A$ for binary
or $-v(t) = -A$ alternately
$s_2(t) = 0$ for binary

$$E_b = \frac{1}{4}\int_0^{T_b} A^2\,dt + \frac{1}{4}\int_0^{T_b} (A)^2\,dt$$



$$+ \frac{1}{2}\int_0^{T_b} 0^2\,dt$$

$$= \frac{A^2 T_b}{2}$$



Energy of basic pulse

$$E_1 = \int_0^{T_b} A^2\, dt = A^2 T_b$$

$$\therefore \quad E_b = \frac{E_1}{2}$$

The 2 thresholds are set at $-\frac{E_1}{2}$ & $\frac{E_1}{2}$

$$BER = \frac{1}{4} P\{\text{error} \mid v(t) \text{ sent}\} + \frac{1}{4} P\{\text{error} \mid -v(t) \text{ sent}\} + \frac{1}{2} P\{\text{error} \mid s_2(t) \text{ sent}\}$$

$$P\{\text{error} \mid v(t) \text{ sent}\} = \int_{-\infty}^{E_1/2} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(r_0 - E_1)^2}{2\sigma_0^2}} dr_0$$

Let $y = \frac{r_0 - E_1}{\sigma_0}$ $\qquad dy = \frac{dr_0}{\sigma_0}$

$r_0 = -\infty \Rightarrow y = -\infty$

$r_0 = \frac{E_1}{2} \qquad y = \left(-\frac{E_1}{2}\right) / \sigma_0$

$$= \int_{-\infty}^{(E_1/2)/\sigma_0} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-y^2/2} \frac{dy}{\sigma_0}$$

$$= Q\left(\frac{E_1}{2\sigma_0}\right)$$

$$P(\text{error} \mid s_2(t) \text{ sent}) = \int_{-\infty}^{-E_1/2} p(r_0 \mid s_2(t))\, dr_0 + \int_{\frac{E_1}{2}}^{\infty} p(r_0 \mid s_2(t))\, dr_0$$

$$= 2\int_{E_1/2}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-r_0^2/2\sigma_0}\, dr_0$$

let $\frac{r_0}{\sigma_0} = y$ $\qquad dr_0 = \sigma_0\, y$

$y_1 = \frac{E_1}{2\sigma_0}$ $\qquad y_2 = 0$

$$= 2Q\left(\frac{E_1}{2\sigma_0}\right)$$

$$\therefore BER = \frac{1}{4} Q\left(\frac{E_1}{2\sigma_0}\right) + \frac{1}{4} Q\left(\frac{E_1}{2\sigma_0}\right) + \frac{1}{2} 2Q\left(\frac{E_1}{2\sigma_0}\right)$$

$$= \frac{3}{2} Q\left(\frac{E_1}{2\sigma_0}\right)$$

$$\sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H_{opt}(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} V^2(t)\, dt$$

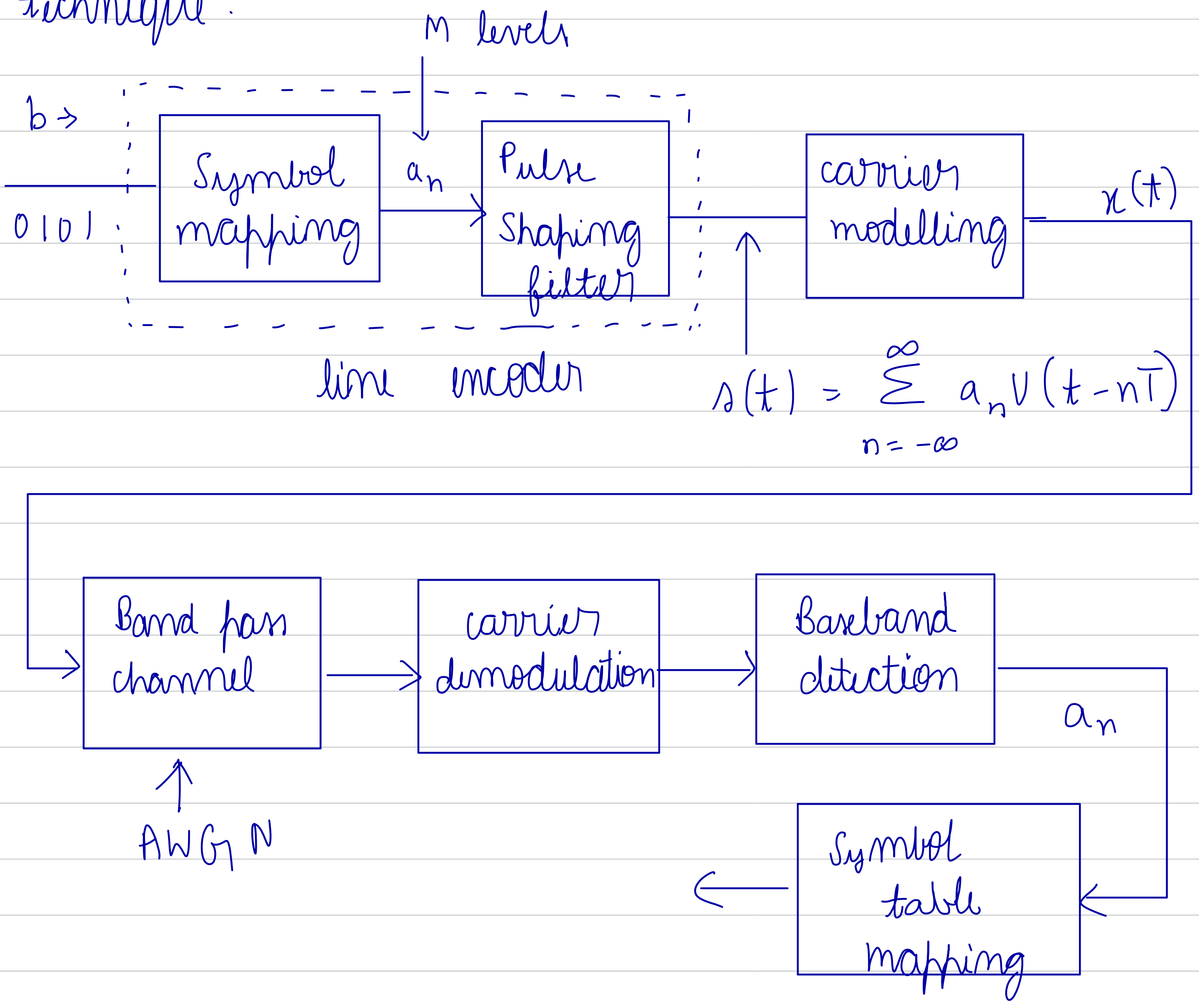
$$= \frac{N_0}{2}\int_0^{T_b} A^2 dt = \frac{N_0}{2} E_1$$

