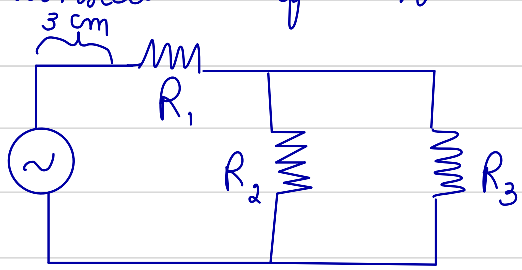
EC301 - RF CIRCUITS

MANVITH PRABHU
211EC228

RF COMPONENTS & CIRCUITS

- If frequency of current is very large then wavelength is comparable to length of circuit.
- So normal KCL & KVL cannot be applied.

- Consider $f = 10$ GHz the $\lambda = 0.03m = 3cm$



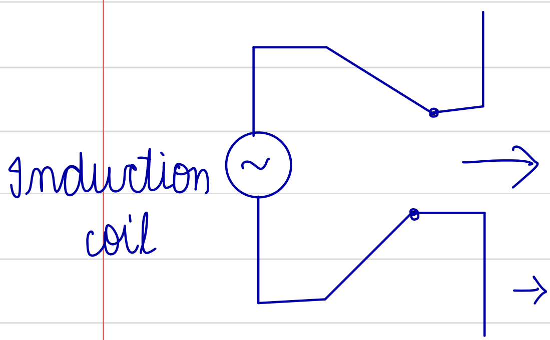
So wire now has inductance & capacitance.

Frequency band	Designation	Application
3 - 30 kHz	Very low frequency (VLF)	Sonar
30 - 300 kHz	Low frequency (LF)	Beacons, Navigation Aid
300 kHz - 3 MHz	Medium frequency (MF)	AM broadcast
3 - 30 MHz	High frequency (HF) HF + Morse code (0, 1)	SW radio & Amateur radio
30 - 300 MHz	Very High frequency (VHF)	Analog TV, FM broadcast (88 - 108 MHz)

300MHz – 3GHz	Ultra high frequency (UHF)	Analog TV, ISM, bluetooth
3GHz – 30 GHz	Super high frequency (SHF)	Applications are being discovered.

Oliver Heaviside reduced the 26 Maxwell's equation to 4.

Experiment by Heinrich Hertz

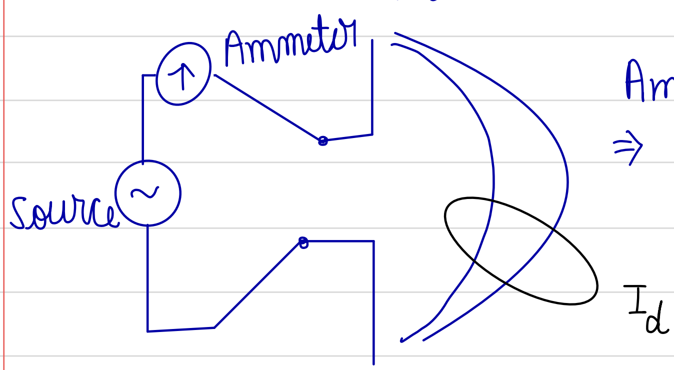


→ Spark is generated

→ Dipole antenna

⇒ Energy is transmitted via EM waves.
⇒ EM waves exists. This was proved by Hertz.

Displacement current:



Ammeter shows a value
⇒ This current is displacement current.

Coordinate system:

1> Rectangular (Cartesian)
2> Cylindrical → used in circular waveguides
3> Spherical
4> Curvilinear

2> Cylindrical: (ρ, ϕ, z)

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \Rightarrow$ Right handed system.

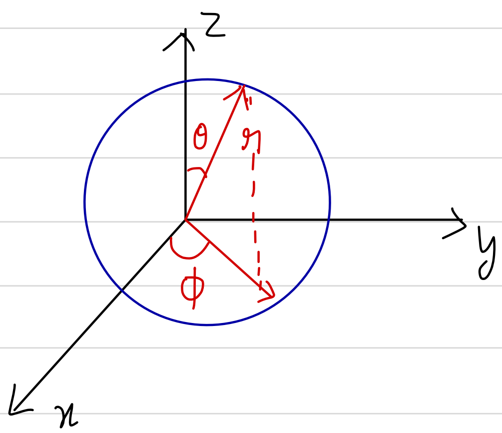
$dV = \rho \, d\rho \, d\phi \, dz$

$x = \rho \cos\phi$ $\quad y = \rho \sin\phi$ $\quad z = z$

$\rho = \sqrt{x^2+y^2}$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

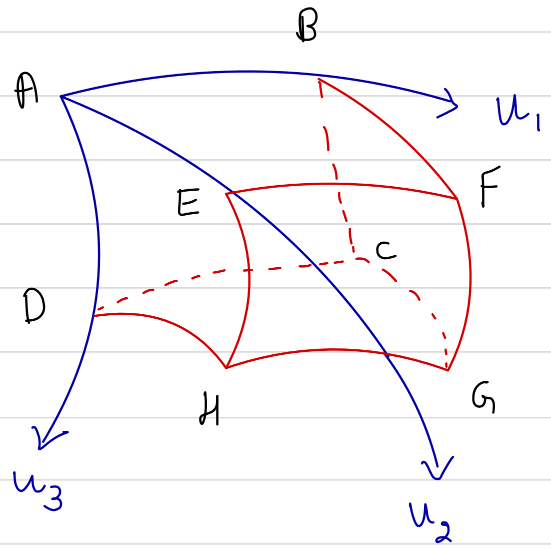
3> Spherical : (r, θ, ϕ)

$r = c \Rightarrow$ sphere
$\theta = c \Rightarrow$ cone
$\phi = c \Rightarrow$ plane



$x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$

4) <u>Curvilinear co-ordinate system:</u>



$A = (u_1, u_2, u_3)$

$B = (u_1 + du_1, u_2, u_3)$

$C = (u_1 + du_1, u_2, u_3 + du_3)$

$D = (u_1, u_2, u_3 + du_3)$

$E = (u_1, u_2 + du_2, u_3)$

$F = (u_1 + du_1, u_2 + du_2, u_3)$

$G = (u_1 + du_1, u_2 + du_2, u_3 + du_3)$

$H = (u_1, u_2 + du_2, u_3 + du_3)$

$AB = h_1 \, du_1$ where h_1, h_2, h_3 are radius of curves.

$AE = h_2 \, du_2$

$AD = h_3 \, du_3$

Gradient : Operates on a scalar quantity and produces vector quantity.

$$\vec{\nabla T} = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

Also $dT = \vec{\nabla T} \cdot \vec{dl}$

where $\vec{dl} = (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$

$dT = |\vec{\nabla T}|\,|\vec{dl}| \cos\theta$

let $|\vec{dl}| = k$

then $\frac{dT}{k} = |\vec{\nabla T}| \cos\theta$

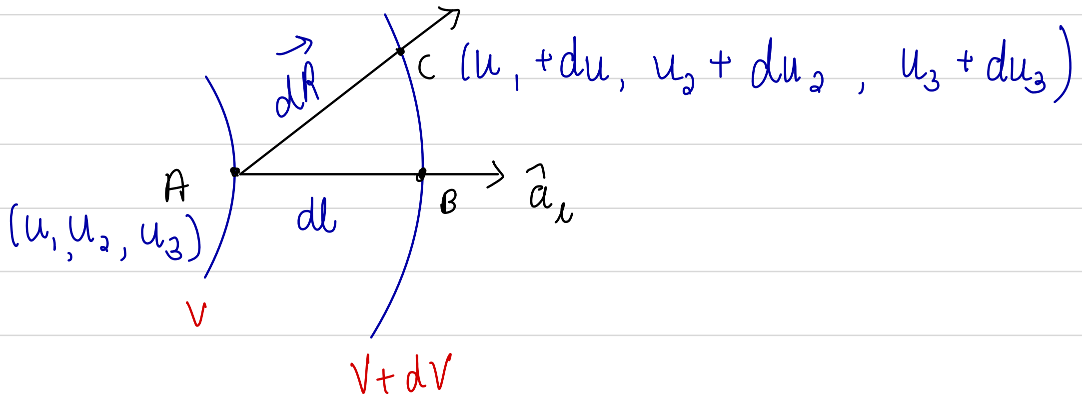
when $\theta = 0°$ $\cos\theta = 1$

$\therefore |\vec{\nabla T}| =$ max spatial rate of change of T.

Direction of $\vec{\nabla T}$ is the direction along which T is changing most rapidly

Gradient in curvilinear surface :

Consider 2 curvilinear surfaces :



$$\vec{\nabla V} = \frac{dV}{dl}\hat{a}_l \rightarrow (1)$$

$$dV = \frac{\partial V}{\partial u_1}du_1 + \frac{\partial V}{\partial u_2}du_2 + \frac{\partial V}{\partial u_3}du_3 \rightarrow (2)$$

$$\vec{\nabla V} = (\nabla V_1)\hat{a}_1 + (\nabla V_2)\hat{a}_2 + (\nabla V_3)\hat{a}_3$$

$$dV = \vec{\nabla V}\cdot\vec{dR} \rightarrow (3)$$

$$= [(\nabla V_1)\hat{a}_1 + (\nabla V_2)\hat{a}_2 + (\nabla V_3)\hat{a}_3]\cdot[h_1 du_1 \hat{a}_1 + h_2 du_2 \hat{a}_2 + h_3 du_3 \hat{a}_3] \rightarrow (4)$$

$$dV = \nabla V_1 h_1 du_1 + \nabla V_2 h_2 du_2 + \nabla V_3 h_3 du_3 \rightarrow (5)$$

From (2) & (5)

$$\nabla V_1 = \frac{1}{h_1}\frac{\partial V}{\partial u_1} \qquad \nabla V_2 = \frac{1}{h_2}\frac{\partial V}{\partial u_2} \qquad \nabla V_3 = \frac{1}{h_3}\frac{\partial V}{\partial u_3}$$

For rectangular coordinates:

$h_1 = h_2 = h_3 = 1$, $u_1 = x$ $u_2 = y$ $u_3 = z$

$$\Rightarrow \nabla V_1 = \frac{\partial V}{\partial x} \qquad \nabla V_2 = \frac{\partial V}{\partial y} \qquad \nabla V_3 = \frac{\partial V}{\partial z}$$

For cylindrical coordinates:

$h_1 = h_3 = 1$ $h_2 = \rho$ $u_1 = \rho$ $u_2 = \phi$ $u_3 = z$

$$\nabla V_1 = \frac{\partial V}{\partial \rho} \qquad \nabla V_2 = \frac{1}{\rho}\frac{\partial V}{\partial \phi} \qquad \nabla V_3 = \frac{\partial V}{\partial z}$$

For spherical system :

$h_1 = 1$ $\quad h_2 = r$ $\quad h_3 = r\sin\theta$

$u_1 = r$ $\quad u_2 = \theta$ $\quad u_3 = \phi$

$\nabla V_1 = \frac{\partial V}{\partial r}$ $\quad \nabla V_2 = \frac{1}{r}\frac{\partial V}{\partial \theta}$ $\quad \nabla V_3 = \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$

Divergence ;

$\nabla\cdot\vec{D} = \rho$

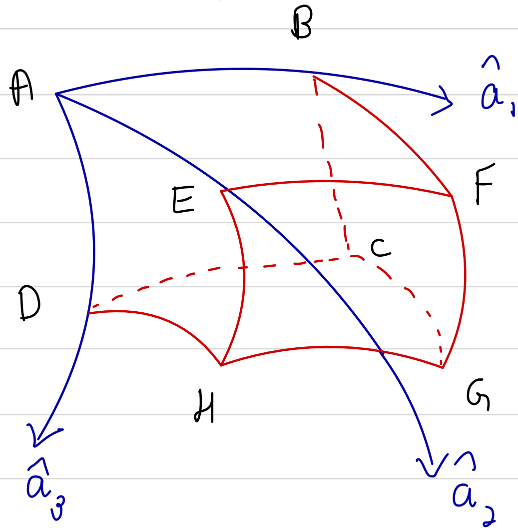
$\oiint \vec{D}\cdot d\vec{S} = \Phi$

For electric field,

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Divergence in curvilinear coordinates

$$\vec{\nabla}\cdot\vec{E} \triangleq \lim_{\Delta V\to 0} \frac{\oiint \vec{E}\cdot d\vec{S}}{\Delta V}$$



(ΔV = volume)

$\vec{E} = E_1\hat{a}_1 + E_2\hat{a}_2 + E_3\hat{a}_3$

Amount flux entering surface AEHD

$$F_E = \vec{E}\cdot d\vec{S}$$
$$= (\hat{a}_1E_1 + \hat{a}_2E_2 + \hat{a}_3E_3)\cdot(\hat{a}_1 \, h_2h_3 \, du_2 \, du_3)$$

$$F_E = E_1 h_2 h_3 \, du_2 du_3$$

Outgoing flux (BFGH) $= F_E + \dfrac{\partial F_E}{\partial u_1} du_1$

Excess flux $= \dfrac{\partial F_E}{\partial u_1} du_1$

$$= \frac{\partial}{\partial u_1}(E_1 h_2 h_3 \, du_2 du_3)$$

$$= \frac{\partial}{\partial u_1}(E_1 h_2 h_3) \, du_1 du_2 du_3$$

In all 3 directions: Excess flux:

$$= \left[\frac{\partial}{\partial u_1}(E_1 h_2 h_3) + \frac{\partial}{\partial u_2}(E_2 h_2 h_3) + \frac{\partial}{\partial u_3}(E_3 h_2 h_3)\right] du_1 du_2 du_3$$

$$\therefore \vec{\nabla} \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oiint \vec{E} \cdot d\vec{s}}{\Delta V}$$

$$= \frac{\cancel{du_1 du_2 du_3}}{h_1 h_2 h_3 \cancel{du_1 du_2 du_3}} \left[\frac{\partial}{\partial u_1}(E_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 E_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 E_3)\right]$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(E_1 h_2 h_3) + \frac{\partial}{\partial u_2}(E_2 h_1 h_3) + \frac{\partial}{\partial u_3}(E_3 h_2 h_1)\right]$$

<u>For rectangular coordinates</u>: $h_1 = h_2 = h_3 = 1$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Spherical co-ordinate system :

$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$

$u_1 = r \qquad u_2 = \theta \qquad u_3 = \phi$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(E_r r^2 \sin\theta\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta E_\theta\right) + \frac{\partial}{\partial\phi}\left(r E_\phi\right)\right]$$

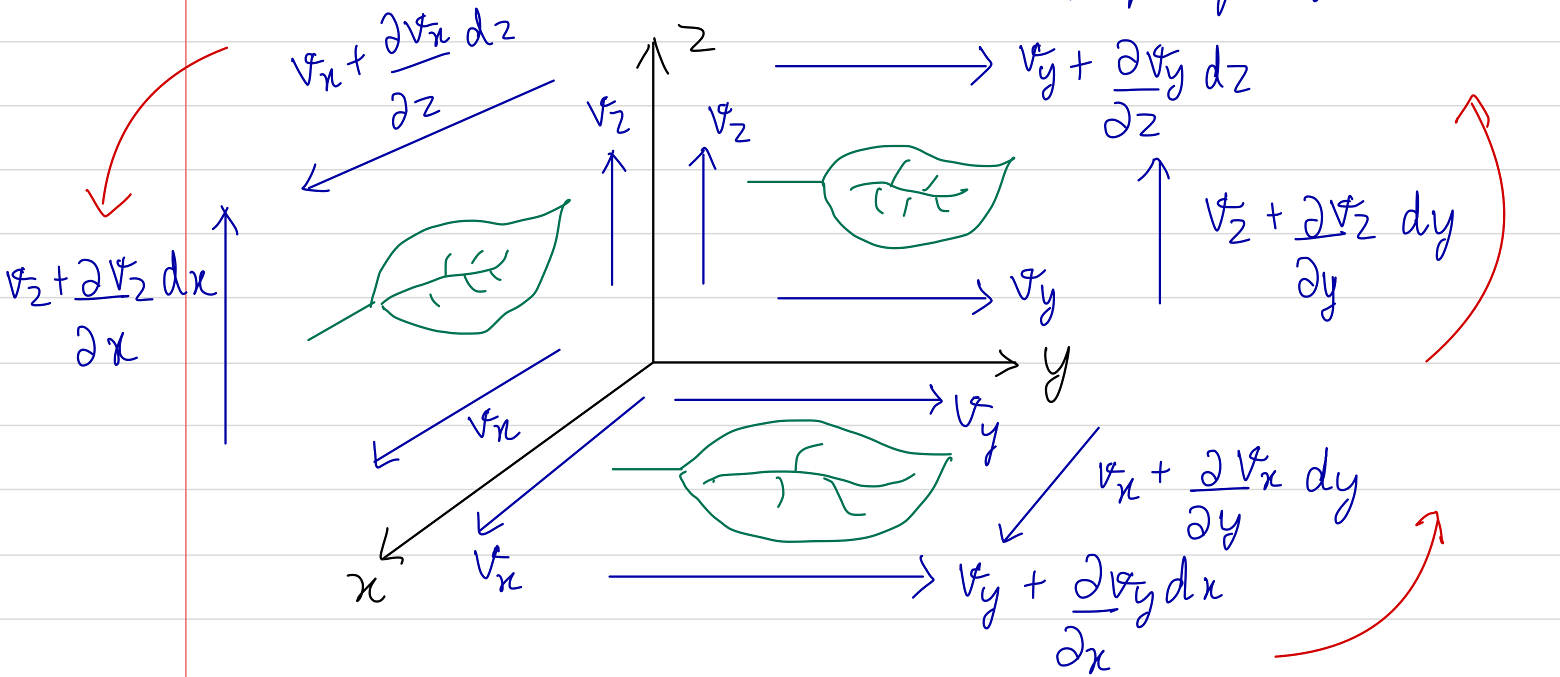
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Cylindrical co-ordinates:

$h_1 = 1 \qquad h_2 = \rho \qquad h_3 = 1 \qquad u_1 = \rho \qquad u_2 = \phi \qquad u_3 = z$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho E_\rho\right) + \frac{\partial}{\partial\phi}\left(E_\phi\right) + \frac{\partial}{\partial z}\left(\rho E_z\right)\right]$$

Curl: Vectors have property of rotation. Curl is used to model the property of rotation.



For leaf to rotate anticlockwise:

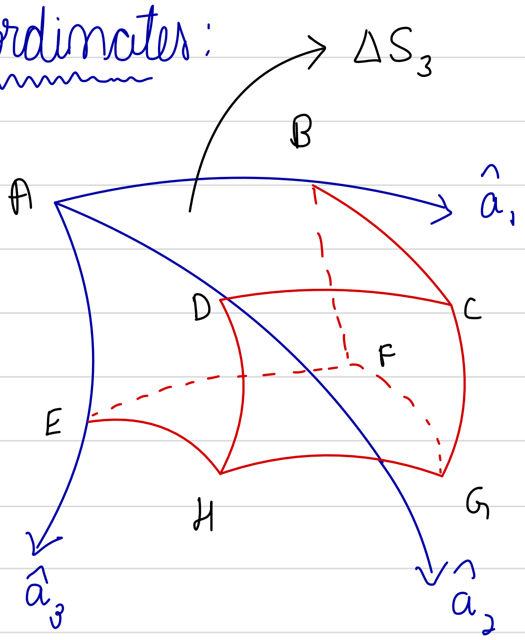
$$(\vec{\nabla} \times \vec{v})_z = \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right]$$

$$(\vec{\nabla} \times \vec{v})_x = \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right]$$

$$(\vec{\nabla} \times \vec{v})_y = \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right]$$

$$\Rightarrow \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ v_x & v_y & v_z \end{vmatrix}$$

Curl in curvilinear coordinates:



$\Delta S_3 = h_1 h_2 \, du_1 \, du_2$

$$(\nabla \times \vec{E})_3 \triangleq \lim_{\Delta S_3 \to 0} \frac{\oint_{ABCDA} \vec{E} \cdot \vec{dl}}{\Delta S_3}$$

$$\oint_{ABCDA} \vec{E} \cdot \vec{dl} = \int_A^B \vec{E} \cdot \vec{dl} + \int_B^C \vec{E} \cdot \vec{dl} + \int_C^D \vec{E} \cdot \vec{dl} + \int_D^A \vec{E} \cdot \vec{dl}$$

$$\int_A^B \vec{E} \cdot \vec{dl} = \vec{E} \cdot h_1 \, du_1 \, \hat{a}_1 = E_1 h_1 \, du_1 \rightarrow ①$$

$$\int_C^D \vec{E} \cdot \vec{dl} = \left(\vec{E} + \frac{\partial \vec{E}}{\partial u_2} du_2 \right) \cdot (h_1 du_1) \, (-\hat{a}_1)$$

$$= -E_1 h_1 \, du_1 - \frac{\partial}{\partial u_2} (E_1 h_1) \, du_1 du_2 \rightarrow ②$$

$$\Rightarrow \int_A^B \vec{E} \cdot \vec{dl} + \int_C^D \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial u_2} \left[E_1 h_1 \right] du_1 du_2 \rightarrow ③$$

$$\int_B^C \vec{E} \cdot \vec{dl} = \left(\vec{E} + \frac{\partial E}{\partial u_1} \right) \cdot h_2 \, du_2 \, \hat{a}_2$$

$$= E_2 h_2 \, du_2 + \frac{\partial}{\partial u_1} (E_2 h_2) \, du_1 du_2 \rightarrow ④$$

$$\int_D^A \vec{E} \cdot \vec{dl} = \vec{E} \cdot (h_2 du_2)(-\hat{a}_2)$$

$$= -E_2 h_2 \, du_2 \rightarrow ⑤$$

$$\therefore \oint \vec{E}\cdot d\vec{l} = \left[\frac{\partial}{\partial u_1}(E_2 h_2) - \frac{\partial}{\partial u_2}(E_1 h_1)\right] du_1 du_2$$

$$\therefore (\vec{\nabla}\times\vec{E})_3 = \lim_{\Delta S_3 \to 0} \frac{\left[\frac{\partial}{\partial u_1}(E_2 h_2) - \frac{\partial}{\partial u_2}(E_1 h_1)\right] \cancel{du_1 du_2}}{h_1 h_2 \cancel{du_1 du_2}}$$

$$= \frac{\left[\frac{\partial}{\partial u_1}(E_2 h_2) - \frac{\partial}{\partial u_2}(E_1 h_1)\right]}{h_1 h_2} \rightarrow (6)$$

Similarly $(\vec{\nabla}\times\vec{E})_1 = \lim_{\Delta S_1 \to 0} \frac{\oint_{ADHEA} \vec{E}\cdot d\vec{l}}{\Delta S_1}$

$$\oint_A^D \vec{E}\cdot d\vec{l} + \oint_H^E \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial u_3}(E_2 h_2)\, du_2\, du_3$$

$$\int_D^H \vec{E}\cdot d\vec{l} + \int_E^A \vec{E}\cdot d\vec{l} = \frac{\partial}{\partial u_2}(E_3 h_3)\, du_2\, du_3$$

$$\therefore (\vec{\nabla}\times\vec{E})_1 = \left[\frac{\partial}{\partial u_2}(E_3 h_3) - \frac{\partial}{\partial u_3}(E_2 h_2)\right] \frac{1}{h_2 h_3}$$

Similarly $(\vec{\nabla}\times\vec{E})_2 = \lim_{\Delta S_2 \to 0} \oint_{AEFBA} \vec{E}\cdot d\vec{l} \Big/ \Delta S_2$

$$\oint_A^E \vec{E}\cdot d\vec{l} + \int_F^B \vec{E}\cdot d\vec{l} = \frac{\partial}{\partial u_3}(E_3 h_3)\, du_1\, du_3$$

$$\int_E^F \vec{E}\cdot d\vec{l} + \int_B^A \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial u_1}(E_1 h_1)\, du_1\, du_3$$

$$\therefore (\vec{\nabla}\times\vec{E})_2 = \left[\frac{\partial}{\partial u_3}(E_3 h_3) - \frac{\partial}{\partial u_1}(E_1 h_1)\right]\frac{1}{h_1 h_3}$$

$$\therefore \vec{\nabla}\times\vec{E} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

For cartesian system: $h_1 = h_2 = h_3 = 1$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ E_x & E_y & E_z \end{vmatrix}$$

For cylindrical coordinates: $h_1 = 1 \quad h_2 = \rho \quad h_3 = 1$

$$\vec{\nabla}\times\vec{E} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \partial\rho & \partial\phi & \partial z \\ E_\rho & \rho E_\phi & E_z \end{vmatrix}$$

For spherical coordinates: $h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$

$$\vec{\nabla}\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \partial r & \partial\theta & \partial\phi \\ E_r & rE_\theta & r\sin\theta E_\phi \end{vmatrix}$$

Laplacian Operator

$$\vec{\nabla} V = \vec{E} \quad \Rightarrow \quad E_1 = (\nabla V)_1 \qquad E_2 = (\nabla V)_2 \qquad E_3 = (\nabla V)_3$$

Laplacian: $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (\vec{\nabla} V) = \nabla^2 V$

$$\vec{\nabla} \cdot \vec{E} \triangleq \frac{1}{h_1 h_2 h_3}\left[h_2 h_3 \frac{\partial E_1}{\partial u_1} + h_1 h_3 \frac{\partial E_2}{\partial u_2} + h_1 h_2 \frac{\partial E_3}{\partial u_3}\right]$$

$$E_1 = (\nabla V)_1 = \frac{1}{h_1}\frac{\partial V}{\partial u_1} \qquad E_2 = (\nabla V)_2 = \frac{1}{h_2}\frac{\partial V}{\partial u_2}$$

$$E_3 = (\nabla V)_3 = \frac{1}{h_3}\frac{\partial V}{\partial u_3}$$

$$\vec{\nabla} \cdot \vec{E} \triangleq \frac{1}{h_1 h_2 h_3}\left[h_2 h_3 \frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial V}{\partial u_1}\right) + h_1 h_3 \frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial V}{\partial u_2}\right) + h_1 h_2 \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial V}{\partial u_3}\right)\right]$$

$$\therefore \nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right]$$

Cartesian:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical: $h_1 = 1 = h_3 \qquad h_2 = \rho$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

For spherical:

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right)\right]$$

For a vector:

$$\nabla^2 \vec{E} = \nabla^2 [\hat{a}_1 E_1 + \hat{a}_2 E_2 + \hat{a}_3 E_3]$$

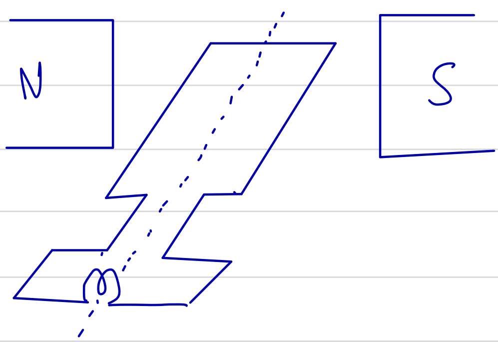
$$= \nabla^2 E_1 \hat{a}_1 + \nabla^2 E_2 \hat{a}_2 + \nabla^2 E_3 \hat{a}_3 \;//$$

Source of emf:

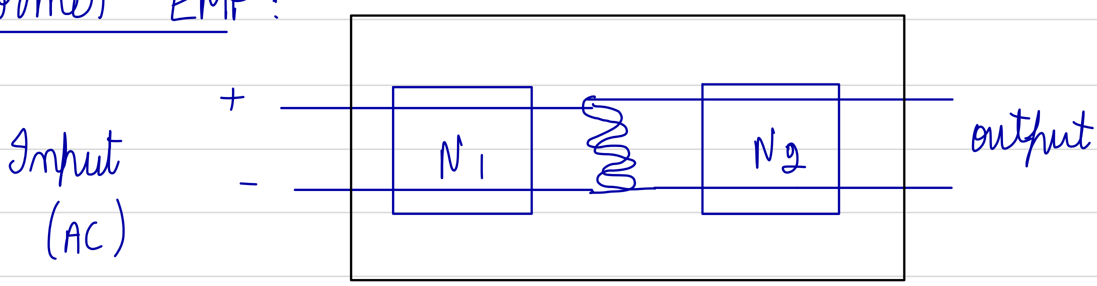
It is a device that provide steady current in circuit while maintaining constant potential difference between its terminals.

Faraday's law: $V = -\frac{d\phi_B}{dt}$

Generator EMF:



Transformer EMF:



Maxwell's 3rd law;

$$V = -\frac{d\phi_B}{dt}$$

$$\phi = \iint \vec{B} \cdot d\vec{S}$$

$$V = \iint \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

Also $V = \oint \vec{E} \cdot d\vec{l}$

$$\therefore \oint \vec{E} \cdot d\vec{l} = \iint \left(\frac{-\partial \vec{B}}{\partial t}\right) \cdot d\vec{S} \rightarrow (1)$$

From Stokes' Law:

$$\oint \vec{E} \cdot d\vec{l} = \iint (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} \rightarrow (2)$$

From (1) & (2),

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Maxwell's first law:

From Gauss' law we have,

$$\oint\!\!\oint \vec{D} \cdot d\vec{S} = Q$$

Also $\oint\!\!\oint \vec{D} \cdot d\vec{S} = \iiint \rho_v \, dv$

Applying divergence theorem:

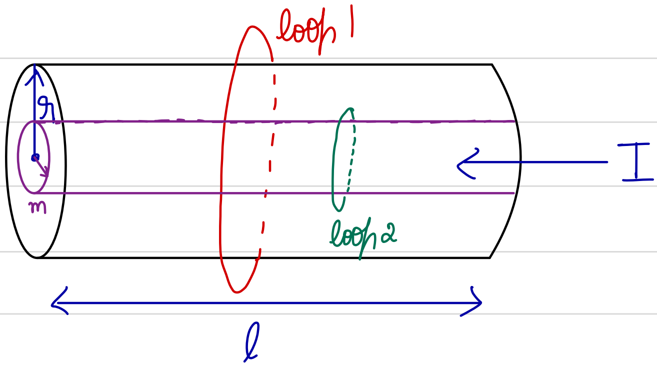
$$\iiint (\vec{\nabla} \cdot \vec{D}) \, ds = \iiint \rho_v \, dv$$

$$\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho_v \Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}}$$

Maxwell's 4th law

From Ampere's law: $\oint \vec{H} \cdot \vec{dl} = I_{enclosed}$

wire :



$J = \dfrac{I}{\pi r^2}$

$\oint \vec{H_1} \cdot \vec{dl} = I$ $\qquad\qquad$ $\oint \vec{H_2} \cdot \vec{dl} = I \dfrac{m^2}{r^2}$

So, $\oint \vec{H} \cdot \vec{dl} = \iint \vec{J} \cdot \vec{ds}$

$\iint (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \iint \vec{J} \cdot \vec{ds}$ (from Stokes' law)

$\Rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J}$ $\rightarrow$ Ampere's law

Now $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$

we have $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{J} = 0$

From continuity equation, $\vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$

For $\vec{\nabla} \cdot \vec{J} = 0$ the $-\dfrac{\partial \rho_v}{\partial t} = 0$. This happens only for DC.

For AC, $-\dfrac{\partial \rho_v}{\partial t} \neq 0$

So $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J_d}$ where $\vec{J_d}$ = displacement current density

& $\vec{J}$ = conduction current

$$\therefore \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{J}_d$$

$$\Rightarrow 0 = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{J}_d$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = -\vec{\nabla}\cdot\vec{J}_d$$

From continuity equation, $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \frac{\partial \rho_v}{\partial t}$$

$$= \frac{\partial(\vec{\nabla}\cdot\vec{D})}{\partial t} \quad \text{(from Gauss' law)}$$

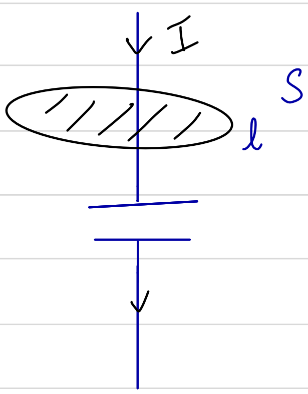
$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \nabla\cdot\frac{\partial\vec{D}}{\partial t} \quad \Rightarrow \quad \vec{J}_d = \frac{\partial\vec{D}}{\partial t}$$

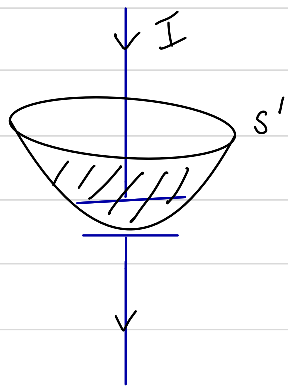
$$\therefore \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \Rightarrow \quad \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$$

Maxwell's 2nd law

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Explanation for Maxwell's 4th law





$$\oint \vec{H}\cdot d\vec{l} = I$$

For S', $\oint \vec{H}\cdot d\vec{l} = I_d$

$$= \iint \vec{J}_d\cdot d\vec{s}'$$

But $\vec{J} = \frac{\partial \vec{D}}{\partial t}$

So $\oint \vec{H} \cdot d\vec{l} = \iint \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$

Uncoupling maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (1)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v \rightarrow (3)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow (4)$$

From (1), $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$

$$\nabla(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\nabla(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left\{ \vec{J} + \frac{\partial D}{\partial t} \right\} \quad \text{(from (2))}$$

$$\nabla\left(\frac{\rho_v}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial \vec{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla\left(\frac{\rho_v}{\varepsilon_0}\right) + \mu_0 \frac{\partial \vec{J}}{\partial t}}$$

In free space, $\rho_v = 0$ & $J = 0$

$$\Rightarrow \boxed{\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \rightarrow \text{wave equation}$$

From ③, $\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{D})$

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = \vec{\nabla}\times\vec{J} + \varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$-\nabla^2\vec{H} = \vec{\nabla}\times\vec{J} + \varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow\quad -\nabla^2\vec{B} = \mu_0(\vec{\nabla}\times\vec{J}) - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow\quad \boxed{\nabla^2\vec{B} - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0(\vec{\nabla}\times\vec{J})}$$

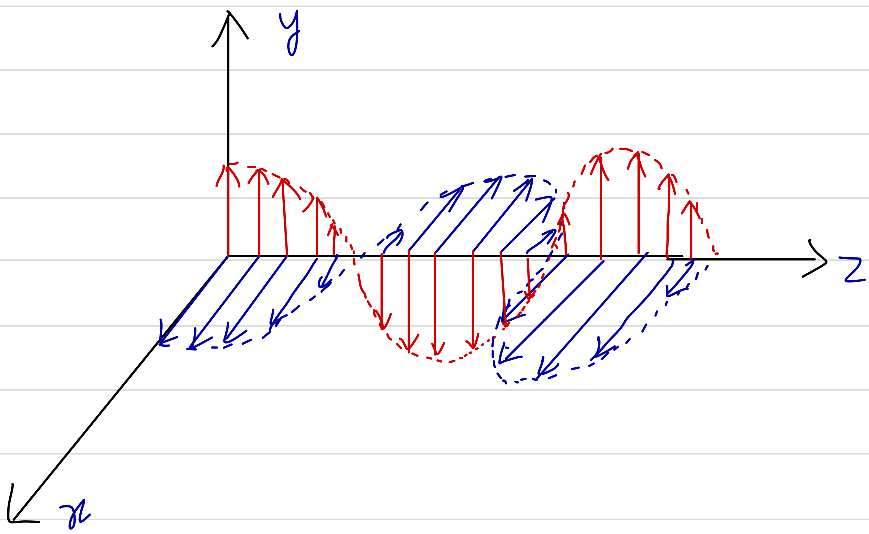
In free space $\vec{J}=0$

$$\Rightarrow\quad \boxed{\nabla^2 B - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0} \rightarrow \text{wave equation}$$

Wave visualization

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{E} = \hat{a}_x E_x$$



$$\text{LHS} = \vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\hat{a}_y - \frac{\partial E_x}{\partial y}\hat{a}_z$$

$$= \frac{\partial E_x}{\partial z}\hat{a}_y$$

$$RHS = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial}{\partial t}\left(\cancel{\hat{a}_x H_x} + \hat{a}_y H_y + \cancel{\hat{a}_z H_z}\right)$$

$$\Rightarrow \quad \hat{a}_y \frac{\partial E_x}{\partial z} = -\hat{a}_y \mu_0 \frac{\partial H_y}{\partial t}$$

$$\Rightarrow \quad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \rightarrow (1)$$

Also $\quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\vec{E} = \hat{a}_x E_x \qquad\qquad \vec{H} = \hat{a}_y H_y$

$\vec{H}$ exhibits variation along z axis only.

