MA111

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10.8 Taylor series

Definition: Let f be function with derivatives of all orders throughout some interval containing a as an interior part. Then the taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Maclaurin series of f is the Taylor series generated by f at $x=a=0$ i.e

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

Q1) Find the taylor series generated by $f(x) = 1/x$ at $x=2$. Where, if anywhere, does the series converges to?

Ans $f(2) = 1/2$

$f'(x) = \frac{-1}{x^2}$ $\quad f''(x) = +\frac{2}{x^3}$ $\quad \dots \quad f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

when $x=2$

$\therefore f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$

Taylor series: $\sum_{k=1}^{\infty} \frac{f^{(k)}(a)(x-2)^k}{k!} = f(2) + f'(2)(x-a) + \dots + \frac{f^{(n)}(2)(x-a)^n}{n!}$

$$= \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 + \dots + \frac{(-1)^n (x-2)^n}{2^{n+1}}$$

For the series to converge,

$$\left| \frac{(-1)(x-2)}{2} \right| < 1$$

$$-1 < \frac{x-2}{2} < 1$$

$$-2 < x-2 < 2$$

$\Rightarrow$ $0 < x < 4$ $\rightarrow$ Interval of convergence

$\therefore$ Radius of convergence is $R = 2$

$\therefore$ Series converges to : $\dfrac{1/2}{1+\frac{(x-2)}{2}} = \dfrac{1}{x}$ for $0 < x < 4$

At $x = 0$ & $x = 4$ series won't converge.

Definition (Taylor polynomial)

Let f be a function with derivatives of order k for $k = 1, 2, 3, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 to N, the taylor polynomial of order n generated by f at $x = a$ is a polynomial,

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = f(a) + f'(a)(x-a) + \dots + \dots + \frac{f^{(k)}(a)(x-a)^k}{k!} + \dots + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Theorem (Taylor's theorem)

If f and it's first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b and $f^{(n)}$ is differentiable on the open interval between a and b then there exists a number c between a & b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)(b-a)^2}{2!} + \ldots + \ldots + \frac{f^{n}(a)(b-a)^n}{n!} + \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}$$

Taylor formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}, \quad (a < c < x)$$

Eg 1: Find the taylor polynomial $f(x) = \sin x$ at $x = 0$

Ans

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)\, x^k}{k!}$$

$f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$

$\therefore f^{(2n+1)}(x) = (-1)^n \cos x$ $\quad f^{(2n)}(x) = (-1)^n \sin x$

$f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = -1 \qquad f^{iv}(0) = 0$

$\therefore f^{(2n+1)}(0) = (-1)^n \qquad f^{(2n)}(0) = 0$

$\therefore P_0(x) = 0 \qquad P_1(x) = x$

$P_2(x) = x \qquad P_3(x) = x - \frac{x^3}{3!}$

$$\therefore P_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n-1}}{(2n-1)!}$$

Also $P_{2n-1} = P_{2n}$

NOTE:
- $P_n(x)$ denotes taylor polynomial of order n and not degree n.
- $f(x) = P_n(x) + R_n(x)$ where $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ where c lies between a & x.

RESULT: If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I = (a, b)$. We say that the taylor series generated by f at $x = a$ converges to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

Theorem (The reminder estimate theorem)

If there is a positive constant M such that $|f^{(n+1)}(t)| < M$ for all t between x and a then the remainder term $R_n(x)$ in Taylor's theorem satisfies the inequality

$$|R_n(x)| \leq \frac{M\,|x-a|^{(n+1)}}{(n+1)!}$$

If this inequality holds to every n and the conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

Eg 2: Show that the Taylor series for $f(x) = \cos x$ at $x = 0$ converges to $f(x)$ for every $x \in \mathbb{R}$

Ans $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + R_n(x)$$

But $$R_n(x) = \frac{f^{(2n+1)}(c)\, x^{2n+1}}{(2n+1)!}$$

$$f^{(2n+1)}(x) = (-1)^n \sin x$$

$|\sin x| \leq 1$ $\quad\therefore\quad$ $$|R_n(x)| \leq \frac{x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty} \frac{x^{2n+1}}{(2n+1)!} = 0, \quad x \in \mathbb{R}$$

$\therefore$ For $\forall x \in \mathbb{R}$, $R_n(x) \to 0$ as $n \to \infty$

$$\therefore \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Eg 3: $f(x) = e^x$

Ans $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

But $$R_n(x) = \frac{f^{(n+1)}(c)\, x^n}{n!}$$

$f^{(n+1)}(x) = e^x \qquad \therefore f^{(n+1)}(c) = e^c \qquad (0 < c < x)$

Consider $M > |x|$ then $e^c < e^M$

$\therefore |R_n(x)| \leq \dfrac{e^M x^{n+1}}{(n+1)!}$

Consider $\displaystyle\lim_{n\to\infty} \dfrac{e^M x^{n+1}}{(n+1)!} = 0 \Rightarrow R_n \to 0 \ \forall x \in \mathbb{R}$ as $n \to \infty$

$\therefore e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Theorem (The alternating series estimation theorem)

If the alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the 3 conditions of the alternating series test then for $n \geq N$

$$S_n = u_1 + u_2 + \dots + u_n.$$

Approximate the sum of the series with an error whose absolute value less than u_{n+1}, the absolute value of the first unused term. Furthermore the sum L lies between any two successive partial sum S_n and S_{n+1} and the remainder $L - S_n$ has the same sign as the first unused term.

Theorem (Series Multiplication of power series)

If $A(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and series $B_n(x) = \displaystyle\sum_{n=0}^{\infty} b_n x^n$ converges absolutely for $|x| < R$, and

$C_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0$

$$= \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)\,B(x)$ for $|x| < R$

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n$$

Eg 4) For what values of x can we replace $\sin x$ by $x - \left(\frac{x^3}{3!}\right)$ and obtain an error whose magnitude is no greater than 3×10^{-4}?

Ans Taylor series generated by $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

This is an alternating series for every non zero values of x.

According to alternating series estimate theorem, the error in truncating $\sin x = x - \frac{x^3}{3!}$ is less than $\left|\frac{x^5}{5!}\right|$

$$\Rightarrow \frac{x^5}{5!} < 3 \times 10^{-4}$$

$$\Rightarrow |x^5| < 3 \times 5! \times 10^{-4}$$

$$\Rightarrow |x| < 0.5143$$

$$\Rightarrow -0.5143 < x < 0.5143$$

Eg 5 Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate value of $\sin x$ at $x = 0.1$

Eg 6 Estimate the error if $P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ is used to estimate value of e^x at $x = 0.5$.

Eg 7 Find the Taylor series generated by the following functions.

i> xe^x ii> $x^4 \sin x$ iii> $e^{-x/2}$ iv> $\frac{1}{2-x}$

v> $\frac{1}{2}(2x + x\cos x)$ vi> $e^x \cos x$ vii> $e^{\sin x}$

viii> $e^x \sin x$ ix> $\cos^2 x \sin x$ x> $\cos x \sin x$

Eg 5 $\sin x = x - \frac{x^3}{3!}$

$\therefore$ Error is less than $\left|\frac{x^5}{5!}\right|$

At $x = 0.1$ error $< \frac{(0.1)^5}{120} = \frac{1}{120} \times 10^{-5}$

$= 8.33 \times 10^{-8}$ //

Eg 6 $P_4(x)$ of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$\therefore$ Error $< \left|\frac{x^5}{5!}\right|$

At $x = 0.5$ error $< \frac{(0.5)^5}{120} = 2.604 \times 10^{-4}$ //

Eg 7 Taylor series of xe^x

i> $xe^x = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots$

$f'(x) = e^x + xe^x$ $f'(0) = 1$

$f''(x) = e^x + e^x + xe^x$ $f''(0) = 2$

|||ly $f^{(n)}(0) = n$

$$\therefore xe^x = x + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} + \dots$$

$$= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots$$

$$\therefore xe^x = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

ii) $x^4 \sin x$

$f'(x) = 4x^3 \sin x + x^4 \cos x$, $f'(0) = 0$

$f''(x) = 12x^2 \sin x + 4x^3 \cos x + 4x^3 \cos x - x^4 \sin x$

$f''(0) = 0$

$f'''(x) = 24x \sin x - 12x^2 \cos x + (12x^2 \cos x - 4x^3 \sin x)2 - 4x^3 \sin x$

$- x^4 \cos x$

$f'''(0) = 0$

iii) $e^{-x/2}$

$f'(x) = -\frac{1}{2} e^{-x/2}$ $\qquad$ $f'(0) = -1/2$

$f''(x) = \frac{1}{4} e^{-x/2}$ $\qquad$ $f''(0) = \frac{1}{4}$

$\therefore f^{(n)}(x) = \frac{(-1)^n}{2^n} e^{-x/2}$ $\qquad$ $\therefore f^{(n)}(0) = \frac{(-1)^n}{2^n}$

$\therefore e^{-x/2} = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{32} + \ldots$

$\therefore e^{-x/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

iv) $1/(2-x)$

$f'(x) = \frac{+1}{(2-x)^2}$ $\qquad$ $f'(0) = \frac{1}{4}$

$f''(x) = \frac{+2}{(2-x)^3}$ $\qquad$ $f''(0) = \frac{1}{4}$

$f'''(x) = \frac{2 \times 3}{(2-x)^4}$ $\qquad$ $f''(0) = \frac{3}{4}$

$f^{iv}(x) = \frac{2 \times 3 \times 4}{(2-x)^5}$ $\qquad$ $f^{iv}(0) = \frac{4!}{2^5}$

$\therefore \frac{1}{2-x} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{4 \times 2!} + \frac{3!}{2^4}\frac{x^3}{3!} + \frac{4!}{2^5}\frac{x^4}{4!} + \ldots$

$\therefore \frac{1}{2-x} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$ //

v) $(1/2)(2x + x\cos x)$

$f'(x) = 1 + \frac{\cos x}{2} - \frac{x \sin x}{2}$ $\qquad$ $f'(0) = 3/2$

$f''(x) = -\frac{\sin x}{2} - \frac{\sin x}{2} - \frac{x \cos x}{2}$ $\qquad$ $f''(0) = 0$ //

$$f'''(x) = -\frac{\cos x}{2} - \frac{\cos x}{2} - \frac{\cos x}{2} + \frac{x\sin x}{2} \qquad f'''(0) = \frac{-3}{2}$$

$$f^{IV}(x) = +\frac{\sin x}{2} + \frac{\sin x}{2} + \frac{\sin x}{2} + \frac{\sin x}{2} + \frac{x\cos x}{2}$$

$$\therefore f^{IV}(0) = 0$$

$$\therefore \frac{2x + x\cos x}{2} = \frac{3x}{2} + \sum_{n=3}^{\infty} \frac{(-1)^n x^{2n-3}}{2(2n-3)!}$$

vi> $e^x \cos x$

$$f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$$

$$f'(0) = 1$$

$$f''(x) = e^x(\cos x - \sin x) - e^x(\sin x + \cos x)$$
$$= -2e^x \sin x \qquad f''(0) = 0$$

$$f'''(x) = -2e^x(\sin x + \cos x) \qquad f'''(0) = -2$$

$$f^{IV}(x) = -2e^x(\sin x + \cos x) - 2e^x(\cos x - \sin x)$$
$$= -4e^x \cos x \qquad f^{IV}(0) = -4$$