$$BER = \frac{3}{2} Q\left(\frac{E_1}{2\sqrt{\frac{N_0}{2} E_1}}\right)$$

$$= \frac{3}{2} Q\left(\sqrt{\frac{E_1}{2N_0}}\right)$$

$$= \frac{3}{2}\left(Q\sqrt{\frac{2E_b}{2N_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Digital Information: Transmission via carrier modulation or Pass band modulation technique.



Digital carrier transmission

Representation of digitally modulated carrier signal

$$x(t) = A(t) \cos[w_c t + \psi(t)]$$

$$= A(t) \cos \psi(t) \cos w_c t - A(t) \sin \psi(t) \sin w_c t$$

$$= I(t) \cos w_c t - Q(t) \sin w_c t$$

$$I(t) = \sum_n a_n \, v(t - nT)$$
$v \rightarrow$ basic pulse shape

$$Q(t) = \sum_n a_n^2 \, w(t - nT)$$

In terms of quadrature components, digitally modulated bandpass random signal can be expressed as:

$$x(t) = A_c \left[I(t) \cos \omega_c t - Q(t) \sin \omega_c t \right]$$

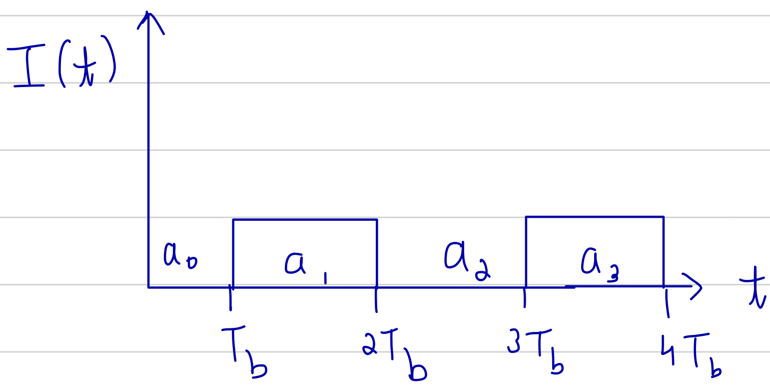
$$= A_c \, R_e \left\{ \tilde{x}(t) \, e^{j\omega_c t} \right\}$$

Binary amplitude shifting

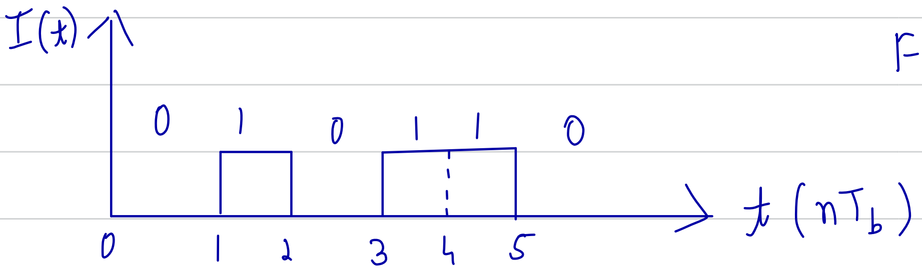
Corresponding to binary inputs 0 & 1, a_n takes the value 0 or 1.