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial x & \partial y & \partial z \\ 0 & H_y & 0 \end{vmatrix} = -\hat{a}_x \frac{\partial H_y}{\partial z} + \hat{a}_z \frac{\partial H_y}{\partial z}$$

$$= -\hat{a}_x \frac{\partial H_y}{\partial z}$$

$$RHS: \quad \cancel{\vec{J}}^{\,0} + \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}\left(\hat{a}_x E_x + \cancel{\hat{a}_y E_y} + \cancel{\hat{a}_z E_z}\right)$$

$$= \varepsilon_0 \frac{\partial}{\partial t}\left(\hat{a}_x E_x\right) //$$

$$\Rightarrow \quad -\hat{a}_x \frac{\partial H_y}{\partial z} = \hat{a}_x \varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \vec{B}_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \rightarrow (2)$$

From (1), $\quad \frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial^2 B_y}{\partial z\, \partial t} \rightarrow (3)$

From (2), $$\frac{\partial^2 B_y}{\partial z^2} = -\mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial z \partial t} \rightarrow (4)$$

From (3) & (4)

$$\frac{\partial^2 B_z}{\partial z \partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\therefore \quad \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\therefore \quad \frac{\partial^2 E_x}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \quad //$$

$\rightarrow$ One dimension wave equation.

Wave equation

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = \underbrace{\nabla\left(\frac{\rho_v}{\varepsilon}\right) + \mu_0 \frac{\partial \vec{J}}{\partial t}}_{\text{Source}}$$

$$\nabla^2 \vec{H} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = -\underbrace{\vec{\nabla} \times \vec{J}}_{\text{Source}}$$

$$E_x = f_1(z - v_0 t) + f_2(z + v_0 t)$$

$f_1(z - v_0 t)$ $\downarrow$ wave travelling along +ve z direction

$f_2(z + v_0 t)$ $\downarrow$ wave travelling along -ve z direction.

$$E_x = C_1 e^{k(z - v_0 t)} + C_2 e^{k(z + v_0 t)}$$

$$\frac{\partial E_x}{\partial z} = kC_1 e^{k(z-v_0 t)} + kC_2 e^{k(z+v_0 t)}$$

$$\frac{\partial^2 E_x}{\partial z^2} = k^2 \left[C_1 e^{k(z-v_0 t)} + C_2 e^{k(z+v_0 t)}\right] \rightarrow ①$$

$$\frac{\partial E_x}{\partial t} = -kv_0 C_1 e^{k(z-v_0 t)} + kv_0 C_2 e^{k(z+v_0 t)}$$

$$\frac{\partial^2 E_x}{\partial t^2} = k^2 v_0^2 \left[C_1 e^{k(z-v_0 t)} + C_2 e^{k(z+v_0 t)}\right] \rightarrow ②$$

$$\frac{1}{v_0^2} \frac{\partial^2 E_x}{\partial t^2} = k^2 \left[C_1 e^{k(z-v_0 t)} + C_2 e^{k(z+v_0 t)}\right] \rightarrow ③$$

From ① & ③ :

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v_0^2} \frac{\partial^2 E_x}{\partial t^2} = 0$$

Poyting's theorem:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow ①$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \rightarrow ②$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) \rightarrow ③$$

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \rightarrow ④$$

From ② & ④

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{H} \cdot \left[-\frac{\partial \vec{B}}{\partial t} \right] - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \rightarrow ⑤$$

$$-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \rightarrow ⑥$$

$$-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D}) + \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})$$

$$-\iiint_{V(A)} \vec{\nabla} \cdot (\vec{E} \times \vec{H})\, dV = \iiint_{V(A)} \vec{E} \cdot \vec{J}\, dV + \frac{1}{2} \iiint_{V(A)} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E})\, dV$$

$$+ \frac{1}{2} \iiint_{V(A)} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H})\, dV$$

V - Volume A - Area

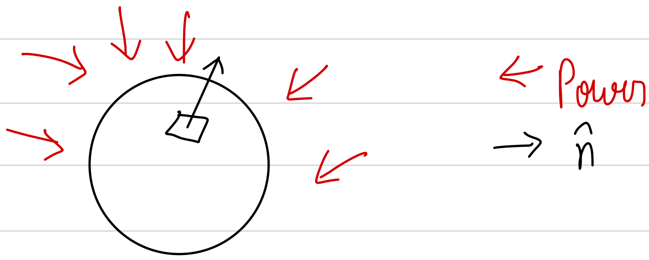
From divergence theorem:

$$-\oiint_{A(V)} (\vec{E} \times \vec{H}) \cdot d\vec{S} = \iiint_{V(A)} \vec{E} \cdot \vec{J} \, dV + \frac{1}{2} \iiint_{vol} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E}) \, dV + \frac{1}{2} \iiint_{vol} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H}) \, dV$$

$\downarrow$ Power

Significance of -ve sign in LHS:

i> Reciever case: Power incoming is opposite to the direction of unit normal to surface.



ii> Transmitter case:

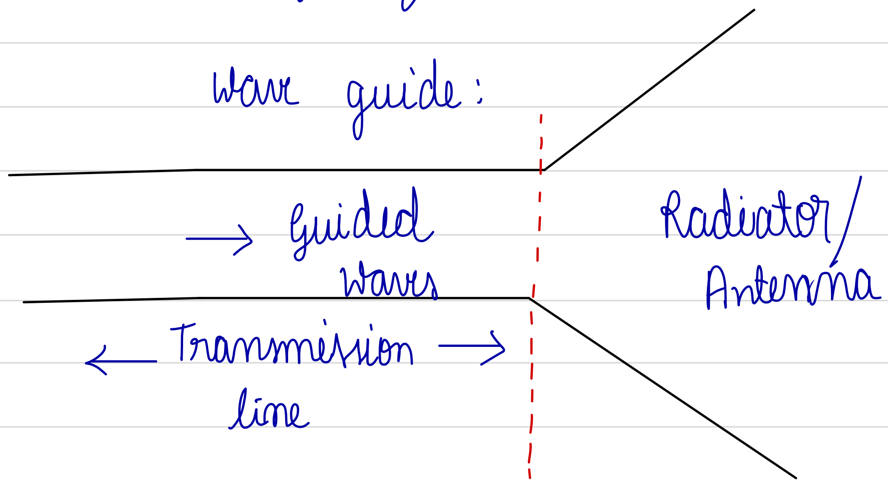
$$-\iiint_{V(A)} \vec{E} \cdot \vec{J} \, dV = \oiint_{A(V)} (\vec{E} \times \vec{H}) \cdot d\vec{S} + \iiint_{vol} \frac{1}{2} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E}) \, dV + \frac{1}{2} \iiint_{vol} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H}) \, dV$$

Negative sign is due to opposite directions of $\vec{E}$ & $\vec{J}$.

Radiation Mechanism

Transducer: Interfacing device

Wave guide:

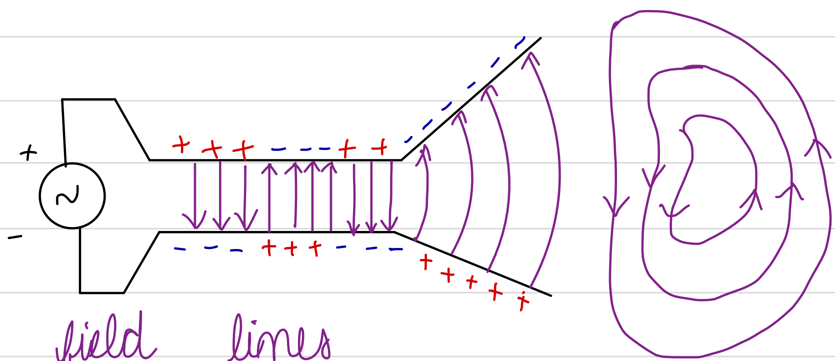


Postulates;

1> Electric field lines start from positive charge and terminate at negative charge or infinity

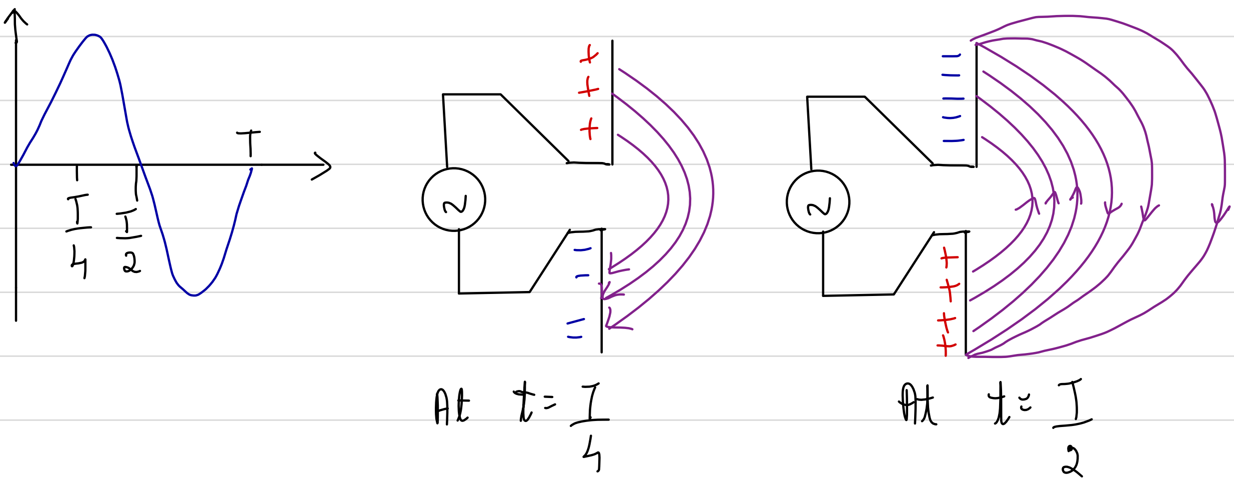
2> Electric field lines can form closed loop, closing on themselves like magnetic field lines.

Open transmittor:



— Electric field lines

Also:



Radiation Pattern:

A sketch of the electric field quantity / magnetic field quantity / Radiated power as a function of angular parameters (θ, ϕ).

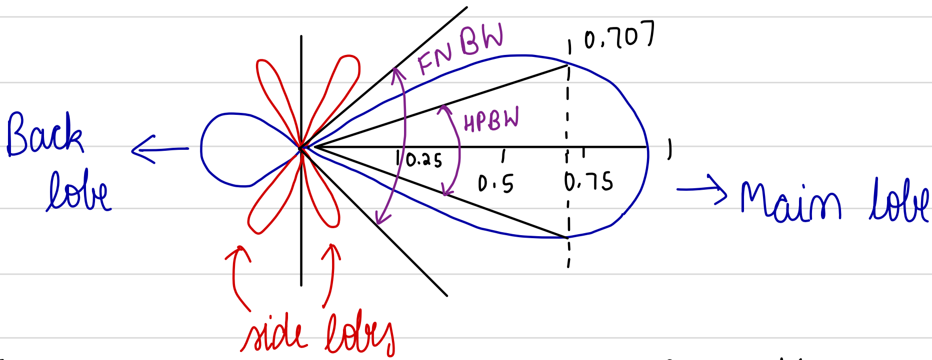


Fig. 2D-Plots of the Radiation plot.

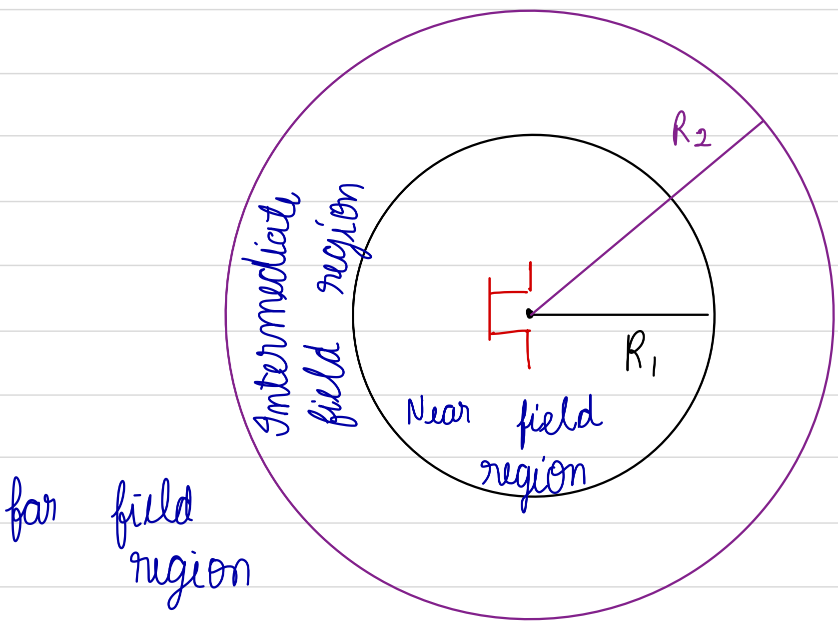
HPBW - Half power Beam width
FNBW - First null Beam width

Omnidirectional Radiating properties

- Omnidirectional: Radiates equally well in all direction.

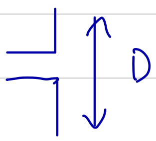
- Ideally such antennas do not exist.

Directional property of antenna:
- Radiates well in certain directions
- Not so well in other directions

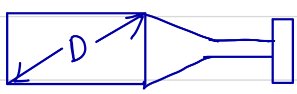


D = largest dimension of antenna

$$R_1 = 0.62\sqrt{\frac{D^3}{\lambda}} \qquad R_2 = \frac{2D^2}{\lambda}$$



or



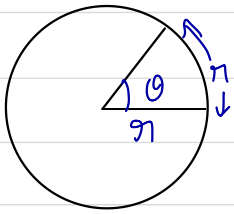
$\vec{W}$ (Poynting vector) & $\vec{P}_{avr}$ are real and directed radially outward. It varies with r as $\frac{1}{r^2}$ in far field region

$\vec{W} = \hat{a}_r \dfrac{A_0 \sin\theta \cos\phi}{r^2} \rightarrow$ some $f(\theta, \phi)$

- In near field region: $\vec{W} = Re + j\,Im$
- In intermediate field region: $\vec{W} \propto \dfrac{1}{r^3}, \dfrac{1}{r^4}, \dfrac{1}{r^5}$ so Im part is negligible.

Solid angle

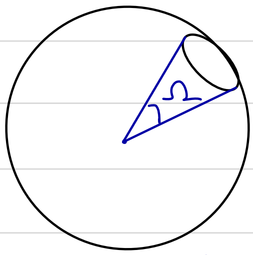
- Radian:



$\theta = 1\,\text{rad} = \dfrac{r}{r}$

$\theta = \dfrac{l}{r}$ (l = length of curve)

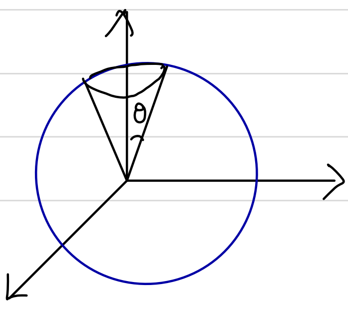
- Steradian:



$\Omega = \dfrac{A}{r^2}$ (Area of circle on sphere)

$\Omega = 1\,sr \Rightarrow \Omega = \dfrac{r^2}{r^2} = 1\,sr$ //

Q> Find the steradian for: $0 \le \theta \le 30^\circ$ ($\pi/6$)
$0 \le \phi \le 2\pi$



Ans: For a sphere $dS = r^2 \sin\theta \, d\theta \, d\phi$

$d\Omega$ = solid angle subtended at the centre of sphere by dS

$$d\Omega = \frac{dS}{r^2} = \sin\theta \, d\theta \, d\phi$$

$$\Omega = \int_0^{2\pi}\int_0^{\pi/6} \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \cdot \left[-\cos\theta\right]_0^{\pi/6}$$

$$\Omega = 2\pi\left(1 - \frac{\sqrt{3}}{2}\right) = \pi\left(2 - \sqrt{3}\right)$$

Phasor notation:

$$i(t) = I_0 \cos(\omega t + \phi)$$

$$i(t) = Re\{I_0 e^{j(\omega t + \phi)}\}$$

$$= Re\{\underbrace{I_0 e^{j\phi}}_{\text{Phasorating (Amplitude \& phase information)}} \cdot \underbrace{e^{j\omega t}}_{\text{Harmonic time variation.}}\}$$

Eg: $i(t) = \cos\omega t + 2\cos(\omega t - \pi/2)$

$$= Re\{e^{j\omega t}\} + 2\,Re\{e^{j(\omega t - \pi/2)}\}$$

$$= Re\{e^{j\omega t}\} + 2\,Re\{e^{j(-\pi/2)} e^{j\omega t}\}$$

$$= Re\{(1 - 2j) e^{j\omega t}\}$$

NOTE: $\vec{E}(x, y, z, t) = \vec{E}_s(x, y, z) e^{j\omega t}$

$\vec{H}(x, y, z, t) = \vec{H}_s(x, y, z) e^{j\omega t}$

Maxwell's equation in Phasor form

1) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\Rightarrow \vec{\nabla} \times [\vec{E}(x, y, z, t)] = -\frac{\partial}{\partial t}(\mu_0 \vec{H}(x, y, z, t))$

$\Rightarrow \vec{\nabla} \times [\vec{E}_s(x, y, z)\, e^{j\omega t}] = -\frac{\partial}{\partial t}(\mu_0 \vec{H}_s(x, y, z)\, e^{j\omega t})$

$\Rightarrow \cancel{e^{j\omega t}}\ \vec{\nabla} \times [\vec{E}(x, y, z)] = -\mu_0 \vec{H}_s(x, y, z)\, j\omega\, \cancel{e^{j\omega t}}$

$\Rightarrow \vec{\nabla} \times \vec{E}_s(x, y, z) = -j\omega\mu_0 \vec{H}_s(x, y, z)$

$\Rightarrow \boxed{\vec{\nabla} \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s}$

2) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

consider $\vec{J} = 0$,

$\Rightarrow \boxed{\vec{\nabla} \times \vec{H}_s = j\omega\varepsilon_0 \vec{E}_s}$

3) $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{E}_s = 0}$ $(Q = 0)$

4) $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{H}_s = 0}$

Time average value of Poynting Vector

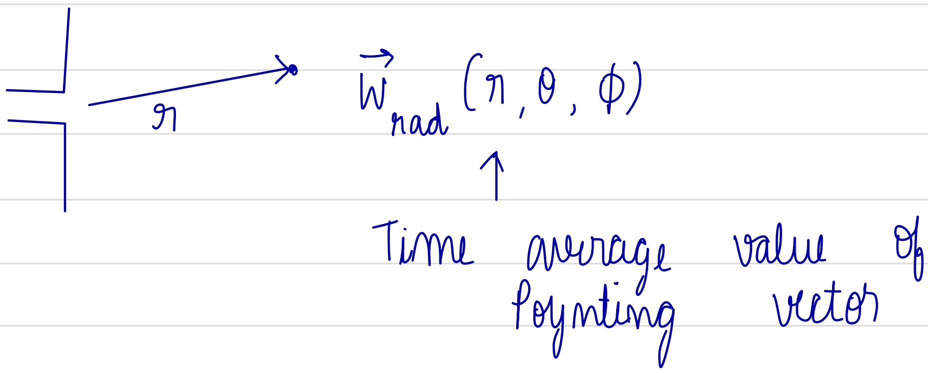
$\vec{P} = \vec{W} \triangleq \vec{E} \times \vec{H}$

$\vec{P}_{ave} = \frac{1}{2} Re [\vec{E} \times \vec{H}^*]$

For $v(t)$, $V_{ave} = \frac{1}{T}\int_0^T v(t) \cdot dt$

Directivity

i> Radiation intensity: Power radiated per unit solid angle.



$$P_{rad} \triangleq \oiint \vec{W}_{rad}(r, \theta, \phi) \cdot d\vec{S}$$

$$= \int_0^{2\pi}\int_0^{\pi} \vec{W}_{rad}(r, \theta, \phi)\, \hat{a}_r \cdot \hat{a}_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} r^2 W_{rad} \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} U(\theta, \phi)\, d\Omega$$

We define $U(\theta, \phi) = r^2 W_{ave}$

- Power passing through infinitesimal area $dS = \vec{W}_{rad}(r, \theta, \phi) \cdot d\vec{S}$

$$\text{Power} = W_{ave}(r, \theta, \phi)\, \hat{a}_r \cdot \hat{a}_r \,(r^2 \sin\theta \, d\theta \, d\phi)$$

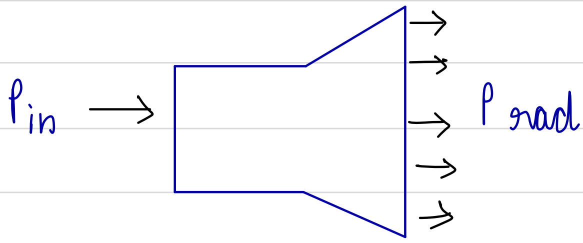
$$= r^2 W_{ave}(r, \theta, \phi) \sin\theta \, d\theta \, d\phi$$

$$\frac{\text{Power radiated}}{\text{unit solid angle}} = \frac{r^2 W_{rad}(r, \theta, \phi) \sin\theta \, d\theta \, d\phi}{\sin\theta \, d\theta \, d\phi}$$

So $U(\theta, \phi) = r^2 W_{rad}(r, \theta, \phi)$

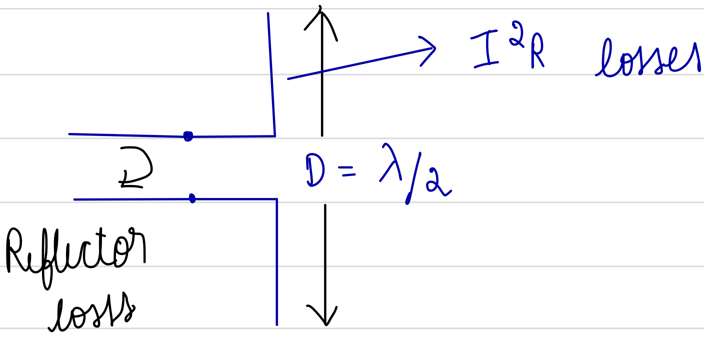
$U(\theta, \phi) = r^2 W_{rad} = r^2 W_{ave}$

Antenna efficiency, $\eta \triangleq \dfrac{P_{rad}}{P_{in}}$



NOTE: • AM Broadcast band : 560 kHz to 1600 kHz

• Losses :



• Gain = Antenna efficiency × Directivity

$G(\theta, \phi) = \eta \cdot D(\theta, \phi)$

$$= \frac{\cancel{P_{rad}}}{P_{in}} \cdot \frac{4\pi\, U(\theta, \phi)}{\cancel{P_{rad}}} = \frac{4\pi\, U(\theta, \phi)}{P_{in}}$$

where $D(\theta, \phi)$ is the directivity along direction (θ, ϕ)

• Maximum directivity, $D_{max} \triangleq \left.\dfrac{4\pi U(\theta,\phi)}{P_{rad}}\right|_{max}$

$\therefore$ Maximum gain, $G_{max} = \left.\dfrac{4\pi U(\theta,\phi)}{P_{in}}\right|_{max}$

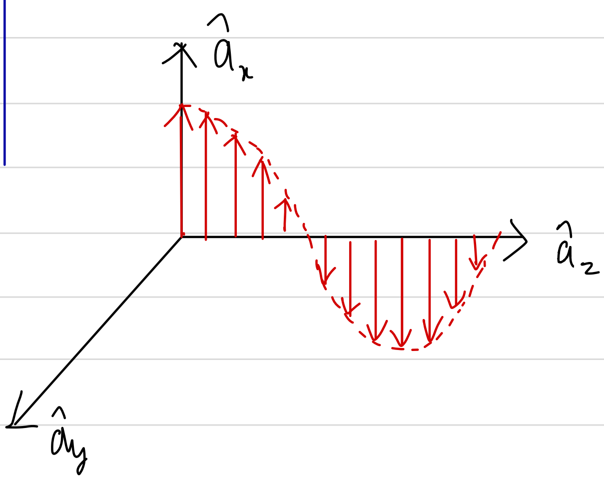
Understanding Maxwell's equation in Phasor form

1) Consider $\vec{\nabla} \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s$

let $\vec{E}_s = \hat{a}_x E_{xs}$ & $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} = 0$

$$\vec{\nabla} \times \vec{E}_s = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial x & \partial y & \partial z \\ E_{xs} & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial E_{xs}}{\partial z} \hat{a}_y$$



$$\Rightarrow \frac{\partial (E_{xs})}{\partial z} \hat{a}_y = -j\omega\mu_0 \left(H_{xs}\hat{a}_x + H_{ys}\hat{a}_y + H_{zs}\hat{a}_y \right)$$

$\Rightarrow H_{xs} = 0$ & $H_{zs} = 0$

And $\dfrac{\partial (E_{xs})}{\partial z} = -j\omega\mu_0 H_{ys}$ $\rightarrow$ ①

Now $E_x(z,t) = E_{x0}^{+} e^{j(\omega t - \beta z)} + E_{x0}^{-} e^{j(\omega t + \beta z)}$

$$= \left[E_{x0}^{+} e^{-j\beta z} + E_{x0}^{-} e^{j\beta z} \right] e^{j\omega t}$$

$$\therefore E_{xs} = E_{x_0}^{+} e^{-j\beta z} + E_{x_0}^{-} e^{j\beta z}$$

$$\frac{\partial E_{xs}}{\partial z} = -j\beta E_{x_0}^{+} e^{-j\beta z} + j\beta E_{x_0}^{-} e^{j\beta z}$$

$$= -j\beta \left[E_{x_0}^{+} e^{-j\beta z} - E_{x_0}^{-} e^{j\beta z} \right]$$

$$\therefore H_{ys} = -\frac{1}{j\omega\mu_0} \cdot (-j\beta) \left[E_{x_0}^{+} e^{-j\beta z} - E_{x_0}^{-} e^{j\beta z} \right]$$

$$= \frac{\omega\sqrt{\mu_0 \varepsilon_0}}{\omega\mu_0} \left[E_{x_0}^{+} e^{-j\beta} - E_{x_0}^{-} e^{j\beta z} \right]$$

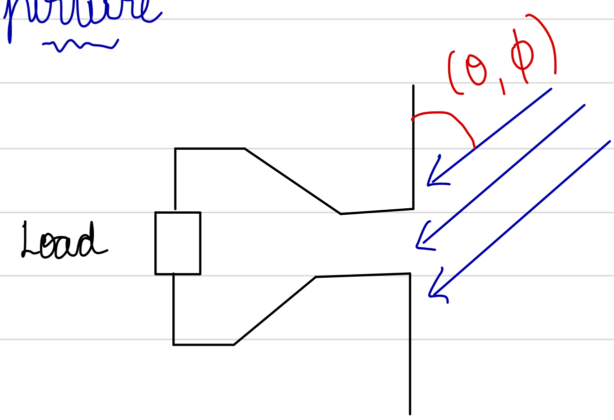
But $\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \cong 377\,\Omega$

$$\therefore H_{ys} = \frac{1}{\eta} \left[E_{x_0}^{+} e^{-j\beta z} - E_{x_0}^{-} e^{j\beta z} \right]$$

$$H_{ys} = \frac{E_{x_0}^{+}}{\eta} e^{-j\beta z} - \frac{E_{x_0}^{-}}{\eta} e^{j\beta z} = H_{y_0}^{+} e^{-j\beta z} + H_{y_0}^{-} e^{j\beta z}$$

$$\therefore H_{y_0}^{+} = \frac{E_{x_0}^{+}}{\eta} \quad \& \quad H_{y_0}^{-} = -\frac{E_{x_0}^{-}}{\eta}$$

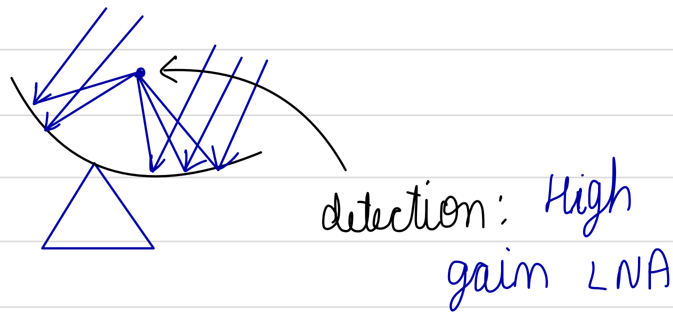
Antenna Aperture



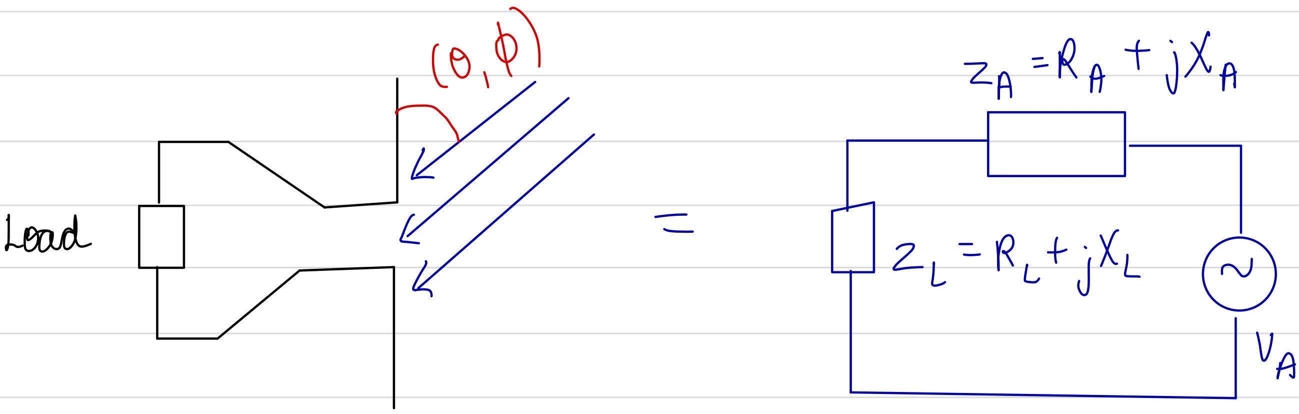
- Power delivered to the load by the Electromagnetic waves arriving from direction (θ, ϕ) is called the aperture.

- Effective aperture of the antenna $\triangleq \dfrac{P_r(\theta,\phi)}{W}\ m^2$

- Parabolic antennas:



LNA = Low noise Amplifier



$R_A = R_{rad} + R_{loss}$

R_{rad} ↓ Radiation resistance; R_{loss} ↓ loss resistance.

→ Half wave dipole ($\lambda/2$), $R_{rad} = 73\,\Omega$

→ Folded dipole antenna, $R_{rad} = 73 \times 4 = 292\,\Omega$,
$R_{loss} \ll R_{rad}$.