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \frac{4x^5}{5!} + \dots$$

$$e^x \cos x = \sum_{n=0}^{\infty} (-4)^n \left(\frac{x^{4n}}{4n!} + \frac{x^{4n+1}}{(4n+1)!} - \frac{2x^{4n+3}}{(4n+3)!} \right)$$

vii> $e^{\sin x}$

$$f'(x) = e^{\sin x} \cos x \qquad f'(0) = 1$$

$$f''(x) = e^{\sin x}(\cos^2 x - \sin x) \qquad f''(0) = 1$$

$$f'''(x) = 0$$

$$f^{IV}(x) = -3$$

$$\therefore e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \dots$$

25-7-22

Fourier series

- $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

This representation of $f(x)$ is called fourier series.

where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$ $\qquad$ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$

Q1) Find the fourier coefficients of

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$

Ans $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$

$= \frac{k}{2\pi}\left[-\pi + \pi\right] = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{k}{\pi}\left[\int_{-\pi}^{0} -\cos nx\,dx + \int_{0}^{\pi} \cos nx\,dx\right]$

$= \frac{k}{\pi}\left(\left[-\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right) = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{k}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx\right]$

$= \frac{k}{\pi}\left(\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n}\right]_{0}^{\pi}\right) = \frac{4k}{\pi n}$ (for odd n)

$$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{4k \sin((2n-1)x)}{(2n-1)\pi}$$

Orthogonality of the trigonometric system

The trigonometric system $\{\cos nx, \sin nx\}$ is orthogonal on the interval $-\pi \leq x \leq \pi$ (hence also on $0 \leq x \leq 2\pi$ or any other interval of length 2π because of periodicity)

a) $\displaystyle\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0$ if $n \neq m$

b) $\displaystyle\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$ if $n \neq m$

c) $\displaystyle\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad \forall\, n, m \in \mathbb{N}$

<u>Proof</u>: a) $\displaystyle\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos (n+m)x + \cos (n-m)x) \, dx$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0 \,//$$

b) $\displaystyle\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{-\pi}^{\pi} \frac{-1}{2} (\cos (n+m)x - \cos (n-m)x) \, dx$

$$= \frac{-1}{2}\left[\frac{\sin(n+m)x}{n+m} - \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0 \,//$$

c) $\displaystyle\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2} (\sin(n+m)x + \sin(n-m)x) \, dx$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)x}{n+m} + \frac{\cos(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0 \,//$$

Theorem

Let $f(x)$ be periodic function with period 2π and piecewise continuous function in the interval $-\pi \le x \le \pi$. Furthermore, let $f(x)$ have a left hand derivative and right hand derivative at each point of the interval. Then the fourier series $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ of $f(x)$ with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ converges. It's sum is $f(x)$.

except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of left hand and right hand limits of $f(x)$ at x_0.

Fundamental period

Let $f(x)$ be a periodic function. The smallest positive period of the function is called fundamental period of the function.

Q1) Find the fundamental period of following functions
$\cos x$, $\sin x$, $\cos nx$, $\sin nx$, $\sin \pi x$, $\cos \pi x$,
$\cos 2\pi x$, $\sin 2\pi x$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$

Ans $\cos x$, $\sin x$, period $= 2\pi$

$\cos nx$, $\sin nx$ $= \frac{2\pi}{n}$

$\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$ $= \frac{2\pi}{2\pi} \times k = k$

$\cos \pi x$, $\sin \pi x$ $= 2\pi/\pi = 2$

$\cos 2\pi x$, $\sin 2\pi x = \frac{2\pi}{2\pi} = 1$

NOTE If $f(x)$ has period p, then $f(ax)$, $a \neq 0$ and $f\left(\frac{x}{b}\right)$, $b \neq 0$ has period $\frac{p}{a}$ & bp respectively.

Periodic function of period $2L$, $L > 0$

Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(a)$ as a function of v have period 2π, if we set $x = \frac{p}{2\pi} v = \frac{L}{\pi} v \Rightarrow v = \frac{\pi}{L} x \rightarrow (1)$ then

$v = \pm \pi$ when $x = \pm L$.

This means that f, as a function of v has period 2π and therefore a fourier series of the form,

$$f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} \theta\right) dv$ $\qquad a_n = \int_{-\pi}^{\pi} \frac{\cos nv}{\pi} f\left(\frac{L}{\pi} v\right) dv$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} \theta\right) \sin nv \, dv.$ $\qquad v = \frac{\pi}{L} x \qquad dv = \frac{\pi}{L} dx \rightarrow (2)$

The fourier series of $f(x)$ of period $2L$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$ $\qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$

Q 3) Find the fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

and $f(x+4) = f(x)$ for all $x \notin [-2, 2]$

Ans $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x) \cos \frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} k \cos \frac{n\pi x}{2}\,dx = \frac{k}{\cancel{2}}\left[\sin \frac{n\pi x}{2}\right]_{-1}^{1} \times \frac{1}{n\pi} \times \cancel{2}$$

$$= \frac{k}{n\pi}\left[\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right] = 2\frac{k}{n\pi} \sin \frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k \sin \frac{n\pi x}{2}\,dx$$

$$= -\frac{k}{2}\left[\cos \frac{n\pi x}{2}\right]_{-1}^{1} \times \frac{2}{n\pi} = 0 //$$

$\therefore$ fourier series : $\frac{k}{2} + \sum_{n=1}^{\infty} \frac{2k}{n\pi} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2}$

NOTE If $f(x) = f(-x)$ $\forall x$ then f is even.
If $f(-x) = -f(x)$ $\forall x$ then f is odd.

If f is odd $\int_{-L}^{L} f(x)\,dx = 0$

If f is even $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

- If $f(x)$ is even function and $f(x+2L) = f(x)$, its fourier series reduces to Fourier cosine series

i.e $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$

$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

If f is odd function and $f(x+2L) = f(x)$ then its fourier series reduces to fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Theorem (Sum and scalar multiple)

The fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 & f_2.

The fourier coefficients of kf are k times the corresponding coefficients of f.

Q4> Find the fourier series of the function $f(x) = x + \pi$ if $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$

Ans $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2} + \pi x\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\pi\cos nx\,dx$$

$$= \frac{2}{n}\sin nx\Big]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[x\int_0^{\pi}\sin nx - \int_0^{\pi}\left(\int_0^{\pi}\sin nx\,dx\right)dx\right]$$

$$= \frac{2}{\pi}\left(\left[\frac{-x\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}\left[\frac{-\cos nx}{n}\right]dx\right)$$

$$= \frac{2}{\pi}\left(\left[-\frac{\pi}{n}\cos n\pi\right] + \left.\frac{\sin nx}{n^2}\right]_0^{\pi}\right)$$

$$= -\frac{2}{n}\cos n\pi \quad = \frac{2}{n}(-1)^{n+1}$$

$$\therefore \quad f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \, dx$$

Q5) Find the two half range expansion of the function.

$$f(x) = \begin{cases} \dfrac{2kx}{M} & \text{if } 0 < x < \dfrac{M}{2} \\ \dfrac{2k(M-x)}{M} & \dfrac{M}{2} < x < M \end{cases}, \quad f(x+M) = f(x)$$

Ans

$$a_0 = \frac{1}{M}\int_0^M f(x)\,dx = \frac{1}{M}\left[\int_0^{M/2} \frac{2kx}{M}\,dx + \int_{M/2}^{M} \frac{2k(M-x)}{M}\,dx\right]$$

$$= \frac{1}{M}\left[\frac{2k}{M} \times \frac{M^2}{4} \times \frac{1}{2} + \frac{2k}{M}\left(\frac{M}{2} \times M - \frac{1}{2}\left(M^2 - \frac{M^2}{4}\right)\right)\right]$$

$$= \frac{1}{M}\left[\frac{kM}{4} + \frac{2k}{M}\left[\frac{M^2}{2} - \frac{3M^2}{8}\right]\right]$$

$$= \frac{1}{M}\left[\frac{kM}{4} + \frac{2k}{M} \times \frac{M^2}{8}\right] = \frac{1}{M} \times \frac{kM}{2} = \frac{k}{2}$$

$$a_n = \frac{2}{M}\int_0^M f(x) \sin\left(\frac{2\pi}{M} nx\right) dx$$

$$= \frac{2}{M}\left[\int_0^{M/2} \frac{2kx}{M} \sin\left(\frac{2\pi}{M} nx\right) dx + \int_{M/2}^{M} \frac{2k(M-x)}{M} \sin\left(\frac{2\pi nx}{M}\right) dx\right.$$

$$a_n = \frac{2}{M}\left[\frac{2k}{M}\int_0^{M/2} x \sin\frac{2\pi n x}{M}\,dx\right] + \frac{2}{M}\left[2k\int_{M/2}^{M} \sin\frac{n2\pi x}{M}\,dx\right]$$
$$+ \frac{2}{M}\left[\frac{2k}{M}\int_{M/2}^{M} -x\sin\left(\frac{2\pi n x}{M}\right)dx\right]$$

$$a_n = \frac{2}{M}\times\frac{2k}{M}\left[\left(\sin\frac{2\pi n x}{M}\right)\times\frac{M^2}{4\pi^2 n^2} - x\cos\left(\frac{2\pi n x}{M}\right)\times\frac{M}{2\pi n}\right]_0^{M/2}$$

$$+ 2k\times\frac{2}{M}\times\frac{-M}{2\pi n}\cos\frac{2\pi n x}{M}\Bigg]_{M/2}^{M}$$

$$-\frac{2}{M}\times\frac{2k}{M}\left[\sin\left(\frac{2\pi n x}{M}\right)\times\frac{M^2}{4\pi^2 n^2} - x\cos\left(\frac{2\pi n x}{M}\right)\times\frac{M}{2\pi n}\right]_{M/2}^{M}$$

$$a_n = \frac{4k}{M^2}\left[-\frac{M}{2}\times\frac{M}{2\pi n}\cos n\pi\right] - \frac{4k}{M}\left[\frac{(1-\cos n\pi)M}{2\pi n}\right]$$
$$-\frac{4k}{M^2}\left[\frac{-M}{2\pi n}\left(M - \frac{M}{2}\cos n\pi\right)\right]$$
$$= -\frac{k\cos n\pi}{n\pi} - \frac{2k}{\pi n}(1-\cos n\pi) + \frac{2k}{\pi n}\left(1 - \frac{\cos n\pi}{2}\right)$$
$$= -\frac{k\cos n\pi}{n\pi} - \frac{2k}{n\pi} + \frac{2k\cos n\pi}{n\pi} + \frac{2k}{n\pi} - \frac{k\cos n\pi}{n\pi}$$
$$= 0$$

$$b_n = \frac{2}{M}\int_0^M f(x)\cos\frac{2\pi n x}{M}\,dx$$
$$= \frac{2}{M}\left[\int_0^{M/2}\frac{2kx}{M}\cos\frac{2\pi n x}{M}\,dx + \int_{M/2}\frac{2k}{M}(M-x)\cos\frac{2\pi n x}{M}\,dx\right.$$

$$= \frac{2}{M}\left[\int_0^{M/2}\frac{2kx}{M}\cos\frac{2\pi n x}{M}\,dx + \int_{M/2}^{M} 2k\cos\frac{2\pi n x}{M}\,dx - \right.$$