$$I(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left[\frac{(t - nT_b)}{T_b} \right]$$

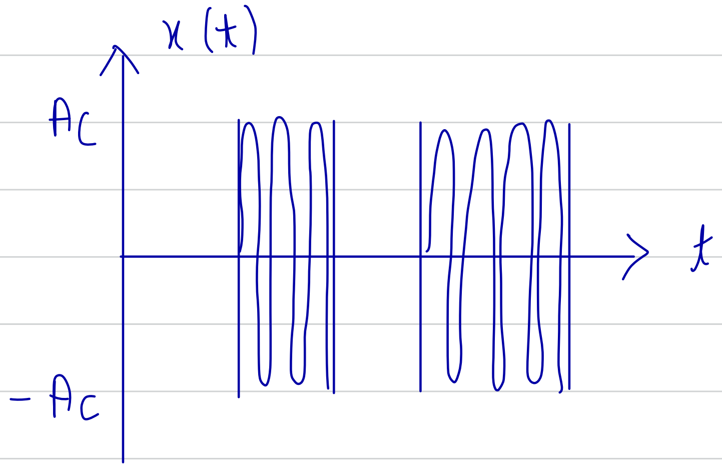
T_b - pulse width

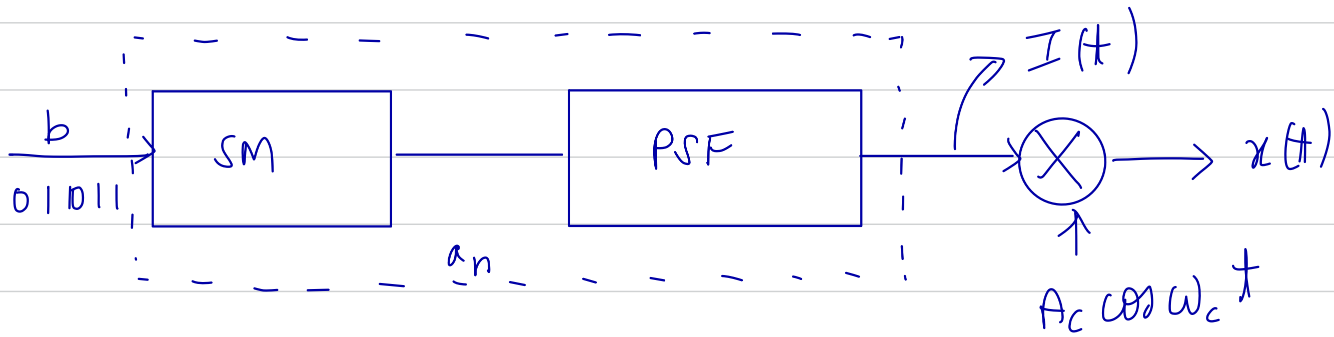


eg:



Find $x(t)$



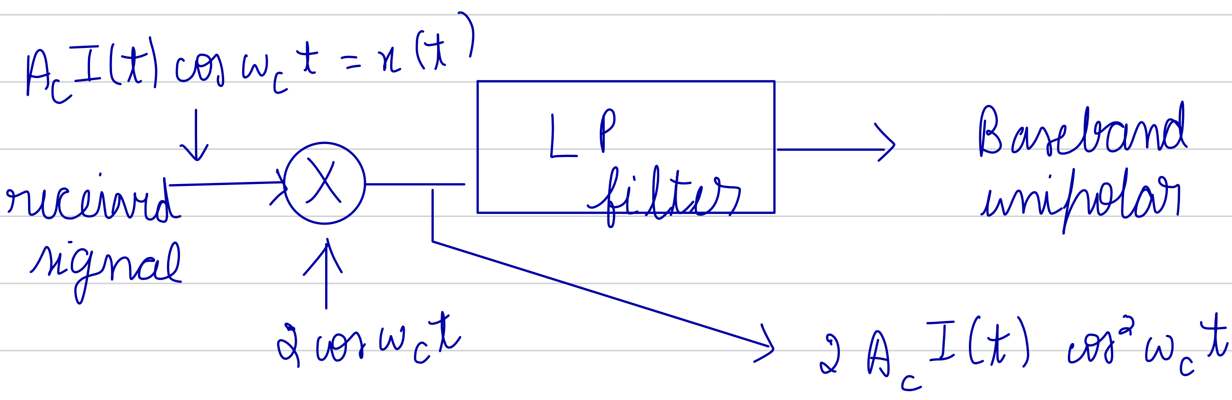


Binary ASK modulator

$0 \leqslant t \leq T_b$

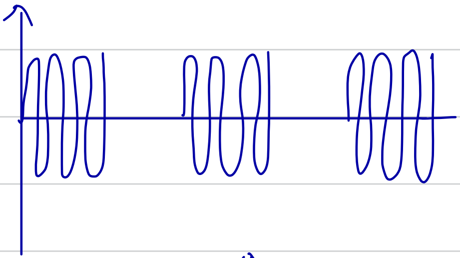
binary 1 : $s_1(t) = \begin{cases} A_c \cos w_c t & 0 \leqslant t \leq T_b \\ 0 & \text{otherwise} \end{cases}$