<u>Conjugate matching:</u> $R_A = R_L$ & $X_A = -X_L$

$\Rightarrow \quad I = \frac{V_A}{2R_A} = \frac{V_A}{2R_L}$

Then power delivered to the load is :

$P_L = |I|^2 \cdot R_L$

$= \frac{1}{2} \frac{V_A^2}{4R_L^2} R_L = \frac{V_A^2}{8R_L}$

$\therefore$ Effective antenna aperture:

$$A_e(\theta, \phi) = \frac{P_L}{W} = \frac{|V_a|^2}{8R_L W}$$

- Scattered (Reradiated) Power $= \frac{1}{2} |I|^2 R_{rad}$

$R_L \cong R_{rad} \quad \Rightarrow \quad P_{rad} = \frac{|V_A|^2}{8R_L}$

$\therefore$ Scattering Aperture, $A_{scat}(\theta, \phi) = \frac{P_{rad}}{W}$

$$= \frac{|V_A|^2}{8R_L^2 W} \cdot R_{rad}$$

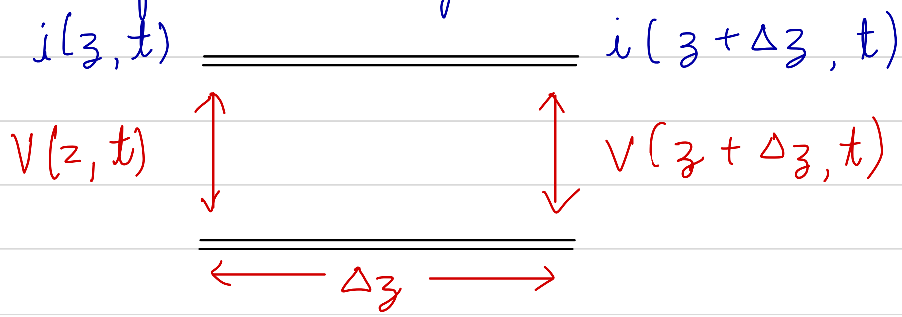
$$\cong A_e(\theta, \phi)$$

Also $\quad A_{loss}(\theta, \phi) = \frac{|V_A|^2}{8R_L^2 W} \cdot R_{loss}$

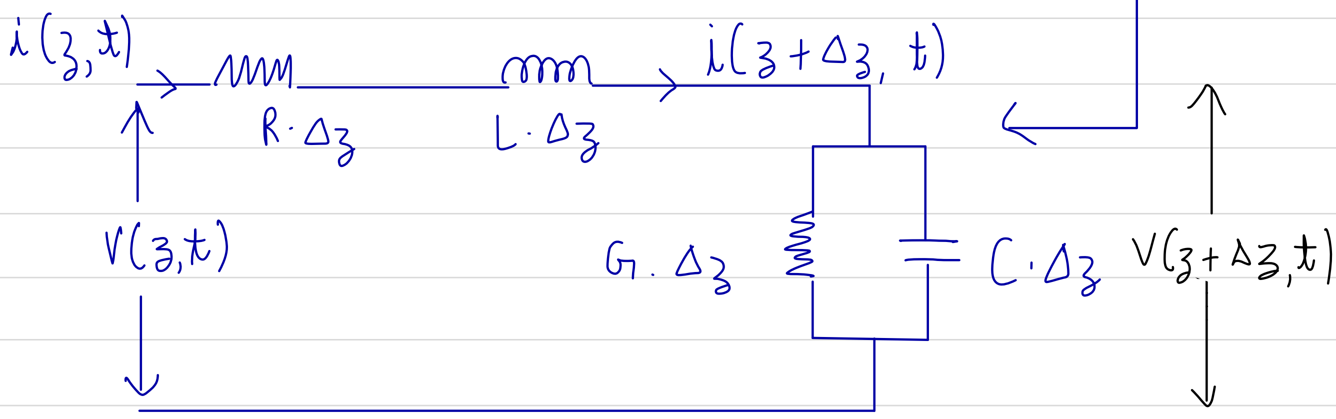
since $R_{loss} \ll R_L$ so this is insignificant.

Transmission Line Theory

This theory is in between Circuit theory and field theory.



R = Resistance / unit length
G = Conductance / unit length
L = Inductance / unit length
C = Capacitance / unit length



Applying KVL to circuit:

$$V(z,t) - R.\Delta z\, i(z,t) - L\cdot\Delta z \frac{\partial(i(z,t))}{\partial t} = V(z+\Delta z,t)$$

$$\Rightarrow \frac{V(z+\Delta z, t) - V(z,t)}{\Delta z} = -\left[R\cdot i(z,t) + L\frac{\partial(i(z,t))}{\partial t}\right]$$

Applying $\lim \Delta z \to 0$

$$\lim_{\Delta z\to 0} \frac{V(z+\Delta z, t) - V(z,t)}{\Delta z} = \lim_{\Delta z \to 0} -\left[R\, i(z,t) + L\frac{\partial(i(z,t))}{\partial t}\right]$$

$$\boxed{\frac{\partial(V(z,t))}{\partial z} + Ri(z,t) + L\frac{\partial(i(z,t))}{\partial t} = 0} \rightarrow (1)$$

Applying KCL :

$$i(z+\Delta z,t) - i(z,t) = -G\Delta z V(z+\Delta z,t) - C\cdot\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -GV(z+\Delta z,t) - C\frac{\partial(V(z+\Delta z,t))}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial(i(z,t))}{\partial z} + GV(z,t) + C\frac{\partial(V(z,t))}{\partial t} = 0} \rightarrow (2)$$

(1) & (2) are known as Telegraphers' equations.

Telegraphers' equation in phasor form :

1) $\frac{dV(z)}{dz} = -RI(z) - j\omega LI(z)$

2) $\frac{dI(z)}{dz} = -GV(z) - j\omega CV(z)$

Differentiating (1) with respect to z

$$\frac{d^2V(z)}{dz^2} = -(R+j\omega L)\frac{dI(z)}{dz}$$

From (2), $\frac{d^2V(z)}{dz^2} = [-(R+j\omega L)]\cdot[-(G+j\omega C)V(z)]$

$$\frac{d^2V(z)}{dz^2} = (R+j\omega L)\cdot(G+j\omega C)\,V(z) \rightarrow (3)$$

Similarly $$\frac{d^2I(z)}{dz^2} = (R+j\omega L)\cdot(G+j\omega C)\,I(z) \rightarrow (4)$$

Complex Propagation constant

$$\gamma \triangleq \sqrt{(R+j\omega L)\cdot(G+j\omega C)}$$

$$\gamma = \sqrt{(RG-\omega^2LC)+j\omega(CR+LG)}$$

$\therefore$ Solution of (3) & (4) are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \rightarrow (5)$$
$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \rightarrow (6)$$

where V_0^+ is a voltage wave propagating along positive direction and V_0^- is a voltage wave propagating along negative wave direction.

From (5),

$$\frac{dV(z)}{dz} = V_0^+ e^{-\gamma z}(-\gamma) + V_0^- e^{\gamma z}\cdot\gamma$$

$$(R+j\omega L)\,I(z) = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}$$

$$I(z) = \frac{\gamma}{(R+j\omega L)}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

$$= \frac{\sqrt{(R+j\omega L)\cdot(G+j\omega C)}}{(R+j\omega L)}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

$$\therefore \; I(z) = \sqrt{\frac{(G+j\omega C)}{(R+j\omega L)}} \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right]$$

$$Z_0 \triangleq \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}} \triangleq \text{characteristic Impedance of Transmission Line.}$$

$$\Rightarrow I(z) = \frac{1}{Z_0} \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right]$$

$$\Rightarrow I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} = \frac{1}{Z_0} \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right]$$

$$\therefore \; I_0^+ \triangleq \frac{V_0^+}{Z_0} \qquad I_0^- \triangleq -\frac{V_0^-}{Z_0}$$

Now consider, $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$

$$V(z) = V_0^+ e^{-(\alpha + j\beta) z} + V_0^- e^{(\alpha + j\beta) z}$$

$$Re\{ V(z) e^{j\omega t} \} = Re\{ V_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)} + V_0^- e^{\alpha z} e^{j(\omega t + \beta z)} \}$$

$$\Rightarrow V(z,t) = V_0^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_0^- e^{\alpha z} \cos(\omega t + \beta z)$$

- β is specified.

NOTE: $\gamma = \alpha + \beta j$

where α = attenuation constant

β = phase constant.

Phase velocity

$$v_p = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad \varepsilon_r \qquad v_p = \frac{1}{\sqrt{\mu \varepsilon}}$$

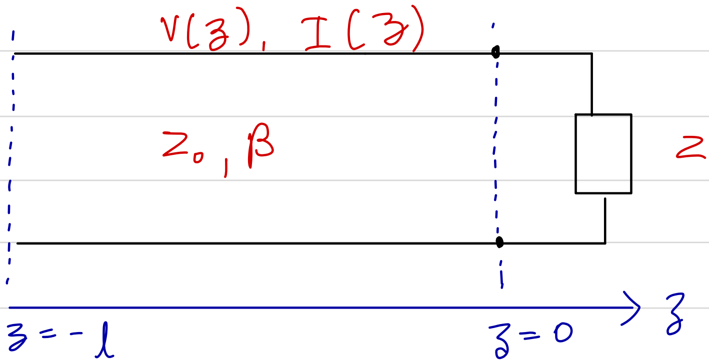
Loss less transmission Line

$R = G = 0$

so $Z_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} = \sqrt{\dfrac{L}{C}}$

$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{-\omega^2 LC} = j\omega\sqrt{LC} = j\beta$

Consider a lossless transmission line:



$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \rightarrow (1)$

$I(z) = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0} e^{j\beta z} \rightarrow (2)$

At the location of load

$V(0) = V_0^+ + V_0^- \rightarrow (3)$

$I(0) = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0} \rightarrow (4)$

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{\dfrac{V_0^+ - V_0^-}{Z_0}} = \frac{Z_0(V_0^+ + V_0^-)}{V_0^+ - V_0^-}$$

Also, $Z_L V_0^+ - Z_L V_0^- = Z_0 V_0^+ + Z_0 V_0^-$

$(Z_L - Z_0) V_0^+ = (Z_L + Z_0) V_0^-$

$\Rightarrow \dfrac{V_0^-}{V_0^+} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = r =$ Voltage reflection coefficient

NOTE: • Transmission line terminated at open circuit: $Z_L = \infty \Rightarrow \Gamma = 1$

• Transmission line short circuit: $Z_L = 0 \Rightarrow \Gamma = -1$

• Also: $-1 \leq \Gamma \leq 1 \Rightarrow |\Gamma| \leq 1$

Consider equation (1),

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$V(z) = V_0^+ \left[e^{-j\beta z} + \frac{V_0^-}{V_0^+} e^{j\beta z} \right]$$

$$\Rightarrow V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma e^{j\beta z} \right] \rightarrow (5)$$

Similarly $I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma e^{j\beta z} \right]$$

∴ Generalised expression for impedance:

$$\boxed{\frac{V(z)}{I(z)} = Z_0 \frac{\left[e^{-j\beta z} + \Gamma e^{j\beta z} \right]}{\left[e^{-j\beta z} - \Gamma e^{j\beta z} \right]}}$$

Time average power flow along a transmission line

$$P_{avg} = \frac{1}{2} \mathrm{Re} \left[V(z) \cdot I^*(z) \right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma e^{j\beta z}\right]$$

$$I^*(z) = \frac{V_0^+}{Z_0}\left[e^{j\beta z} - \Gamma^* e^{-j\beta z}\right]$$

$$V(z)\cdot I^*(z) = \frac{|V_0^+|^2}{Z_0}\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]\cdot\left[e^{j\beta z} - \Gamma^* e^{-j\beta z}\right]$$

$$= \frac{|V_0^+|^2}{Z_0}\left[1 - \Gamma^* e^{2j\beta z} + \Gamma e^{-2j\beta z} - |\Gamma|^2\right]$$

$$P_{ave} = \frac{1}{2} Re\left[V(z)\cdot I^*(z)\right]$$

$$= \frac{|V_0^+|^2}{2} Re\left[1 - \underbrace{\Gamma^* e^{2j\beta z} + \Gamma e^{-2j\beta z}}_{\text{complex}} - |\Gamma|^2\right]$$

$A = \Gamma e^{2j\beta z}$ $\qquad A^* = \Gamma^* e^{-2j\beta z}$

$A - A^* = 2\,Im(A) \longrightarrow$ neglected

$$\therefore P_{ave} = \frac{|V_0^+|^2}{2}\left[1 - |\Gamma|^2\right]$$

$$P_{ave} = \frac{|V_0^+|^2}{2} - |\Gamma|^2\frac{|V_0^+|^2}{2}$$

$\frac{|V_0^+|^2}{2}$ ↓ Power travelling forward

$|\Gamma|^2\frac{|V_0^+|^2}{2}$ → Contribution to reflected power from mismatched load.

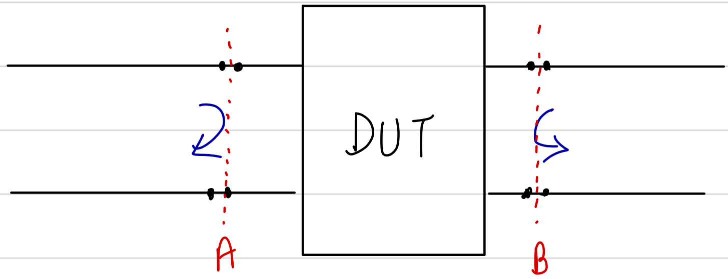
- Return loss, $RL = -20\log_{10}|\Gamma|$ dB

If $|\Gamma| = 1 \Rightarrow RL = 0$ & it is open circuit

If $|\Gamma| = 0.5 \Rightarrow RL = -20\log_{10}(0.5) \cong 6$ dB

As $|\Gamma| \to 0$, $RL \to \infty$ & it is short circuit

Insertion loss in Transmission Loss :



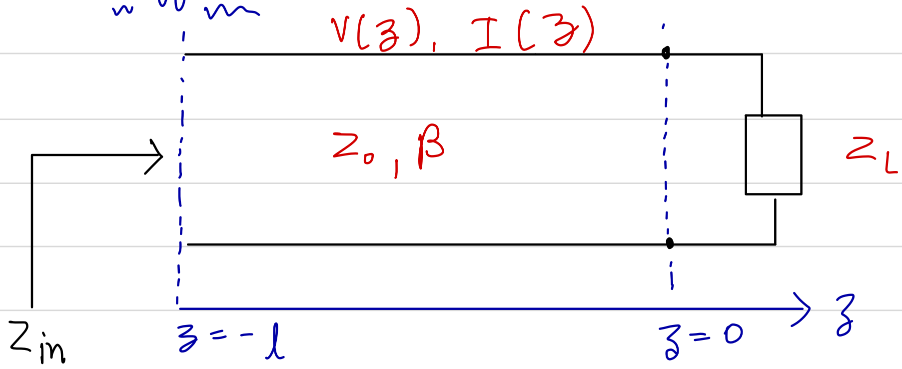
DUT = Design under test.

Standing wave ratio

$$SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad 1 \leq SWR \leq \infty$$

- Once a standing wave is setup, the distance between 2 consecutive maximas or minimas $= \frac{\lambda}{2}$
- Distance between consecutive maxima and minima $= \frac{\lambda}{4}$
- Slotted line is used to measure SWR

Reflection coefficient



$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$V(z=-l) = \underbrace{V_0^+ e^{+j\beta l}}_{\text{Forward wave}} + V_0^- e^{-j\beta l} \rightarrow \text{Reverse travelling wave.}$$

$$\Gamma' = \Gamma(z=-l)$$
$$= \frac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}}$$
$$= \frac{V_0^-}{V_0} e^{-j2\beta l} = \Gamma e^{-j2\beta l}$$

Impedance Transformation equation

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma e^{j\beta z}\right] \rightarrow ①$$
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma e^{j\beta z}\right] \rightarrow ②$$

$$V(z=-l) = V_0^+\left[e^{j\beta l} + \Gamma e^{-j\beta l}\right]$$
$$I(z=-l) = \frac{V_0^+}{Z_0}\left[e^{j\beta l} - \Gamma e^{-j\beta l}\right]$$

Input impedance, $Z_{in} = \dfrac{V(-l)}{I(-l)}$

$$\Rightarrow Z_{in} = \frac{Z_0\left[e^{j\beta l} + \Gamma e^{-j\beta l}\right]}{\left[e^{j\beta l} - \Gamma e^{-j\beta l}\right]}$$

$$= \frac{Z_0\left[e^{j\beta l} + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-j\beta l}\right]}{\left[e^{j\beta l} - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-j\beta l}\right]}$$

$$= \frac{Z_0\left[(Z_L + Z_0) e^{j\beta l} + (Z_L - Z_0) e^{-j\beta l}\right]}{\left[(Z_L + Z_0) e^{j\beta l} - (Z_L - Z_0) e^{-j\beta l}\right]}$$

$$\Rightarrow Z_{in} = Z_0 \left[\frac{Z_L (e^{j\beta l} + e^{-j\beta l}) + Z_0 (e^{j\beta l} - e^{-j\beta l})}{Z_L (e^{j\beta l} - e^{-j\beta l}) + Z_0 (e^{j\beta l} + e^{-j\beta l})} \right]$$

$$= Z_0 \left[\frac{Z_L \cos\beta l + j Z_0 \sin\beta l}{j Z_L \sin\beta l + Z_0 \cos\beta l} \right]$$

$$\therefore \boxed{Z_{in} = Z_0 \left[\frac{Z_L + j Z_0 \tan\beta l}{Z_0 + j Z_L \tan\beta l} \right]} \rightarrow \text{Impedance transformation equation.}$$

<u>case I:</u> Short circuited transmission line.

i.e $Z_L = 0$

$\Gamma = -1$ //

$$Z_{in} = Z_0 \left[\frac{0 + j Z_0 \tan\beta l}{Z_0 + 0} \right]$$

$$\Rightarrow Z_{in} = j Z_0 \tan\beta l \;//$$

Generalized impedance, Z at any z,

$$Z_{in} = \frac{V(z)}{I(z)} = -Z_0 \frac{[e^{-j\beta z} - e^{j\beta z}]}{[e^{-j\beta z} + e^{j\beta z}]}$$

$$= -j Z_0 \frac{\sin\beta z}{\cos\beta z} = -j Z_0 \tan\beta z \;//$$

$$V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z})$$

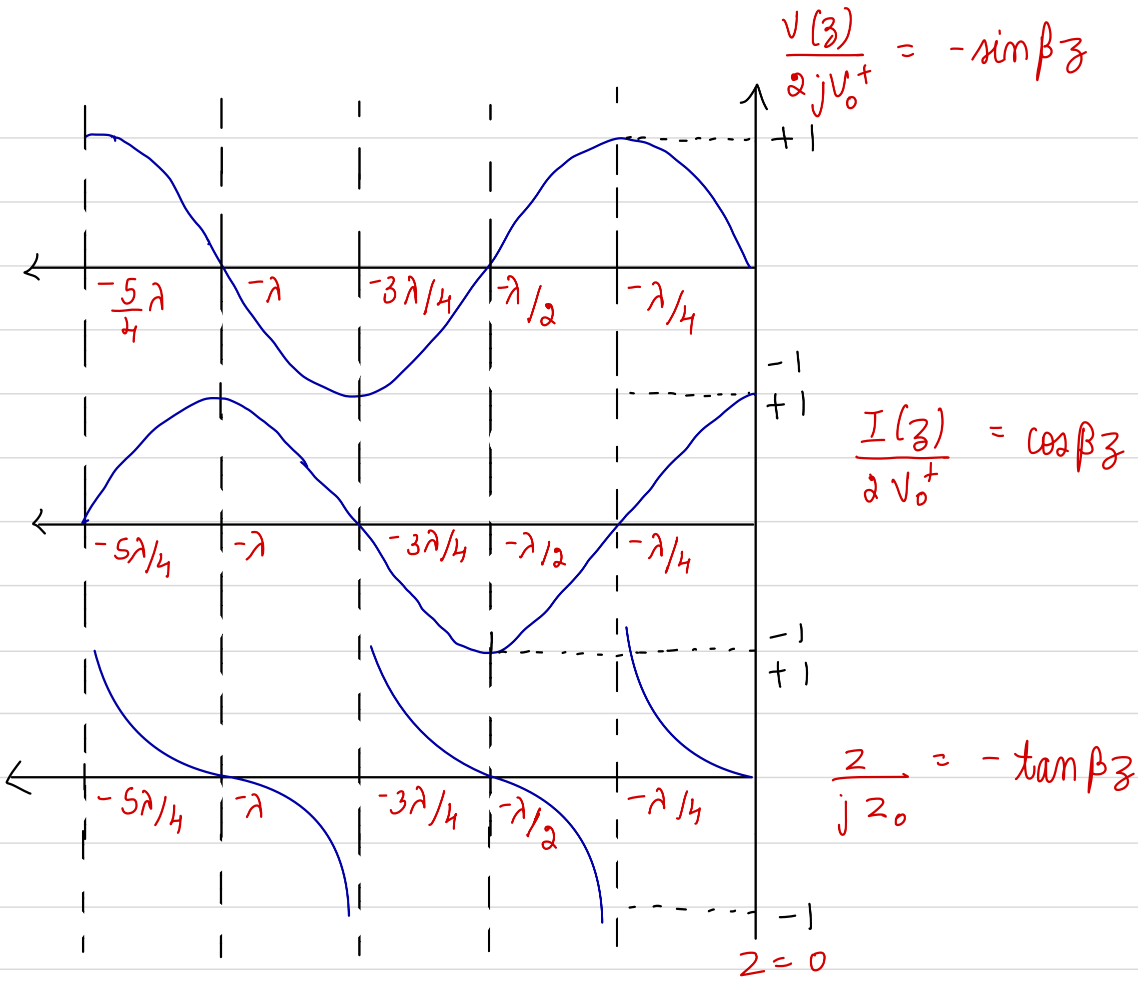
$\Gamma = -1$,

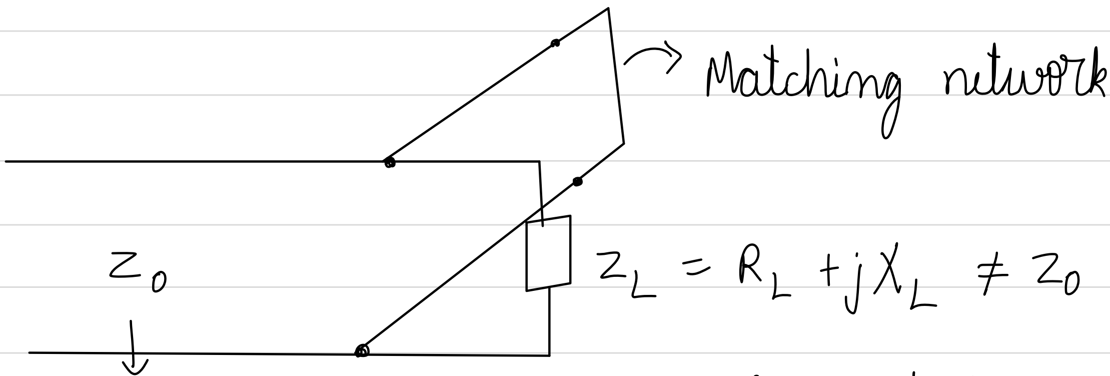
$$V(z) = V_0^+ (e^{-j\beta z} - e^{j\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} + e^{j\beta z})$$

$$\Rightarrow V(z) = -2j V_0^+ \sin\beta z$$

$$I(z) = \frac{2V_0}{Z_0} \cos\beta z$$

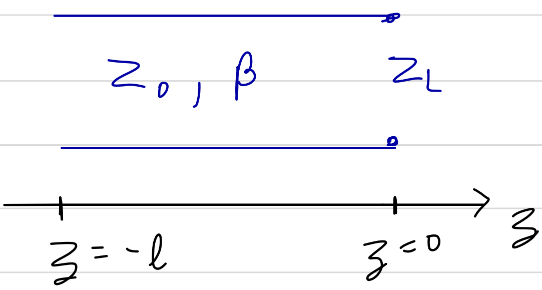




Assuming to be real (considering lossless)

→ Matching network is used to tune Z_L to make $Z_L = Z_0$

<u>case II</u> : open circuit : $Z_L \to \infty$



As $Z_L \to \infty$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq 1$$

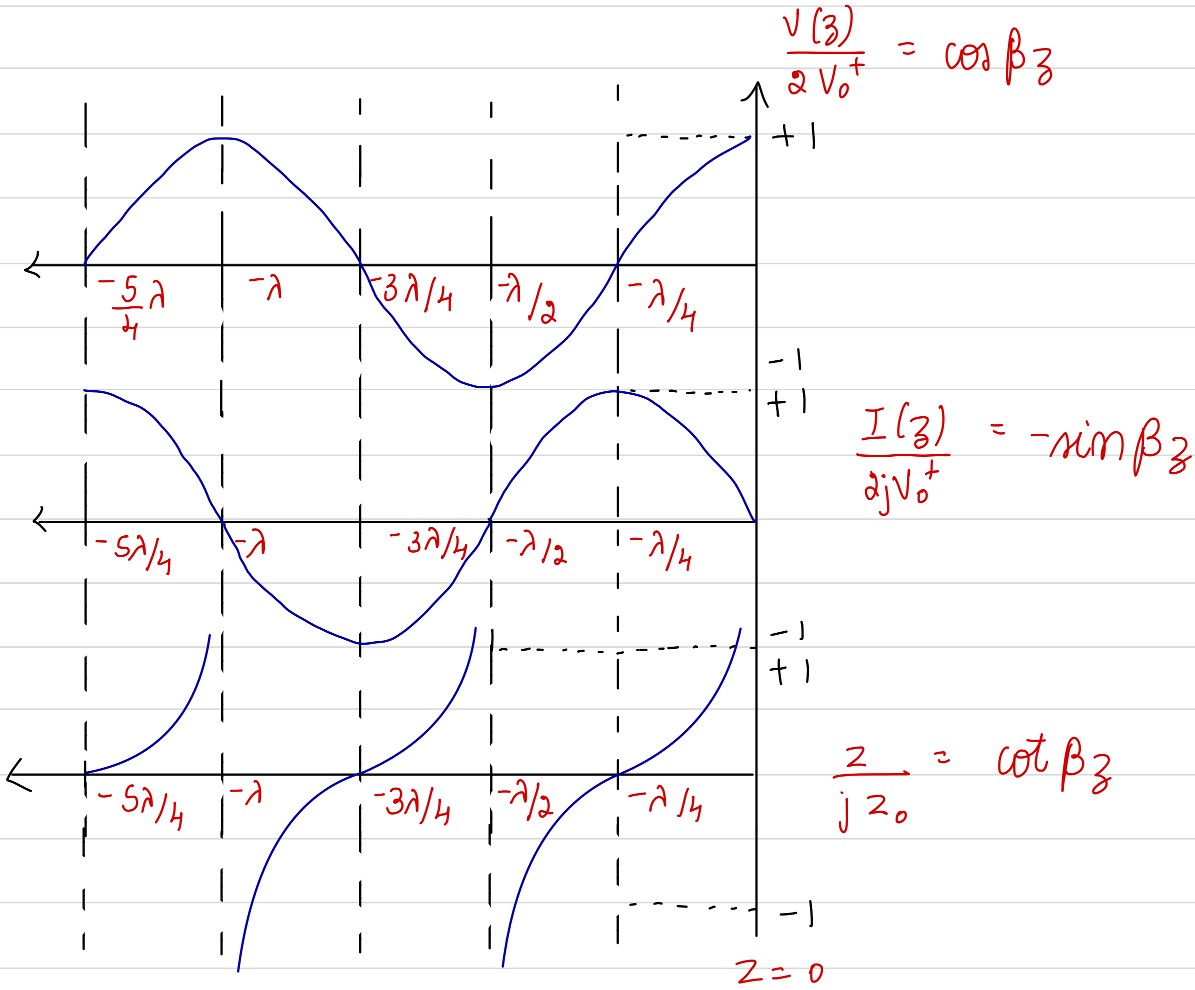
$$V(z) = V_0^+ \left[e^{-j\beta z} + r\, e^{j\beta z} \right]$$
$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - r\, e^{j\beta z} \right]$$

Putting $r = 1$

$$V(z) = V_0^+ \left[e^{-j\beta z} + e^{j\beta z} \right] = 2V_0^+ \cos\beta z$$
$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - e^{j\beta z} \right] = -2\frac{V_0^+}{Z_0} j \sin\beta z$$

Generalized impedance

$$Z = \frac{V(z)}{I(z)} = \frac{2V_0^+ \cos\beta z}{-2\frac{V_0^+}{Z_0} j \sin\beta z} = j Z_0 \cot\beta z$$



2 important types of transmission lines:

i) $\lambda/4$ line $\rightarrow$ Impedance inverter:

- It can transform short circuit to open circuit & vice versa

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right] , \quad \beta = \frac{2\pi}{\lambda}$$

When $l = \lambda/4$, $\beta\lambda = \frac{\pi}{2}$ $\Rightarrow$ $\tan\beta l \rightarrow \infty$

Put $x = \tan\beta l$

$$\therefore Z_{in} = \lim_{x\to\infty} Z_0 \left[\frac{Z_L + jxZ_0}{Z_0 + jxZ_L} \right]$$

$$= \lim_{x\to\infty} Z_0 \frac{x}{x} \left[\frac{(Z_L/x) + jZ_0}{(Z_0/x) + jZ_L} \right]$$

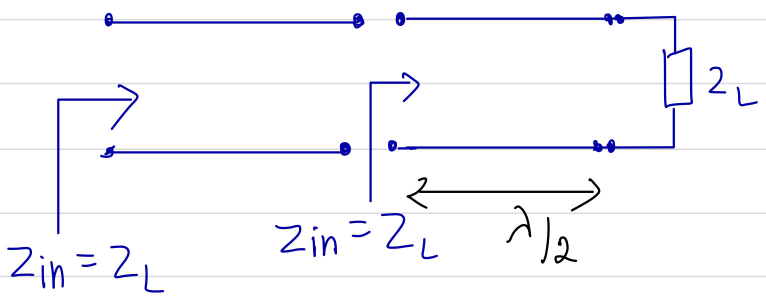
$\Rightarrow Z_{in} = \frac{Z_0^2}{Z_L}$ i.e $Z_{in} \propto \frac{1}{Z_L}$ } Impedance inverter

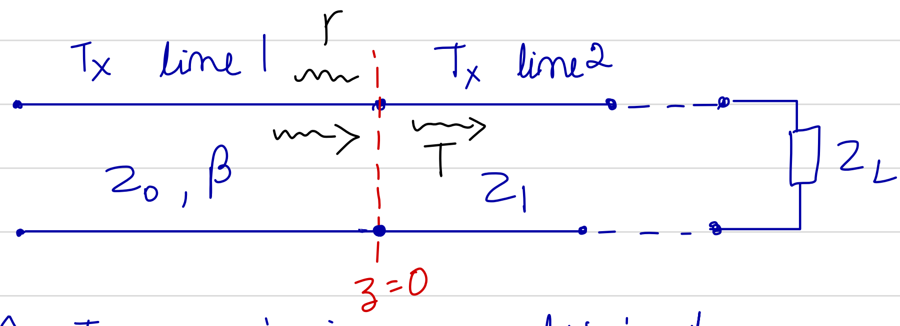
ii) $\lambda/2$ line $\rightarrow$ Impedance repeater:

$l = \frac{\lambda}{2}$, $\beta = \frac{2\pi}{\lambda}$ $\Rightarrow$ $\beta l = \pi$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\pi}{Z_0 + jZ_L \tan\pi} \right] = Z_L$$

So impedance repeats itself:





$T \triangleq$ Transmission coefficient

$\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$; $V_1(z) = V_0^+ [e^{-j\beta z} + \Gamma e^{j\beta z}]$

Behaviour of voltage wave along transmission line 1

wave in the 2nd transmission line:

$V_2(z) = T V_0^+ e^{-j\beta z}$

At $z = 0$

$V_0^+ [1 + \Gamma] = T V_0^+ \Rightarrow T = 1 + \Gamma$

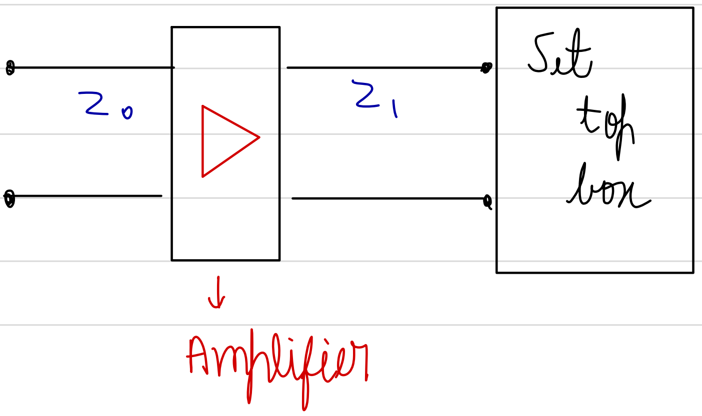
$\therefore$ $\boxed{T = 1 + \Gamma = 1 + \dfrac{Z_1 - Z_0}{Z_1 + Z_0} = \dfrac{2Z_1}{Z_1 + Z_0}}$

NOTE · Insertion loss $\triangleq -20 \log_{10} |T|$ dB

→ If $Z_1 = Z_0$, T_1 $\Rightarrow$ $IL = 0$ dB

→ If $Z_1 = 10Z_0$, $T = \dfrac{20Z_0}{10Z_0 + Z_0} = \dfrac{20}{11} = 1.818$

$\therefore IL \cong -5.2$ dB



NOTE: We measure power rather than voltage in high frequency circuits.