$$- \int_{M/2}^{M} \frac{2k}{M} x \cos \frac{2\pi n x}{M} \, dx \Bigg]$$

$$= \frac{2}{M} \times \left[\frac{2k}{M} \left(x \sin \left(\frac{2\pi n x}{M} \right) \times \frac{M}{2\pi n} + \cos \left(\frac{2\pi n x}{M} \right) \times \frac{M^2}{4\pi^2 n^2} \right) \right]_0^{M/2}$$

$$+ \frac{2}{M} \times \left[2k \sin \left(\frac{2\pi n x}{M} \right) \times \frac{M}{2\pi n} \right]_{M/2}^{M}$$

$$- \frac{2}{M} \times \left[\frac{2k}{M} \left(x \sin \left(\frac{2\pi n x}{M} \right) \times \frac{M}{2\pi n} + \cos \left(\frac{2\pi n x}{M} \right) \times \frac{M^2}{4\pi^2 n^2} \right] \right._{M/2}^{M}$$

$$b_n = \frac{2}{M} \times \frac{M^2}{4\pi^2 n^2} \times \frac{2k}{M} \left(\cos n\pi - 1 \right) - \frac{2}{M} \times \frac{2k}{M} \times \frac{M^2}{4\pi n^2} \left(1 - \cos n\pi \right)$$

$$= - \frac{4k}{M^2} \times \frac{M^2}{4\pi^2 n^2} \times 2 \left(1 - \cos n\pi \right)$$

$$= - \frac{2k}{n^2 \pi^2} \left(1 - \cos n\pi \right)$$

$$\therefore f(x) = \frac{k}{2} - \frac{2k}{\pi^2} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)}{n^2} \cos \frac{2n\pi x}{M}$$

$$= \frac{k}{2} - \frac{4k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left(\frac{2\pi (2n-1) x}{M} \right)$$

ORDINARY DIFFERENTIAL EQUATIONS

A function $F(x, y, y', y'', \dots, y^{(n)}) = 0$ is a ordinary differential equation. Where x is independent variable $y(x)$ is a function of x.

→ First order ODE:-

$F(x, y, y') = 0$ → Implicit form
$y' = f(x, y)$ → explicit form.

- Order is the highest differential of independent variable that appears in the equation.

- Solution containing arbitrary constant is general solution of the ODE. If fixed constant is present then it is called particular solution of ODE.

<u>Initial value problem:</u>
An ODE together with an initial condition $y(x_0) = y_0$ is called an initial value problem. If ODE is explicit $y' = f(x, y)$, then IVP is of the $y(x_0) = y_0$

<u>Eg 1:</u> Solve $y' = 3y$, $y(0) = 2$
Ans
$$\int \frac{dy}{y} = \int 3\,dx$$
$$\ln y = 3x$$
$$y = C e^{3x}$$
$$2 = C e^{3x}$$
$$\therefore y(x) = 2e^{3x}$$

2> $y' + 2\sin 2\pi x = 0$

Ans $\int dy = \int -2\sin 2\pi x \, dx$

$y = \dfrac{\cos 2\pi x}{\pi} + C$

3> $y' + x e^{-x^2/2} = 0$

Ans $\int dy = -\int x e^{-x^2/2} \, dx$

$y = \int e^{t} dt$ where $t = \dfrac{-x^2}{2}$

$y = e^{t} + C$

$y = e^{-x^2/2} + C$

4> $y' = 4e^{-x} \cos x$

Ans $\int dy = 4 \int e^{-x} \cos x \, dx$

$= 4 \left[-\cos x \, e^{-x} - \int e^{-x} \sin x \, dx \right]$

$y = 4\left(-e^{-x} \cos x + e^{-x} \sin x - \int \cos x \, e^{-x} dx \right)$

$y = 4 e^{-x} (\sin x - \cos x) - 4y$

$y = \dfrac{4}{5} e^{-x} (\sin x - \cos x) + C$ //

5> $y'' = -y$

Ans $y = \sin x + C$ //

Separable ODE

$$g(y)\,y' = f(x)$$

$$\int g(y)\,y'\,dx = \int f(x)\,dx$$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$$

Eg 1: $y' = 1 + y^2$

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c)$$

Eg 2: $y' = (1+x)e^{-x}y^2$

Ans

$$\int \frac{dy}{y^2} = \int e^{-x}\,dx + \int x e^{-x}\,dx + c$$

$$-\frac{1}{y} = -e^{-x} + (-xe^{-x} - e^{-x}) + c$$

$$-\frac{1}{y} = -(2+x)e^{-x} + c$$

$$\Rightarrow y = \frac{1}{(2+x)e^{-x} + c'} \qquad \text{where } c' = -c$$

Eg 3: $y' = -2xy, \quad y(0) = 3$

$$\int \frac{dy}{y} = \int -2x\,dx$$

$$\ln y = -x^2 + C$$
$$y = C' e^{-x^2}$$
$$3 = C' \qquad \Rightarrow \qquad y(x) = 3e^{-x^2}$$

Q1) Find general solution for:

i) $y^3 y' + x^3 = 0$ ii) $y' = \sec^2 y$

iii) $y' \sin 2\pi x = \pi x \cos 2\pi x$

Ans i) $\int y^3 \, dy = \int x^3 \, dx + c$

$$\frac{y^4}{4} = -\frac{x^4}{4} + C$$

$$y^4 = C' - x^4 \qquad \text{where} \quad C' = 4C$$

ii) $\int \cos^2 y \, dy = \int dx + C$

$$\int \frac{1 + \cos 2y}{2} \, dy = x + C$$

$$\frac{y}{2} + \frac{\sin 2y}{4} = x + C$$

iii) $\int dy = \int \pi x \cot 2\pi x \, dx + C$

$=$

Homogeneous method

If $y' = f\left(\frac{y}{x}\right)$

put $y = ux$ $\Rightarrow y' = u + x\frac{du}{dx}$

$$\Rightarrow u + x\frac{du}{dx} = f(u)$$

$$\Rightarrow \int \frac{du}{f(u)-u} = \int \frac{dx}{x} + C$$

Q1> Solve, $y' = \dfrac{y^2 - x^2}{2xy}$

Ans $y' = \dfrac{y^2/x^2 - 1}{2(y/x)}$

Put $y = ux$

$$u + x\frac{du}{dx} = \frac{u^2-1}{2u}$$

$$x\frac{du}{dx} = \frac{u^2 - 1 - 2u^2}{2u} = -\frac{(u^2+1)}{2u}$$

$$\Rightarrow \int \frac{2u\,du}{u^2+1} = -\int \frac{dx}{x} + C$$

$$\ln(u^2+1) = -\ln x + C$$

$$u^2 + 1 = e^{-\ln x} \cdot e^{C}$$

$$u^2 + 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} = \frac{C-x}{x} \Rightarrow y = \pm\sqrt{x(C-x)}$$

Also $\quad x^2 - cx + y^2 = 0$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

Q2> Find a general solution of following ODE,

i> $y' = e^{(2x-1)} y^2$ $\qquad$ ii> $xy' = x + y$

iii> $xy' = y^2 + y$ $\qquad$ iv> $xy' + y = 0$, $\quad y(4) = 6$

v> $y' = -\frac{4x}{y}$ $\qquad y(2) = 3$

Ans i> $$\int \frac{dy}{y^2} = \int e^{(2x-1)} dx + C$$

$$\frac{-1}{y} = \frac{e^{2x-1}}{2} + C$$

$$y = \frac{2}{C' - e^{2x-1}} \qquad C' = -2C$$

ii> $$\frac{dy}{dx} = 1 + \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = 1$$

$y = ux$

$$\cancel{u} + x\frac{du}{dx} = 1 + \cancel{u}$$

$$\int du = \int \frac{dx}{x} + C$$

$$u = \ln x + C$$

$$y = x(\ln x + C)$$

iii) $\int \frac{dy}{y(y+1)} = \int \frac{dx}{x} + C$

$\int \frac{dy}{y} - \frac{dy}{y+1} = \ln x + C$

$\ln \frac{y}{y+1} = \ln x + C'$

$\frac{y}{y+1} = Ce^{x} \qquad C = e^{C'}$

$y = yCe^{x} + Ce^{x}$

$y = \frac{Ce^{x}}{1-Ce^{x}}$ //

Exact ODE

$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow (1)$

For the ODE to be exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

An ODE is said to be exact if there is a function $u(x,y)$ such that $du = M\,dx + N\,dy \rightarrow (2)$

But $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \rightarrow (3)$

$\Rightarrow M = \frac{\partial u}{\partial x}$ & $N = \frac{\partial u}{\partial y}$

But from (1) $du = 0 \Rightarrow u(x,y) = C.$

We have $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

But $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Q3) Verify whether is $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Ans Here $M = x^2 - 4xy - 2y^2$

$N = y^2 - 4xy - 2x^2$

$\dfrac{\partial M}{\partial y} = -4x - 4y$ $\qquad \dfrac{\partial N}{\partial x} = -4y - 4x$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ It is exact.

ii) $(x+y)^2 dx - (y^2 - 2xy - y^2)\,dy = 0$

$\dfrac{\partial M}{\partial x} = 2(x+y)$ $\qquad \dfrac{\partial N}{\partial y} = -(-2)(x+y) = 2(x+y)$

$\therefore$ It is exact.

If given ODE is exact, then we solve ODE by following method:

1) Compare the given equation with $Mdx + Ndy = 0$ and find out M & N.

2) Find out $\dfrac{\partial M}{\partial y}$ & $\dfrac{\partial N}{\partial x}$. If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then we conclude that given equation is exact.

3) Integrate M with respect to x treating y as a constant.

4) Integrate with respect y only those terms of N which do not contain any x.

5> Equate the sum of these two integrals with an arbitrary constant. Thus we obtain required solution.

i.e $\int M\,dx + \int \begin{pmatrix}\text{terms in } N \text{ not} \\ \text{containing } x\end{pmatrix} dy = C$

(treating y as constant)

Q4> Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Ans This is exact

$$\int M\,dx = \int (x^2 - 4xy - 2y^2)\,dx$$

$$= \frac{x^3}{3} - 2x^2y - 2y^2x$$

$$\int N\,dy = \int y^2\,dy = \frac{y^3}{3}$$

$\therefore$ Solution is $\frac{x^3}{3} + \frac{y^3}{3} - 2xy(x+y) = C$

Integrating factor: If an equation of the form $M\,dx + N\,dy = 0$ is not exact, it can always be made exact by multiplying by some function $F(x,y)$. Such a multiplier i.e $F(x,y)$ is called integrating factor.