binary 0 : $s_2(t) = 0 \qquad , \quad 0 \leqslant t \leqslant T_b$

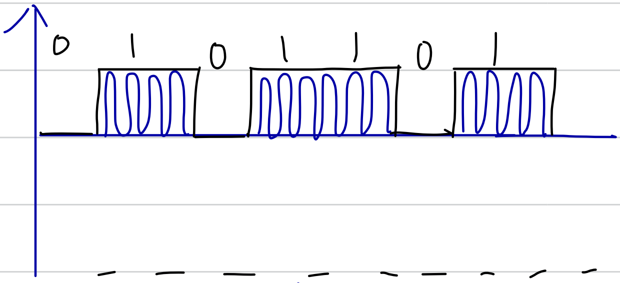


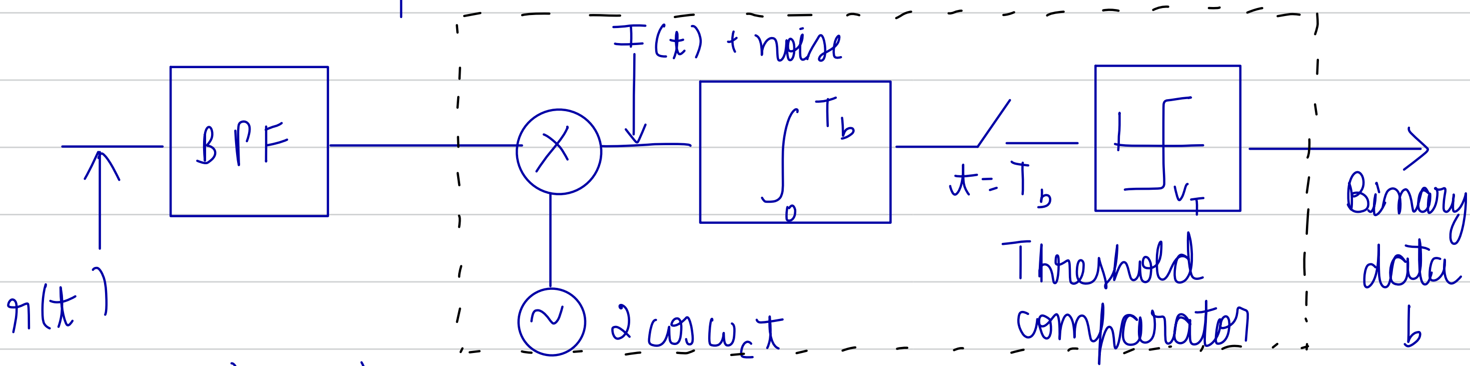
$A_c I(t) (1 + \cos 2 w_c t)$

$x(t) = a_n \ \text{rect}\left[\frac{(t - T_b)}{T_b}\right] \cos w_c t$



output :



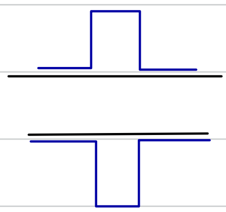


$r(t) = x(t) + n(t)$

Structure of reciever

Binary Phase shift keying

Information	bits	a_n
$b_n \rightarrow$	1	1
	0	-1



Output of PSF (pulse shaping filter) is Polar NRZ pulse train in which amplitude modulates the carrier

$s_1(t) = A_c \cos\omega_c t \qquad 0 \leq t \leq T_b$

$s_2(t) = -A_c \cos\omega_c t$

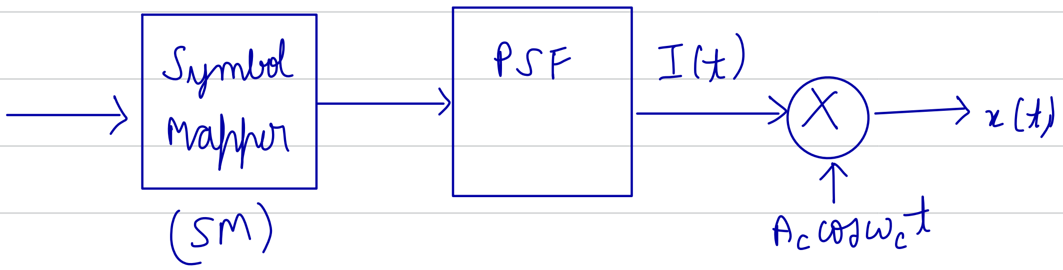
BPSK signal can be represented as:

$$x(t) = A_c \sum_{n=-\infty}^{\infty} a_n \, \mathrm{rect}\left[\frac{(t - T_b)}{T_b}\right] \cos\omega_c t$$