$P_1 \rightarrow$ Power at point 1
$P_2 \rightarrow$ Power at point 2

Power ratio $\triangleq \frac{P_1}{P_2}$

Power ratio in decibels $= 10 \log_{10} \frac{P_1}{P_2}$

dBm = decibels w.r.t mW

$$dBm = 10 \log_{10}\left(\frac{P_1}{1mW}\right) \text{ dBm}$$

Point A : Power is 1mW $\rightarrow$ 0 dBm
Point B : Power is $1\mu W$ $\rightarrow$ -30 dBm
Point C : Power is 10 W $\rightarrow$ 40 dBm

$P = \frac{V^2}{R}$, if R is normalised to 1Ω

$P = V^2$

$$\frac{V_1}{V_2} = \sqrt{\frac{P_1}{P_2}}$$

$$\ln \frac{V_1}{V_2} = \ln \sqrt{\frac{P_1}{P_2}} = \frac{1}{2} \ln \frac{P_1}{P_2}$$

Suppose $\frac{P_1}{P_2} = e^2$

$$\frac{1}{2} \ln \frac{P_1}{P_2} = \frac{1}{2} \ln e^2 = 1 \text{ Np (Neper)}$$

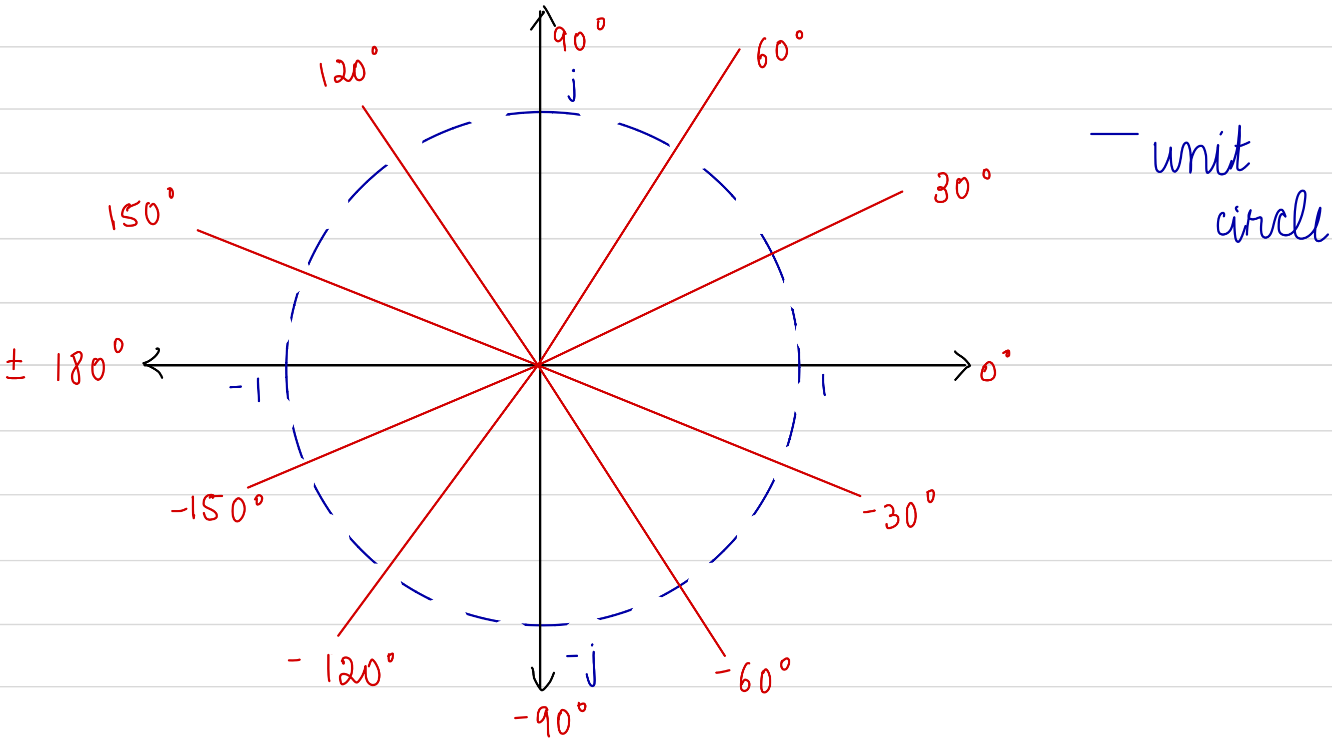
$$1 \text{ dB} = 8.686 \text{ dB}$$

Smith Chart

→ Polar plot of reflection coefficient

$$\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta} \quad , \quad |\Gamma| \leq 1$$

$$= \Gamma_r + j\Gamma_i$$



Normalised $\Gamma = \dfrac{z_l - 1}{z_l + 1}$ where $z_l = \dfrac{Z_L}{Z_0}$

$$|\Gamma| e^{j\theta} = \frac{z_l - 1}{z_l + 1}$$

$$\Rightarrow z_l |\Gamma| e^{j\theta} + |\Gamma| e^{j\theta} = z_l - 1$$

$$z_l = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{-j\theta}}$$

$$z_l = r_l + jx_L = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}}$$

$$|r|e^{j\theta} = r_r + j r_i$$

$$\Rightarrow z_L = \frac{1 + (r_r + j r_i)}{1 - r_r - j r_i}$$

$$= \frac{(1+r_r) + j r_i}{(1-r_r) - j r_i} \cdot \frac{(1-r_r) + j r_i}{(1-r_r) + j r_i}$$

$$= \frac{1 - r_r^2 + j(1+r_r)r_i + j(1-r_r)r_i - r_i^2}{(1-r_r)^2 + r_i^2}$$

$$= \frac{(1 - r_r^2 - r_i^2) + j(2r_i)}{(1-r_r)^2 + r_i^2}$$

$$r_L = \frac{(1 - r_r^2 - r_i^2)}{(1-r_r)^2 + r_i^2} \rightarrow (1) \qquad x_L = \frac{2r_i}{(1-r_r)^2 + r_i^2} \rightarrow (2)$$

From equation (1) :

$$r_L[1 + r_r^2 - 2r_r + r_i^2] = 1 - r_r^2 - r_i^2$$

$$r_L - 2r_L r_r + r_r^2(1+r_L) + r_i^2(1+r_L) = 1$$

$$r_r^2(1+r_L) - 2r_L r_r + r_i^2(1+r_L) = 1 - r_L$$

$$r_r^2 - \frac{2r_L}{1+r_L} r_r + r_i^2 = \frac{1-r_L}{1+r_L}$$

$$r_r^2 - \frac{2r_L}{1+r_L} r_r + \left(\frac{r_L}{1+r_L}\right)^2 + (r_i - 0)^2 = \frac{1-r_L}{1+r_L} + \frac{r_L^2}{(1+r_L)^2}$$

$$\left[r_r - \frac{r_L}{1+r_L}\right]^2 + \left[r_i - 0\right]^2 = \left(\frac{1}{1+r_L}\right)^2$$

↓

Equation of the constant resistance circle

Centre $= \left(\frac{r_L}{1+r_L}, 0\right)$ radius $= \frac{1}{1+r_L}$

From equation ②:

$$(1-r_r)^2 + r_i^2 = \frac{2r_i}{x_L}$$

$$1 - 2r_r + r_r^2 + r_i^2 = \frac{2r_i}{x_L}$$

$$(r_r - 1)^2 + r_i^2 - \frac{2r_i}{x_L} + \frac{1}{x_L^2} = \frac{1}{x_L^2}$$

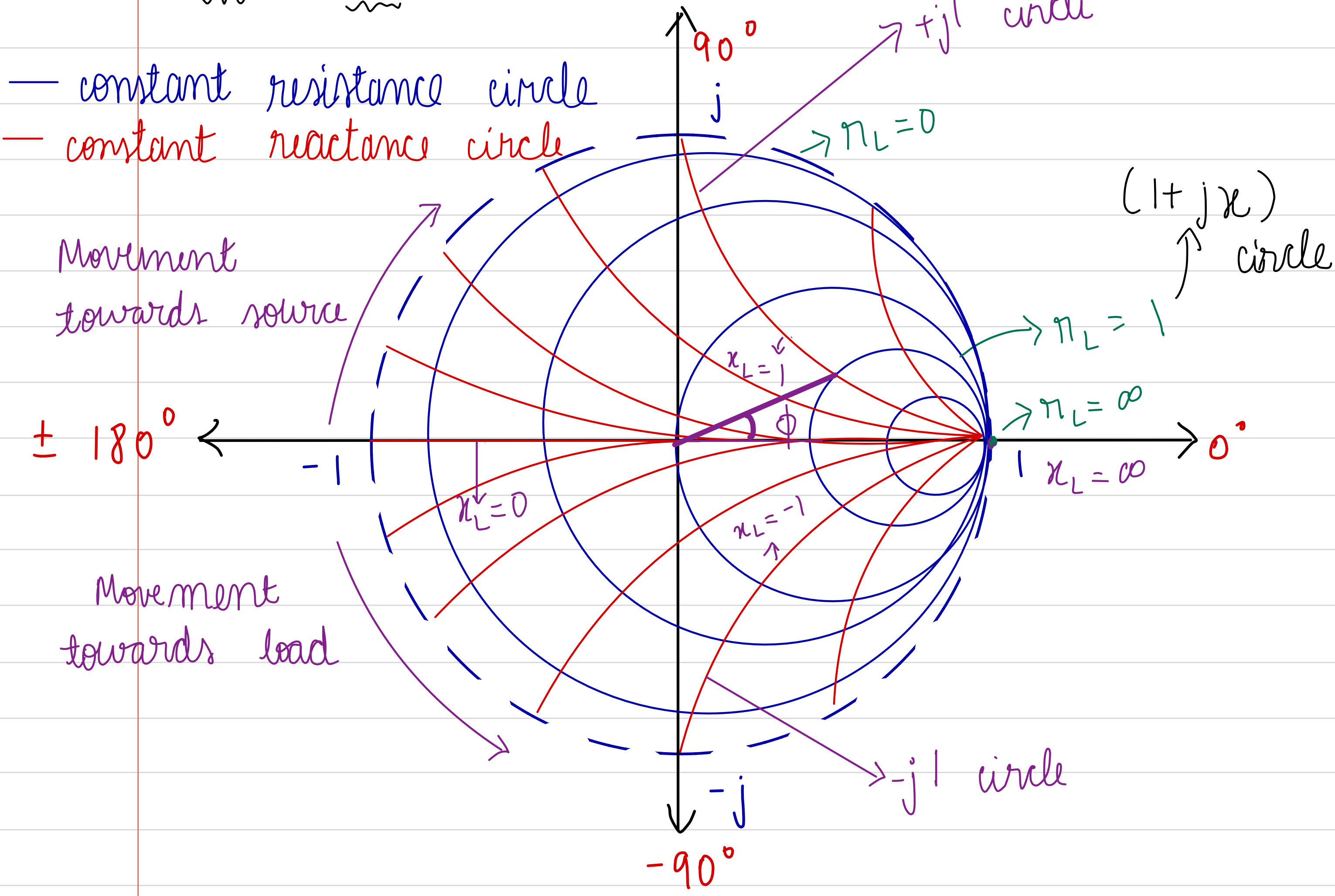
$$\Rightarrow (r_r - 1)^2 + \left(r_i - \frac{1}{x_L}\right)^2 = \frac{1}{x_L^2}$$

↓

Equation of constant reactance circle.

Centre $= \left(1, \frac{1}{x_L}\right)$ radius $= \frac{1}{x_L}$

Smith chart:



We have (generalised impedance) $Z_s = Z_0 \dfrac{[e^{-j\beta z} + r e^{j\beta z}]}{[e^{-j\beta z} - r e^{j\beta z}]}$

Normalised impedance, $z_s = \dfrac{Z_s}{Z_0} = \dfrac{e^{-j\beta z} + r e^{j\beta z}}{e^{-j\beta z} - r e^{j\beta z}}$

At $z = -l$

$$z_{in} = \frac{1 + |r| e^{j(\phi - 2\beta l)}}{1 - |r| e^{j(\phi - 2\beta l)}}$$

If $l = 0$, $z_{in} = z_l = \dfrac{1 + |r| e^{j\phi}}{1 - |r| e^{j\phi}}$

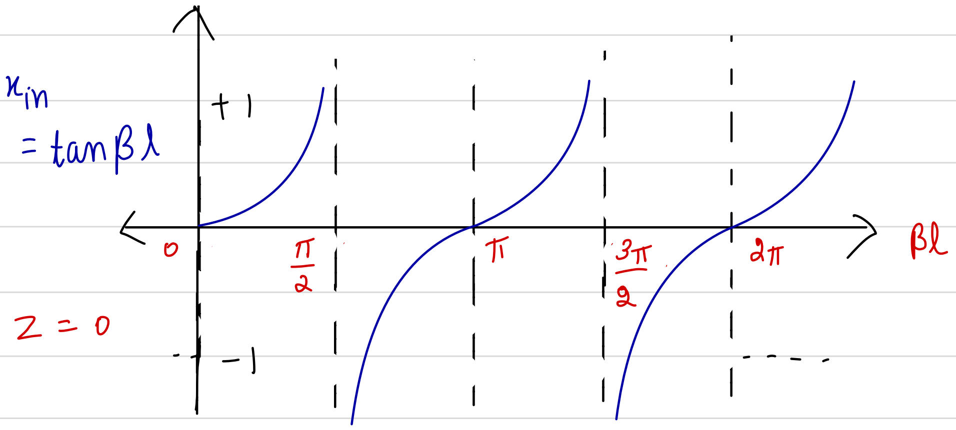
Load admittance: $Y_L = \dfrac{1}{Z_L}$

Characteristic admittance, $Y_0 \triangleq \dfrac{1}{Z_0}$

$$y_L = \frac{1+\Gamma}{1-\Gamma}$$

If $Z_{in} = jZ_0 \tan\beta l = jX_{in}$ for short circuit

$$\frac{X_{in}}{Z_{in}} = x_{in} = \tan\beta l$$



$$X_{in} = Z_0 \tan\beta l$$

Suppose $\beta l < 0.5$ rad

$$\tan\beta l \simeq \beta l$$

$$X_{in} = Z_0 \beta l$$

$$\omega L_{eq} = Z_0 \frac{2\pi}{\lambda} l$$

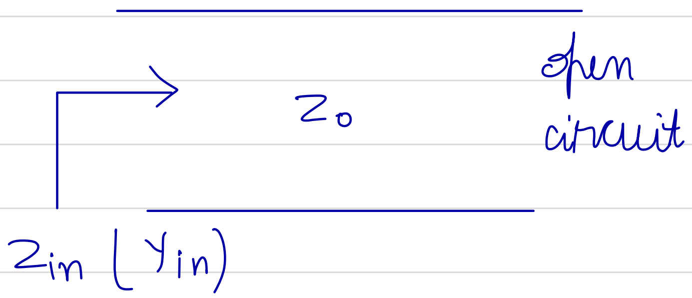
$$2\pi f L_{eq} = Z_0 \frac{2\pi l}{\lambda_0}$$

$$L_{eq} = \frac{Z_0 l}{f\lambda} = \frac{Z_0 l}{v}$$

Suppose $0 < \beta l < \pi/2$

$$\omega L_{eq} = Z_0 \tan\beta l$$

$$L_{eq} = \frac{Z_0 \tan\beta l}{\omega} = \frac{Z_0 \tan\beta l}{2\pi f}$$



$$Z_{in} = -jZ_0 \cot\beta l$$

$$Y_{in} = \frac{1}{Z_{in}} = \frac{1}{-jZ_0 \cot\beta l} = jY_0 \tan\beta l$$

$$B_{in} = Y_0 \tan\beta l \qquad \rightarrow B_{in} = \text{susceptibility}$$

normalised form: $\quad b_{in} = \dfrac{B_{in}}{Y_0} = \tan\beta l$

NOTE:

$$Z_{sc} = jZ_0 \tan\beta l$$

$$Z_{oc} = -jZ_0 \cot\beta l$$

So $\quad Z_0 = \sqrt{Z_{sc} \cdot Z_{oc}}$

$$\frac{Z_{sc}}{Z_{oc}} = -\tan^2\beta l$$

$$\beta l = \tan^{-1}\left[\sqrt{\frac{-Z_{sc}}{Z_{oc}}}\right]$$

- Suppose $\quad l < \lambda/4$

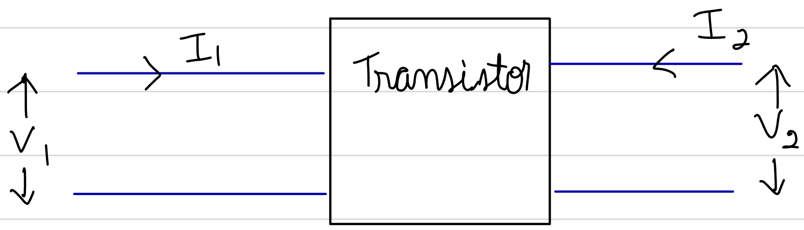
$$Z_{sc} = j\omega L$$

$$Z_{oc} = \frac{1}{j\omega C}$$

then $\quad \beta l = \tan^{-1}\sqrt{\dfrac{-j\omega L}{1/j\omega C}}$

$$= \tan^{-1}(\omega\sqrt{LC})$$

2 point model of transistor



$$V_1 = h_{11} I_1 + h_{12} I_2$$
$$I_2 = h_{21} V_1 + h_{22} V_2$$
$$\rightarrow \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

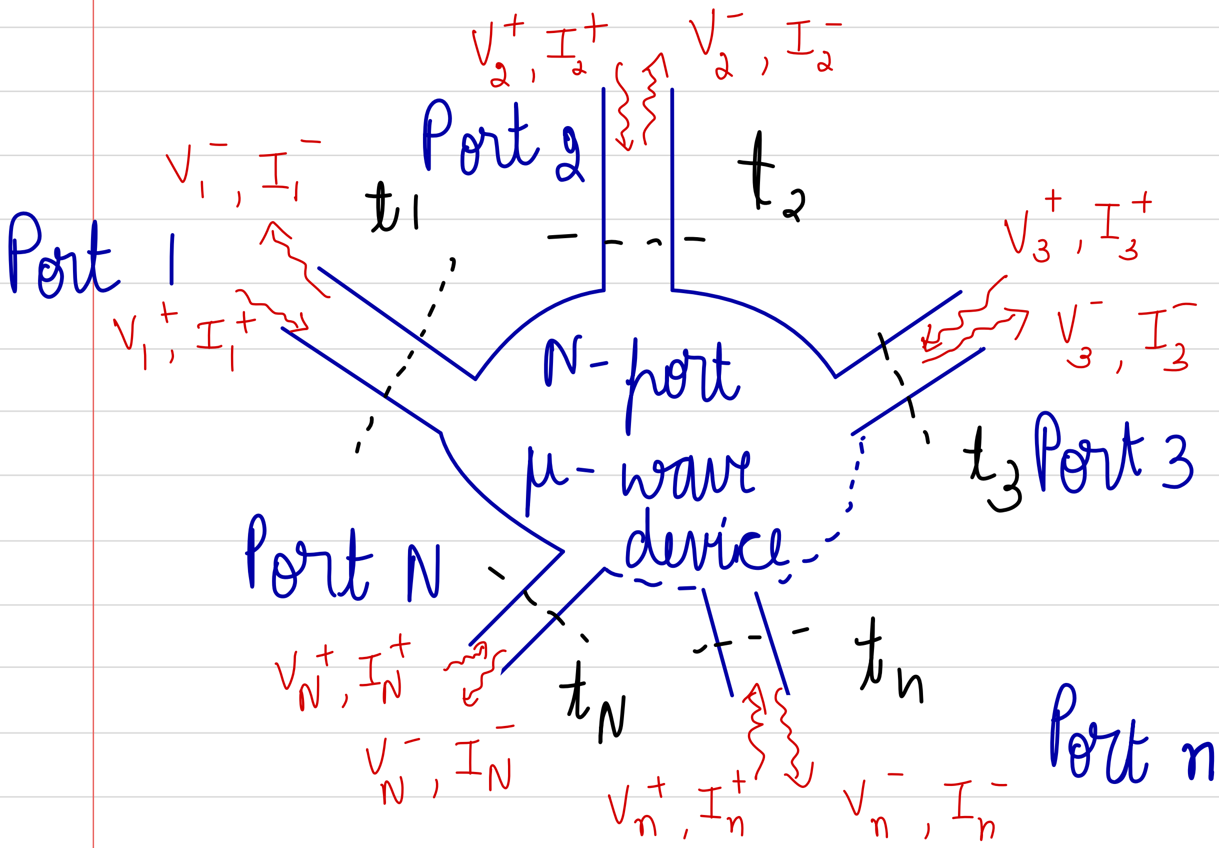
$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2 = 0}$$ → Input impedance with output short.

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0}$$ → Reverse amplification with input open circuit

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0}$$ → Current gain with output short.

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0}$$ → output conductance with input open circuit.

Modeling of microwave (μ- wave) devices using the theory of N- port networks



General term : $V_n = V_n^+ + V_n^-$

$I_n = I_n^+ + I_n^-$

Z parameters

$$V_1 = I_1 Z_{11} + I_2 Z_{12} + I_3 Z_{13} + \dots\dots + I_N Z_{1N}$$

$$V_2 = I_1 Z_{21} + I_2 Z_{22} + \dots\dots + I_N Z_{2N}$$

$\vdots$

$$V_n = I_1 Z_{n1} + I_2 Z_{n2} + \dots\dots + I_N Z_{nN}$$

$\vdots$

$$V_N = I_1 Z_{N1} + I_2 Z_{N2} + \dots\dots + I_N Z_{NN}$$

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} & \cdots & Z_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nn} & \cdots & Z_{nN} \\ \vdots & \vdots & & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{Nn} & \cdots & Z_{NN} \end{bmatrix}_{N \times N} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \\ \vdots \\ I_N \end{bmatrix}$$

So $Z_{ij} = \left. \dfrac{V_i}{I_j} \right|_{I_k = 0}$ for $k = 1, 2, \ldots N$ & $k \neq j$

Y parameters

$$I_1 = Y_{11} V_1 + Y_{12} V_2 + \cdots + Y_{1N} V_N$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2 + \cdots + Y_{2N} V_N$$
$$\vdots \qquad \vdots$$
$$I_n = Y_{n1} V_1 + Y_{n2} V_2 + \cdots + Y_{nN} V_N$$
$$\vdots \qquad \vdots$$
$$I_N = Y_{N1} V_1 + Y_{N2} V_2 + \cdots + Y_{NN} V_N$$

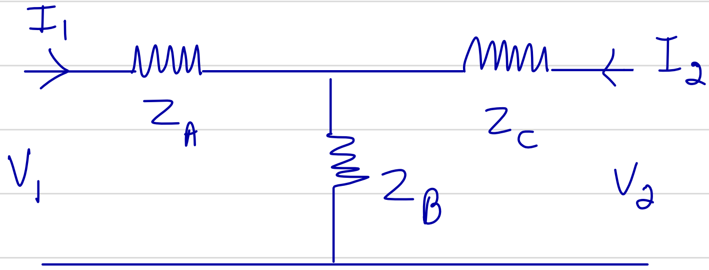
i.e:

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} & \cdots & Y_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} & \cdots & Y_{nN} \\ \vdots & \vdots & & \vdots & & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{Nn} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ \vdots \\ V_N \end{bmatrix}$$

i.e $[\underline{I}] = [\underline{\underline{Y}}] [\underline{V}]$

$Y_{ij} = \left. \dfrac{I_i}{V_j} \right|_{V_k = 0}$ where $k \in N - \{j\}$

Q)



Find z parameters

Ans

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

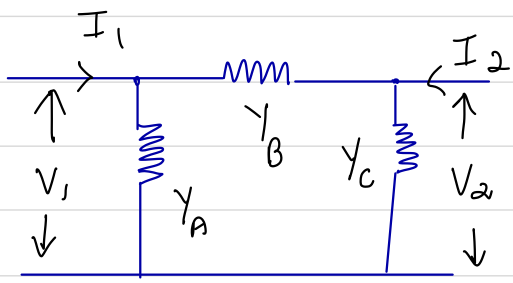
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_A + Z_B$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_B$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_B \qquad V_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_C + Z_B$$

$$\Rightarrow Z = \begin{bmatrix} Z_A + Z_B & Z_B \\ Z_B & Z_B + Z_C \end{bmatrix}$$

Admittance Parameters of a π network



$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

$$I_1 = Y_A V_1 + (V_1 - V_2) Y_B \rightarrow (1)$$
$$I_2 = Y_C V_2 + (V_2 - V_1) Y_B \rightarrow (2)$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_A + Y_B$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_B$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -Y_B \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_C + Y_B$$

$$Y = \begin{bmatrix} Y_A + Y_B & -Y_B \\ -Y_B & Y_C + Y_B \end{bmatrix}$$

Scattering parameters : (S - parameters)

$$V_n = V_n^+ + V_n^- \qquad I_n = I_n^+ + I_n^-$$

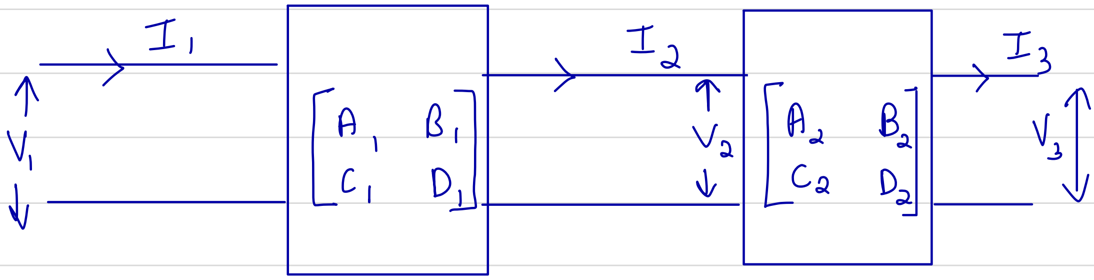
$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_n^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2n} & \cdots & S_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} & \cdots & S_{nN} \\ \vdots & \vdots & & \vdots & & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{Nn} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_n^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

$$V_n^- = S_{n1} V_1^+ + S_{n2} V_2^+ \cdots + S_{nn} V_n^+ + \cdots S_{Nn} V_N^+$$

i.e $S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k=0}$ for all $k \in N - \{j\}$

NOTE: To find S_{ij} all except j^{th} terminal should be terminated with Z_0

ABCD parameters



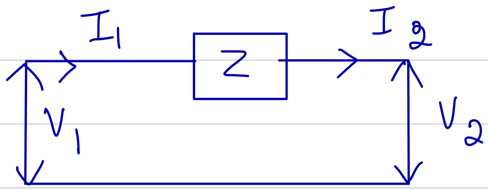
$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

i.e $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

Eg:



Find ABCD parameters

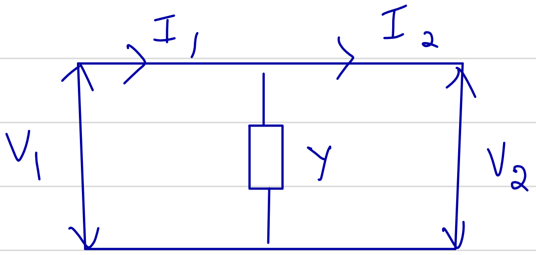
Ans

$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

$A = \left.\dfrac{V_1}{V_2}\right|_{I_2=0} = 1$ $\qquad B = \left.\dfrac{V_1}{I_2}\right|_{V_2=0} = Z$

$C = \left.\dfrac{I_1}{V_2}\right|_{I_2=0} = 0$ $\qquad D = \left.\dfrac{I_1}{I_2}\right|_{V_2=0} = 1$

Eg:

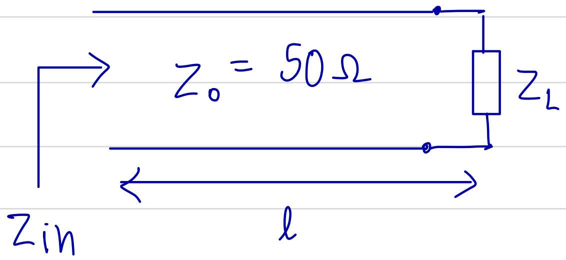


Find ABCD parameters

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1 \qquad B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = 0$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = Y \qquad D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = 1$$

Q>



$Z_L = 50 + j25$, $l = \lambda/2$

Find $|r|$, $\angle\phi$, SWR

Ans $$z_L = \frac{50 + j25}{50} = 1 + j0.5$$

From smith chart $|r| = 0.25$ & $SWR = 1.65$

$$y_{in} = \frac{z_L + j\tan\beta l}{1 + j z_L \tan\beta l}$$

$l = \frac{\lambda}{2}$ $\quad \tan\beta l \rightarrow \infty$

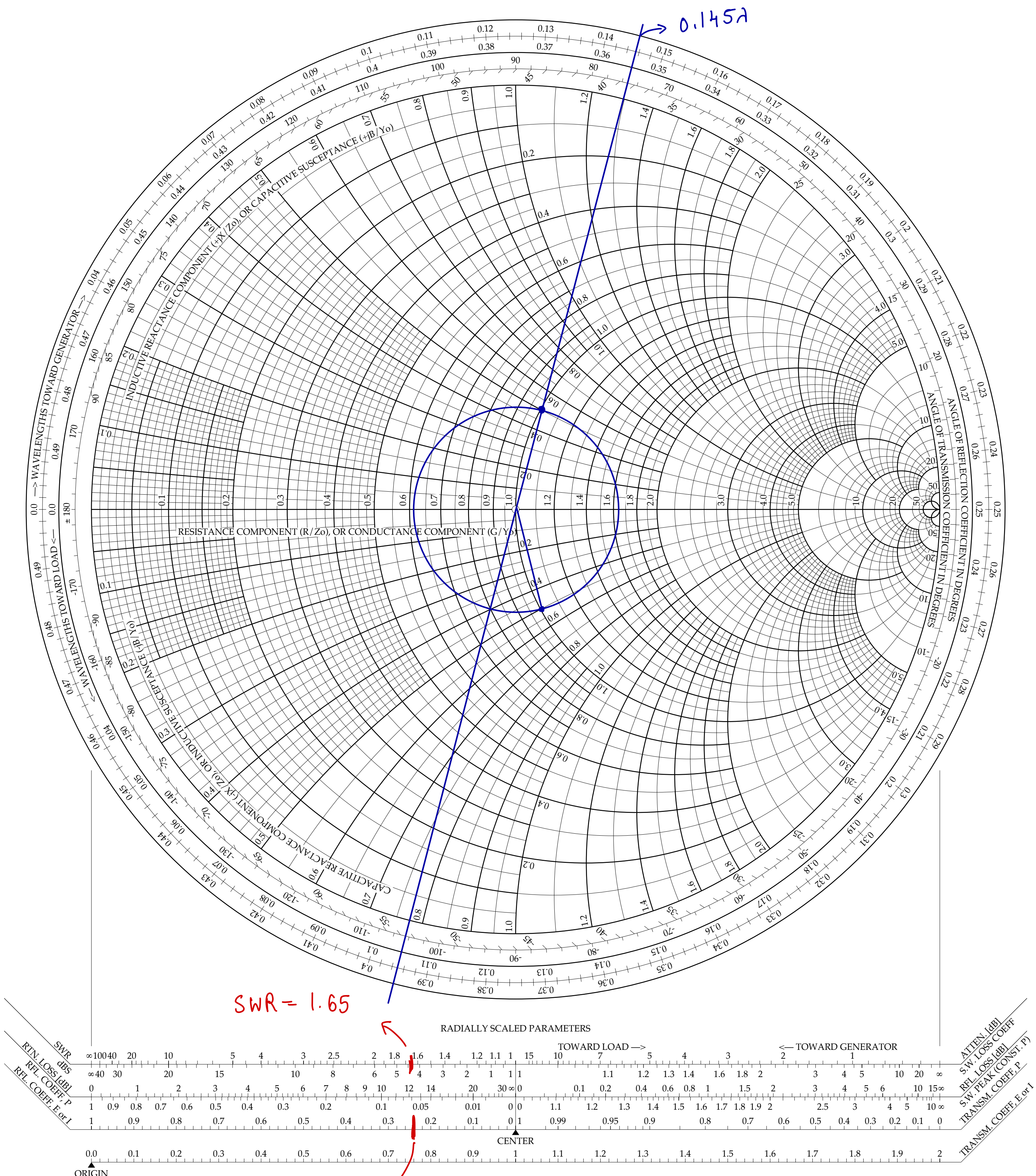
$$\Rightarrow y_{in} = \frac{1}{z_L} = y_L$$

$$z_L = 1 + j0.5 = 1.118 \angle 26.56^\circ$$

$$y_L = \frac{1}{1.118 \angle 26.56^\circ} = 0.894 \angle -26.56^\circ$$

$$= 0.799 - j0.4$$

The Complete Smith Chart



RFL coeff = 0.25

Q2> If $Z_L = 30 + j50\,\Omega$ $\quad l = 5.2$ cm

$Z_0 = 100\,\Omega$ $\quad f = 750$ MHz

Find $|\Gamma|$, $\angle\phi$, SWR

Ans $z_l = \dfrac{30 + j50}{100} = 0.3 + 0.5j$

SWR = 4.2 $\quad |\Gamma| = 0.61$ → from smith chart.