Consider, $P\,dx + Q\,dy = 0 \rightarrow (1)$ (not exact)

Now, $FP\,dx + FQ\,dy = 0$ is exact

$$\Rightarrow \frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x}$$

$$F_y P + F P_y = F_x Q + F Q_x \to (3)$$

If $F(x,y) = F(x)$, then (3) becomes

$$F P_y = F' Q + F Q_x \to (4)$$

Dividing by $1/FQ$ we get

$$\frac{P_y}{Q} = \frac{F'}{F} + \frac{Q_x}{Q}$$

$$\Rightarrow \frac{dF}{F} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx \to (5)$$

Integrating (5)

$$\int \frac{dF}{F} = \int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx$$

$$\Rightarrow \ln F = \int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx$$

$$\therefore F = e^{\int R(x)dx} \quad \text{where } R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) //$$

If $F(x,y) = F(y)$ then

$$F(y) = e^{\int R(y)dy} \qquad R(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) //$$

Q5) Find an integrating factor and solve the ODE

$$(e^{x+y} + ye^y)\,dx + (xe^y - 1)\,dy = 0, \quad y(0) = -1$$

Ans $\dfrac{\partial P}{\partial y} = e^{x+y} + e^y(y+1) \qquad \dfrac{\partial Q}{\partial x} = e^y$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = e^{x+y} + ye^y$$

$$F(y) = e^{\int R(y)\,dy}$$

$$\int R(y)\,dy = \int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy$$

$$= \int \frac{1}{e^{x+y}+ye^{y}} (-1)(e^{x+y}+ye^{y})\,dy$$

$$= \int -1\,dy = -y$$

$$\therefore F(y) = e^{-y}$$

$\therefore$ Exact ODE = $(e^{x}+y)dx + (x-e^{-y})dy = 0$

Solution : $\int (e^{x}+y)\,dx + \int -e^{-y}\,dy = c$

$$\Rightarrow e^{x}+yx+e^{-y} = c$$

Given $y(0) = -1$

$$1+0+e^{+1} = c$$

$$\therefore e^{x}+yx+e^{-y} = 1+e$$

Q6> Solve : i> $y(2xy+e^{x})dx - e^{x}dy = 0$

ii> $y\sin 2x\,dx = (1+y^{2}+\cos^{2}x)\,dy$

iii> $(x^{2}+y^{2})dx - 2xy\,dy = 0$

Linear ODE

A first order ODE is said to be linear if it can be brought into the form $y'+P(x)y = Q(x)$ and non linear if it cannot be brought into this format.

If $Q(x) = 0$ then $y' + P(x)y = 0$ is a homogeneous linear first order ODE.
If $Q(x) \neq 0$ then it is called non homogeneous linear first order ODE.
The general solution of linear ODE is

$$(I.F)\, y(x) = \int (I.F) \cdot Q(x)\, dx + C$$

where $I.F = e^{\int P(x)\, dx}$

Q7> $y' + y\tan x = \sin 2x$, $y(0) = 1$

Ans $I.F = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x$

$$\Rightarrow \sec x \cdot y = \int \sec x \sin 2x \, dx + C$$

$$\sec x \cdot y = 2\int \sin x \, dx + C$$

$$y \sec x = -2\cos x + C$$

$$1 = -2 + C \Rightarrow C = 3$$

$$\therefore y = -2\cos^2 x + 3\cos x$$

<u>Bernoulli's equation</u>

$$y' + P(x)\, y = g(x)\, y^a \rightarrow (1)$$

If $a = 0$ or $a = 1$, (1) becomes linear ODE.
If $a \neq 0$ and $a \neq 1$
take $u = y^{1-a}$

Then $u' = (1-a)\, y^{-a}\, y' \rightarrow$ ②

Substitute $y' = (g(x)\, y^{a} - p(x)\, y)$ in ②, we get

$u' = (1-a)\, y^{-a}\, (g(x)\, y^{a} - p(x)\, y)$

$\Rightarrow u' = (1-a)\, (g(x) - p(x)\, y^{1-a})$

$\Rightarrow u' + (1-a)\, u\, p(x) = (1-a)\, g(x)$

Which is a linear ODE

Q8) Solve : $y' = Ay - By^{2}$, $A, B \in \mathbb{R}$

Ans

$$\frac{y'}{y^{2}} - \frac{A}{y} = -B$$

Put $\frac{1}{y} = u \Rightarrow \frac{-1}{y^{2}} \frac{dy}{dx} = \frac{du}{dx}$

$$\Rightarrow \frac{du}{dx} + Au = B$$

$$IF = e^{\int A\, dx} = e^{Ax}$$

$$\Rightarrow u e^{Ax} = \int B e^{Ax}\, dx + C$$

$$u e^{Ax} = \frac{B}{A} e^{Ax} + C$$

$$u = \frac{B}{A} + C e^{-Ax}$$

$$\Rightarrow y = \frac{1}{\left(\frac{B}{A} + C e^{-Ax}\right)}$$ //

Consider :

i) $y' = 2x \quad , \quad y(0) = 1$

$\Rightarrow \int dy = \int 2x\,dx + C$

$\Rightarrow \quad y = x^2 + C \qquad$ Also $\quad y(0) = 1 \Rightarrow \quad 1 = 0 + C$

$\therefore \quad y = x^2 + 1 \quad \Rightarrow$ unique solution.

ii) $xy' = y - 1 \qquad y(0) = 1$

$$\Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x} + C$$

$\Rightarrow \quad \ln(y-1) = \ln x + C$

$\Rightarrow \quad y - 1 = C'x \qquad$ where $\quad C' = e^C$

$y(0) = 1 \quad \Rightarrow \quad$ This has infinite solution.

Existence theorem

Let the function $f(x, y)$ of the ODE, of the Initial value problem :

$$y' = f(x, y) \quad \& \quad y(x_0) = y_0 \quad \rightarrow (1)$$

be continuous at all points (x, y) in some rectangle $R : |x - x_0| < a \; , \; |y - y_0| < b$ and bounded on R such that, there is a number k such that $f(x, y) \leq k \; , \; \forall \; (x, y) \in R$.

Then the IVP (1) has atleast one solution $y(x)$. The solution exists atleast for all x in the subinterval $|x - x_0| < \alpha$

where $\alpha = \text{Min}\left\{ a, \frac{b}{k} \right\}$

Uniqueness theorem:

Let f and f_y be continuous for all (x, y) in R and bounded.

$$|f(x,y)| < K \quad , \quad |f_y(x,y)| < M \quad , \quad f(x,y) \in R.$$

Then the IVP has a unique solution $y(x)$ in the interval $|x - x_0| < \alpha$ where $\alpha = \text{Min}\left\{a, \frac{b}{k}\right\}$

$$y' = f(x,y) \qquad y(x_0) = y_0$$

$$dy = f(x,y)\, dx$$

$$y = \int f(x,y)\, dx + C$$

$$f = y(x) = \int_{x_0}^{x} f(x, y(t))\, dt + C$$

$$y_0 = \int_{x_0}^{x_0} f(x, y(x_0))\, dt + C$$

$$\Rightarrow \quad C = y_0$$

$$\Rightarrow \quad y(x) = y_0 + \int_{x_0}^{x} f(x, y(t))\, dt$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

Consider the IVP,

$$y' = f(x,y) \quad , \quad y(x_0) = y_0 \quad -(1)$$

with all conditions in existance theorem, we may write the IVP (1) in the integral form:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Consider the sequence of functions $(y_m(x))_{m=0}^{\infty}$ defined recursively : $y_0(x) = y_0 \longrightarrow (2)$ and

$$y_{m+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_m(t))\, dt \quad -(3)$$

Then $y(x) = \lim_{m\to\infty} y_n(x)$, $\forall x$, $|x-x_0|<a$ is the solution of the IVP. $\rightarrow$ (1)

Eg 1: Given that $\frac{dy}{dx} = x+y^2$ and $y=0$ when $x=0$.

Determine the value of y when $x=0.3$ correct to four places of decimals.

Ans Given, $f(x,y) = x+y^2$, $y_0=0$ at $x_0=0$

$$y(x) = \int_0^x (x+y^2)dx$$

$$y_{m+1}(x) = \int_{x_0}^{x} f(t, y_m(t))\,dt, \quad m = 0, 1, 2, \ldots$$

$\rightarrow$ This method is called Picard's Iteration method.

$$y_1(x) = \int_0^x (x+y_0^2)\,dx = \int_0^x x\,dx = \frac{x^2}{2}$$

at $x = 0.3$, $y_1(x) = \frac{(0.3)^2}{2} = 0.0450$

Second iteration,

$$y_2(x) = \int_0^x (x+y_1^2)\,dx = \int_0^x \left(x+\frac{x^4}{4}\right)dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

$$y_2(0.3) = \frac{(0.3)^2}{2} + \frac{(0.3)^2}{20} = 0.0451$$

$$y_3(x) = \int_0^x (x + y_2^2)\,dx = \int_0^x \left(x + \left(\frac{x^2}{2} + \frac{x^5}{20} \right)^2 \right) dx$$

$$= \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{x^7}{20} \right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^{11}}{4400} + \frac{x^8}{1600}$$

$$y_3(0.3) = \frac{(0.3)^2}{2} + \frac{(0.3)^5}{20} + \frac{(0.3)^8}{1600} + \frac{(0.3)^{11}}{4400} \cong 0.0451$$

Hence $y = 0.0451$ correct to four decimal places at $x = 0.3$.

Eg 2: If $\frac{dy}{dx} = 2 - \frac{y}{x}$ and $y(1) = 2$. Perform three iteration of Picard's method to estimate value of y when $x = 1.2$. Work to 4 decimal places.

Ans $f(x,y) = 2 - \frac{y}{x}$ $\qquad y_0(x_0) = 2$ at $x_0 = 1$

$$y_1(x) = 2 + \int_1^x f(x, y_0)\,dx = 2 + \int_1^x \left(2 - \frac{2}{x}\right) dx$$

$$= 2 + 2x - 2\ln x - 2 = 2(x - \ln x)$$

$$y_1(1.2) = 2.4 - 2\ln(1.2) = 2.0353$$

$$y_2(x) = 2 + \int_1^x \left(2 - \frac{y_1(x)}{x}\right) dx$$

$$= 2 + \int_1^x 2 - \left(\frac{2(x - \ln x)}{x}\right) dx$$

$$= 2 + \int_1^x \frac{2\ln x}{x}\,dx$$

$$y_2(x) = 2 + \cancel{2}\frac{(\ln x)^2}{\cancel{2}} = 2 + (\ln x)^2$$

$$y_2(1.2) = 2 + (\ln(1.2))^2 = 2.0332$$

$$y_3(x) = 2 + \int_1^x \left(2 - \frac{y_2}{x}\right) dx$$

$$= 2 + \int_1^x \left[2 - \left(\frac{2 + (\ln x)^2}{x}\right)\right] dx$$

$$= 2 + 2x - 2\ln x - (\ln x)^3/3 - 2$$

$$y_3(1.2) = 2.4 - 2\ln(1.2) - \frac{\ln(1.2)^3}{3}$$

$$= 2.4 - 0.3646 - 0.00202$$

$$\cong 2.03742$$

$\therefore$ $y(1.2) = 2.0374$ correct to 4 decimal places.

11-8-22

Theorem: (Fundamental Theorem for the second order homogeneous linear ODE)

For a homogeneous linear ODE $y'' + p(x)y' + q(x)y = 0$ any linear combination of 2 solutions on an open interval I is again a solution of given equation. In particular for such an equation sums and constant multiple of solutions are again solutions.