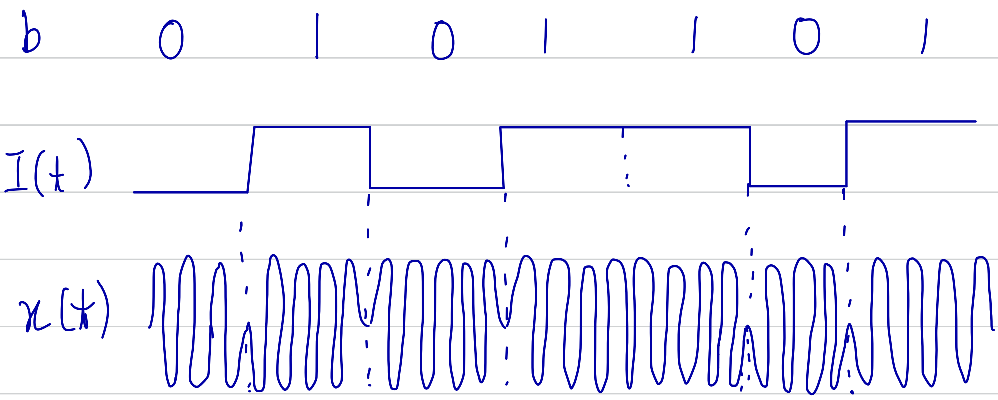
where $a_n \in \{1, -1\}$

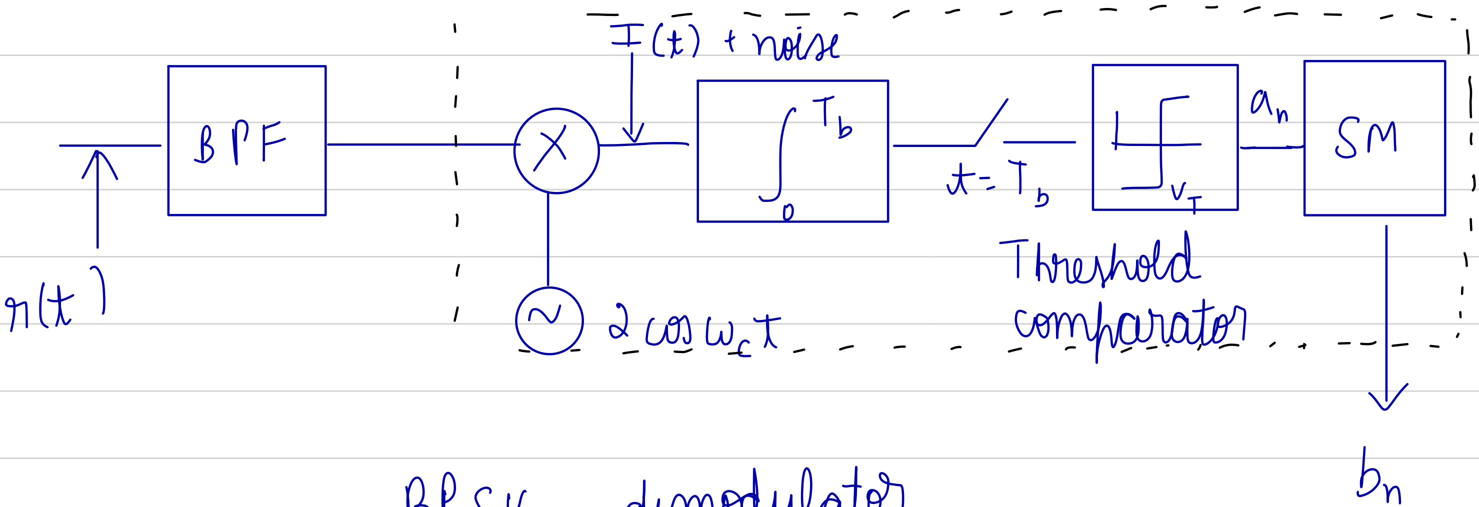
$$I(t) = \sum_{n=-\infty}^{\infty} a_n \, \mathrm{rect}\left[\frac{(t - T_b)}{T_b}\right]$$ (base band signal)

$Q(t) = 0.$



BPSK modulator





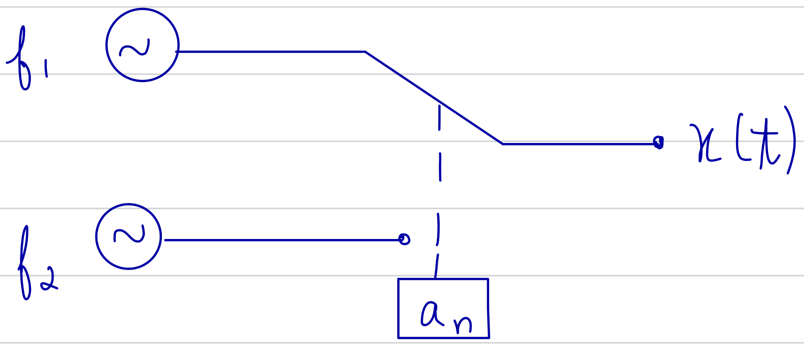
BPSK demodulator

Corresponding b_n (binary information sequence) $= 0$ or 1. We use 2 distinct frequencies