$\lambda = \dfrac{c}{f} = 0.4$ m

$\dfrac{l}{\lambda} = \dfrac{52}{40} = 0.13$

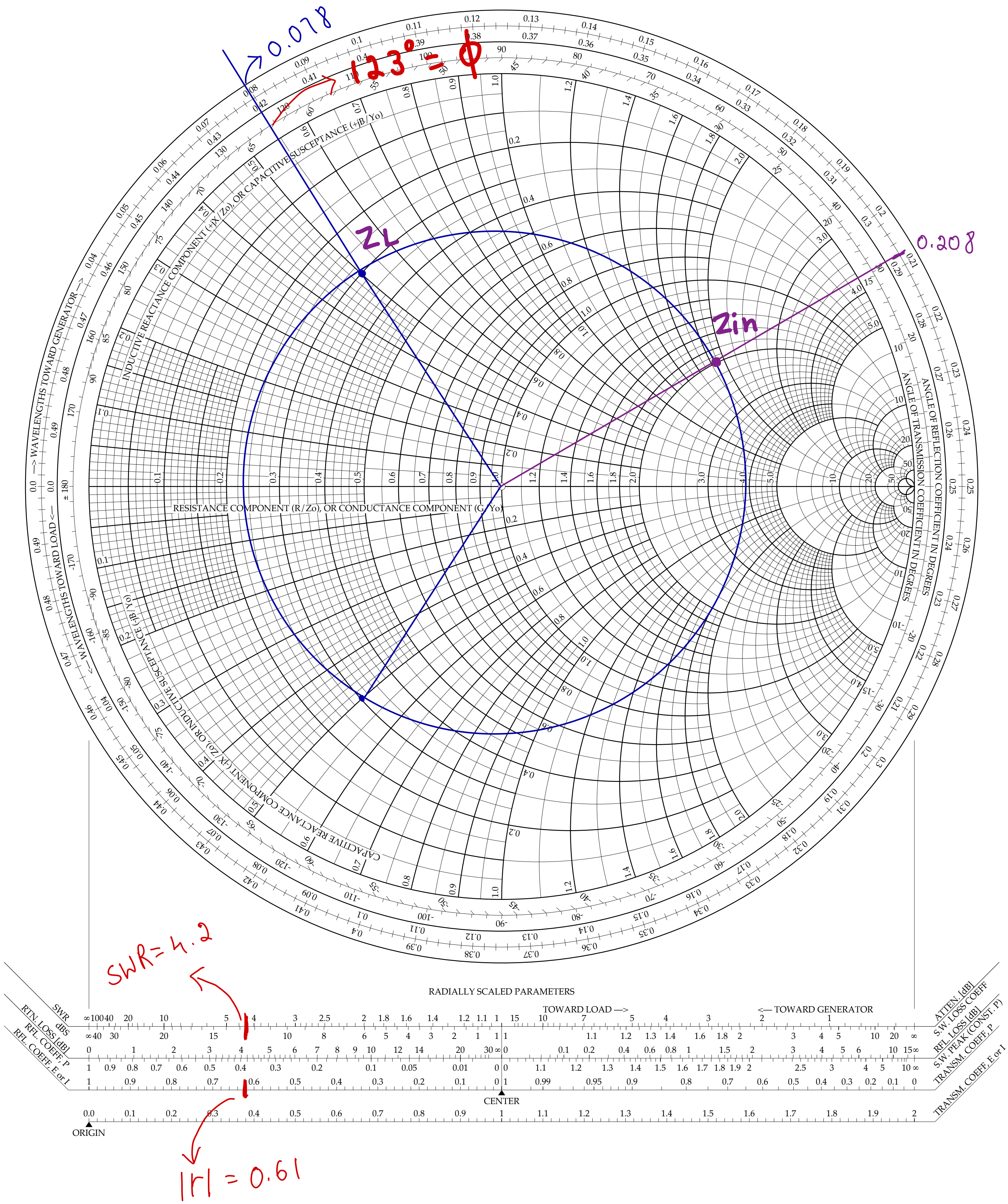
$l = 0.13\lambda$

$z_{in}(l) = 0.13 + 0.078$
$= 0.208$

From smith chart $z_{in} = 2 + j1.9$

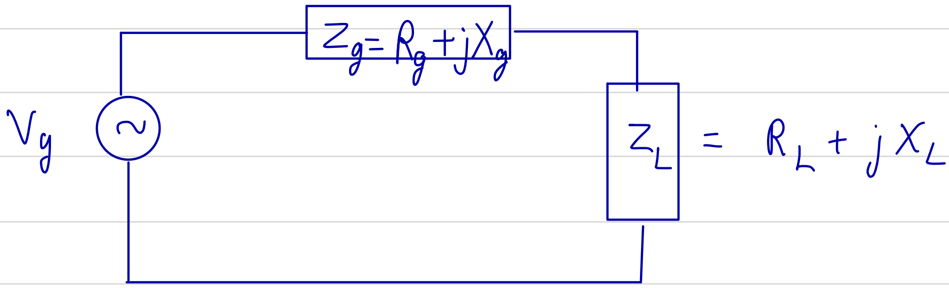
$\angle\phi \cong 123^\circ$ // (from smith chart)

The Complete Smith Chart



Impedance matching techniques:

1) Conjugate matching
2) Z_0 matching

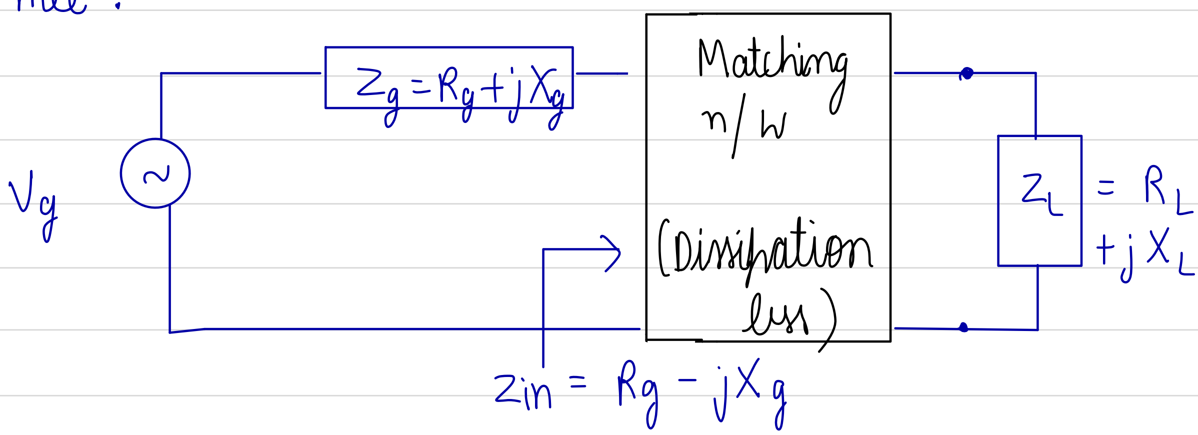


1) **Conjugate matching:** $R_g = R_L$ & $X_g = -X_L$

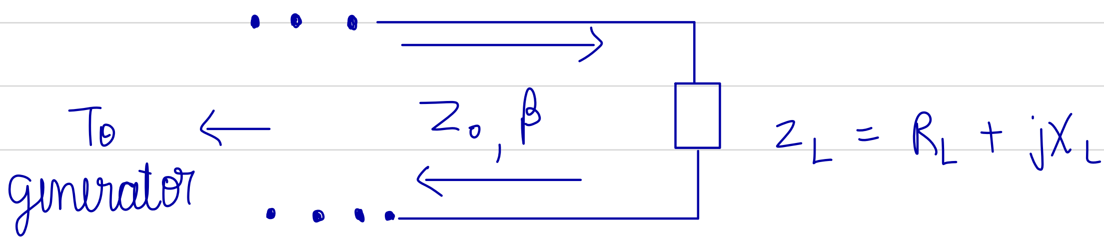
$$I = \frac{V_g}{2R_g} = \frac{V_g}{2R_L}$$

Power delivered to load, $P_L = \frac{|V_g|^2}{4R_g}$

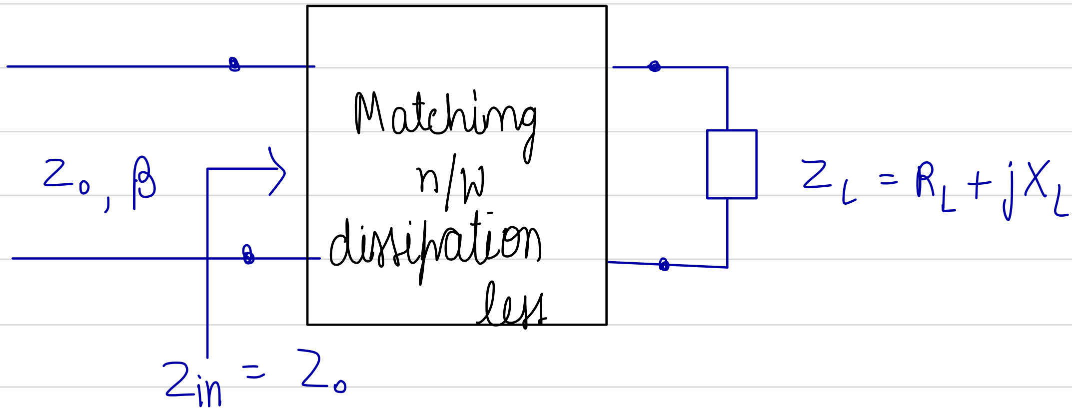
- In practice, conjugate matching condition is not met.



2) **Z_0 matching:**



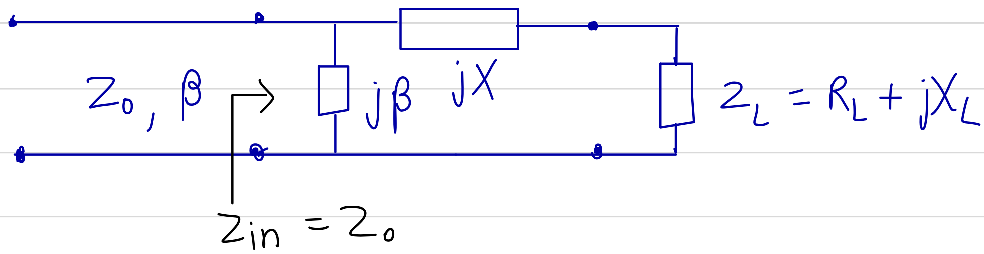
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$



Types of matching network :

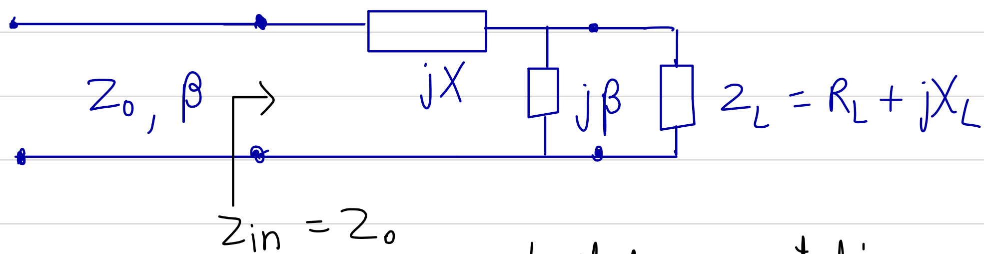
1) Series Reactance based matching networks
2) Shunt reactance based matching networks
3) L type matching networks

3) L type matching network :



L type matching n/w 1

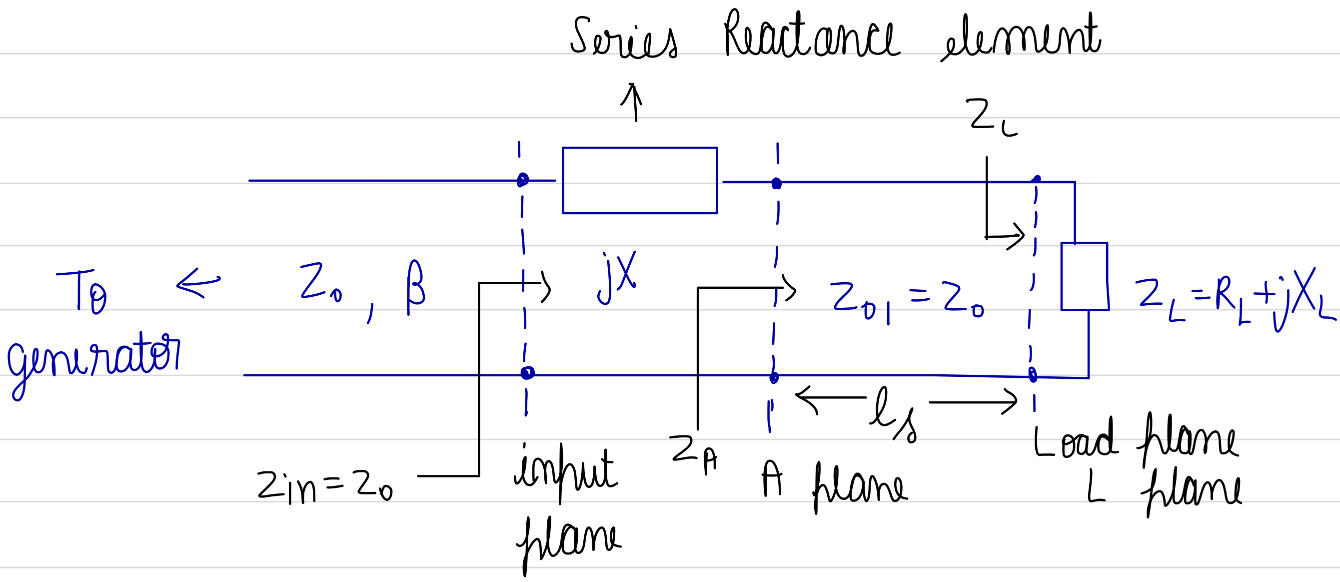
Employed when $z_L = \frac{Z_L + jX_L}{Z_0}$ falls outside the $(1+jX)$ circle.



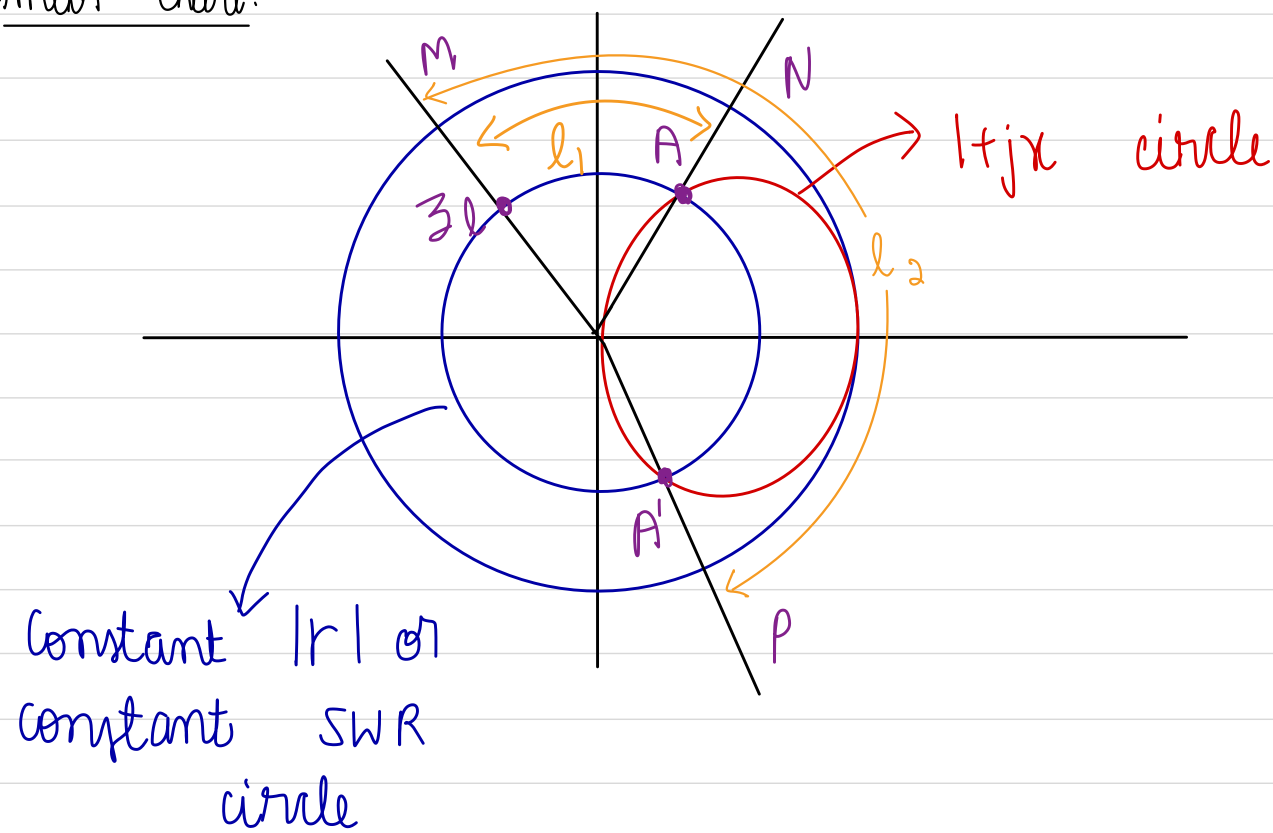
L type matching n/w 2

This network is employed when $z_L = \frac{Z_L}{Z_0}$ lies inside $(1+jx)$ circle.

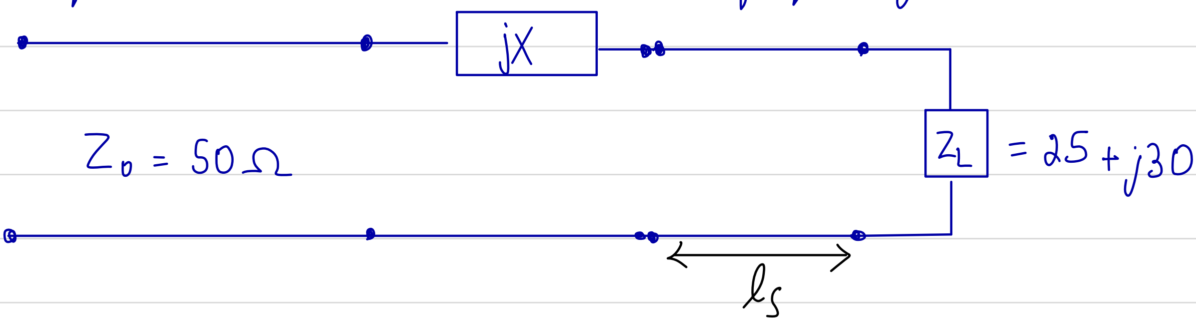
1) Series reactance network



Smith chart:



Q3) Given: $f_0 = 2000$ MHz (critical frequency)



Ans Given: $f_0 = 2$ GHz

$$\lambda = \frac{3 \times 10^8}{2 \times 10^9} = 15 \text{ cm}$$

$$z_L = \frac{25 + j30}{50} = 0.5 + j0.6$$

From smith chart:

$$\frac{l_1}{\lambda} = (0.165 - 0.1) = 0.065$$

$$\frac{l_2}{\lambda} = (0.334 - 0.1) = 0.234$$

$\therefore l_1 = 0.975$ cm & $l_2 = 3.51$ cm

Also from smith chart: $X = -1.1$ when $l_s = l_1$
$X = 1.1$ when $l_s = l_2$

Normalized $= j1.1$

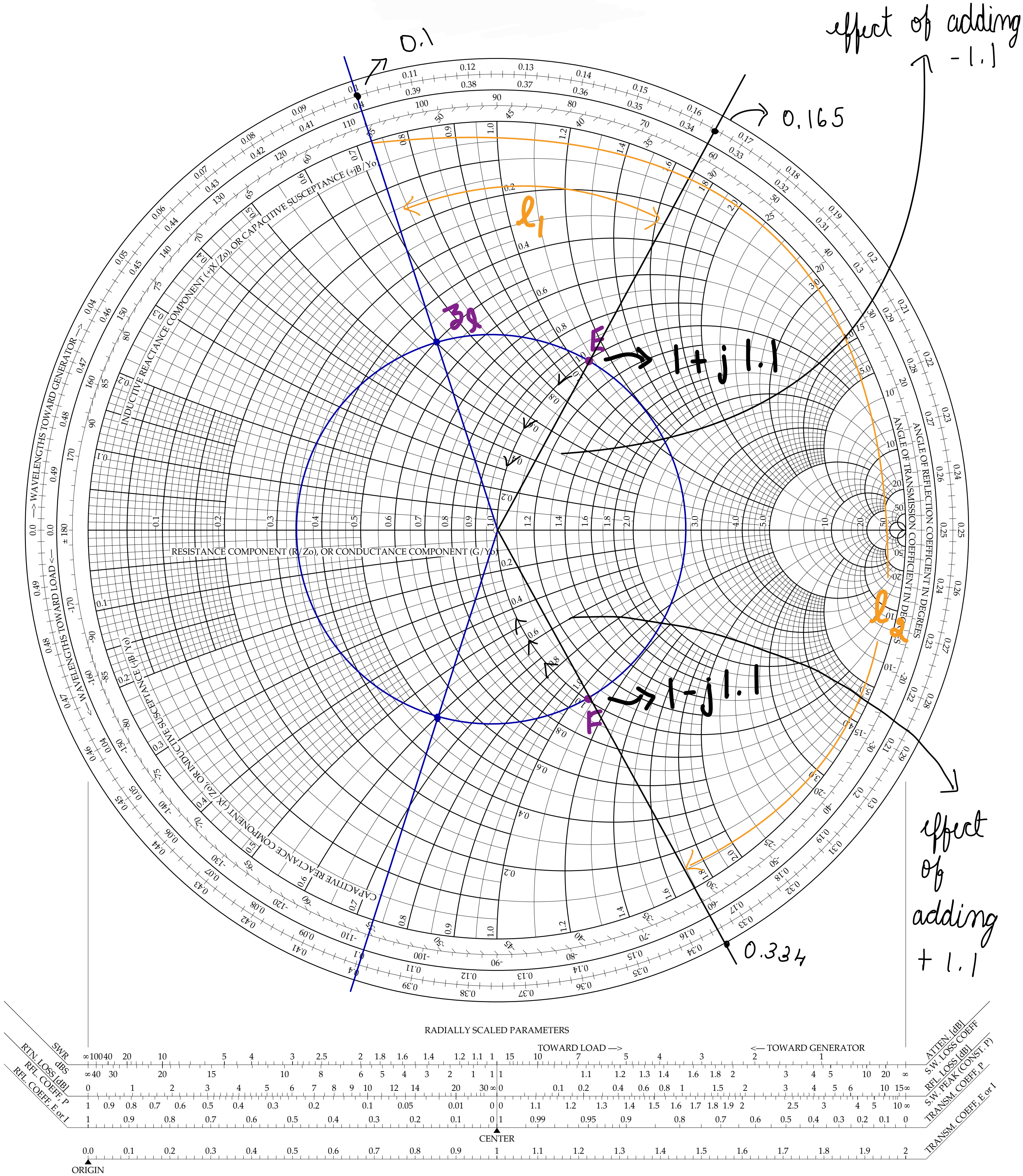
Required capacitive reactance $= j\,1.1 \times 50 = j55$

$$\Rightarrow \frac{1}{j\omega C_{eq}} = -j55$$

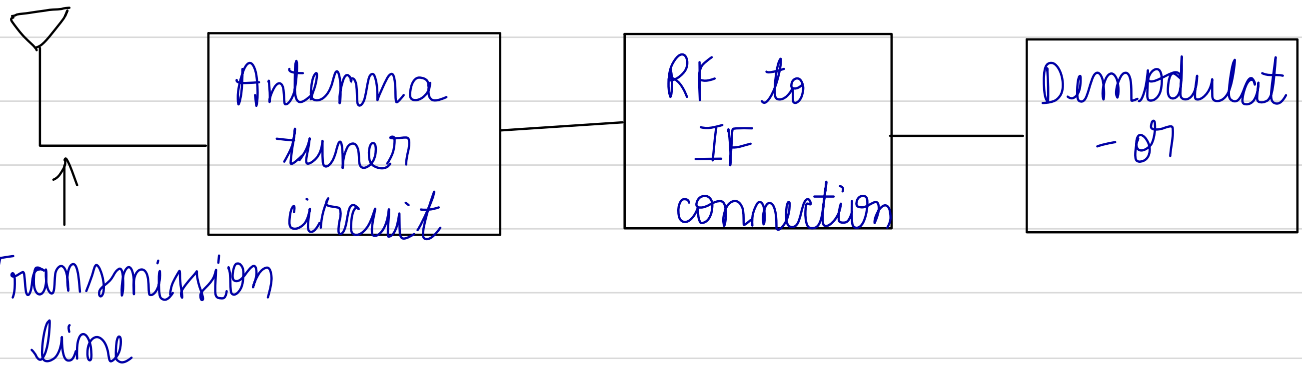
$$\Rightarrow C_{eq} = 1.45 \text{ pF}$$

Similarly $j\omega L = j55 \Rightarrow L_{eq} = 4.376$ nH

The Complete Smith Chart



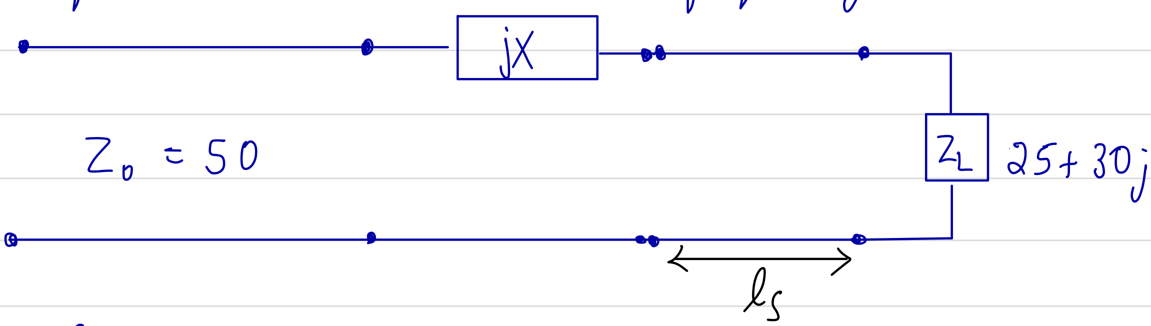
NOTE: FM Radios work at a frequency of 88 - 108 MHz



- Impedance of antenna is a function of frequencies.
- So the antennas are designed to such a frequency such that the errors occurred over the range is acceptable.

Eg: 88 MHz - 108 MHz designed at somewhere middle of 2 frequencies.

Q4) Given: $f_0 = 1800$ MHz (critical frequency), $l_s = 3.51$ cm



Ans $\lambda = \dfrac{3 \times 10^8}{1.8 \times 10^9} = 16.66$ cm $\qquad z_L = 0.5 + 0.6j$

$\dfrac{l_s}{\lambda} = \dfrac{3.51}{16.66} = 0.21$

$j\omega L = j(1800 \times 10^6 \times 4.376 \times 10^{-9}) = j49.491$

upon normalization : $j\dfrac{49.491}{50} = 0.98$

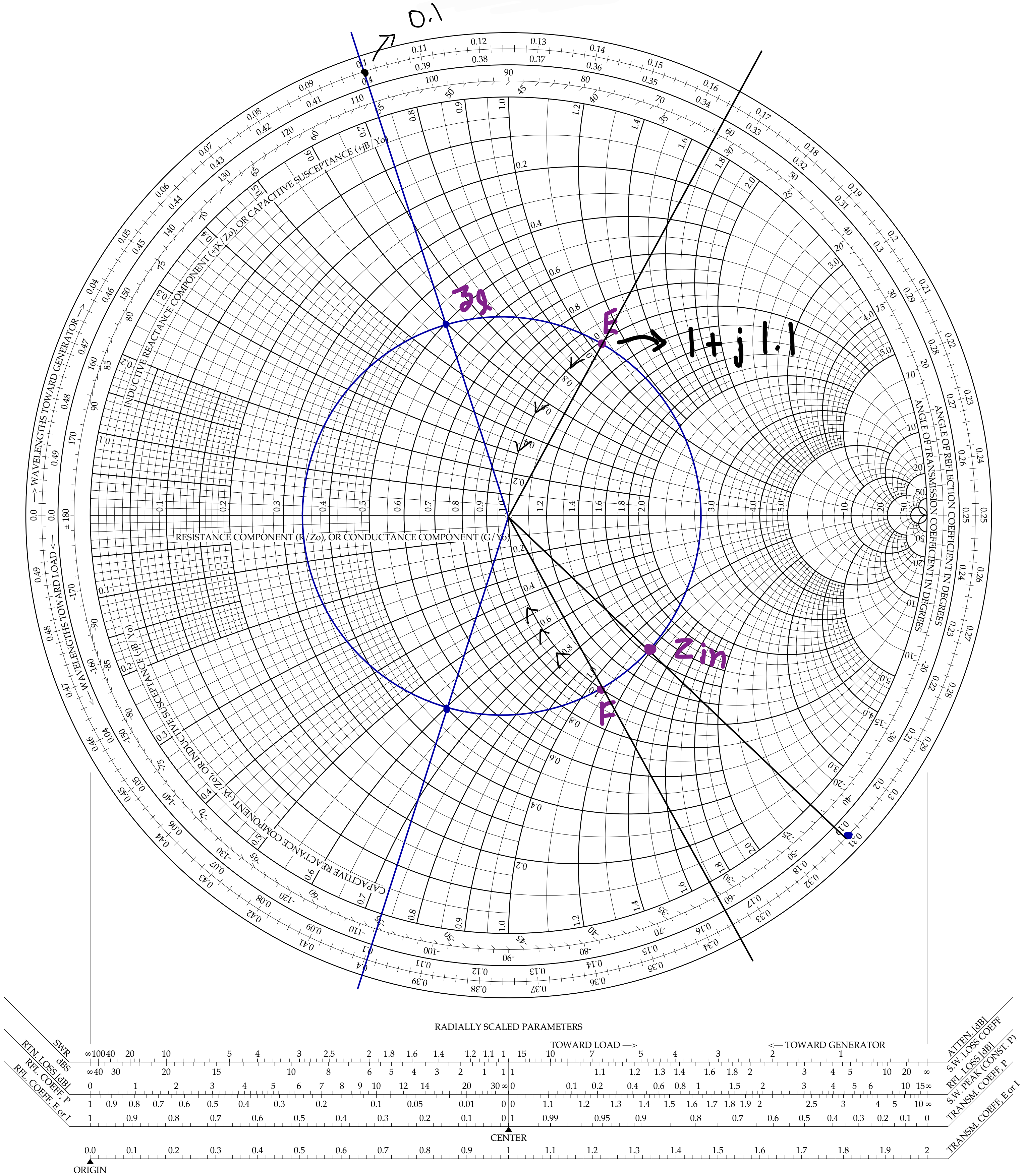
$z_{in}(l) = 0.1 + 0.21$
$= 0.3$

$Z_{in} = 1.45 - j(1.25) + j(0.98) = 1.45 - 0.26j$ //

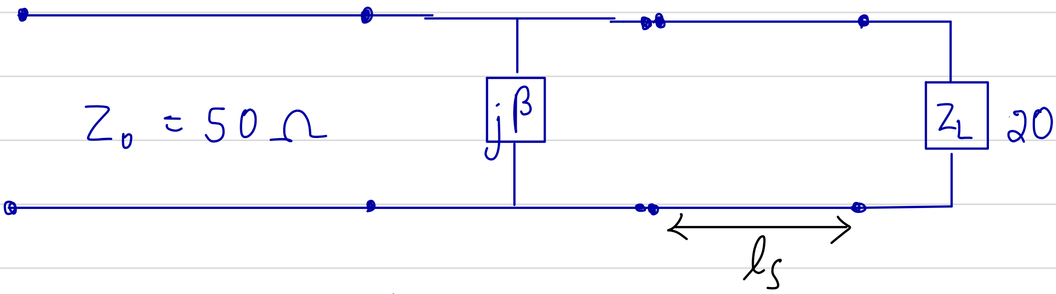
$Z_A = (1.45 - j\,0.26) \times 50$

$= 72.5 - 13j$ //

The Complete Smith Chart



Q5> $f_0 = 3\ GHz \qquad \lambda = 10\ cm$



Ans $\quad z_l = \frac{20}{50} = 0.4 \qquad \lambda = \frac{3\times10^8}{3\times10^9} = 10\ cm$

$Y_0 = 0.02$ mho

$Y_A = y_A \cdot Y_0 = (1 - j\,0.98) \times 0.02$
$= 0.02 - j\,0.0196$

if $f_0 \rightarrow f_0' = 2.5\ GHz$
$\lambda_0' = 12.5\ cm$

$l_s = 0.088 \times \lambda = 0.088 \times 10 = 0.88\ cm$ //

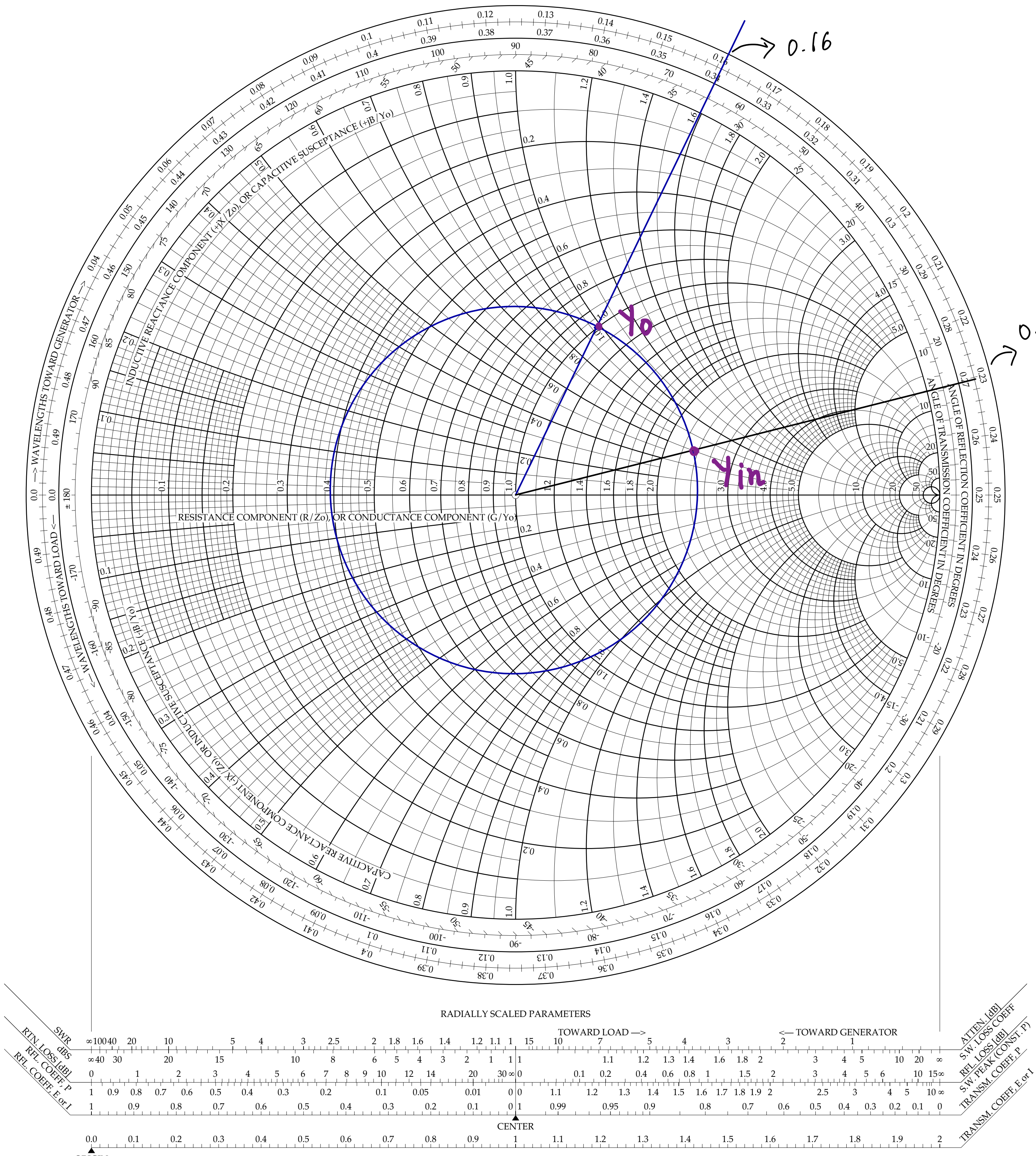
$\frac{l_s}{\lambda_0'} = \frac{0.88}{12.5} = 0.07$

$Y_{in}(l) = 0.16 + 0.07$
$= 0.23$

$Y_{in} = 2.4 + 0.6j$

The Complete Smith Chart

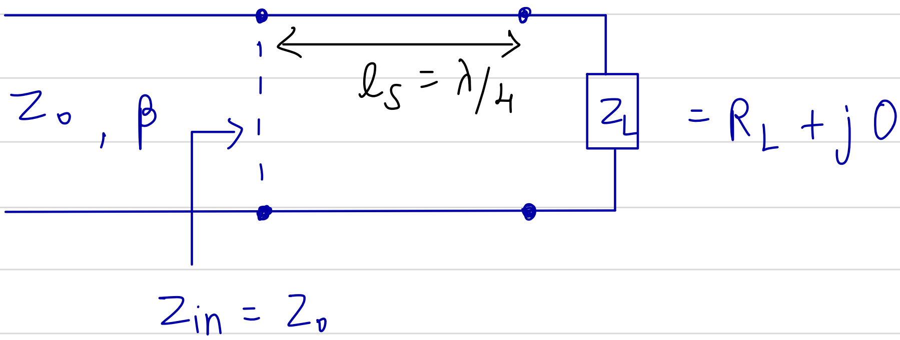
Black Magic Design



$\lambda/4$ line as an impedance matching network

1) Load is real, (no complex load)

2) $Z_0 \rightarrow$ real



Radiation

Electric charges - $\rho_e \rightarrow \vec{J}$ [Electric Conduction current density]

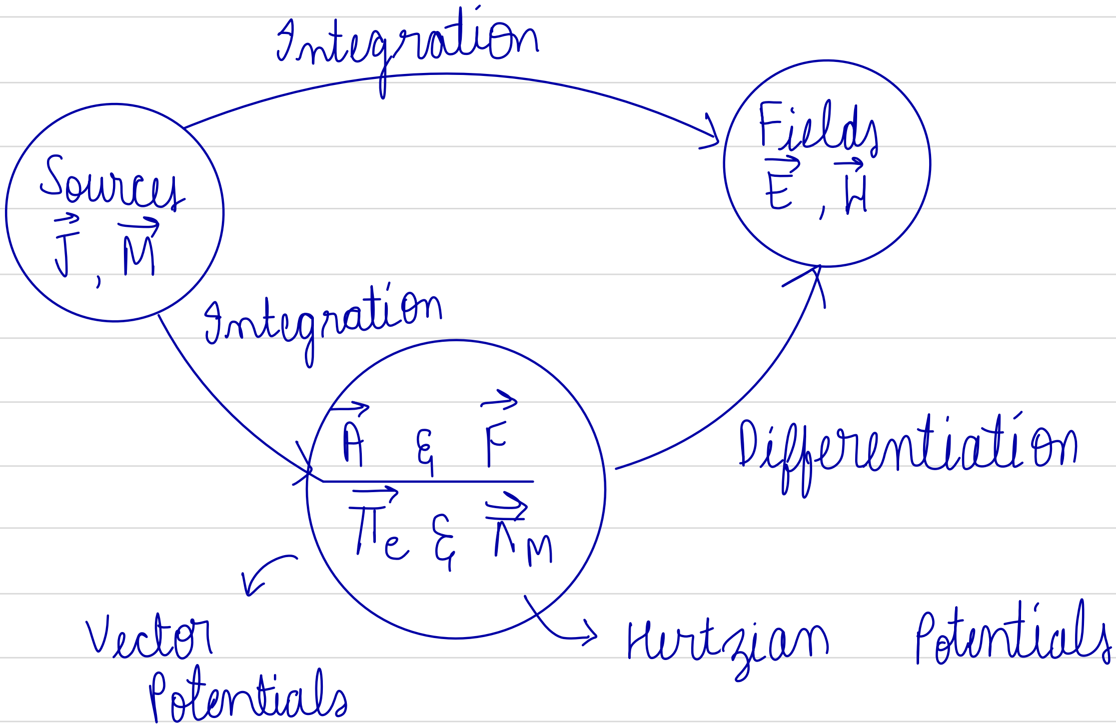
Magnetic charges - $\rho_m \rightarrow \vec{M}$ [Magnetic conduction current density]

Maxwell's equation (modified)

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\vec{M} - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_e \qquad \vec{\nabla} \cdot \vec{B} = \rho_m$$



where $\vec{A}$ is magnetic vector Potential

$\vec{J} \neq 0$
$\vec{M} = 0$ $\Rightarrow \vec{A} \rightarrow \begin{cases} \vec{E}_A \\ \vec{H}_A \end{cases}$

$\vec{F}$ is electric vector Potential.