- A second order ODE is linear if equation of form $y'' + p(x)y' + q(x)y = r(x)$ has $r(x) = 0$

for all x. Then $y'' + p(x)\, y' + q(x) = 0$ is called second order linear homogeneous ODE

• The solutions y_1 & y_2 of $y'' + p(x)\, y' + q(x)\, y = 0$ are called proportional if for all x
$y_1(x) = k\, y_2(x)$ or $y_2(x) = m(y_1)$

• Linearly Independent: 2 functions $y_1(x)$ & $y_2(x)$ are called linearly independent on an interval I if $k\, y_1(x) + k\, y_2(x) = 0$ for all x in I implies $k_1 = k_2 = 0$.
If $y_1(x) = \frac{k_1}{k_2}\, y_2(x) \Rightarrow$ linearly dependent.
(proportional)

Basis: A basis of solutions of $y'' + p(x) y' + q(x) y = 0$ on an open interval I is a pair of linearly independent solution of $y'' + p(x) y' + q(x) y = 0$ on I. i.e $y(x) = C_1 y_1 + C_2 y_2$

Finding second solution:

Consider $y''(x) + p(x)\, y' + q(x) = 0$
Let $y_1(x)$ be a solution.
To find $y_2(x)$ which is linearly independent of $y_1(x)$ we substitute $y_2(x) = u(x)\, y_1(x)$
$\left(\because \frac{y_2(x)}{y_1(x)} \text{ is some function } u(x) \right)$

$y_2' = uy_1' + u'y_1 \rightarrow ①$

$y_2'' = 2u'y_1' + uy_1'' + u''y_1 \Rightarrow ②$

$\because$ y_2 is solution of given equation,

$y_2'' + p(x)\,y_2' + y_2 = 0 \rightarrow ③$

Substitute ① & ② in ③

$uy_1'' + u''y_1 + 2u'y_1' + (uy_1' + u'y_1)\,p(x) + q(x)\,(uy_1) = 0$

$\Rightarrow u\,(y_1'' + p(x)\,y_1' + q(x)\,y_1) + u''y_1 + 2u_1'y_1' + p(x)\,u'y_1 = 0$

Since y_1 is a solution, $y_1'' + p(x)y_1' + q(x)\,y_1 = 0$

$\Rightarrow u''y_1 + 2u_1'y_1' + p(x)\,u'y_1 = 0 \rightarrow ④$

let $v = u'$ $\quad v' = u''$

$\Rightarrow v'y_1 + 2vy_1' + p(x)\,v\,y_1 = 0$

$\div\ y_1$

$\Rightarrow v' + 2\dfrac{y_1'}{y_1}v + p(x)\,v = 0$

$\Rightarrow \dfrac{v'}{v} = -\left(\dfrac{2y_1'}{y_1} + p(x)\right)$

$\Rightarrow \displaystyle\int \frac{dv}{v} = -\int\left(2\frac{y_1'}{y_1} + p(x)\right)dx$

$\ln v = -\left(2\ln|y_1| + \displaystyle\int p(x)\,dx\right)$

$v = \dfrac{1}{y_1^2}\,e^{-\int p(x)\,dx}$

$\therefore\ u = \displaystyle\int \frac{1}{y_1^2}\,e^{-\int p(x)\,dx}\,dx = \int v\,dx$

Q1) Find basis solution of $(x^2 - x)\,y'' - xy' + y = 0$ given $y_1(x) = x$

Ans $p(x) = \dfrac{-x}{x^2 - x} = \dfrac{-1}{x-1}$ $\quad y_1(x) = x$

$$V = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

$$= \frac{1}{x^2} e^{\int +\frac{1}{x-1} dx} = \frac{1}{x^2} e^{\ln(x-1)} = \frac{x-1}{x^2}$$

$$u = \int v\,dx = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$= \ln x + \frac{1}{x}$$

$$\therefore\ y_2(x) = \left(\ln x + \frac{1}{x}\right) x = x \ln x + 1 \,//$$

Q2) Find the second solution of $xy'' - (x+1)y' + y = 0$ if $y_1(x) = e^x$

Ans $p(x) = -(1 + 1/x)$ $\quad y_1(x) = e^x$

$$V = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

$$= \frac{1}{e^{2x}} e^{-\int -\left(1+\frac{1}{x}\right) dx} = \frac{1}{e^{2x}} e^{(x + \ln x)} = \frac{xe^x}{e^{2x}} = xe^{-x}$$

$$u = \int v\,dx = \int xe^{-x}\,dx$$

$$u = -(x+1)e^{-x}$$

$$\therefore\ y_2(x) = uy_1 = -(x+1) \,//$$

Second order linear homogeneous ODE with constant coefficients

Consider $y'' + p(x)y' + q(x)y = 0 \rightarrow (1)$

Let $y_1(x)$, $y_2(x)$ be solution of (1) such that $y_2 = u y_1$

where $u = \int v\,dx$, $v = \frac{1}{y_1^2} e^{-\int p\,dx}$

Now if p, q are constants i.e $y'' + ay' + b = 0$, $a, b \in \mathbb{R}$

let $y(x) = e^{\lambda x}$ λ is a constant

then $y'' = \lambda^2 e^{\lambda x}$ & $y' = \lambda e^{\lambda x}$

$\Rightarrow (\lambda^2 + a\lambda + b) e^{\lambda x} = 0 \rightarrow (2)$

Conclusion: $y(x) = e^{\lambda x}$ is a solution of given differential equation if and only if λ is a root of $\lambda^2 + a\lambda + b = 0$

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} , \quad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

$\rightarrow$ 2 distinct real roots:

If $a^2 - 4b > 0$ then $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $y_1(x) = e^{\lambda_1 x}$ $y_2(x) = e^{\lambda_2 x}$ are basis solutions of given differential equation.

$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$ is a general solution where $C_1, C_2 \in \mathbb{R}$.

$\rightarrow$ Real double roots: If $a^2 - 4b = 0$, then $\lambda_1 = \lambda_2 = -a/2 \in \mathbb{R}$ &

$y_1(x) = e^{\alpha x}$, $y_2(x) = u(x) e^{\alpha x}$ where $\alpha = -\frac{a}{2}$

Here $P(x) = a$

$$V = \frac{1}{y_1^2} e^{-\int P dx} = \frac{1}{e^{-ax}} e^{-ax} = 1$$

$$u = \int V dx = \int dx = x$$

$\therefore$ $y_2(x) = x e^{\alpha x}$

$\Rightarrow$ $y_1(x) = e^{\alpha x}$ $\quad y_2(x) = x e^{\alpha x}$

$y(x) = e^{\alpha x}(A + Bx)$

$\Rightarrow$ **Complex conjugate roots:** If $a^2 - 4b < 0$ then

$$\lambda_1 = \frac{-a + i\sqrt{4b-a^2}}{2} \qquad \lambda_2 = \frac{-a - i\sqrt{4b-a^2}}{2}$$

$\lambda_1 = \alpha + i\beta$ $\qquad \lambda_2 = \alpha - i\beta$

where $\alpha = -a/2$ $\qquad \beta = \sqrt{4b-a^2}/2$

$$e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

$$e^{\lambda_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$y_1(x) = \frac{e^{\lambda_1 x} + e^{\lambda_2 x}}{2} = e^{\alpha x}\cos\beta x$$

$$y_2(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{2i} = e^{\alpha x}\sin\beta x$$

$\therefore$ $y_1(x) = e^{\alpha x}\cos\beta x$ and $y_2(x) = e^{\alpha x}\sin\beta x$ are basis solutions of given differential equation and $y(x) = e^{\alpha x}[C_1\cos\beta x + C_2\sin\beta x]$ is a general solution

Ex1 Solve $y'' - 4y' + 5y = 0$

Ans $a = -4$ $\qquad b = 5$

$a^2 - 4b = 16 - 20 = -4 < 0$

$\Rightarrow$ $\lambda_1 = 2 + i$ $\qquad \lambda_2 = 2 - i$

$\Rightarrow \quad y_1(x) = e^{2x}\cos x \qquad\qquad y_2(x) = e^{2x}\sin x$

$\therefore \; y(x) = e^{2x}[C_1\cos x + C_2 \sin x] \quad , \quad C_1, C_2 \in \mathbb{R}.$

Solve the following ODE:

1) $y' + 25y = 0$

Ans $a = 0 \qquad b = 25$

$\lambda_1 = 5i \qquad \lambda_2 = -5i$

$y(x) = [C_1 \cos 5x + C_2 \sin 5x] \quad , \quad C_1, C_2 \in \mathbb{R}$

2) $y'' + y' - 2y = 0 \quad , \quad y(0) = 4 \quad , \quad y'(0) = -5$

Ans $a = 1 \qquad b = -2$

$\lambda_1 = \frac{-1+\sqrt{1+8}}{2} = 1 \qquad \lambda_2 = -2$

$y(x) = C_1 e^{x} + C_2 e^{-2x}$

We have, $y(0) = 4$ & $y'(0) = -5$

$\Rightarrow 4 = C_1 + C_2 \rightarrow (1)$

$y'(x) = C_1 e^{x} - 2C_2 e^{-2x}$

$-5 = C_1 - 2C_2 \rightarrow (2)$

$(1) - (2) \Rightarrow \quad 9 = 3C_2 \quad \Rightarrow \quad C_2 = 3 \qquad \therefore C_1 = 1$

$\therefore \; y(x) = e^{x} + 3e^{-2x}$ //

3) $y'' + y' + \frac{1}{4}y = 0 \quad , \quad y(0) = 3$ & $y'(0) = 3.5$

Ans $a = 1 \qquad b = 1/4$

$\lambda_1 = \frac{-1+\sqrt{1-4\times 1/4}}{2} \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -1/2$ //

$y_1(x) = e^{-x/2} \qquad\qquad y_2(x) = xe^{-x/2}$

$y(x) = e^{-x/2}[C_1 + x C_2]$

Given $y(0) = 3$ and $y'(0) = 3.5$

$y(0) = 3 = e^0 [C_1 + 0 \times C_2] \Rightarrow C_1 = 3$ //

$y'(x) = -\frac{e^{-x/2}}{2}[C_1 + x C_2] + e^{-x/2} C_2$

$y'(0) = 3.5 = -1.5 + C_2 \Rightarrow C_2 = 5$

$\therefore y(x) = e^{-x/2}[3 + 5x]$ //

12-8-22

Euler - Cauchy Equations:

Euler - Cauchy equations are ODEs of the form:

$x^2 y'' + axy' + by' = 0 \rightarrow (1)$

with given constants a & b and unknown function $y(x)$.

Let $y(x) = x^m$, then $y'(x) = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

Substitute y, y', y'' into (1), we get

$x^2 m(m-1)x^{m-2} + ax\, m x^{m-1} + b x^m = 0$

$x^m (m^2 + (a-1)m + b) = 0$

if $y(x) = x^m \neq 0$, for all x then $(m^2 + (a-1)m + b) = 0 \rightarrow (2)$

Hence $y = x^m$ is a solution of (1) if and only if m is a root of (2).

Let m_1 and m_2 are roots of (2), then

$m_1 = \frac{(1-a) + \sqrt{(a-1)^2 - 4b}}{2}$ $m_2 = \frac{(1-a) - \sqrt{(a-1)^2 - 4b}}{2}$

→ Real and distinct roots: If $(a-1)^2 - 4b > 0$, then $m_1 \neq m_2 \in \mathbb{R}$

and $y_1(x) = x^{m_1}$, $y_2(x) = x^{m_2}$ are basis solution of 1 and $y(x) = C_1 x^{m_1} + C_2 x^{m_2}$ is the general solution of 1.