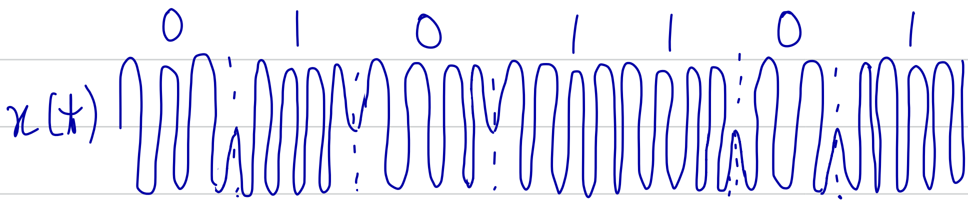
$$f_1 = f_0 + \frac{\Delta f}{2} \qquad f_2 = f_0 - \frac{\Delta f}{2}$$

Correspondingly the signals are:

$$\left.\begin{aligned} b=1 &\rightarrow s_1(t) = A\cos(\omega_1 t + \phi_1) \\ b=0 &\rightarrow s_2(t) = A\cos(\omega_1 t + \phi_2) \end{aligned}\right\} \quad 0 \le t \le T_b$$



Phase discontinuity leads to undesirable broadening of the signal spectrum



<u>Continuous phase FSK</u>: Even when frequency is changed, there is continuity in phase of the output waveform

$$\left.\begin{aligned} s_1(t) &= A_c\cos(\omega_1 t + \phi_1) \\ s_2(t) &= A_c\cos(\omega_2 t + \phi_2) \end{aligned}\right\} \quad 0 \le t \le T_b$$

$s_1(t) = A_c \cos\left(2\pi\left(f_c + \frac{\Delta f}{2}\right)t + \phi_1\right) \qquad 0 \le t \le T_b$

$s_2(t) = A_c \cos\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t + \phi_2\right) \qquad 0 \le t \le T_b$

$$\left.\begin{aligned} s_1(t) &= A_c \cos\left[2\pi f_c + \pi \Delta f t + \phi_1\right] \\ s_2(t) &= A_c \cos\left[2\pi f_c - \pi \Delta f t + \phi_2\right] \end{aligned}\right\} 0 \le t \le T_b$$

Continuous phase FSK

$f = f_c + K_f I(t)$ or $f = f_c + \frac{\Delta f}{2} I(t)$

$\Delta f = f_1 - f_2$ for $I(t) = +1$

$f = f_c + \frac{\Delta f}{2}$

for $I(t) = -1$, $f = f_c - \frac{\Delta f}{2}$

↓

instantaneous.

orthogonality of BFSK signal

We want to choose f_1 & f_2 such that $s_1(t)$ & $s_2(t)$ are orthogonal

Let $\phi_1 = \phi_2 = \phi$ (using some V_{CO})

For orthogonality, $\int_0^{T_b} s_1(t)\, s_2(t)\, dt = 0$

$$\Rightarrow \int_0^{T_b} A_c \cos(2\pi f_c t + \pi \Delta f t + \phi)\, A_c \cos(2\pi f_c t - \pi \Delta f t + \phi)\, dt = 0$$

$$\Rightarrow \frac{A_c^2}{2}\int_0^{T_b}\left[\cos(4\pi f_c t + 2\phi) + \cos(2\pi \Delta f t)\right] dt$$

$$\Rightarrow \frac{A_c^2}{2}\left[\frac{\sin(4\pi f_c t + 2\phi)}{4\pi f_c}\Bigg|_0^{T_b} + \frac{\sin(2\pi \Delta f t)}{2\pi\Delta f}\Bigg|_0^{T_b}\right] = 0$$

$$\Rightarrow \frac{A^2}{2}\left[\frac{\sin(2\pi\Delta f t)}{2\pi\Delta f} + \frac{\sin[4\pi f_c T_b + 2\phi] - \sin 2\phi}{4\pi f_c}\right] = 0$$

if $T_b = NT_c = \frac{N}{f_c}$ and $T_b \Delta f = \frac{n}{2}$

$$\int_0^{T_b} s_1(t)\cdot s_2(t)\, dt = \frac{A_c^2}{2}\,\frac{\sin 2\pi\Delta f t}{2\pi\Delta f}\Bigg|_0^{T_b}$$

$$= \frac{A_c^2}{2} T_b \frac{\sin 2\pi\Delta f T_b}{2\pi\Delta f T_b} = \frac{A_c^2 T_b}{2}\operatorname{sinc}(2\Delta f T_b) \;//$$

Gram - Schmidt orthonormalization process

- To find a set of orthonormal vectors from a set of vectors that need not be orthogonal or normalised.

- The first basis function can be anyone of $s_i(t)$, $i = 1, 2, \dots m$.