$\vec{J} = 0$
$\vec{M} \neq 0$ $\Rightarrow$ $\vec{F} \rightarrow \begin{cases} \vec{E}_F \\ \vec{H}_F \end{cases}$

By superposition: $\vec{E} = \vec{E}_A + \vec{E}_F$
$\vec{H} = \vec{H}_A + \vec{H}_F$

1) Case 1: $\vec{J} \neq 0$
$\vec{M} = 0$

$$\vec{\nabla} \times \vec{H}_A = \vec{J} + \frac{\partial \vec{D}_A}{\partial t} = \vec{J} + j\omega\varepsilon \vec{E}_A$$

$$\vec{\nabla} \times \vec{E}_A = -\frac{\partial \vec{B}}{\partial t} = -j\omega\mu \vec{H}_A$$

$$\vec{\nabla} \cdot \vec{D}_A = \rho_e \qquad \vec{\nabla} \cdot \vec{B}_A = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad \Rightarrow \quad \vec{B}_A = \vec{\nabla} \times \vec{A}$$

$$\vec{H}_A = \frac{1}{\mu}(\vec{\nabla} \times \vec{A})$$

Also $\vec{\nabla} \times \vec{E}_A = -j\omega\mu \frac{1}{\mu}(\vec{\nabla} \times \vec{A})$

$$\Rightarrow \vec{\nabla} \times [\vec{E}_A + j\omega \vec{A}] = 0$$

$$\Rightarrow \vec{\nabla} \times (\overrightarrow{\nabla \phi_e}) = 0 \qquad \text{Also } \vec{\nabla} \times (-\overrightarrow{\nabla \phi_e}) = 0$$

$$\vec{E}_A + j\omega\vec{A} = -\overrightarrow{\nabla \phi_e} \qquad \vec{E}_A = -j\omega\vec{A} - \vec{\nabla}\phi_e$$

where ϕ_e is Scalar electric potential

Also $\vec{E}_A = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\vec{\nabla} \cdot \vec{A})$

$\vec{\nabla}\cdot\vec{B}=0$

Vector Magnetic potential $\vec{A}$

$$\vec{B}_A = \vec{\nabla}\times\vec{A} \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$$

$$\vec{\nabla}\times\vec{E}_A = -j\omega\vec{B}_A = -j\omega(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times[\vec{E}_A + j\omega\vec{A}]=0$$

$$\vec{\nabla}\times[-\nabla\phi_e]=0$$

$$\vec{\nabla}\times\frac{1}{\mu}(\vec{\nabla}\times\vec{A}) = \vec{J}+j\omega\varepsilon\vec{E}_A$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\vec{J}+j\omega\mu\varepsilon\vec{E}_A$$

$$\nabla(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A} = \mu\vec{J}+j\omega\varepsilon\mu[-\nabla\phi_e - j\omega\vec{A}]$$

$$\nabla(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A} = \mu\vec{J}-\nabla(j\omega\mu\varepsilon\phi_e)+\omega^2\mu\varepsilon\vec{A}$$

Now $k^2 \triangleq \omega^2\mu\varepsilon$

$$\Rightarrow \nabla(\vec{\nabla}\cdot\vec{A}+j\omega\mu\varepsilon\phi_e) = \mu\vec{J}+k^2\vec{A}+\nabla^2\vec{A}$$

$$\Rightarrow \underbrace{\nabla^2 A + k^2\vec{A} = -\mu\vec{J} + \nabla(\vec{\nabla}\cdot\vec{A}+j\omega\mu\varepsilon\phi_e)}_{\text{Lorentz Gauge condition}}$$

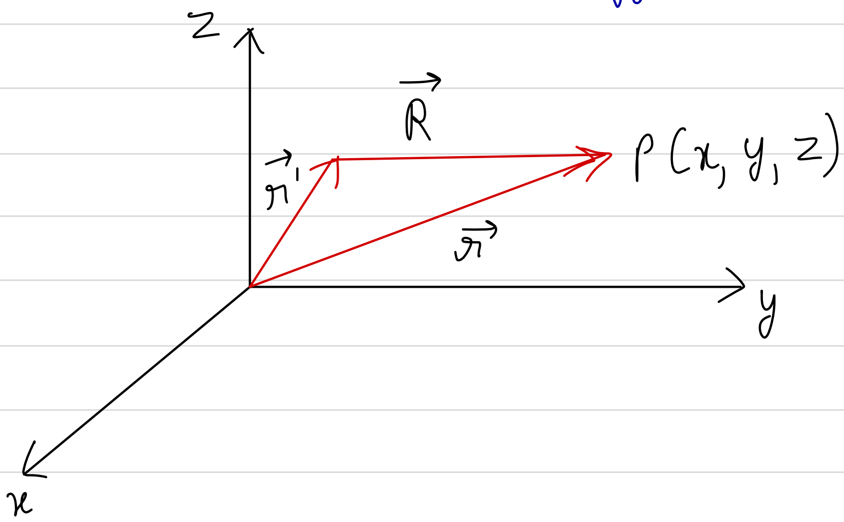
$\vec{A}\rightarrow$ i> $\vec{\nabla}\times\vec{A} = \vec{B}_A$

ii> $\vec{\nabla}\cdot\vec{A} = -j\omega\mu\varepsilon\phi_e \Rightarrow \phi_e = \dfrac{-1}{j\omega\mu\varepsilon}\vec{\nabla}\cdot\vec{A}$

$$\Rightarrow \phi_e = j\frac{1}{\omega\mu\varepsilon}\vec{\nabla}\cdot\vec{A}$$

Also $\boxed{\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}}$ → Vector Helmholtz equation

So $$\vec{A} = \frac{\mu}{4\pi} \iiint_{vol} \frac{\vec{J}\, e^{-jkR}}{R}\, dV$$



Case 2 : $\vec{J} = 0$, $\vec{M} \neq 0$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla} \times \vec{E} = -\vec{M} - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \cdot \vec{B} = \rho_m$$

$\vec{M}$ → $\vec{F}$ ⟶ $\vec{E}_F$ & $\vec{H}_F$

vector electric potential

$$\vec{\nabla} \cdot \vec{D}_F = 0$$

$$\vec{D}_F = -(\vec{\nabla} \times \vec{F}) \Rightarrow \vec{E}_F = -\frac{1}{\varepsilon} (\vec{\nabla} \times \vec{F})$$

$$\vec{\nabla} \times \vec{H}_F = j\omega\varepsilon \vec{E}_F$$

$$\vec{\nabla} \times \vec{H}_F = j\omega\varepsilon \left[-\frac{1}{\varepsilon} (\vec{\nabla} \times \vec{F}) \right]$$

$$\vec{\nabla} \times [\vec{H}_F + j\omega \vec{F}] = 0$$

$$\Rightarrow \quad \vec{H}_F + j\omega \vec{F} = -\nabla \phi_m$$

Also $\quad \vec{\nabla} \times \vec{E}_F = -\vec{M} - j\omega\mu \vec{H}_F$

$$\vec{\nabla} \times \left[-\frac{1}{\varepsilon} (\vec{\nabla} \times \vec{F}) \right] = -\vec{M} - j\omega\mu \vec{H}_F$$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{F}] = \varepsilon \vec{M} + j\omega\mu\varepsilon \vec{H}_F$$

$$\nabla(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F} = \varepsilon \vec{M} + j\omega\mu\varepsilon [-j\omega \vec{F} - \nabla \phi_m]$$

$$\nabla(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F} = \varepsilon \vec{M} + \omega^2 \mu\varepsilon \vec{F} - \nabla(j\omega\mu\varepsilon\phi_m)$$

$$\Rightarrow \underbrace{\nabla(\vec{\nabla} \cdot \vec{F} + j\omega\mu\varepsilon\phi_m) + \varepsilon \vec{M} = \nabla^2 \vec{F} + k^2 \vec{F}}$$

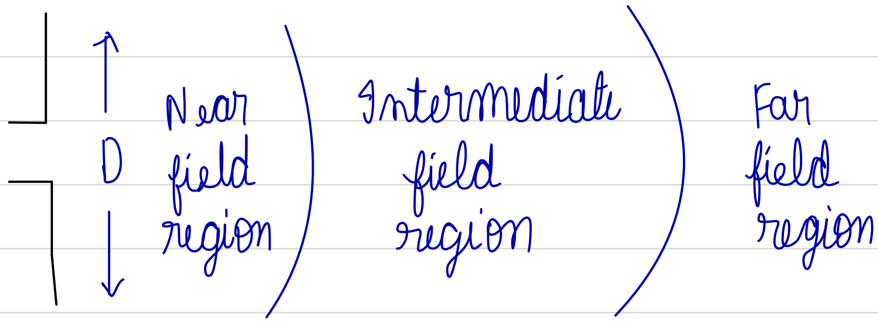
Lorentz gauge condition i.e $\nabla(\vec{\nabla} \cdot \vec{F} + j\omega\mu\varepsilon\phi_m) = 0$

$\boxed{\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}} \rightarrow$ Vector Helmholtz equation

So $\quad \vec{\nabla} \cdot \vec{F} = -j\omega\varepsilon\mu\phi_m \quad \Rightarrow \quad \phi_m = j \dfrac{1}{\omega\mu\varepsilon} \vec{\nabla} \cdot \vec{F}$

Similarly $\quad \vec{F} = \dfrac{\varepsilon}{4\pi} \displaystyle\iiint_{vol} \frac{\vec{M} \, e^{-jkR}}{R} \, dV$

$\vec{E}$, $\vec{H}$ in the far field region



$$\vec{A}(r,\theta,\phi) = \frac{\mu}{4\pi}\iiint_{vol} \frac{\vec{J}e^{-jkR}}{R}\, dV$$

$$\vec{A} = \left(\hat{a}_r A_r(r,\theta,\phi) + \hat{a}_\theta A_\theta(r,\theta,\phi) + \hat{a}_\phi A_\phi(r,\theta,\phi)\right)e^{-jkR}$$

NOTE:

- Variation of components of $\vec{A}$ w.r.t r is as $\frac{1}{r^n}$ where $n = 1, 2, 3, 4, \ldots$

$$\Rightarrow \hat{a}_r A_r(r,\theta,\phi) = \hat{a}_r\left[A_r'(\theta,\phi)\frac{1}{r} + A_r''(\theta,\phi)\frac{1}{r^2} + A_r'''(\theta,\phi)\frac{1}{r^3} + \ldots\right]$$

Similarly $\hat{a}_\theta A_\theta(r,\theta,\phi) = \hat{a}_\theta\left[A_\theta(\theta,\phi)\frac{1}{r} + A_\theta''(\theta,\phi)\frac{1}{r^2} + \ldots\right]$

Similar for $\hat{a}_\phi A_\phi$

But in far field region, $\frac{1}{r^2}$ and higher terms are neglected.

So

$$\vec{A} = \left[\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)\right]\frac{e^{-jkR}}{r}$$

$$\Rightarrow \vec{A} = \hat{a}_r A_r + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{A_r'(\theta,\phi)\, e^{-jkr}}{r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\, e^{-jkr} A_r'(\theta,\phi)\right)$$

$$= \frac{1}{r^2}\left[e^{-jkr} A_r'(\theta,\phi) - jkr\, e^{-jkr} A_r'(\theta,\phi)\right]$$

$$= \underbrace{\cancel{\frac{A_r'(\theta,\phi)\, e^{-jkr}}{r^2}}}_{\text{neglected}}{}^{\,\to 0} - \frac{jk A_r'(\theta,\phi)\, e^{-jkr}}{r}$$

$\therefore$ First term $= \dfrac{-jk A_r'(\theta,\phi)\, e^{-jkr}}{r}$

Second term $\triangleq \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left[\dfrac{\sin\theta\ A_\theta'(\theta,\phi)\ e^{-jkr}}{r}\right]$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\ A_\theta'(\theta,\phi)\, e^{-jkr}\right]$$

$$\simeq 0$$

Third term $\triangleq \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}\left[\dfrac{A_\phi(\theta,\phi)\, e^{-jkr}}{r}\right]$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left[A_\phi(\theta,\phi)\, e^{-jkr}\right]$$

$$\simeq 0$$

$$\therefore\ \vec{\nabla}\cdot\vec{A} = \psi = \frac{-jk A_r'(\theta,\phi)\, e^{-jkr}}{r}$$

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{a}_\phi$$

First term $\triangleq \frac{\partial \psi}{\partial r} = \frac{\partial}{\partial r}\left\{\frac{-jk\, A_r'(\theta,\phi)\, e^{-jkr}}{r}\right\}$

$$= -jk\, A_r'(\theta,\phi)\left\{\frac{-e^{-jkr}}{r^2} - \frac{jk e^{-jkr}}{r}\right\}$$

$$= \underbrace{j\frac{k A_r'(\theta,\phi) e^{-jkr}}{r^2}}_{\text{neglected}} - k^2\frac{A_r(\theta,\phi)e^{-jkr}}{r}$$

$\therefore$ First term $= -\frac{k^2 A_r'(\theta,\phi)\, e^{-jkr}}{r}$

Second term $= \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{-jk\, A_r'(\theta,\phi)e^{-jkr}}{r}\right\}$

$\simeq 0$

Third term $= \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left\{\frac{-jk\, A_r'(\theta,\phi)\, e^{-jkr}}{r}\right\}$

$\simeq 0$

$$\vec{E}_A = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\vec{\nabla}\cdot\vec{A}) \qquad (k^2 = \omega^2\mu\varepsilon)$$

Now, $j\frac{1}{\omega\mu\varepsilon}\nabla(\vec{\nabla}\cdot\vec{A}) = -j\frac{\omega^2\mu\varepsilon}{\omega\mu\varepsilon}\frac{A_r'(\theta,\phi)\, e^{-jkr}}{r}\hat{a}_r$

$$= -j\omega\frac{A_r'(\theta,\phi)\, e^{-jkr}}{r}\hat{a}_r$$

$$= -j\omega A_r(r,\theta,\phi)\ \hat{a}_r$$

$$\vec{E}_A = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\vec{\nabla}\cdot\vec{A})$$

$$= -j\omega\left[A_r(r,\theta,\phi)\hat{a}_r + A_\theta(r,\theta,\phi)\hat{a}_\theta + A_\phi(r,\theta,\phi)\hat{a}_\phi\right]$$

$$+ j\omega A_r(r,\theta,\phi)\hat{a}_r$$

$$\therefore \vec{E}_A = -j\omega\left[A_\theta(r,\theta,\phi)\hat{a}_\theta + A_\phi(r,\theta,\phi)\hat{a}_\phi\right]$$

$$\vec{E}_A = E_{Ar}\hat{a}_r + E_{A\theta}\hat{a}_\theta + E_{A\phi}\hat{a}_\phi$$

$$\therefore E_{Ar} = 0 \,//$$

$$E_{A\theta} = -j\omega A_\theta \,// \quad \& \quad E_{A\phi} = -j\omega A_\phi \,//$$

Now $\vec{H}_A = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})$

$$\vec{\nabla}\times\vec{A} = \hat{a}_r(\vec{\nabla}\times\vec{A})_r + \hat{a}_\theta(\vec{\nabla}\times\vec{A})_\theta + \hat{a}_\phi(\vec{\nabla}\times\vec{A})_\phi$$

$$\therefore (\vec{\nabla}\times\vec{A})_r \triangleq \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left\{\frac{A'_\phi(\theta,\phi)e^{-jkr}\sin\theta}{r}\right\} - \frac{\partial}{\partial\phi}\left\{\frac{A'_\theta(\theta,\phi)e^{-jkr}}{r}\right\}\right]$$

$$\simeq 0$$

$$(\vec{\nabla}\times\vec{A})_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$

$$= \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left\{\underbrace{\frac{A_r(\theta,\phi)e^{-jkr}}{r}}_{\text{neglected}}\right\} - \frac{\partial}{\partial r}\left\{\frac{rA_\phi(\theta,\phi)e^{-jkr}}{r}\right\}\right]$$

$$\therefore (\vec{\nabla}\times\vec{A})_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left\{A'_\phi(\theta,\phi)\,e^{-jkr}\right\}$$

$$= \frac{jk}{r}\,A_\phi'(\theta,\phi)\,e^{-jkr}$$

Similarly, $(\vec{\nabla}\times\vec{A})_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{r\,A_\theta(\theta,\phi)\,e^{-jkr}}{r}\right\} - \frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{A_r'(\theta,\phi)\,e^{-jkr}}{r}\right\}$$

$\hookrightarrow$ neglected

$$= \frac{-jk\,A_\theta(\theta,\phi)\,e^{-jkr}}{r}$$

$$\vec{H}_A = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})$$

But $H_{A_r} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_r = 0$

$$H_{A\theta} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_\theta = \frac{1}{\mu}\,\frac{j\omega\sqrt{\mu\varepsilon}\;A_\phi'(\theta,\phi)\,e^{-jkr}}{r}$$

$$= j\frac{\omega}{\eta}\,A_\phi'(\theta,\phi)\,\frac{e^{-jkr}}{r}$$

$$H_{A\phi} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_\phi = \frac{1}{\mu}(-jk)\,A_\theta'(\theta,\phi)\,\frac{e^{-jkr}}{r}$$

$$= -j\frac{\omega}{\eta}\,A_\theta'(\theta,\phi)\,\frac{e^{-jkr}}{r}$$

$\therefore\; H_{A\theta} = j\frac{\omega}{\eta}A_\phi = -\frac{E_\phi}{\eta}$ $\qquad H_{A\phi} = -j\frac{\omega}{\eta}A_\theta = \frac{E_\theta}{\eta}$

$$\vec{H}_A = \frac{1}{\eta}\left(\hat{a}_r \times \vec{E}_A\right)$$

$$= \frac{1}{\eta}\left(\hat{a}_r \times \left[0\hat{a}_r + (-j\omega A_\theta)\hat{a}_\theta - j\omega A_\phi \hat{a}_\phi\right]\right)$$

$$= \frac{1}{\eta}\left[-j\omega A_\theta \hat{a}_\phi + j\omega A_\phi \hat{a}_\theta\right] //$$

$\vec{H}$ in terms of $\vec{F}$:

$$\vec{E}_F = -\frac{1}{\varepsilon}\left(\vec{\nabla} \times \vec{F}\right)$$

$$\vec{H}_F = -j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\vec{\nabla}\cdot\vec{F}\right)$$

So $H_r \cong 0$

$H_\theta = -j\omega F_\theta$

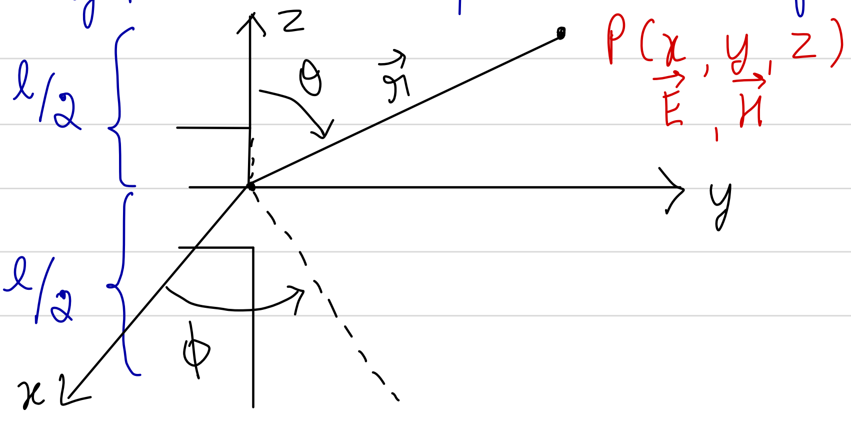
$H_\phi = -j\omega F_\phi$

Then $E_r \cong 0$ $\quad E_\theta = -j\omega\eta F_\phi = \eta H_\phi$

$E_\phi = j\omega\eta F_\theta = -\eta H_\theta$

Infinitesimal dipole:

Overall length $l \lll \lambda$, diameter of wire 'a' $\to 0$



If $l = \lambda$, then the voltage at any instance varies spacially. When $l <<< \lambda$ we assume that spacial variations do not exist.

$$\vec{A} \triangleq \frac{\mu}{4\pi} \iiint_{vol} \frac{\vec{J}\, e^{-jkr}}{R}\, dv$$

$$\vec{I}_e(x', y', z') = \begin{cases} I_0\, \hat{a}_z & ; \quad 0 \leq z' < l/2 \\ I_0(-\hat{a}_z) & ; \quad -l/2 \leq z' < 0 \end{cases}$$

$$\therefore \vec{A} = \frac{\mu}{4\pi} \int_C \frac{I_e(x', y', z')\, e^{-jkr}}{r}\, dz'$$

$$= \frac{\mu}{4\pi} \int_{-l/2}^{l/2} I_0 \frac{e^{-jkr}}{r}\, \hat{a}_z\, dz'$$

$$\therefore \vec{A} = \frac{\mu}{4\pi r} I_0 l e^{-jkr}\, \hat{a}_z$$

Rectangular to Sperical transformation:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

But here $A_x = A_y = 0$

$$\therefore A_r = A_z\cos\theta = \mu_0 \frac{Il\cos\theta\; e^{-jkr}}{4\pi r}$$

$$A_\theta = -A_z \sin\theta = -\frac{\mu I_0 l \sin\theta\, e^{-jkr}}{4\pi r}$$

$$A_\phi = 0$$

$$\boxed{\therefore\ \vec{A} = \frac{\mu I_0 l\, e^{-jkr}}{4\pi r}\left[\hat{a}_r(\cos\theta) + \hat{a}_\theta(-\sin\theta) + \hat{a}_\phi(0)\right]}$$

Now to find $\vec{H} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})$

And $\vec{E} = \frac{1}{j\omega\varepsilon}[\vec{\nabla}\times\vec{H}]$

$$H_{A_r} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_r = 0$$

$$H_{A_\theta} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_\theta = 0$$

$$H_{A_\phi} = \frac{1}{\mu}(\vec{\nabla}\times\vec{A})_\phi = \frac{jk I_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

Also $\vec{\nabla}\times\vec{H} = (\vec{\nabla}\times\vec{H})_r\hat{a}_r + (\vec{\nabla}\times\vec{H})_\theta\hat{a}_\theta + (\vec{\nabla}\times\vec{H})_\phi\hat{a}_\phi$

$$(\vec{\nabla}\times\vec{H})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \underbrace{\cancel{\frac{\partial H_\theta}{\partial\phi}}}_{0}\right]$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left\{\frac{jk I_0 l \sin^2\theta}{4\pi r}\left(1 + \frac{1}{jkr}\right)e^{-jkr}\right\}\right]$$

$$= \frac{jk I_0 l\cos\theta}{2\pi r^2}\left\{1 + \frac{1}{jkr}\right\}e^{-jkr}$$

$$E_r = \frac{1}{j\omega\varepsilon}(\vec{\nabla}\times\vec{H})_r$$

$$= \frac{1}{j\omega\varepsilon}\frac{j\omega\sqrt{\mu\varepsilon}\ I_0 l\cos\theta}{2\pi r^2}\left\{1+\frac{1}{jkr}\right\}e^{-jkr}$$

$$= \frac{\eta I_0 l\cos\theta}{2\pi r^2}\left\{1+\frac{1}{jkr}\right\}e^{-jkr}$$

$$(\vec{\nabla}\times\vec{H})_\theta = \frac{1}{r}\left[\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\theta}} - \frac{\partial}{\partial r}(rH_\phi)\right]$$

$$= \frac{1}{r}\left[-\frac{\partial}{\partial r}\left\{\frac{r\,jkI_0 l\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right.$$

$$= -\frac{e^{-jkr}}{r}\left[\frac{k^2 I_0 l\sin\theta}{4\pi} + \frac{k^2 I_0 l\sin\theta}{4\pi jkr} - \frac{jkI_0 l\sin\theta}{(jkr)^2 4\pi}jk\right]$$

$$= -\frac{k^2 I_0 l\sin\theta\, e^{-jkr}}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right]$$

$$\therefore \vec{E}_\theta = \frac{1}{j\omega\varepsilon}[\vec{\nabla}\times\vec{H}]_\theta$$

$$= \frac{-1}{j\omega\varepsilon}\frac{k\,\omega\sqrt{\mu\varepsilon}\ I_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right]e^{-jkr}$$

$$= \frac{jk\eta I_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right]e^{-jkr}$$

$$(\vec{\nabla}\times\vec{H})_\phi = \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right] = 0$$

$$E_\phi = \frac{1}{j\omega\varepsilon}(\vec{\nabla}\times\vec{H})_\phi \quad \Rightarrow \quad E_\phi = 0$$

NOTE: $\therefore H_r = H_\theta = 0$

$$H_\phi = \frac{jkI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$E_r = \frac{\eta I_0 l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = \frac{jk\eta I_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{k^2 r^2}\right]e^{-jkr}$$

$$E_\phi = 0$$

Poynting Vector

$$\vec{W} \triangleq \frac{1}{2}\left[\vec{E}\times\vec{H}^*\right]$$

$$= \frac{1}{2}\left[(E_r\hat{a}_r + E_\theta\hat{a}_\theta)\times(H_\phi^*\hat{a}_\phi)\right]$$

$$= \frac{1}{2}E_r H_\phi^*(-\hat{a}_\theta) + \frac{1}{2}E_\theta H_\phi^*\hat{a}_r$$

$$\therefore \vec{W} = W_r\hat{a}_r - W_\theta\hat{a}_\theta$$

where $W_r = \frac{1}{2}E_\theta H_\phi^*$ $\quad W_\theta = \frac{1}{2}E_r H_\phi^*$

W_θ is neglected

Also $H_\phi^* = \left[\dfrac{I_0 l \sin\theta}{4\pi r^2} - \dfrac{jkI_0 l \sin\theta}{4\pi r}\right] e^{jkr}$

$$\Rightarrow H_\phi^* = \frac{I_0 l \sin\theta}{4\pi r}\left[\frac{1}{r} - jk\right] e^{jkr}$$

$$\therefore W_r = \frac{1}{2} E_\theta H_\phi^*$$

$$= \frac{1}{2} \frac{j\eta k I_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{k^2 r^2}\right] e^{-jkr} \cdot \frac{I_0 l \sin\theta}{4\pi r}\left[\frac{1}{r} - jk\right] e^{jkr}$$

$$= \frac{1}{2} \frac{j\eta k |I_0 l|^2 \sin^2\theta}{16\pi^2 r^2}\left[\cancel{\frac{1}{r}} - jk + \cancel{\frac{1}{jkr^2}} - \cancel{\frac{1}{r}} - \frac{1}{k^2 r^3} + \cancel{\frac{jk}{k^2 r^2}}\right]$$

$$= \frac{1}{2} \frac{j\eta k |I_0 l|^2 \sin^2\theta}{16\pi^2 r^2}\left[-jk - \frac{1}{k^2 r^3}\right]$$

$$= \frac{1}{2} \frac{(-jk)\, j\eta k |I_0 l|^2 \sin^2\theta}{16\pi^2 r^2}\left[1 + \frac{1}{jk^3 r^3}\right]$$

$$= \frac{1}{2} \frac{\eta k^2 |I_0 l|^2 \sin^2\theta}{16\pi^2 r^2}\left[1 - \frac{j}{k^3 r^3}\right]$$

Now $k^2 = 4\pi^2 f^2 \mu\varepsilon = \dfrac{4\pi^2 f^2}{c^2} = \dfrac{4\pi^2}{\lambda^2}$

$$\therefore W_r = \frac{1}{2}\,\frac{4\pi^2\eta}{\lambda^2}\,\frac{|I_0 l|^2}{16\pi^2 r^2}\sin^2\theta\left[1 - \frac{j}{(kr)^3}\right]$$

$$= \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left[1 - \frac{j}{(kr)^3}\right]$$

$$Re\{W_r\} = \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}$$

$$Im\{W_r\} = \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left[\frac{-j}{k^3 r^3}\right] \rightarrow \text{negligible}$$

Also $W_\theta = -\frac{1}{2} E_r H_\phi^*$

$$= \frac{j\eta k\,|I_0 l|^2 \sin\theta\cos\theta}{16\pi^3 r^3}\left[1 + \frac{1}{(kr)^2}\right] \rightarrow \text{negligible}$$

Power radiated by infinitesimal dipole

$$P_{rad} \triangleq \oiint \vec{W}\cdot d\vec{s}$$

$$= \int_0^{2\pi}\int_0^{\pi} [\hat{a}_r W_r + \hat{a}_\theta W_\theta]\cdot \hat{a}_r\,(r^2 \sin\theta\, d\theta\, d\phi)$$

$$= \int_0^{2\pi}\int_0^{\pi} W_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi \int_0^{\pi} \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left[1 - \frac{j}{k^3 r^3}\right] r^2 \sin^3\theta\, d\theta$$

$$= \frac{\pi\eta}{4}\left|\frac{I_0 l}{\lambda}\right|^2 \left[1 - \frac{j}{k^3 r^3}\right]\int_0^{\pi} \sin^3\theta\, d\theta$$

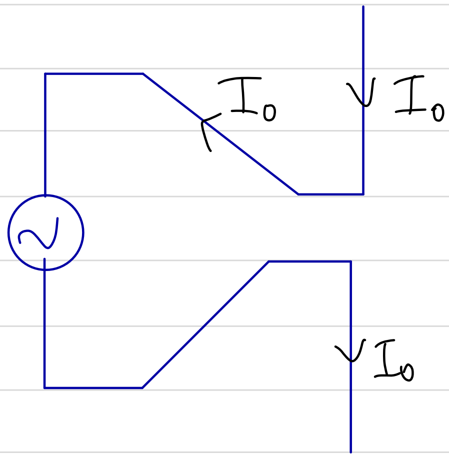
$$\int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}$$

$$P_{rad} = \frac{\pi\eta}{4} \left|\frac{I_0 l}{\lambda}\right|^2 \left[1 - \frac{j}{(kr)^3}\right] \frac{4}{3}$$

$$P_{rad} = \frac{\pi\eta}{3} \left|\frac{I_0 l}{\lambda}\right|^2 \left[1 - \frac{j}{(kr)^3}\right]$$

$\hookrightarrow$ neglected

$$\therefore P_{rad} = \frac{\pi\eta}{3} \left|\frac{I_0 l}{\lambda}\right|^2$$



$$P_{rad} = \frac{1}{2} |I_0|^2 R_r$$

Radiation resistance, $R_r = \frac{\pi\eta}{3} \frac{2}{|I_0|^2} \left|\frac{I_0 l}{\lambda}\right|^2$

$$= \frac{\pi\eta}{3} \, 2 \left|\frac{l}{\lambda}\right|^2$$

$$\Rightarrow R_r = \frac{2\pi\eta}{3} \left|\frac{l}{\lambda}\right|^2$$

$$= \frac{2\pi \; 120\pi}{3} \left|\frac{l}{\lambda}\right|^2 = 80\pi^2 \left|\frac{l}{\lambda}\right|^2$$

Taking $\pi^2 \cong 10$

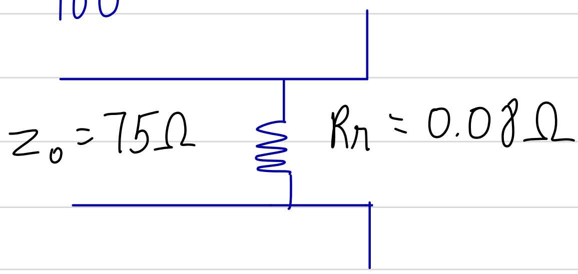
So $R_r = 800 \left|\frac{l}{\lambda}\right|^2$

For infinitesimal dipole : $l <<< \lambda$

So $\frac{l}{\lambda} < \frac{1}{50}$. $\rightarrow$ let $\frac{l}{\lambda} = \frac{1}{100}$

$\therefore \; R_r = \frac{800}{100^2} = 0.08\,\Omega$ //

$\Rightarrow$



$\Rightarrow$ There is a heavy mismatch. So infinitesimal dipoles are highly inefficient in power transfer.

$$Z_{in} = Z_0 \frac{[Z_L + jZ_0 \tan\beta l]}{Z_0 + jZ_L \tan\beta l}$$

As $Z_L \rightarrow \infty$ $\quad Z_{in} = \frac{Z_0}{j\tan\beta l} = -jZ_0 \cot\beta l$

NOTE: So input impedance of infinitesimal dipole is predominantly capacitive.