→ <u>A real double root:</u> If $(a-1)^2 - 4b = 0$, then $m_1 = m_2 = \frac{1-a}{2}$

$\Rightarrow y_1(x) = x^{(1-a)/2}$

Here $P(x) = a/x$, $V = \frac{1}{y_1^2} e^{-\int P dx} = \frac{1}{x^{(1-a)}} e^{-\int \frac{a}{x} dx}$

$V = \frac{1}{x^{(1-a)}} x^{-a}$ $\Rightarrow V = \frac{1}{x}$ also $u = \int V dx = \int (1/x) dx$
$= \ln x$

$\therefore y_2 = u y_1 = \ln x \cdot x^{(1-a)/2}$

$\therefore y(x) = x^{(1-a)/2} (C_1 + C_2 \ln x)$, $C_1, C_2 \in \mathbb{R}$

→ <u>Complex conjugate roots:</u> If $(a-1)^2 - 4b < 0$ then

$m_1 = \frac{(1-a) + i\sqrt{4b - (1-a)^2}}{2}$ & $m_2 = \frac{(1-a) - i\sqrt{4b - (1-a)^2}}{2}$

Now let $\alpha = \frac{1-a}{2}$ & $\beta = \frac{\sqrt{4b - (1-a)^2}}{2}$

$\Rightarrow m_{1,2} = \alpha \pm i\beta$

$\Rightarrow x^{m_1} = x^{\alpha} \cdot x^{i\beta} = x^{\alpha} e^{i(\beta \ln x)}$

$= x^{\alpha} (\cos(\beta \ln x) + i \sin(\beta \ln x))$

IIIly $x^{m_2} = x^{\alpha} (\cos(\beta \ln x) - i \sin(\beta \ln x))$

$y_1 = \frac{x^{m_1} + x^{m_2}}{2} = x^{\alpha} \cos(\beta \ln x)$

$y_2 = \frac{x^{m_1} - x^{m_2}}{2i} = x^{\alpha} \sin(\beta \ln x)$

Here y_1 and y_2 form basis solution of (I) &

$y(x) = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$, $C_1, C_2 \in \mathbb{R}$

is the general solution, where $\alpha = \dfrac{1-a}{2}$

$\beta = \dfrac{\sqrt{4b-(1-a)^2}}{2}$

Q) Find the general solution of following ODE:

i) $x^2y'' - 3xy' + 10y = 0$

$a = -3 \qquad b = 10$

$(a-1)^2 - 4b = 4^2 - 40 = 16 - 40 = -24$ //

$\alpha = 2 \qquad \beta = \sqrt{6}$

$\Rightarrow \quad y(x) = x^2\left[C_1 \cos(\sqrt{6} \ln x) + C_2 \sin(\sqrt{6} \ln x)\right]$ //

ii) $x^2y'' - 20y = 0$

$a = 0 \qquad b = -20$

$(a-1)^2 - 4b = 1 + 80 = 81 > 0$

$\therefore \alpha = 1/2 \qquad \beta = 9/2$

$y(x) = C_1x^5 + C_2x^{-4}$ //

iii) $xy'' + 2y' = 0 \quad \Rightarrow \quad x^2y'' + 2xy' = 0$

$a = 2 \qquad b = 0$

$m_1 = -1 \qquad m_2 = 0$

$\therefore \quad y(x) = C_1x^{-1} + C_2 = C_2 + C_1/x$ //

iv) $4x^2y'' + 5y = 0 \quad \Rightarrow \quad x^2y'' + (5/4)y = 0$

$a = 0 \qquad b = 5/4$

$(a-1)^2 - 4b = -4 < 0 \quad \Rightarrow \quad \alpha = 1/2 \qquad \beta = 1$

$y(x) = \sqrt{x}\,(C_1 \cos \ln x + C_2 \sin \ln x)$ //

24-8-22

Consider a homogeneous linear second order ODE

$$y'' + p(x)y' + q(x)y = 0 \rightarrow (1)$$

with variable coefficients $p(x)$ and $q(x)$ and two initial conditions

$$y(x_0) = k_0 \quad , \quad y'(x)_0 = k_1 \rightarrow (2)$$

with given x_0, k_0 and k_1.

Existence and uniqueness theorem for Second order ODE

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then IVP consisting of (1) & (2) has a unique solution $y(x)$ on interval I.

NOTE: Two functions f_1 and f_2 are said to be linearly dependent on an interval if $\frac{f_1}{f_2} = C$ for all x in the interval.

Let y_1 and y_2 be two solutions of (1)

Theorem

Let ODE (1) have continuous coefficients $p(x)$ & $q(x)$ on an open interval I. Then 2 solutions $y_1(x)$ & $y_2(x)$ of (1) on the interval I are linearly dependent on I if and only if their Wronskian is zero

i.e $W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$ on I

or $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$

Furthermore $W(y_1, y_2)(x_0) = 0$ for some $x_0 \in I$ if and only if $W(y_1, y_2)(x) = 0 \quad \forall\, x \in I$

i.e $W(y_1, y_2)(x_0) = 0 \iff W(y_1, y_2)(x) = 0$

Also $W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\, g(x)$, $g(x) \neq 0 \quad \forall x \in I$

We have $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ and therefore

$$W'(y_1, y_2)(x) = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$
$$= y_1 y_2'' - y_2 y_1'' \longrightarrow 3$$

Since y_1 & y_2 are solutions of 1

$\Rightarrow W'(y_1, y_2)(x) = y_1(-p(x) y_2' - q(x) y_2) + y_2 (p(x) y_1' + q(x) y_1)$

$= -p(x)(y_1 y_2' - y_2 y_1')$

$= -p(x)\, W(y_1, y_2)(x)$

$\Rightarrow W'(y_1, y_2)(x) + p(x)\, W(y_1, y_2)(x) = 0 \rightarrow 4$

This is a linear ODE in $W(y_1, y_2)$

$\therefore \quad W(y_1, y_2)(x) = C e^{-\int p(x)\, dx}, \quad C \in \mathbb{R}$

Taking $x = x_0$, we get

$C = W(y_1, y_2)(x_0)$

For all $x \in I$ $\quad \therefore W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\, e^{-\int p(x)\, dx}$

<u>NOTE</u>: $W(y_1, y_2) = \left(\dfrac{y_2}{y_1}\right)' y_1^2 \quad$ if $\quad y_1 \neq 0$

or $\quad W(y_1, y_2) = -\left(\dfrac{y_1}{y_2}\right)' y_2^2 \quad$ if $\quad y_2 \neq 0$

Theorem (Existence of a General solution)

If $p(x)$ and $q(x)$ are continuous functions on an open interval I, then $y'' + p(x)y' + q(x)y = 0$ has a general solution

Theorem (A General solution includes all solutions)

If the ODE $y'' + p(x)y' + q(x)y = 0$ has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution $Y(x)$ of $y'' + p(x)y' + q(x)y = 0$ on I is of the form $Y(x) = C_1 y_1(x) + C_2 y_2(x)$ where y_1, y_2 is any basis of solutions of $y'' + p(x)y' + q(x)y = 0$ on I and C_1 and C_2 are constants.

25-8-22

Q1) Show the linear independence by finding the Wronskian of the following:

i) $y_1 = \cos\alpha x$ $\quad y_2 = \sin\alpha x$

ii) $y_1 = xe^x$ $\quad y_2 = x$

iii) $y_1 = x$ $\quad y_2 = 1/x$

iv) $e^{-x}\cos\alpha x$, $e^{-x}\sin\alpha x$

v) $x^k\cos(\ln x)$, $x^k\sin(\ln x)$

Ans i) $W = \begin{vmatrix} \cos\alpha x & \sin\alpha x \\ -\alpha\sin\alpha x & \alpha\cos\alpha x \end{vmatrix}$

$= \alpha\cos^2\alpha x + \alpha\sin^2\alpha x = \alpha \neq 0 \Rightarrow$ linearly independent

ii> $W = \begin{vmatrix} xe^x & x \\ e^x(x+1) & 1 \end{vmatrix} = xe^x - xe^x(x+1)$
$= xe^x(1-x-1) = -x^2e^x \neq 0$ provided interval does not contain zero.

iii> $W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = \frac{-2}{x}, \neq 0 \Rightarrow$ linearly independent

Q2> Find a second order ODE for which the given functions are solutions, Show the linear independence by finding Wronskian.

i> $\cos 5x$, $\sin 5x$ $\quad y(0)=3$, $y'(0)=5$

ii> x^2, $x^2 \ln x$ $\quad y(1)=4$, $y'(1)=6$

iii> 1, e^{-2x} $\quad y(0)=1$, $y'(1)=-1$

Non - homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

Definition: A General solution of a non-homogeneous ODE is a solution of the form $y(x) = y_h(x) + y_p(x) \rightarrow (2)$ where $y_h(x) = C_1y_1(x) + C_2y_2(x)$ is a solution of homogeneous ODE $y'' + p(x)y' + q(x)y = 0$ on I & $y_p(x)$ is any solution of (1) on I containing no arbitrary constant.

Let $y'' + p(x)y' + q(x)y = 0 \rightarrow (3)$

Theorem: (Relation between $y'' + p(x)y' + q(x)y = r(x)$ and $y'' + p(x)y' + q(x) = 0$)

a) The sum of a solution y of ① on some open interval I and a solution $\bar{y}$ of ③ on I is a solution of ① on I

b) The difference of 2 solution of ① on I is a solution of ① on I.

<u>Theorem</u>: If the coefficients $p(x)$ and $q(x)$ and the function $r(x)$ in $y'' + p(x)y' + q(x)y = r(x)$ are continuous on some open interval I then every solution of ① on I is obtained by assigning suitable values to the arbitrary constants C_1 and C_2 in $y_h(x)$.

Method of Undetermined constants

Terms in $r(x)$	Choice of $y_p(x)$
$Ke^{\alpha x}$	$Ce^{\alpha x}$
Kx^n	$K_n x^n + K_{n-1}x^{n-1} + \ldots + K_1 x + K_0$
$K\cos\beta x$ / $K\sin\beta x$	$A\cos\beta x + B\sin\beta x$
$Ke^{\alpha x}\cos\beta x$ / $Ke^{\alpha x}\sin\beta x$	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Choice Rule for the method of undetermined constants

1) <u>Basic rule</u>: If $R(x)$ in $y'' + ay' + by = r(x)$ is one of the function given in above table's left column then choose $y_p(x)$ accordingly and find

the constant by substituting y_p and its derivatives into $y'' + ay' + by = r(x)$. $\longrightarrow$ 4

2) Modified rule: If a term in your choice for $y_p(x)$ happens to be a solution of the homogeneous ODE $y'' + ay' + by = 0$, multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation $y'' + ay' + by = 0$)

3) Sum rule: If $r(a)$ is sum of functions given in left column of the table choose $y_p(x)$ as the sum of the functions in the corresponding functions in right column.