Let us take $s_1(t)$

The unit energy function $\phi_1(t)$ is obtained by dividing $s_1(t)$ by $\sqrt{E_1}$

i.e $\phi_1(t) = s_1(t)/\sqrt{E_1}$

$$||s_1(t)|| = \sqrt{\int |s_1(t)|^2 dt} = \sqrt{E_1}$$

To find second basis function $\phi_2(t)$, we subtract from $s_2(t)$ it's projection into $\phi_1(t)$.

$$\theta_2(t) = s_2(t) - \phi_1(t)\left(s_2(t) * \phi_1(t)\right)$$

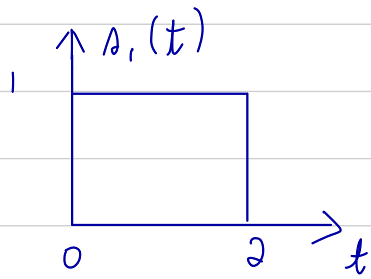
where $\left(s_2(t) * \phi_1(t)\right) = \int_T s_2(t)\, \phi_1(t)\, dt$

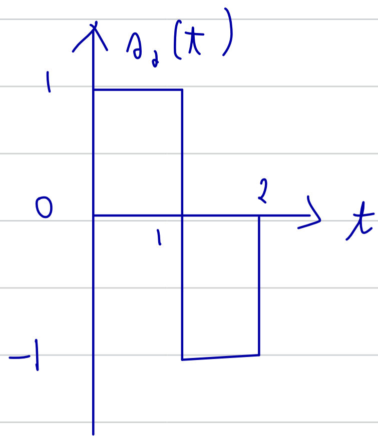
$\theta_2(t)$ is orthogonal to $\phi_1(t)$. The second basis function is $\phi_2(t) = \dfrac{\theta_2(t)}{||\theta_2(t)||} = \dfrac{\theta_2(t)}{\sqrt{E_{\theta_2}}}$

where $||\theta_2(t)|| = \sqrt{\int_T \theta_2^2(t)\, dt} = \sqrt{E_{\theta_2}}$

This procedure can be extended.

Eg





Find ϕ_1 & ϕ_2

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$||s_1(t)|| = \sqrt{\int_0^2 1^2 dt} = \sqrt{2}$$

$$\phi_1 = \frac{s_1(t)}{\sqrt{2}}$$

$$s_1 * \phi_1 = \int_0^2 s_2 \phi_1 \, dt$$

$$= \int_0^2 s_2(t) \frac{s_1(t)}{\sqrt{2}} \, dt$$

$$= \int_0^1 1 \times \frac{1}{\sqrt{2}} \, dt + \int_1^2 \frac{-1}{\sqrt{2}} \, dt$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0 \,//$$

$$\theta_2(t) = s_2(t) - \phi_1(t)\,(s_2(t) * \phi_1(t))$$

$$\theta_2(t) = s_2(t) - \phi_1(t) \times 0$$

$$= s_2(t)$$

$$\|\theta_2(t)\| = \sqrt{\int_0^2 s_2^2(t)\,dt} = \sqrt{\int_0^2 1\,dt} = \sqrt{2} \,//$$

$$\phi_2(t) = \frac{\theta_2(t)}{\|\theta_2(t)\|} = \frac{s_2(t)}{\sqrt{2}} \,//$$

Vector representation of NWGN

Vector representation of waveform:

- Suppose we have finite energy signals $s_i(t)$, $i = 1, 2, \ldots M$ defined over $[t_0, t_0 + T]$
- Using G S orthonormalisation procedure we can find orthonormal basis functions. $\phi_j(t)$, $j = 1, 2, \ldots N$ $(N \leq M)$ such that M waveforms can be represent-ed exactly as

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t) \text{ where}$$

$$s_{ij} = (s_i(t) * \phi_j(t)) = \int_T s_i(t)\, \phi_j(t)\, dt$$

The wave $s_i(t)$ can be represented by N tuples

$$s_i = (s_{i1}, s_{i2}, s_{i3}, \ldots s_{iN}), \quad i = 1, 2, 3, \ldots M$$

In the subspace spanned by the orthonormal basis is called signal space.

- The concept of length & distance for vector space proves quite useful.

- The set of waveforms $s_i(t)$, $i = 1, 2, \ldots M$ is the signal set for modulation scheme.

The set of M vectors $s_i(t)$ is called a signal constellation. It is unique representation for signal set in the signal space determined by the basis function

Vector space representation of noise

Consider representation of WAGN $n_o(t)$ with spectral density $\frac{N_0}{2}$

$$n_o(t) = \sum_{n=1}^{N} n_i \phi_i(t) + n'(t)$$

with $n_i = n(t) * \phi_i(t)$ is a projection of $n(t)$ on the axis $\phi_i(t)$

$n'(t)$ is defined by $n(t) - \sum_{i=1}^{N} n_i \phi_i(t)$ and represents the difference between WGN & its finite dimension representation in vector space spanned by

$$\phi_i(t), \quad i = 1, 2, \ldots N$$

It can be shown that $n'(t)$ is irrelavant about the decision of which signal was transmitted.