Radian Distance & Radian sphere

$kr = 1$

$\Rightarrow \frac{2\pi}{\lambda} r = 1 \quad \Rightarrow \quad r = \frac{\lambda}{2\pi} = r_1$

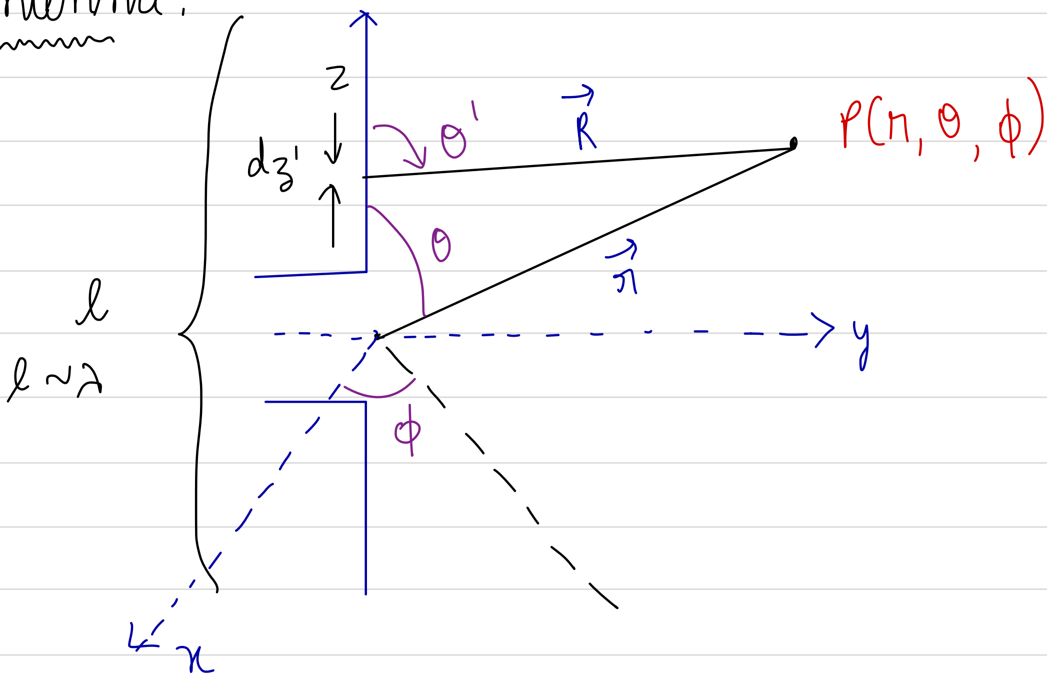
A sphere of radius $r_1 = \frac{\lambda}{2\pi} \rightarrow$ Radian sphere

$kr <<< 1$ $\quad$ $kr = 1$ $\quad$ $kr > 1$ $\quad$ $kr >> 1$

NF $\quad$ IF $\quad$ FF

NF = near field IF = Intermediate field
FF = Far field

Larger antenna:



$$\vec{A} = \frac{\mu}{4\pi} \int \frac{I_e(x', y', z')\, e^{-jkR}\, dz'}{R}$$

$R \cong r$

So $$\vec{A} = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \frac{I_0\, \vec{a}_z\, e^{-jkr}}{r}\, dz'$$

Here $H_r = H_\theta = 0$

$$H_\phi = \frac{jk I_0 l \sin\theta}{4\pi r} \left[1 + \frac{1}{jkr} \right] e^{-jkr}$$

$$E_r = \frac{\eta I_0 l \cos\theta}{2\pi r^2} \left[1 + \frac{1}{jkr} \right] e^{-jkr}$$

$$E_\theta = \frac{j\eta k I_0 l \sin\theta}{4\pi r^2} \left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2} \right] e^{-jkr}$$

In Near field region: $(kr \ll 1)$

$H_r = H_\theta = 0$

$$H_\phi = \frac{jkI_0 l \sin\theta}{4\pi r} \frac{1}{jkr} e^{-jkr} = \frac{I_0 l \sin\theta}{4\pi r^2} e^{-jkr}$$

$$E_r = \frac{\eta I_0 l \cos\theta}{2\pi r^2}\left[1 + \frac{1}{jkr}\right] e^{-jkr} \simeq -j\frac{\eta I_0 l \cos\theta}{2\pi r^3 k} e^{-jkr}$$

$$E_\theta = \frac{j\eta k I_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]$$

$$= \frac{j\eta k I_0 l \sin\theta}{4\pi r}\left[-\frac{1}{(kr)^2}\right] = -j\eta \frac{I_0 l \sin\theta}{4\pi k r^3} e^{-jkr}$$

$$\vec{W} = \frac{1}{2}\left[\vec{E} \times \vec{H}^*\right]$$

$$= \frac{1}{2}\left[(\hat{a}_r E_r + \hat{a}_\theta E_\theta) \times \hat{a}_\phi H_\phi^*\right]$$

$$W_r = \frac{1}{2} E_\theta H_\phi^*$$

$$= \frac{1}{2}\left(\frac{-j\eta I_0 l \sin\theta e^{-jkr}}{4\pi k r^3}\right)\left(\frac{I_0 l \sin\theta}{4\pi r^2} e^{+jkr}\right)$$

$$= \frac{-j\eta |I_0 l|^2 \sin^2\theta}{32\pi^2 k r^5}$$

$$W_\theta = -\frac{1}{2} E_r H_\phi^*$$

$$= -\frac{1}{2}\left(\frac{-j\eta I_0 l \cos\theta e^{-jkr}}{2\pi r^3 k}\right) \cdot \left(\frac{I_0 l \sin\theta e^{jkr}}{4\pi r^2}\right)$$

$$W_\theta = \frac{j\eta |I_0 l|^2 \sin\theta \cos\theta}{16\pi^2 r^5 k}$$

In intermediate field region : $(kr > 1)$

$$H_\phi = \frac{jk I_0 l \sin\theta}{4\pi r} e^{-jkr}$$

$$E_r = \frac{\eta I_0 l \cos\theta}{2\pi r^2} e^{-jkr}$$

$$E_\theta = \frac{j\eta k I_0 l \sin\theta}{4\pi r} e^{-jkr}$$

$$W_r = \frac{1}{2} E_\theta H_\phi^*$$

$$= \frac{1}{2} \left\{ \frac{j\eta k I_0 l \sin\theta}{4\pi r} e^{-jkr} \right\} \left\{ \frac{-jk I_0 l \sin\theta}{4\pi r} e^{jkr} \right\}$$

$$= \frac{\eta k^2 |I_0 l|^2 \sin^2\theta}{32\pi^2 r^2}$$

$$W_\theta = -\frac{1}{2} E_r H_\phi^*$$

$$= -\frac{1}{2} \left\{ \frac{\eta I_0 l \cos\theta}{2\pi r^2} e^{-jkr} \right\} \left\{ \frac{-jk I_0 l \sin\theta \, e^{jkr}}{4\pi r} \right\}$$

$$= \frac{j\eta k |I_0 l|^2 \sin\theta \cos\theta}{16\pi^2 r^3}$$

In Far field region:

$H_r = H_\theta = E_\phi = 0$ Also $E_r \cong 0$

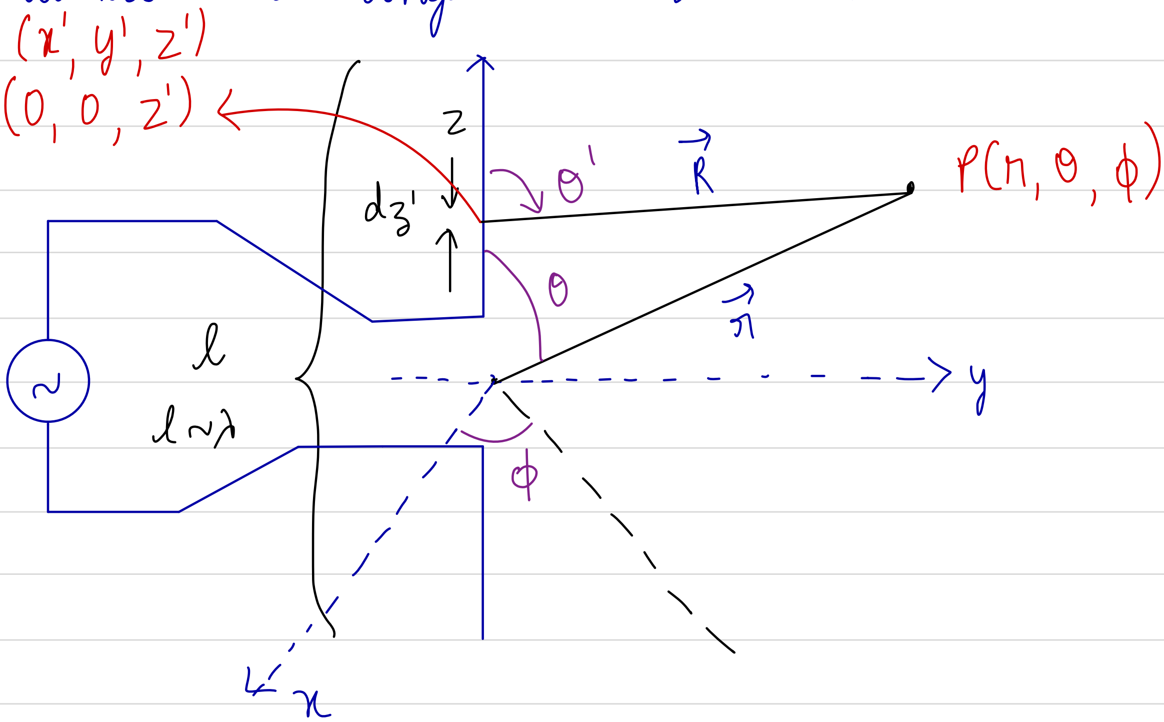
$$H_\phi = \frac{jk I_0 l \sin\theta}{4\pi r} e^{-jkr}$$

$$E_\theta = \frac{j\eta k I_0 l \sin\theta}{4\pi r} e^{-jkr}$$

} TEM (Transverse electro magnetic waves)

Also $W_\theta = 0$ and $W_r = \frac{\eta k^2 |I_0 l|^2 \sin^2\theta}{32\pi^2 r^2}$

Consider a longer antenna as shown in figure:



$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$= \sqrt{x^2 + y^2 + (z-z')^2} \qquad z = r\cos\theta$$

$$= \sqrt{x^2 + y^2 - 2rz'\cos\theta + z^2 + z'^2}$$

$$= \sqrt{r^2 - 2rz'\cos\theta + z'^2}$$

So $R = \sqrt{\underbrace{r^2}_{b} + \underbrace{(z'^2 - 2rz'\cos\theta)}_{x}} = \sqrt{b+x}$

From binomial theorem:

$$(b+x)^m = b^m + m b^{m-1} + \frac{m(m-1)}{2!} + b^{m-2} x^2 + \dots$$

Here $b = r^2$ $\quad x = (z')^2 - 2rz'\cos\theta$ $\quad m = 1/2$

So $(b+x)^{1/2}$; 1) $b^{1/2} = (r^2)^{1/2} = r$

2) $m\, b^{m-1} x = \frac{1}{2} (r^2)^{-1/2} \cdot \left((z')^2 - 2rz'\cos\theta\right)$

$$= \frac{(z')^2}{2r} - z'\cos\theta$$

3) $\frac{m(m-1)}{2} b^{m-2} x^2 = \frac{1}{2} \cdot \left(\frac{-1}{2}\right) \cdot \frac{1}{2} (r^2)^{-3/2} \left((z')^2 - 2rz'\cos\theta\right)^2$

$$= \frac{-1}{8r^3} \left[(z')^4 + 4r^2(z')^2\cos^2\theta - 4rz'^3\cos\theta\right]$$

$$= \frac{-(z')^4}{8r^3} + \frac{1}{2r^2}(z')^3\cos\theta - \frac{(z')^2}{2r}\cos\theta$$

4) $\frac{m(m-1)(m-2)}{3!} b^{m-3} x^3$

$$= \frac{1}{6} \cdot \frac{1}{2} \cdot \left(\frac{-1}{2}\right) \cdot \left(\frac{-3}{2}\right) (r^2)^{-5/2} \left[(z')^2 - 2rz'\cos\theta\right]^3$$

$$= \frac{1}{16} \frac{1}{r^5} \left[(z')^6 - 8r^3(z')^3\cos^3\theta - 6r(z')^5\cos\theta + 12r^2(z')^4\cos^2\theta\right]$$

$$= \frac{(z')^6}{16r^5} - \frac{3}{8}\frac{(z')^5}{r^4}\cos\theta + \frac{3}{4}\frac{1}{r^3}(z')^4\cos^2\theta - \frac{1}{2}\frac{(z')^3}{r^2}\cos^3\theta$$

Neglecting other terms;

$$(b+x)^m = r + \frac{(z')^2}{2r} - z'\cos\theta - \frac{(z')^4}{8r^3} + \frac{1}{2r^2}(z')^3\cos\theta$$
$$-\frac{(z')^2}{2r}\cos^2\theta + \frac{(z')^6}{16r^5} - \frac{3}{8}\frac{(z')^5}{r^4}\cos\theta + \frac{3}{4r^3}(z')^4\cos^2\theta$$
$$-\frac{1}{2r^2}(z')^3\cos^3\theta + \ldots$$

$$R = (r - z'\cos\theta) + \frac{(z')^2\sin^2\theta}{2r} + \frac{(z')^3\cos\theta\sin^2\theta}{2r^2}$$
$+ \ldots$ (neglect higher order terms, $n \geq 3$)

In far field region $R \cong r - z'\cos\theta \Rightarrow$ Phase term
$R \cong r$: Amplitude terms.

$$\vec{A} = \frac{\mu}{4\pi}\int_{-l/2}^{l/2} \frac{I_0(x', y', z')\, e^{-jkR}}{R}\, dl'$$

Substitute R according to amplitude or phase.

Eg: Let $l = 7\lambda$, $\theta = 90°$, $r = 100\lambda$, $z' = \frac{l}{2}$

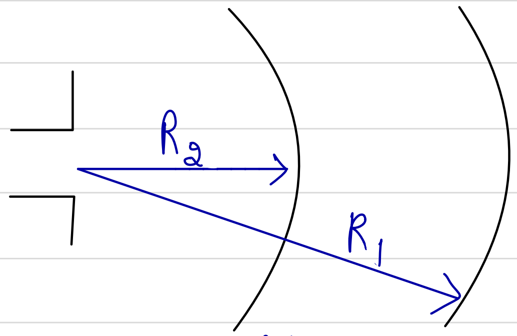
$$R = \sqrt{r^2 - 2rz'\cos\theta + (z')^2}$$
$$= \sqrt{(100\lambda)^2 + (3.5\lambda)^2}$$
$$= 100.06\lambda$$

$\Delta R = R - r = 0.06\lambda$

$$\vec{A} = \frac{\mu}{4\pi} \int \frac{I_o(x', y', z')\, e^{-jkR}}{\cancel{R}\ r}\, dl$$

$$\Delta\phi = \frac{2\pi}{\lambda} \Delta R = \frac{2\pi}{\lambda} \cdot 0.06\lambda \text{ rad} = 21.6^\circ$$

We can replace R by r in denominator as ΔR is small, but in e^{-jkR} we cannot as $\Delta\phi$ is large.



For far field:

Most significant ignored term: $\left(\frac{(z')^2}{2r}\right) \sin^2\theta$

whose max value is $\frac{(z')^2}{2r}$ when $\theta = 90^\circ$

$\frac{(z')^2}{2r}$ has max value when $z' = l/2$

i.e $= \frac{l^2}{8r}$

$\Delta e = \frac{l^2}{8r} \rightarrow$ Phase error $\Delta\phi = \frac{2\pi}{\lambda} \frac{l^2}{8r}$

NOTE: A max phase error of 22.5° or $\pi/8$ is acceptable.

$$\frac{2\pi}{\lambda}\frac{l^2}{8r}\bigg|_{r=R_1} \leq \frac{\pi}{8} \rightarrow \frac{2l^2}{\lambda R_1} \leq 1$$

$$\therefore \quad R_1 \geq \frac{2l^2}{\lambda}$$

l is the largest physical dimension of radiator.

For near field: Most significant term: $(z')^3 \frac{\cos\theta \sin^2\theta}{2r^2}$

$\cos\theta \sin^2\theta \rightarrow$ has max at $\theta = 54.735°$

$R_2 \geq 0.62\sqrt{\frac{l^3}{\lambda}}$ → This is derived from:

$$\frac{2\pi}{\lambda}\left[\frac{l^3}{8R_2^2} \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3}\right] \leq \frac{\pi}{8}$$

In intermediate field region:

$$E_\theta = \frac{j\eta k I_0 l \sin\theta \, e^{-jkr}}{4\pi r} \qquad H_\phi = \frac{jk I_0 l \sin\theta \, e^{-jkr}}{4\pi r}$$

$$E_r = \frac{\eta k I_0 l \sin\theta \, e^{-jkr}}{2\pi r^2}$$

$$W_r = \frac{1}{2} E_\theta H_\phi^* = \frac{1}{2\eta}|E_\theta|^2$$

$$= \frac{1}{2\eta} \frac{\eta^2 |k I_0 l|^2 \sin^2\theta}{16\pi^2 r^2}$$

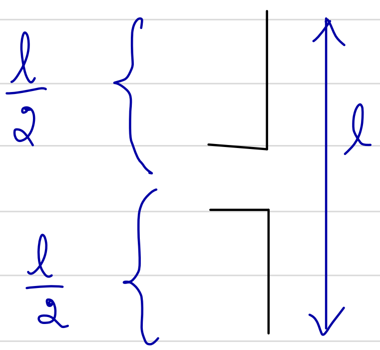
$$= \frac{\eta |k I_0 l|^2 \sin^2\theta}{32\pi^2 r^2}$$

$$W_\theta = -\frac{1}{2} E_r H_\phi^*$$

$$= -\frac{1}{2} \frac{\eta I_o l \cos\theta \, e^{-jkr}}{2\pi r^2} \left[\frac{-jk I_o l \sin\theta \, e^{jkr}}{4\pi r} \right]$$

$$= \frac{j\eta k \, |I_o l|^2 \sin\theta\cos\theta}{16\pi^2 r^2}$$

Finite length dipole



$l \sim \lambda$

$\vec{I}_l \, (x'=0, \, y'=0, \, z')$

NOTE: If $l \leq \frac{\lambda}{50} \Rightarrow$ infinitesimal dipole.

$$\vec{I}_l = \begin{cases} I_o \sin\left[k\left(\frac{l}{2} - z'\right)\right] \hat{a}_z & 0 \leq z' \leq \frac{l}{2} \\ I_o \sin\left[k\left(\frac{l}{2} + z'\right)\right] \hat{a}_z & -\frac{l}{2} \leq z' \leq 0 \end{cases}$$

If $l = \frac{\lambda}{2}$, then I_l at $z' = 0$

$$I_l = I_o \sin\left[\frac{2\pi}{\lambda}\frac{\lambda}{4} - 0\right] = I_o$$

At $z' = l/2$ & $z' = -l/2$

$I_e(l/2) = 0$ $I_e(-l/2) = 0$

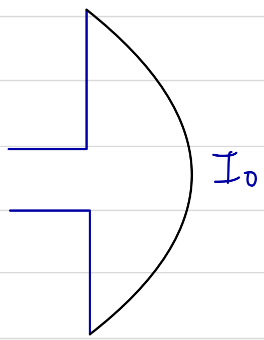
If $l = \lambda$

$I_e(0) = I_0 \sin\left[\frac{2\pi}{\lambda}\frac{\lambda}{2}\right] = 0$

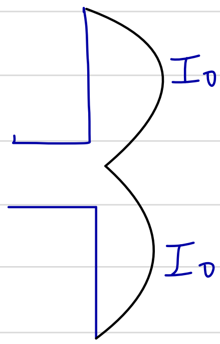
$I_e(l/2) = I_0 \sin\left[\frac{2\pi}{\lambda}\left(\frac{\lambda}{2} - \frac{\lambda}{2}\right)\right] = 0$

III^{ly} $I_e(-l/2) = 0$

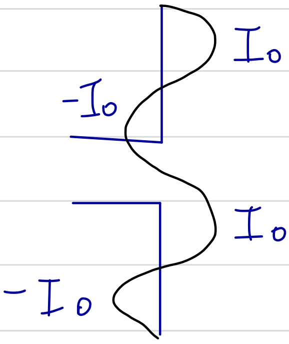
For $l = \lambda/4$



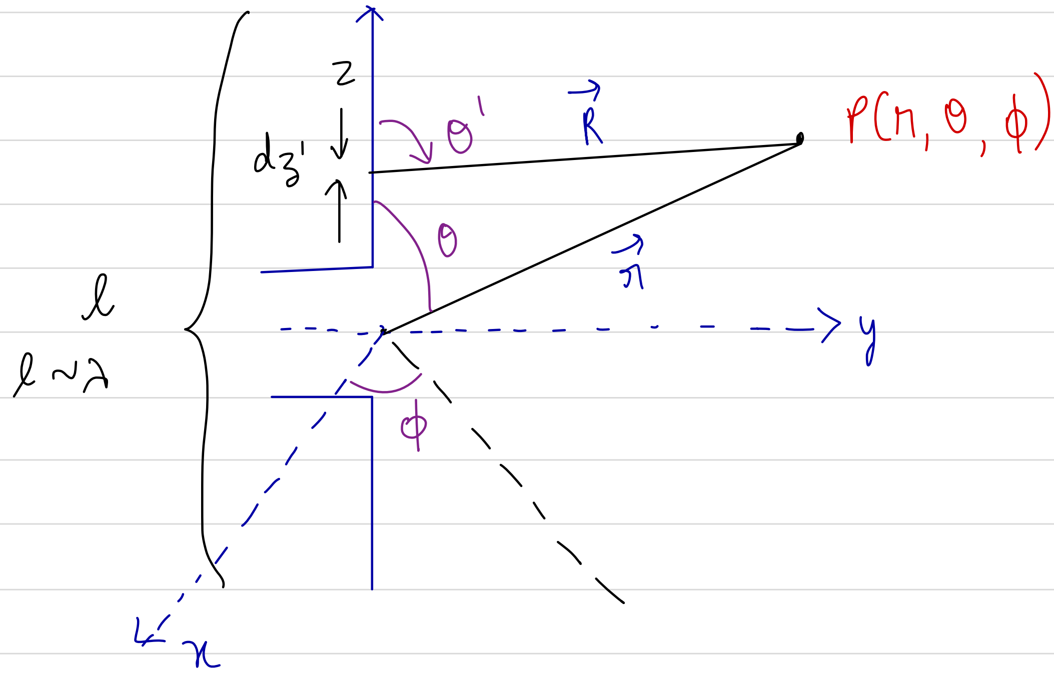
For $l = \lambda$



For $l = 2\lambda$



In far field region:



$R = r - z'\cos\theta$: Phase terms

$R = r$: Amplitude terms

$$dz' \longrightarrow dE_\theta = \frac{j\eta k I_0 l \sin\theta \, e^{-jkR}}{4\pi R}$$

$$dE_\theta = \frac{j\eta k I_0 l \sin\theta \; e^{-jk(r - z'\cos\theta)}}{4\pi r}$$

$$= \frac{j\eta k I_0 l \; e^{jkz'\cos\theta} \sin\theta}{4\pi r} \; e^{-jkr}$$

$$E_\theta = \underbrace{\frac{j\eta k \sin\theta \, e^{-jkr}}{4\pi r}}_{\text{Element factor}} \underbrace{\int_{-l/2}^{l/2} I_e(x', y', z') \, e^{jkz'\cos\theta} \, dz'}_{\text{Space factor}}$$

$$E_\theta = \frac{j\eta k \sin\theta\, e^{-jkr}}{4\pi r}\left[\int_{-\frac{l}{2}}^{0} I_0 \sin\left[k\left(\frac{l}{2}+z'\right)\right] e^{jkz'\cos\theta}\, dz' \right.$$

$$\left. + \int_{0}^{l/2} I_0 \sin\left[k\left(\frac{l}{2}-z'\right)\right] e^{jkz'\cos\theta}\, dz'\right]$$

NOTE: $\displaystyle\int e^{\alpha x}\sin(\beta x+\gamma)\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta x+\gamma) - \beta\cos(\beta x+\gamma)\right]$

$\alpha = jk\cos\theta \qquad \beta = k \qquad \gamma = \frac{kl}{2}$

$$I_1 = \int e^{(jk\cos\theta)z'} \sin\left[kz' + \frac{kl}{2}\right] dz'$$

$$= \frac{e^{(jk\cos\theta)z'}}{k^2\sin^2\theta}\left[jk\cos\theta\sin\left(kz'+\frac{kl}{2}\right) - k\cos\left(kz'+\frac{kl}{2}\right)\right]$$

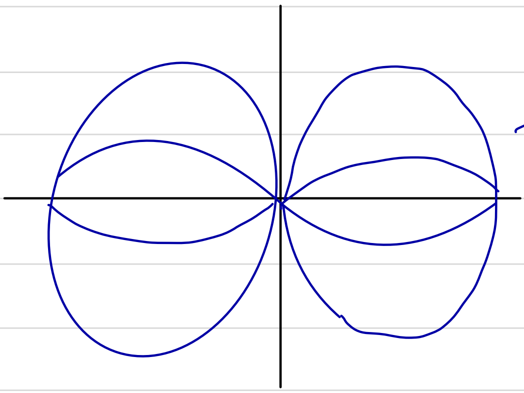
Applying limits from $-l/2$ to 0

$$I_1\Big]_{-l/2}^{0} = \frac{1}{k^2\sin^2\theta}\left[jk\cos\theta\sin\frac{kl}{2} - k\cos\frac{kl}{2}\right] - \frac{e^{-jk\frac{l\cos\theta}{2}}}{k^2\sin^2\theta}\left[-k\right]$$

$$= \frac{1}{k^2\sin^2\theta}\left[jk\cos\theta\sin\left(\frac{kl}{2}\right) - k\cos\left(\frac{kl}{2}\right) + k e^{-jk\frac{l\cos\theta}{2}}\right]$$

$$E_\theta = \frac{j\eta \sin\theta\, e^{-jkr}}{4\pi r} \left[I_1 \right]_{-l/2}^{0} + \left[I_2 \right]_{0}^{l/2}$$

$$E_\theta = \frac{j\eta\, I_0 e^{-jkr}}{2\pi r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]$$



→ omnidirectional pattern

$$H_\phi = \frac{E_\theta}{\eta} = \frac{j\, I_0 e^{-jkr}}{2\pi r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]$$

$$\vec{W} = \hat{a}_r \frac{1}{2\eta} |E_\theta|^2$$

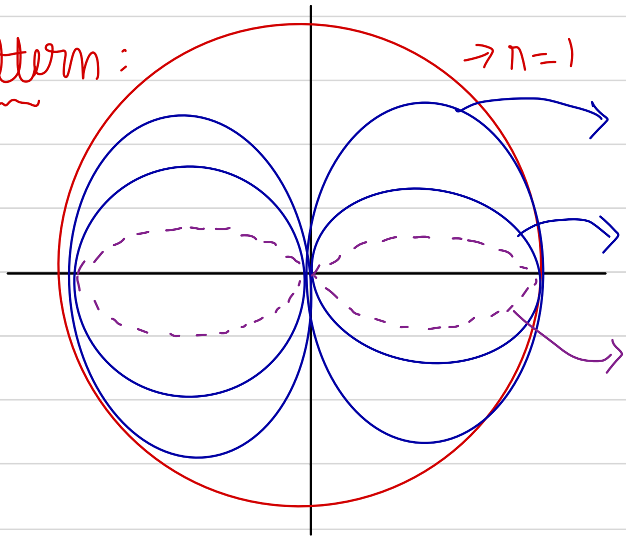
$$= \hat{a}_r \frac{1}{2\eta} \frac{I_0^2 \eta^2}{4\pi^2 r^2} \left[\frac{\cos((kl/2)\cos\theta) - \cos(kl/2)}{\sin\theta} \right]^2$$

$$= \frac{I_0^2 \eta}{8\pi r^2 \sin^2\theta} \left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right) \right]^2$$

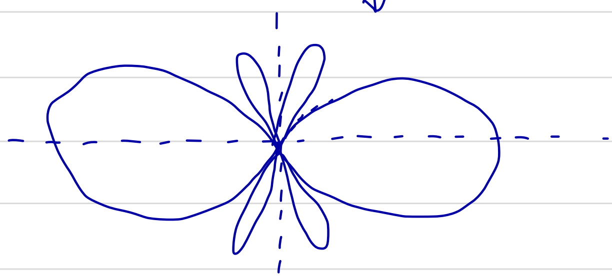
$$P_{rad} = \oiint \vec{W} \cdot d\vec{S} = \oiint U \cdot d\Omega$$

where $U = r^2 W_r$

Power Pattern:



If L increases we get side lobes



$l = \frac{\lambda}{2} \to$ Half wave dipole

HF communication $\to$ 3 - 30 MHz

$l = \lambda/2$ has inverted V configuration.

$$\frac{kl}{2} = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$$

$$\therefore E_\theta = \frac{j\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\pi/2 \cos\theta\right)}{\sin\theta}\right], \quad H_\phi = \frac{E_\theta}{\eta}$$

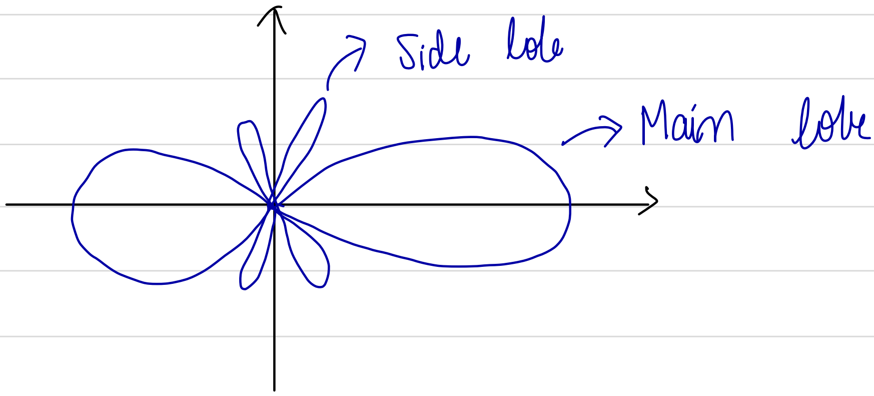
$$P_{rad} = \frac{1}{2}|I_0|^2 R_{rad}$$

$$R_{rad} = \frac{2P_{rad}}{|I_0|^2} = 73\,\Omega$$

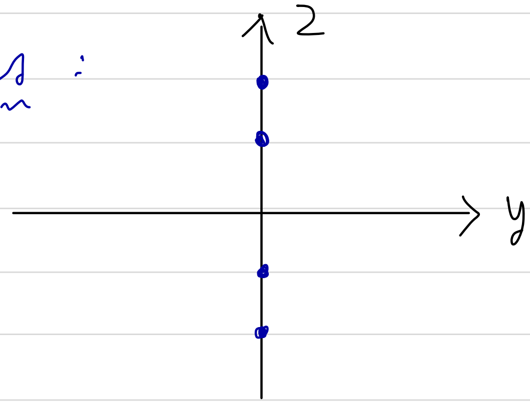
Also $Z_{in} = (73 + j42.5)\Omega$

More directive patterns:

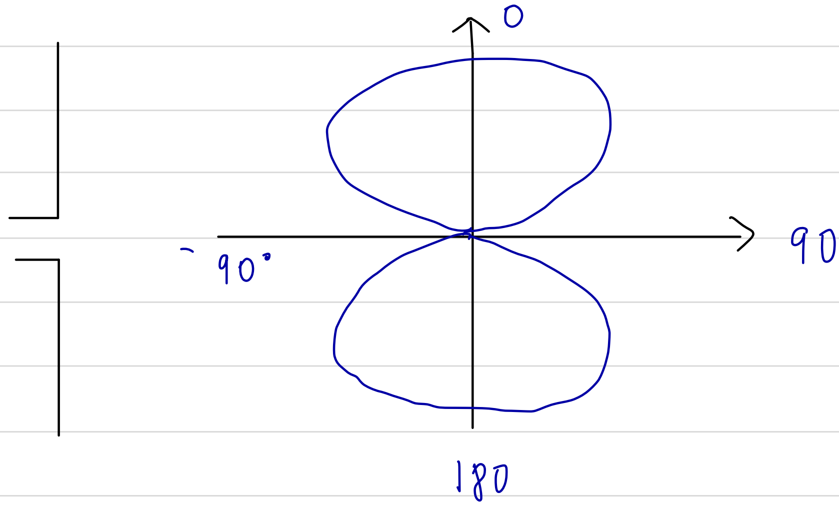
Inverse l relative to λ

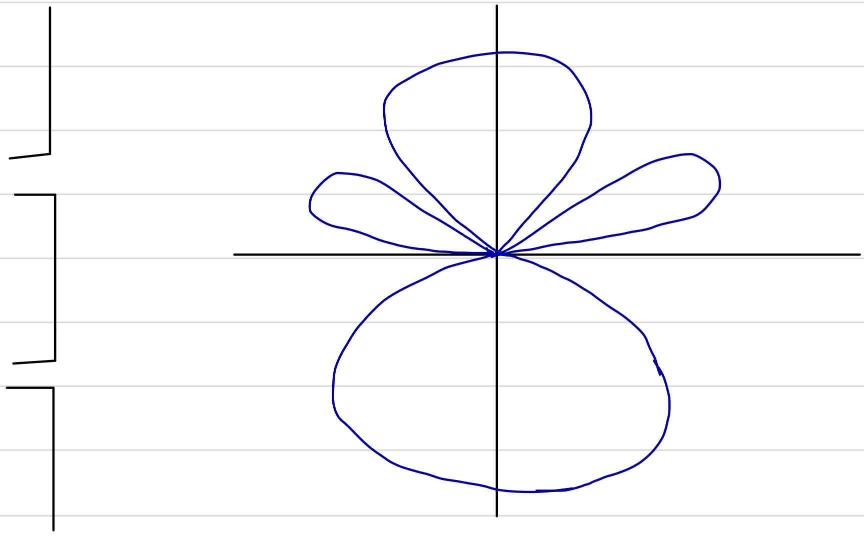


Design of antenna arrays:

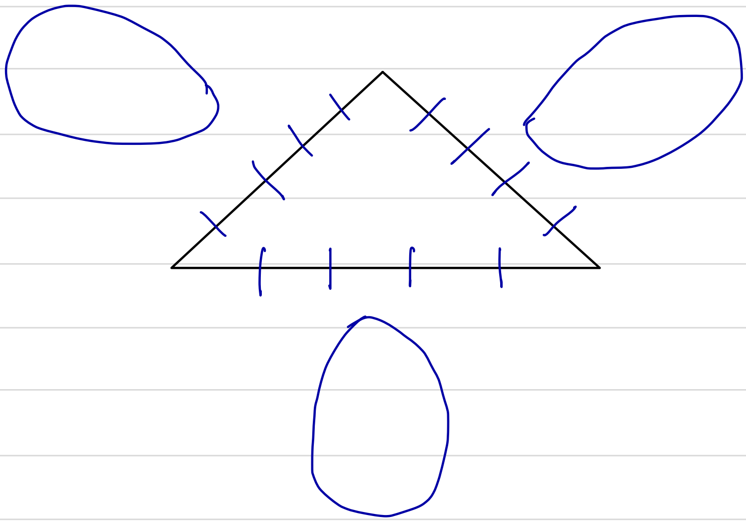


Radiation due to one antenna:

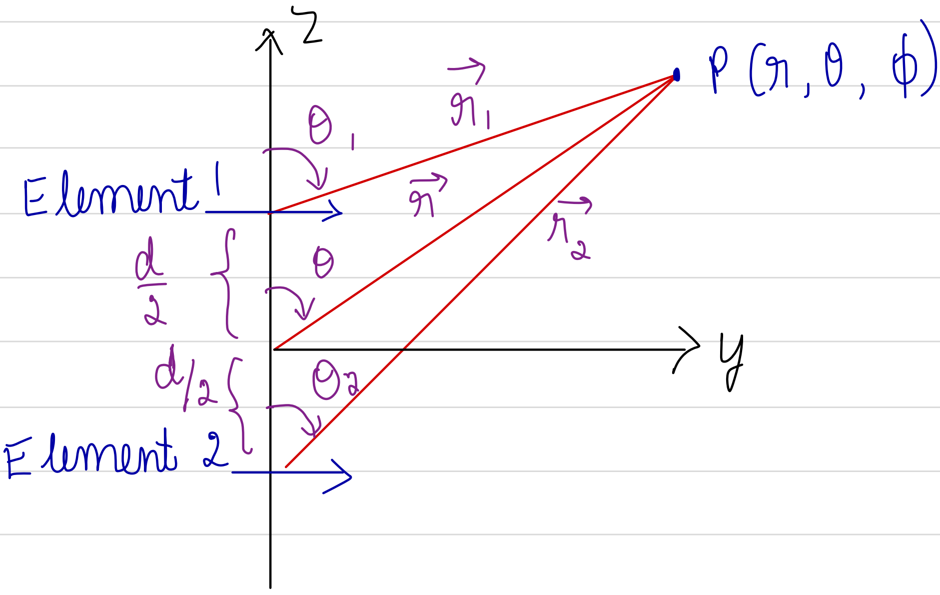




In mobile base station



Consider:



$$\vec{E}_1 = \hat{a}_\theta \frac{j\eta k I_0 l e^{-jkr_1}}{4\pi r_1} \cos\theta_1$$

$$\vec{E}_2 = \hat{a}_\theta \frac{j\eta k I_0 l e^{-jkr_2}}{4\pi r_2} \cos\theta_2$$

Array parameters:

i> The geometrical shape of the array:

- → Linear
- → Rectangular
- → Triangular
- → Circular

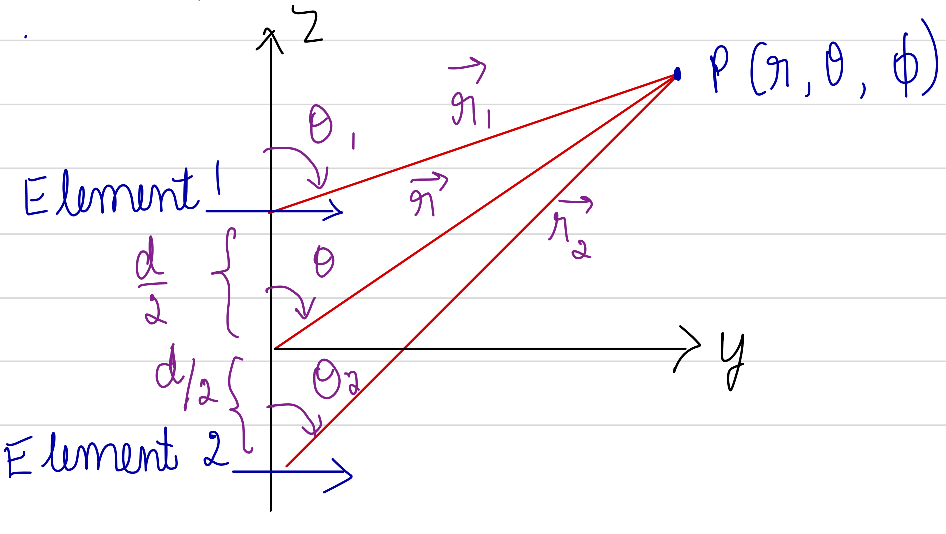
ii> Spacing 'd' between elements

iii> Phase difference in the excitation applied to elements

iv> Excitation amplitudes applied to elements of array

v> Non identical array elements

Antenna array: Arrangement of multiple radiators in a certain geometric pattern with addition degrees of freedom and radiation pattern.



• When P is too far away then $\theta_1 \cong \theta \cong \theta_2$ and r_1, r_2, r are almost parallel.

• So for phase terms:

$$r_1 = r - \frac{d}{2}\cos\theta$$

$$r_2 = r + (d/2)\cos\theta$$

For amplitude term: $r_1 \cong r \cong r_2$

• Let β be the phase difference in excitation applied to element 1 & 2.

• Then excitation applied to element 1 leads the excitation applied to element 2 by β.

$$\vec{E}_1 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-j(kr_1 - \beta/2)}$$

$$\vec{E}_2 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-j(kr_2 + \beta/2)}$$

$$\therefore \vec{E}_1 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-j\left(kr - k\frac{d}{2}\cos\theta - \beta/2\right)}$$

$$\Rightarrow \vec{E}_1 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-jkr} e^{j\left(\frac{kd\cos\theta}{2} + \beta/2\right)} \hat{a}_\theta$$

|||ly $$\vec{E}_2 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-j\left(kr + k\frac{d}{2}\cos\theta + \beta/2\right)}$$

$$\vec{E}_2 = \frac{j\eta k I_0 l \cos\theta}{4\pi r} e^{-jkr} + e^{-j\left(\frac{kd\cos\theta}{2} + \beta/2\right)} \hat{a}_\theta$$

$$\vec{E}_t = \vec{E}_1 + \vec{E}_2$$

$$= \frac{j\eta k I_0 l \cos\theta \; e^{-jkr}}{4\pi r} \hat{a}_\theta \left(e^{j\left(\frac{kd\cos\theta+\beta}{2}\right)} + e^{-j\left(\frac{kd\cos\theta+\beta}{2}\right)} \right)$$

$$= \underbrace{\frac{j\eta k I_0 l \cos\theta \; e^{-jkr}}{4\pi r}}_{\text{Element factor (EF)}} \cdot \underbrace{2\cos\left(\frac{kd\cos\theta+\beta}{2}\right)}_{\text{Array factor (AF)}} \hat{a}_\theta$$

Element factor (EF) Array factor (AF)

$\vec{E}_t$ is the electric field created by a single element placed at the origin.

This is called principle of pattern multiplication

eg: let $d = \lambda/4$

i> $\beta = 0$

$$E_t = \frac{j\eta k I_0}{2\pi r} \cos\theta \cos\left(\frac{kd\cos\theta+\beta}{2}\right)$$

normalised, = 1.

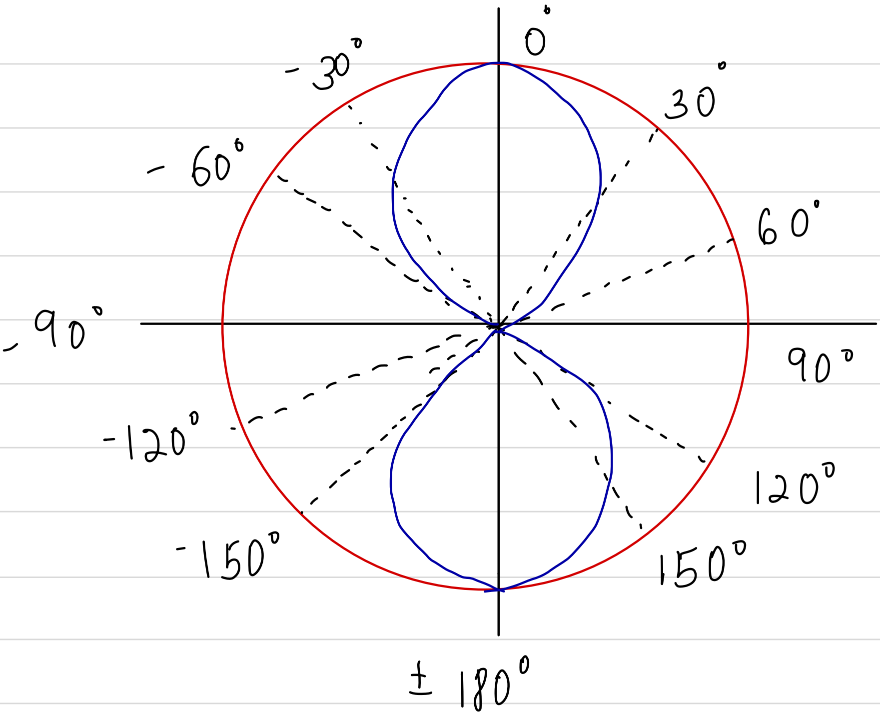
$$E_t = \cos\theta \cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{4} \times \frac{1}{2} \cos\theta\right)$$

$$= \cos\theta \cos\left(\frac{\pi}{4}\cos\theta\right)$$

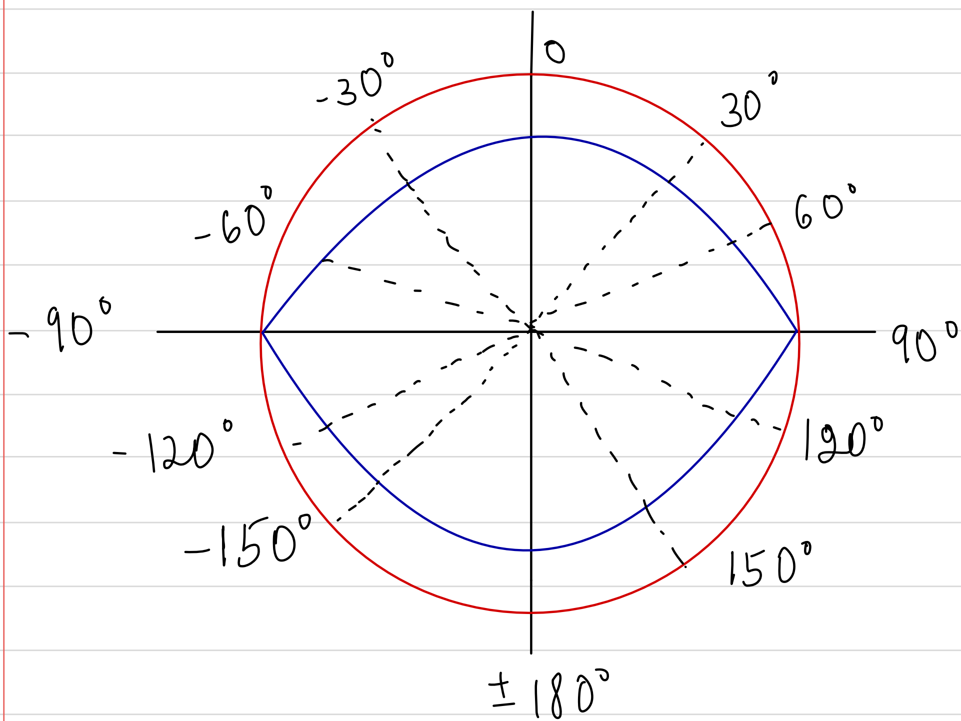
Nulls of expressions $\Rightarrow$ $E_t = 0$

$\cos\theta_n \cos\left(\frac{\pi}{4}\cos\theta_n\right) = 0 \quad \Rightarrow \quad \theta_n = 90°$

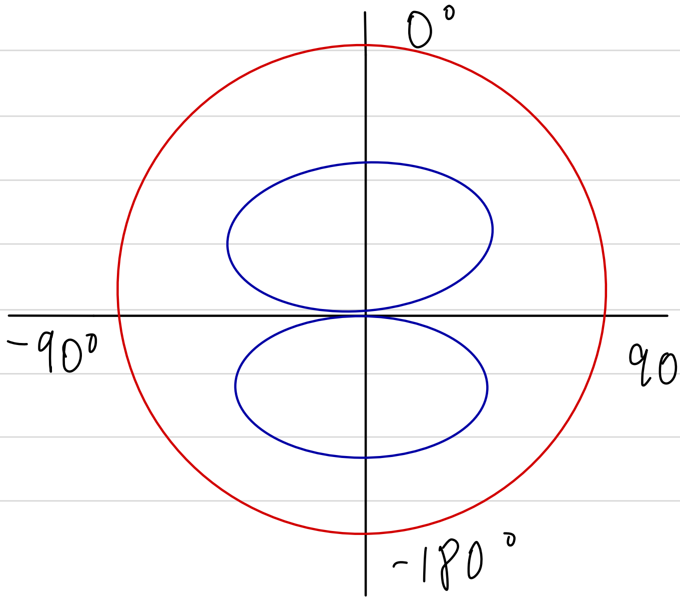
or $\frac{\pi}{4}\cos\theta_n = \frac{\pi}{2} \quad \Rightarrow \quad \cos\theta_n = 2$ (not possible)



Element factor



Array factor



Total

Pattern has been modified but new nulls are not introduced.

$\therefore$ $d = \frac{\lambda}{4}$ not adequate for creation of nulls.

$$E_{t_n} = \cos\theta_n \cos\left(\frac{kd\cos\theta_n + \beta}{2}\right)$$

if $d = \lambda/4$, $\beta = \pi/2$

$$E_{t_n} = \cos\theta_n \cos\left[\frac{\pi}{4}(1+\cos\theta_n)\right] = 0$$

$\cos\theta_n = 0 \Rightarrow \theta_n = 90^\circ$

Also $\frac{\pi}{4}(1+\cos\theta_n) = \pm\pi/2$

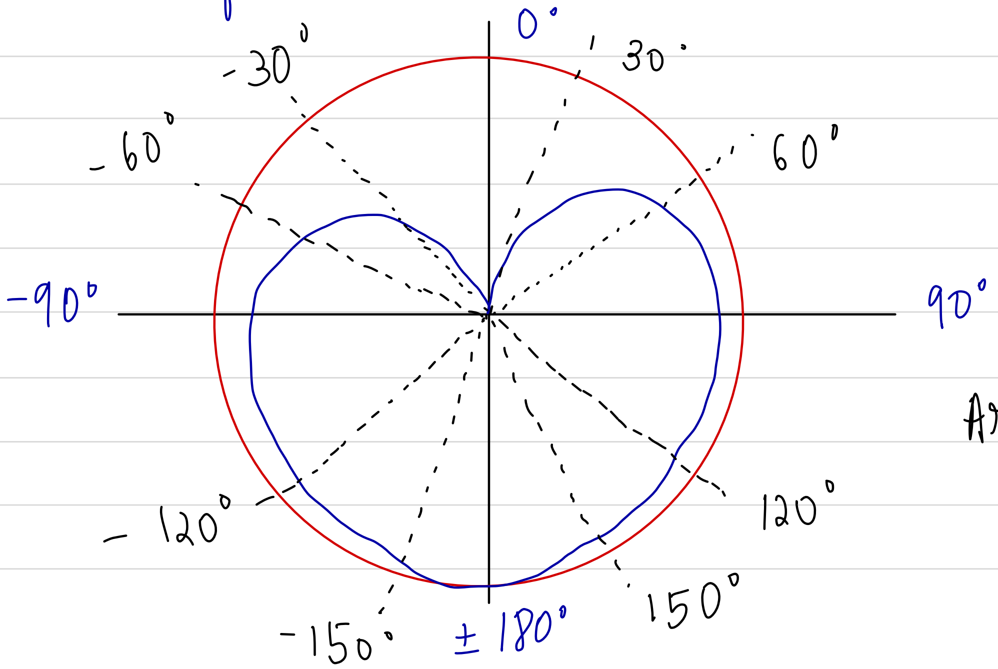
$\Rightarrow 1+\cos\theta_n = 2$ or $1+\cos\theta_n = -2$

$\cos\theta_n = 1$ $\qquad$ $\cos\theta_n = -3$

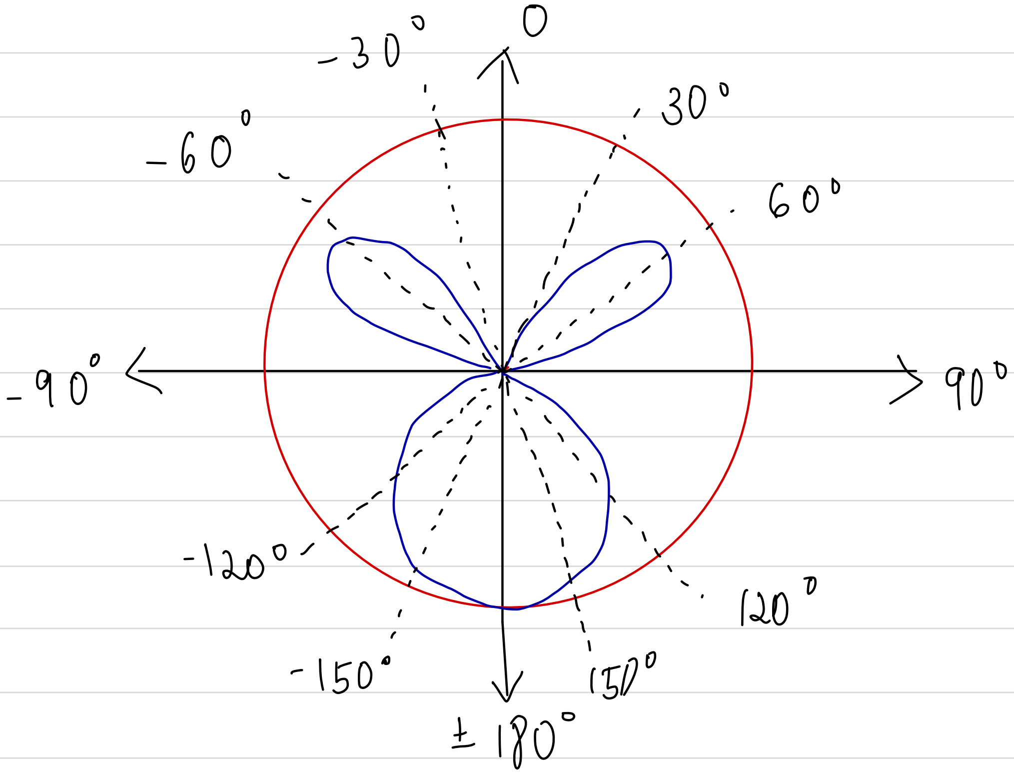
(not possible)

$\Rightarrow \theta_n = 0^\circ$ //

Element factor remains same as last one.

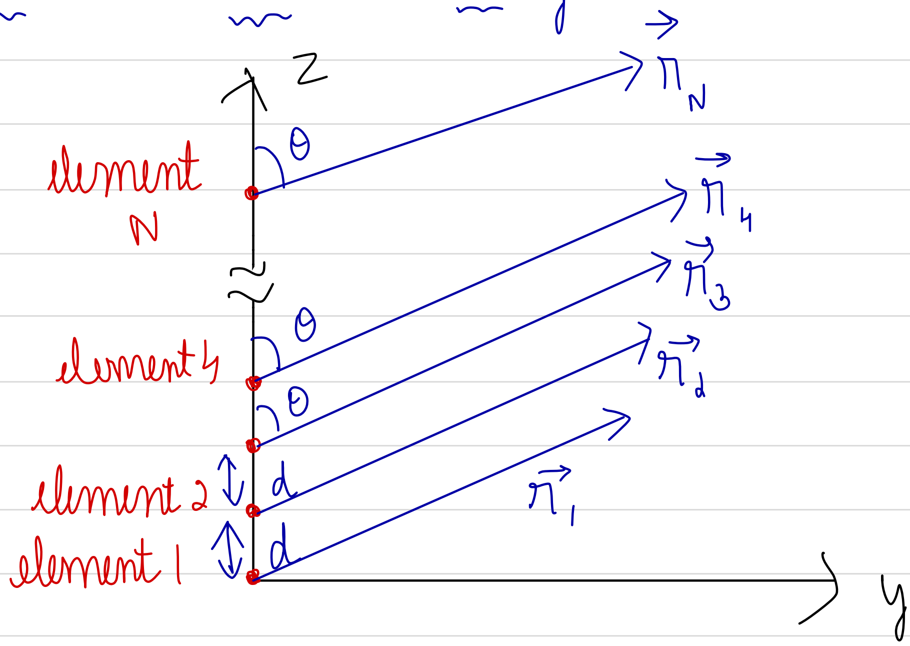


Array factor



Total pattern

N-element linear array



$r_1 = r_2 + d\cos\theta$
$r_2 = r_3 + d\cos\theta$
$r_3 = r_4 + d\cos\theta$
$\vdots$
$r_{N-1} = r_n + d\cos\theta$

Excitation applied to element i leads excitation applied to element i-1 by 'β' rad.

Array factor, $AF = 1 + e^{j(kd\cos\theta+\beta)} + e^{j2(kd\cos\theta+\beta)}$
$+ e^{j3(kd\cos\theta+\beta)} + \ldots e^{j(N-1)(kd\cos\theta+\beta)}$

let $kd\cos\theta + \beta = \psi$

$$AF = 1 + e^{j\psi} + e^{j2\psi} + e^{3j\psi} + \ldots + e^{j(N-1)\psi}$$

Also

$$(AF)\cdot e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \ldots + e^{jN\psi}$$

$$AF \cdot e^{j\psi} = e^{jN\psi} + AF - 1$$

$$\Rightarrow AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1}$$

$$\Rightarrow AF = \frac{e^{j\frac{N\psi}{2}}}{e^{j\psi/2}} \cdot \frac{\left[e^{j\frac{N\psi}{2}} - e^{-j\frac{N\psi}{2}}\right]}{\left[e^{j\psi/2} - e^{-j\psi/2}\right]}$$

$$= e^{j(N-1)\psi/2} \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

So magnitude $= 1$.

$$AF = \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

Applying L'Hospitals rule:

let $\frac{f(\psi)}{g(\psi)} = \frac{\sin(N\psi/2)}{\sin(\psi/2)}$

$$\lim_{\psi\to 0} \frac{f(\psi)}{g(\psi)} = \lim_{\psi\to 0} \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

$$= \lim_{\psi\to 0} \frac{(N/2)\cos(N\psi/2)}{(1/2)\cos\psi/2} = N$$

So max value of $AF = N$, at $\psi = 0$

Normalised value of AF :

$$(AF)_N = \frac{1}{N} \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

For small value of ψ, $\sin\psi \cong \psi$

$$AF = \frac{1}{N} \frac{\sin(N\psi/2)}{\psi/2} = \frac{2}{N\psi} \sin\left(\frac{N\psi}{2}\right)$$

Minimas, Null location:

$$\sin\left(\frac{N\psi}{2}\right)\Big|_{\theta=\theta_n} = 0 \Rightarrow \frac{N\psi}{2} = \pm n\pi, \quad n = 1, 2, \ldots, N-1$$

So $$\frac{N}{2}\left[kd\cos\theta_n + \beta\right] = \pm n\pi$$

$$\Rightarrow kd\cos\theta_n + \beta = \pm \frac{2n\pi}{N}$$

$$\therefore \cos\theta_n = \frac{\lambda}{2\pi d}\left(\pm\frac{2n\pi}{N} - \beta\right)$$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left\{-\beta \pm \frac{2n\pi}{N}\right\}\right]$$

So if number of antennas increase, number of nulls increases.

For maximum $\sin \frac{\psi}{2} \Big|_{\theta = \theta_m} = 0$

$$\frac{\psi}{2} = \pm m\pi \quad , \quad m = 0, 1, 2, \dots$$

$$\frac{1}{2}(kd\cos\theta_m + \beta) = \pm m\pi$$

$$\cos\theta_m = \frac{\lambda}{2\pi d}(-\beta \pm 2m\pi)$$

$$\therefore \theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left\{-\beta \pm 2m\pi\right\}\right]$$

NOTE: For half power point $\frac{\sin x}{x} = 0.707$ at $x = \pm 1.391$

$$\therefore \frac{N\psi}{2}\Big|_{\theta = \theta_h} = \pm 1.391$$

$$\frac{N}{2}(kd\cos\theta_h + \beta) = \pm 1.391$$

$$\therefore \theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left\{-\beta \pm \frac{2.782}{N}\right\}\right]$$

Antenna Polarization

(Only in far field region)

- It is the polarisation associated with electric field radiated by antenna

→ Orientation of electric field (behaviour of tip of $\vec{E}$ field vector as a function of time)

$$\vec{E} = E_\theta \hat{a}_\theta + E_\phi \hat{a}_\phi$$

1) Linear Polarization : Tip of the electric field vector moving along a straight line.

$E_\theta = A e^{j\alpha}$ $\qquad E_\phi = B e^{j\beta}$

$$\vec{\mathcal{E}}(r,\theta,\phi,t) \triangleq Re\{\vec{E} e^{j\omega t}\}$$
$$= Re\{(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) e^{j\omega t}\}$$
$$= Re\{(\hat{a}_\theta A e^{j\alpha} + \hat{a}_\phi B e^{j\beta}) e^{j\omega t}\}$$
$$= Re\{A e^{j(\omega t+\alpha)} \hat{a}_\theta + B e^{j(\omega t+\beta)}\}$$
$$\therefore \vec{\mathcal{E}}(r,\theta,\phi,t) = \hat{a}_\theta A \cos(\omega t+\alpha) + \hat{a}_\phi B \cos(\omega t+\beta)$$

$E_\phi = 0$: Infinitesimal dipole

$$\therefore \vec{\mathcal{E}}(r,\theta,\phi,t) = A\cos(\omega t+\alpha)\hat{a}_\theta$$

$$E_\theta = \frac{j\eta k I_0 l \sin\theta\, e^{-jkr}}{4\pi r} \qquad I_0 = \text{Phasor} = |I_0| e^{j\alpha}$$

$$= j \frac{\eta k |I_0| l}{4\pi} e^{-j(kr-\alpha)} \frac{\sin\theta}{r}$$

$$= jE_0\, e^{-j(kr-\alpha)} \frac{\sin\theta}{r}$$

$$\vec{\mathcal{E}}(r,\theta,\phi,t) = Re\left\{ \frac{jE_0 \sin\theta}{r} e^{-j(kr-\alpha)} e^{j\omega t} \right\}$$

$$= Re\left\{ \frac{jE_0 \sin\theta}{r} e^{j(\omega t-(kr-\alpha))} \right\}$$

$$= -\frac{E_\theta \sin\theta}{r} \sin[\omega t - (kr - \alpha)]$$

For far field region without loss of generality:

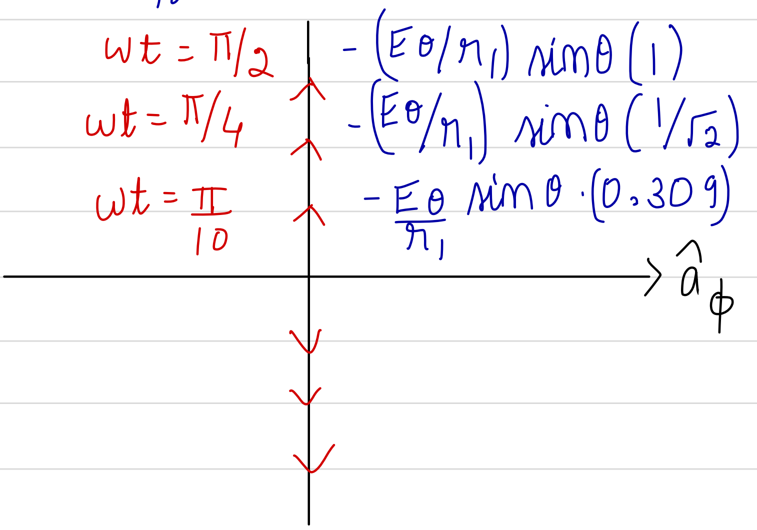
we assume, $kr_1 - \alpha = m(2\pi)$ where $m \in \mathbb{Z}$

$r_1 \rightarrow$ point in far field region

At $r = r_1$

$$\vec{\mathcal{E}}(r_1, \theta, \phi, t) = -\frac{E_\theta \sin\theta}{r_1} \sin(\omega t)$$

when $\omega t = \frac{\pi}{10}$ $\sin\omega t = 0.309$



$\Rightarrow$ Linear polarisation

2) Circular polarisation :

$A = B$ $\alpha - \beta = \pi/2$ $\Rightarrow$ $\beta = \alpha - \pi/2$

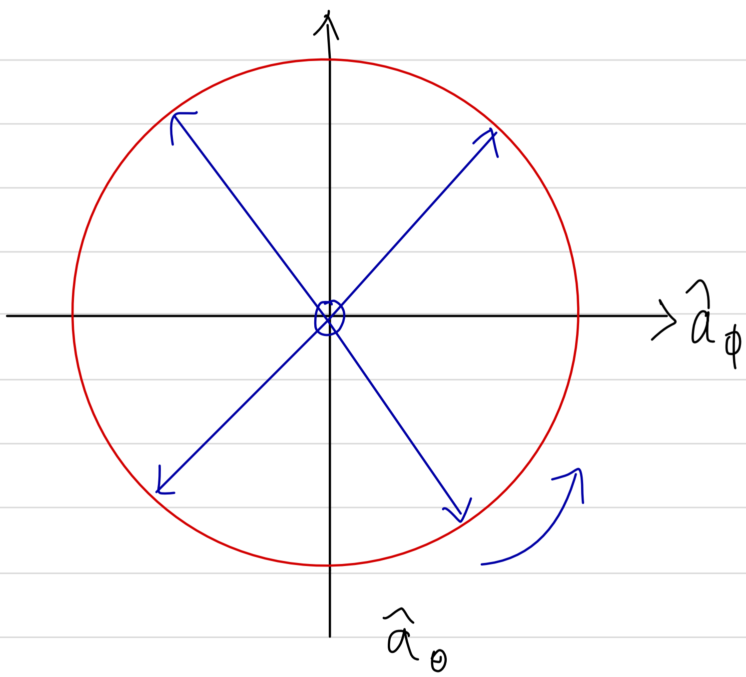
$$\vec{\mathcal{E}}(r, \theta, \phi, t) = \hat{a}_\theta A \cos(\omega t + \alpha) + \hat{a}_\phi B \cos(\omega t + \beta)$$

$$= A[\cos(\omega t + \alpha)\, \hat{a}_\theta + \cos(\omega t + \alpha - \pi/2)\, \hat{a}_\phi]$$

$$= A\left[\cos(\omega t + \alpha)\, \hat{a}_\theta + \cos\left(\frac{\pi}{2} - (\omega t + \alpha)\right) \hat{a}_\phi\right]$$

$$\therefore \vec{\mathcal{E}}(r, \theta, \phi, t) = A[\hat{a}_\theta \cos(\omega t + \alpha) + \hat{a}_\phi \sin(\omega t + \alpha)]$$

	$\omega t + \alpha$	$\vec{\mathcal{E}}(r, \theta, \phi, t)$
1>	0	$A\hat{a}_\theta$
2>	$\pi/4$	$\frac{A}{\sqrt{2}}(\hat{a}_\theta + \hat{a}_\phi)$
3>	$\pi/3$	$\frac{A}{2}(\hat{a}_\theta + \sqrt{3}\hat{a}_\phi)$
4>	$\pi/2$	$A\hat{a}_\phi$
5>	$3\pi/4$	$\frac{A}{\sqrt{2}}(-\hat{a}_\theta + \hat{a}_\phi)$
6>	π	$-A\hat{a}_\theta$
7>	$5\pi/4$	$\frac{-A}{\sqrt{2}}(\hat{a}_\theta + \hat{a}_\phi)$
8>	$3\pi/2$	$-A\hat{a}_\phi$
9>	$7\pi/4$	$\frac{A}{\sqrt{2}}(\hat{a}_\theta - \hat{a}_\phi)$
10>	2π	$A\hat{a}_\theta$



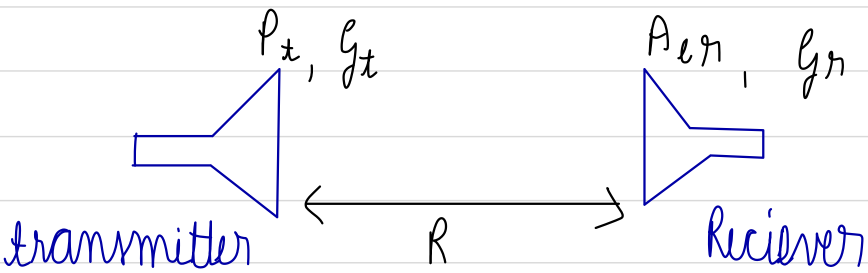
This is Left circular Polarisation (Counter clockwise) (LCP)

If $\alpha = \beta - \pi/2 \Rightarrow$ Right circular polarization (RCP) $\rightarrow$ clockwise

3) Elliptical Polarization:

$A \neq B$. No phase relationship is imposed.

Frie's transmission equation



$W \Rightarrow$ Power density at transmit antenna
$G_t \Rightarrow$ Gain factor of transmit antenna

$$W = \frac{P_t \cdot G_t}{4\pi R^2}$$

G_r = Gain of reciever
A_{er} = Aperture of reciever

$$P_r = A_{er} W = \frac{A_{er} P_t G_t}{4\pi R^2}$$

$$A_{er} = \frac{G_r \lambda^2}{4\pi} \qquad \therefore P_r = \frac{P_t G_t G_r \lambda^2}{16 \pi R^2}$$

$$\therefore P_r = P_t \cdot G_t \cdot G_r \left(\frac{\lambda}{4\pi R}\right)^2$$

$\hookrightarrow$ Free space loss