Solve the IVP

1) $y'' + 3y' + 2.25y = -10e^{-1.5x}$, $y(0) = 1$, $y'(0) = 0$

Ans Step 1: Find $y_h(x)$

Homogeneous equation: $y'' + 3y' + 2.25y = 0$

$a^2 - 4b = 9 - 4 \times 2.25 = 0$ $\qquad \lambda = -a/2 = -1.5$

$\therefore \quad y_h(x) = e^{-1.5x}(C_1 + C_2 x)$

Step 2: $\qquad y_p(x) = C_3 x^2 e^{-1.5x}$

$y_p' = C_3\left(2x e^{-3/2 x} - \frac{3}{2}x^2 e^{-1.5x}\right)$

$y_p''(x) = C_3\left[2e^{-1.5x} - 3x e^{-1.5x} - 3x e^{-1.5x} + \frac{9}{4}x^2 e^{-1.5x}\right]$

Substitute y_p, y_p', y_p'' in given ODE

$$C_3\left(2-3x-3x+\frac{9}{4}x^2\right)+3C_3\left(2x-\frac{3}{2}x^2\right)+2.2\,Cx^2=-10$$

By comparing coefficients we get $C=-5$

$y_p(x) = -5x^2 e^{-1.5x}$

$\therefore\ y(x) = e^{-1.5x}\left(C_1 + C_2 x - 5x^2\right)$

Put $y(0)=1 \Rightarrow C_1 = 1$

$y'(x) = e^{-1.5x}\left(C_2 - 10x\right) - e^{-1.5x}(1.5)\left(C_1 + C_2 x - 5x^2\right)$

Put $y'(0)=0$

$0 = C_2 - 1.5C_1 \Rightarrow C_2 = 1.5\times 1 = 1.5$

$y(x) = e^{-1.5x}\left(1 - 1.5x - 5x^2\right)$

Q2) Solve the IVP

$$y'' + 2y' + \frac{3}{4}y = 2\cos x - \frac{\sin x}{4} + 0.09x \qquad y(0)=2.78,\ y'(0)=-0.43$$

Ans $y'' + 2y' + \frac{3}{4}y = 0$

$a^2 - 4b = 4 - 3 = 1$

$y_h(x) = C_1 e^{-0.5x} + C_2 e^{-1.5x}$

$$y_p(x) = \underbrace{A\cos x - B\sin x}_{y_{p_1}} + \underbrace{k_1 x + k_0}_{y_{p_2}}$$

$y_{p_1}' = -A\sin x - B\cos x \qquad y_{p_2}' = k_1$

$y_{p_1}'' = -A\cos x + B\sin x \qquad y_{p_2}'' = 0$

Substitute y_{p_1}, y_{p_1}', y_{p_1}'' in given ODE

$$-A\cos x + B\sin x - 2A\sin x - 2B\cos x + \frac{3}{4}A\cos x - \frac{3}{4}B\sin x = 2\cos x - \frac{\sin x}{4} + 0.09x$$

$$-A + \frac{3}{4}A + 2B = 2 \qquad -\frac{B}{4} - 2A = -\frac{1}{4}$$

$$\Rightarrow \quad A + 8B = 8 \qquad B + 8A = +1 \rightarrow (2)$$

$$\Rightarrow \quad A = 8 - 8B \rightarrow (1)$$

Substitute (1) in (2) $\Rightarrow$ $B - 64B + 64 = 1$

$B = 1$ $\Rightarrow$ $A = 0$

Substitute y''_{p_2}, y'_{p_2}, y_{p_2} in ODE,

$$2k_1 + \frac{3}{4}(k_1 x + k_0) = 2\cos x - \frac{1}{4}\sin x + 0.09x$$

$$0.75 k_1 = 0.09 \quad = 0.12 \qquad \Rightarrow k_0 = -0.32$$

$$\therefore y(x) = C_1 e^{-1.5x} + C_2 e^{-0.5x} + \sin x + 0.12x - 0.32$$

Put $y(0) = 2.78$

$$2.78 = C_1 + C_2 - 0.32$$

$$3.1 = C_1 + C_2 \rightarrow (3)$$

$$y'(x) = -1.5C_1 e^{-1.5x} - 0.5C_2 e^{-0.5x} + \cos x + 0.12$$

Put $y'(0) = -0.43$

$$-0.43 = -1.5C_1 - 0.5C_2 + 1 + 0.12$$

$$-1.55 = -1.5C_1 - 0.5C_2$$

$$3C_1 + C_2 = 3.1 \rightarrow (5)$$

$$(4) - (3) \Rightarrow \quad 2C_1 = 0 \qquad \Rightarrow C_1 = 0.$$

$$\Rightarrow C_2 = 3.1$$

$$\therefore y(x) = 3.1e^{-0.5x} + \sin x + 0.12x - 0.32$$

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Method of variation of parameters

Consider a second order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \rightarrow ①$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions on I.

$$y(x) = y_h(x) + y_p(x)$$

$$y_p(x) = -y_1(x)\int \frac{y_2(x)\, r(x)}{W(y_1, y_2)}\, dx + y_2(x)\int \frac{y_1(x)\, r(x)}{W(y_1, y_2)}\, dx$$

$$= -y_1 \int \frac{y_2\, r(x)}{W(y_1, y_2)}\, dx + y_2 \int \frac{y_1\, r(x)\, dx}{W(y_1, y_2)}$$

where y_1, y_2 are obtained from:

$y_h(x) = C_1 y_1 + C_2 y_2$ (C_1, C_2 are arbitrary constants)

Ex1 Solve $y'' - 2y' = e^x \sin x$

Ans Consider $y'' - 2y' = 0$

$$y_h(x) = C_1 + C_2 e^{2x}$$

Let $y_1 = 1$ $\quad y_2 = e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

$$y_p(x) = -y_1 \int \frac{y_2\, r(x)\, dx}{W(y_1, y_2)} + y_2 \int \frac{y_1\, r(x)\, dx}{W(y_1, y_2)}$$

$$= -1 \int \frac{e^{2x} \cdot e^x \sin x}{2e^{2x}}\, dx + e^{2x} \int \frac{e^x \sin x\, dx}{2e^{2x}}$$

$$= -\frac{1}{2} \int e^{x} \sin x\, dx + \frac{e^{2x}}{2} \int e^{-x} \sin x\, dx$$

NOTE = $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

$\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

$$-\frac{1}{2}\int e^{x}\sin x\,dx = -\frac{1}{2}\,\frac{e^{x}}{2}(\sin x+\cos x) = \frac{-e^{x}}{4}(\sin x-\cos x)$$

$$\frac{1}{2}\int e^{-x}\sin x\,dx = \frac{e^{-x}}{2\times 2}(-\sin x-\cos x) = -\frac{\bar{e}^{x}}{4}(\sin x+\cos x)$$

$$\Rightarrow \quad -\frac{1}{2}\int e^{x}\sin x\,dx + \frac{e^{2x}}{2}\int e^{-x}\sin x\,dx = -\frac{e^{x}}{4}(2\sin x) = -\frac{e^{x}}{2}\sin x$$

$$\therefore\ y(x) = C_1 + C_2e^{2x} - \frac{e^{x}}{2}\sin 2x \,/\!/$$

2) Solve $\quad y'' + a^2y = \operatorname{cosec} ax$

Ans $\quad y_h(x) \Rightarrow y'' + a^2 y = 0$

$$y_h(x) = C_1\sin ax + C_2\cos ax$$

$$W(y_1, y_2) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$y_p(x) = -y_1\int \frac{y_2\,r(x)\,dx}{W(y_1,y_2)} + y_2\int \frac{y_1\,r(x)\,dx}{W(y_1,y_2)}$$

$$\int \frac{y_2\,r(x)}{W(y_1,y_2)}\,dx = \int \frac{\cos ax\ \operatorname{cosec} ax\ dx}{a} = \frac{\ln|\sin ax|}{a^2}$$

$$\int \frac{y_1\,r(x)}{W(y_1,y_2)}\,dx = \int \frac{\sin ax\ \operatorname{cosec} ax\,dx}{a} = \frac{x}{a}$$

$$y_p(x) = -\frac{\sin ax\ \ln|\sin ax|}{a^2} + \frac{x\cos ax}{a}$$

$$\therefore\ y(x) = C_1\cos ax + C_2\sin ax - \cos ax\left(\frac{x}{a}\right) + \frac{\sin ax\ \ln|\sin ax|}{a^2}$$

Higher order linear ODE

Consider the n^{th} order linear ODE :

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = r(x) \rightarrow (1)$$

where $y^{(n)} = \dfrac{d^n y}{dx^n}$

Homogeneous linear ODE:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = 0$$

For a homogeneous linear ODE sums and constant multiples of solutions of (2) on some open interval I are again solutions of (2) on I.

General solution

General solution of the homogeneous equation (2) on I, is solution of (2) on I and is of the form,

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x) \rightarrow (3)$$

where $y_1, y_2 \dots y_n$ basis of solution of (2) on I. (i.e, these solutions are linearly independent)

Linearly independent:

Consider $y_1, y_2, \dots y_n$ defined on some interval I, then these functions are called linearly independent on I, if the equation :

$$k_1 y_1(x) + k_2 y_2(x) + k_3 y_3(x) + \dots + k_n y_n(x) = 0$$

implies : $k_1 = k_2 = k_3 = \dots = k_n = 0$ // $\forall x \in I$

Q1> Show that $y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x^3$ are linearly independent on any interval

Q2> Show that the functions $y_1(x) = x^2$, $y_2(x) = 5x$, $y_3(x) = 2x$ are linearly dependent on any interval.

Ans 2> $k_1 x^2 + k_2 5x + k_3 2x = 0$

Put $x = -1 \Rightarrow k_1 - 5k_2 - 2k_3 = 0 \rightarrow (1)$

Put $x = 1 \Rightarrow k_1 + 5k_2 + 2k_3 = 0 \rightarrow (2)$

Put $x = 2 \Rightarrow 4k_1 + 10k_2 + 4k_3 = 0 \rightarrow (3)$

Now $\begin{vmatrix} 1 & -5 & -2 \\ 1 & 5 & 2 \\ 4 & 10 & 4 \end{vmatrix} = 0$

$\Rightarrow$ Linearly dependent.

1> $k_1 x + k_2 x^2 + k_3 x^3 = 0$

Put $x = 1$, $k_1 + k_2 + k_3 = 0 \rightarrow (1)$

Put $x = -1$, $-k_1 + k_2 - k_3 = 0 \rightarrow (2)$

Put $x = 2$, $2k_1 + 4k_2 + 8k_3 = 0 \rightarrow (3)$

From (1) & (2) $k_2 = 0$

$\Rightarrow k_1 = -k_3 \rightarrow$ Put in (3)

$2k_1 - 8k_1 = 0$

$-6k_1 = 0 \Rightarrow k_1 = 0 \Rightarrow k_3 = 0$

Initial value problem

An IVP for the ODE (2) consists of (2) and n intial conditions:

$y(x_0) = a_0$, $y'(x_0) = a_1$, ... $y^{(n-1)}(x_0) = a_{n-1}$ → 5
with the given x_0 in the open interval I and $a_0, a_1, \ldots, a_{n-1}$ are constants.

Theorem: (Existence & Uniqueness of the IVP):

If the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ in 2 are continuous on some open interval I and $x_0 \in I$, then the IVP 2 with 5 has a unique solution $y(x)$ on I.