The random variables $\{n_i\}$ are uncorrelated, each has a mean square value $\frac{N_0}{2}$

We can represent WGN, $n(t)$ as a Gaussian random vector. In the signal space spanned by $\{\phi_i(t), i = 1, 2, \dots N\}$ where the components are Gaussian Random variables between 0 & $N_0/2$.

Signal detection in WGN:

The recieved signal is $r(t) = s_i(t) + n(t)$, $0 \leq t \leq T$.

$r = s_i + N$

Decision regions: The detector partitions the signal space into M disjoint decision regions using the decision rule.

i.e $D_i = \{ r_i : \| r - s_i \| \leq \| r - s_j \|$ for all $j \neq i \}$ $i = 1, 2, \dots M$

if the recieved signal vector r falls in region D_i it is closer to signal vector s_i in the signal space. The decision rule is

- if r in $D_i \Rightarrow s_i$ is transmitted.

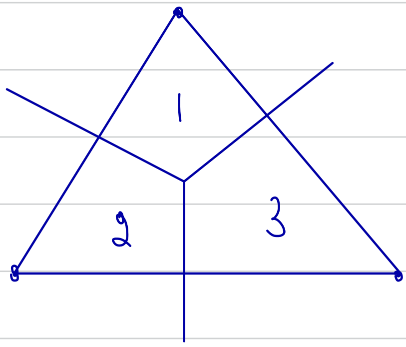
To partition the signal into decision region.

1) We connect adjacent pairs of signal points by a line.

2) We draw the perpendicular bisector of the line.

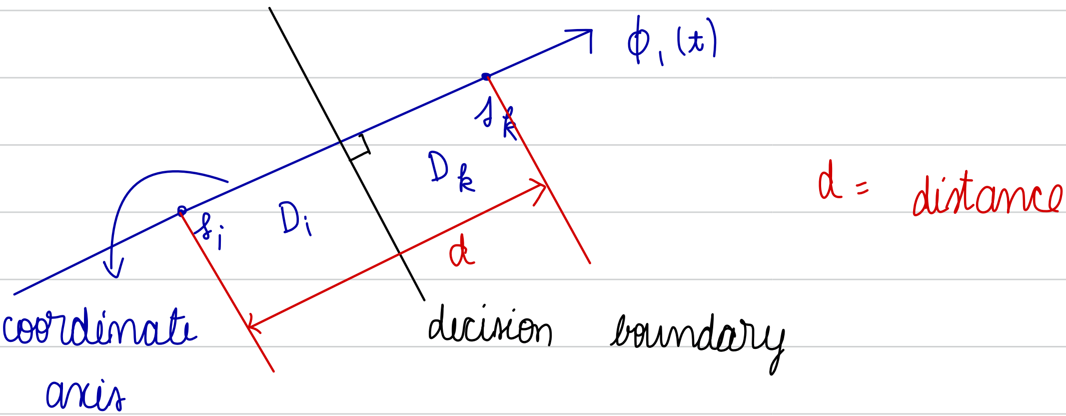
3) The boundary of the decision region is formed by the bisector of the line joining the points.

Eg:



2 Signal error probability

The signal constellation with 2 signal s_i & s_k is shown in figure



We select the signal basis so that the first orthonormal basis function $\phi_1(t)$ lies along the line connecting the vectors $\vec{s_i}$ & $\vec{s_k}$

Then the dimension of signal space is 1. We consider only the projection n_1 of WGN on $\phi_1(t)$

n_1 is a Gaussian random variable with mean = 0 and variance = $N_0/2$

If vector $\vec{s}_i$ is transmitted, the detector decides $\vec{s}_k$ if $r = s_i + n_1$ is beyond the perpendicular bisector of the line $\vec{s}_k - \vec{s}_i$, i.e this occurs if $n_1 > d/2$ where d is distance $\|\vec{s}_i - \vec{s}_k\|$

$$P(s_k \text{ decided} \mid s_i \text{ sent}) = P(r \text{ lies in } D_k)$$

$$= P(n_1 > d/2)$$

$$= \int_{d/2}^{\infty} p_{n_1}(n_1)\, dn_1$$

$$= \int_{d/2}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-\frac{(n_1 - 0)^2}{2\frac{N_0}{2}}} dn_i$$

$$= \int_{d/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{2}\left(\frac{n_1}{\sqrt{\frac{N_0}{2}}}\right)^2} dn_i$$

let $y = \dfrac{n_1}{\sqrt{\frac{N_0}{2}}}$ $\qquad dn_i = \sqrt{\dfrac{N_0}{2}}\, dy$

$$= \int_{\frac{d}{\sqrt{2N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{y^2/2} dy = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\frac{\|s_k - s_i\|}{\sqrt{2N_0}}\right)$$