Q3) Solve : $y^4 - 5y^2 + 4y = 0$

Ans Let $y = e^{\lambda x}$

then $\lambda^4 - 5\lambda^2 + 4 = 0$

$\Rightarrow \lambda = 1, -1, 2, -2$

$\Rightarrow y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-x} + C_4 e^{-2x}$

Q4) Solve: $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$, $y(1) = 2$
$y'(1) = 1$, $y''(1) = -4$

Ans Let $y = x^m$

$(m(m-1)(m-2) - 3m(m-1) + 6m - 6)\, x^m = 0$

$m = 1, 2, 3$

$\therefore y(x) = C_1 x + C_2 x^2 + C_3 x^3$ //

$y(1) = 2$

$\Rightarrow 2 = C_1 + C_2 + C_3$

$y'(1) = 1 \Rightarrow 1 = C_1 + 2C_2 + 3C_3$

$y''(1) = -4 \Rightarrow -4 = 2C_2 + 6C_3$

$C_1 = 2 \qquad C_2 = +1 \qquad C_3 = -1$

$y = 2x + x^2 - x^3$

Linear Independence of solution

The Wronskian W of n solutions $y_1, y_2, \dots, y_n$ of (2) defined as the n^{th} order determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem: (Linear independence & dependence of solution)

Let the ODE (2) have continuous coefficients $p_0(x), \dots, p_{n-1}(x)$ on an open interval I, then n solutions $y_1, y_2, \dots y_n$ of (2) on I are linearly dependent if and only if their Wronskian is zero for some point $x = x_0$.

Theorem: If the coefficients $p_0(x), \dots, p_{n-1}(x)$ of (2) are continuous on some open interval then (2) has a general solution.

Theorem: If the ODE (2) has continuous coefficients

$P_0(x)$, $P_1(x)$, ... $P_{n-1}(x)$ on some open interval I, then every solution $Y(x)$ of ② on I is of the form $Y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$ where $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ form basis of solutions of ② and $C_1, \dots, C_n$ are suitable constants.

Homogeneous linear ODE with constant coefficients

Consider the n^{th} order homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$$

Let $y = e^{\lambda x}$, substitute $y, y', \dots, y^n$ in ① we get $e^{\lambda x}(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0) = 0$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \rightarrow ②$$

The function $y = e^{\lambda x}$ is a solution of ODE if and only if λ is a root of ②

Real & distinct Root

If all the roots $\lambda_1, \lambda_2, \dots \lambda_n$ of 2 are real & distinct then n solutions of ① is

$$y_1(x) = e^{\lambda_1 x} \quad y_2(x) = e^{\lambda_2 x} \quad \dots \quad y_n(x) = e^{\lambda_n x} \rightarrow ③$$

Linear independence of $y_1 = e^{\lambda_1 x}$ ... $y_n = e^{\lambda_n x}$

Let $E = \exp[(\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n)x]$, then the Wronskian of $y_1, y_2, \dots\dots y_n$ is

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots\cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots\cdots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} = E \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

Here $E \neq 0$ for all $x \in I$. Here $W = 0$ if and only if the determinant of this matrix is 0. This is called Vandermode or Cauchy determinants. The determinant is equals to $(-1)^{n(n-1)/2} V$. Where V is the product of all factors $\lambda_j - \lambda_k$. $j < k \leq n$

Complex simple roots

If $\lambda = \alpha + i\beta$ is a simple root of (2) $\bar{\lambda} = \alpha - i\beta$ is also a root. The real solutions are

$y_1 = e^{\alpha x} \cos\beta x \qquad y_2 = e^{\alpha x} \sin\beta x$

Complex multiple root:

If $\lambda = \alpha + i\beta$ (twice) a root of (2), then $\bar{\lambda} = \alpha - i\beta$ (twice) is also a root of (1).

Then real solutions of (1) are

$y_1 = e^{\alpha x} \cos\beta x \qquad y_2 = e^{\alpha x} \sin\beta x \qquad y_3 = x e^{\alpha x} \cos\beta x$

$y_4 = x e^{\alpha x} \sin\beta x$

Real Multiple root

If λ is a root of ② of order m. Then the linearly independent solutions of ① corresponding to λ is

$$e^{\lambda x}, \quad x e^{\lambda x}, \quad \dots, \quad x^{m-1} e^{\lambda x}$$

Solve the IVP

$$y^{(4)} - 9(y)^2 - 400y = 0, \quad y(0) = 0, \quad y'(0) = 0$$
$$y''(0) = \frac{5}{2} \qquad y'''(0) = -\frac{7}{2}$$

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Higher order linear Non homogeneous ODE

Consider the n^{th} order linear non homogeneous ODE

$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \dots + p_1(x) y' + p_0(x) y = r(x) \to$ ① where $p_{n-1}(x)$, $p_1(x)$, $p_0(x)$ and $r(x)$ are functions of x and $r(x)$ is not identically zero.

The general solution of ① is of the form $y(x) = y_h(x) + y_p(x)$ where $y_h(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$ is a general solution of the corresponding homogeneous equation and $y_p(x)$ is any solution of ① containing no arbitrary constant.

If $p_{n-1}(x), \dots p_1(x)$, $p_0(x)$ and $r(x)$ are continuous in ① then ① has the general solution which include all the solution of ①.

Homogeneous equation of ①:

$$y^{(n)} + p_{n-1} y^{(n-1)} + \dots \dots + p_1(x) y' + p_0(x) y = 0 \to ②$$

Method of undetermined coefficients

1> Basic rule is same as second order ODE

2> Modification rule

If a term in our choice $y_p(x)$ is a solution of the homogeneous equation (2), then multiply this term by x^k, where k is the smallest positive integer such that this term x^k is not a solution (2)

3> Sum rule is same as second order ODE.

Q> Solve the IVP

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \quad y(0) = 3, \quad y'(0) = -3, \quad y'' = -47$$

Ans Homogeneous equation:

$$e^{\lambda x}(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = 0$$

$\lambda = -1$ is a triple root of characteristic equation:

$\Rightarrow$ General solution: $e^{-x}(1 + C_1x + C_2x^2)$

Let $y_p(x) = C x^3 e^{-x}$

$y_p' = C e^{-x}(3x^2 - x^3)$

$y_p'' = C(6xe^{-x} - e^{-x}(3x^2) - (x^3e^{-x}) + x^3e^{-x})$

$y_p''' = C(6e^{-x} - 6xe^{-x} + 12xe^{-x} + 6x^2e^{-x} + 3x^2e^{-x} - x^3e^{-x})$

Substitution y_p, y_p', y_p'', y_p''' in given ODE.

We get $C = 5$

$\Rightarrow y(x) = (C_1 + C_2x + C_2x^3)e^{-x} + 5x^3e^{-x}$

$y(0) = 3 \Rightarrow c_1 = 3$

$y' = -3e^{-x} + C_2 e^{-x} - C_2 x e^{-x} + 2C_3 x e^{-x} - C_3 x^2 e^{-x} + 15x^2 e^{-x}$
$- 5x^3 e^{-x} = 0$

$y'(0) = 3 \Rightarrow C_2 = 0.$

Similarly we find that $C_3 = -25$.

$\therefore y(x) = (5x^3 - 25x^2 - x)e^{-x}$ //

Method of variation of parameters

Let $y_1, y_2, \ldots y_n$ are linear independent solutions. of the homogeneous equation ②.

Then $y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)\, r(x)\, dx}{W(x)}$

$= y_1(x) \int \frac{W_1(x)\, r(x)\, dx}{W(x)} + \ldots + y_n(x) \int \frac{W_n(x)\, r(x)\, dx}{W(x)}$

where $W(x)$ is the Wronskian of $y_1, \ldots, y_n$ and $W_k(x)$ is obtained from $W(x)$ by replacing the k^{th} column of $W(x)$ by $[0, 0, \ldots 0, 1]^T$

Q Solve non homogeneous Euler - cauchy equation
$x^3 y''' - 3x^2 y'' + 6x y' - 6y = x^4 \ln x$

Ans Consider $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

$x^m (m(m-1)(m-2) - 3m(m-1) + 6m - 6) = 0$

$x^m (m(m^2 - 3m + 2) - 3m^2 + 3m + 6m - 6) = 0$

$x^m (m^3 - 3m^2 + 2m - 3m^2 + 9m - 6) = 0$

$x^m (m^3 - 6m^2 + 11m - 6) = 0$

$m = 1, \ m = 2, \ m = 3$

$\Rightarrow \quad y_h(x) = C_1 x + C_2 x^2 + C_3 x^3$

Now consider $\quad y''' - \frac{3}{x} y + \frac{6}{x^2} y' - \frac{6}{x^3} y = x \ln x \rightarrow ①$

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x\left(12x^2 - 6x^2\right) - 1\left(6x^3 - 2x^3\right)$$

$$= 6x^3 - 4x^3 = 2x^3 \quad \neq 0$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = 3x^4 - 2x^4 = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -\left(3x^3 - x^3\right) = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$y_p(x) = x \int \frac{x^4 \, x \ln x}{2x^3} \, dx + x^2 \int \frac{(-2x^3) \, x \ln x}{2x^3} \, dx + x^3 \int \frac{x^2 x \ln x \, dx}{2x^3}$$

$$= \frac{x}{2} \int x^2 \ln x \, dx - x^2 \int x \ln x \, dx + \frac{x^3}{2} \int \ln x \, dx$$

$$I_1 = \int x^2 \ln x \, dx$$

$$= \ln x \, \frac{x^3}{3} - \int \frac{1}{x} \frac{x^3}{3} \, dx = \left[\frac{x^3 \ln x}{3} - \frac{x^3}{9} \right]$$

$$I_2 = \int x \ln x \, dx = \ln x \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$I_3 = \int \ln x \, dx = x(\ln x + 1)$$

$$\therefore \quad y_p = \frac{x}{2}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] - x^2\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] + \frac{x^3}{2}\left[x(\ln x + 1)\right]$$

$$= \frac{x^4}{6}\ln x - \frac{x^4}{18} - \frac{x^4}{2}\ln x + \frac{x^4}{4} + \frac{x^4}{2}\ln x + \frac{x^4}{2}$$

$$= \frac{x^4}{6}\left(\ln x - \frac{11}{6}\right) \mathbin{/\!/}$$

$$\therefore \quad y(x) = y_h(x) + y_p(x)$$

$$= C_1 x + C_2 x^2 + C_3 x^3 + \frac{x^4}{6}\left(\ln x - \frac{11}{6}\right) \mathbin{/\!/}$$

Series solution of ODE

Let $y(x) = \sum_{m=0}^{\infty} a_m x^m$ then,

$y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1}$, $y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

Now consider ODE , $y' = 4xy$

then $y = Ce^{2x^2}$

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$

Now $y' = 4xy$

$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots) = 4x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$

$\Rightarrow a_1 = 0$, $2a_2 = 4a_0 \Rightarrow a_2 = 2a_0$

$a_3 = 0$, $4a_4 = 4a_2 \Rightarrow a_4 = a_2$

$\therefore a_{2m+1} = 0$

$$y(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2m} x^{2m}$$

$$= a_0 + 2a_0 x^2 + 2a_0 x^4 + \frac{4}{3} a_0 x^6 + \frac{2}{3} a_0 x^8 + \dots$$

$$= a_0 \left(1 + 2x^2 + 2x^4 + \frac{4}{3} x^6 + \frac{2}{3} x^8 + \dots\right)$$

$$y(x) = a_0 e^{2x^2}$$