CONTROL SYSTEMS

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Metric space

Matrix space is a set of Metrics
Consider a set of vectors X and distance be d.

then $d: X \times X \longrightarrow \mathbb{R}^+$,
It satisfies

1> $d(x, y) \geq 0 \quad \forall \; x, y \in X$
2> $d(x, y) = d(y, x)$
3> $d(x, y) = d(y, x) = 0$ iff $x = y$
4> $d(x, y) + d(y, z) \geq d(x, z)$

Ex: 1> $X = \mathbb{R}$, $\quad d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ $\quad d(x, y) = |x - y|$

2> $X = \mathbb{R}^2$, $\quad d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$,

$$d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad x = (x_1, x_2), \; y = (y_1, y_2)$$

$l_1: |x_1 - y_1| + |x_2 - y_2|$

$l_2: \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$l_3: \sqrt[3]{(x_1 - y_1)^3 + (x_2 - y_2)^3}$

$\vdots$

$l_\infty: \max(|x_1 - y_1|, |x_2 - y_2|)$

3> Any set X

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Metric subspace

$Y \subset X$

(Y, d_Y) (X, d_X)

Y is a metric subspace of X

Ex: $(\mathbb{R}, d)$ is a metric subspace of $(\mathbb{C}, d)$ where $\mathbb{C}$ is a set of complex numbers.

$d(x, y) \leq \max\{d(x,z), d(y,z)\}$ $\forall x, y \in X$ defined on certain set X. This is not true for all X.

NOTE • Set of all $m \times n$ matrices over some field is a metric space with respect to distance.

$d(A, B) = \text{rank}(B - A)$

Q1> Check whether $d(x, y) = |\log(y/x)|$ can be a distance function.

Ans Here $d(x, y) \neq d(y, x)$. $\therefore$ It cannot be distance function.

NOTE • Any normed vector space defined as $d(x, y) = \|x - y\|$ is a metric space.

Properties of norm in vector space

$\| \ \| : V \rightarrow \mathbb{R}$

1> $\|\vec{x}\| \geq 0$ $\forall \vec{x} \in V$

$= 0$ iff $\vec{x} = \vec{0}$

2) $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

3) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Inner Product space:

$\langle \vec{x}, \vec{y} \rangle \triangleq \sum_i x_i y_i$

$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{F}$

V : Set of vector
$\mathbb{F}$: Set of scalars

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

NOTE:
- A normed linear space $(X, \| \, \|)$ is a linear space equipped with norm
- A normed linear space is a metric space when defined as $d(x, y) = \|x - y\|$
- Translation invariant:
  $\forall \, x, y, z \in X$ & $\alpha \in \mathbb{F}$
  i) $d(x+z, y+z) = \|x - y\|$
  ii) $d(\alpha x, \alpha y) = |\alpha| \, \|x - y\|$, $\forall \, \alpha \in \mathbb{F}$

Basis: Independent vectors that can be spanned throughout space.

No. of basis vector = No of dimension.

Polynomials: $\mathbb{F}(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$p(x) + (-p(x)) = \theta$

$\theta \rightarrow$ zero polynomial

$F(x) = a_0 \equiv \mathbb{R}^1$

$F(x) = a_0 + a_1 x \equiv \mathbb{R}^2$

Cardinality: No. of elements in a set.

Sequence & series

If $s_1, s_2,$ are terms in sequence then: Absolute summable series:

$$\ell_1 : \left\{ f : S \to R \quad \sum_{j=1}^{\infty} |f(s_j)| < \infty \right\}$$

$$\ell_2 : \left\{ f : S \to R \quad \sum_{j=1}^{\infty} |f(s_j)|^2 < \infty \right\}$$

$$L_1 : \left\{ f : \int_I |f(t)|\, dt < \infty \right\}$$

$$L_2 : \left\{ f : \int_I |f(t)|^2\, dt < \infty \right\}$$

Fourier transform

Consider $L_2[\mathbb{R}, \mathbb{C}] = \left\{ f : \int_{\mathbb{R}} |f(t)|^2\, dt < \infty \right\}$

$$\mathcal{F}(\omega) = \int_{\mathbb{R}} f(t)\, e^{-j\omega t}\, dt \quad , \quad \omega \in \mathbb{R}$$

Now let Ω be a +ve real number $(\mathbb{R}^+)$

$B_\Omega = \{ f \in L_2[\mathbb{R} ; \mathbb{C}] ; \ \mathcal{F}(\omega) = 0 , \ |\omega| \geq 0 \}$ support $[-\Omega, \Omega]$

with B_Ω as a vector space over $\mathbb{C}$

$\mathbb{C} \rightarrow$ set of complex number

$L_2 \rightarrow$ Set of energy signal

$B_\Omega \rightarrow$ Set of band limited signals

Discrete bandlimited signal :

$x = \{ k \in \mathbb{Z} : x(k) \}$ where $x \in \mathbb{C}$

$x = \{ k_1, k_2 \in \mathbb{Z} : x(k_1, k_2)$

Eg : $\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$

$\delta(k_1, k_2) = \begin{cases} 1 & (k_1, k_2) = (0, 0) \\ 0 & \text{otherwise} \end{cases}$

Linear systems

Let T be a linear transform from V to W

i.e $T: V \rightarrow W$, then

1) $T(x_1 + x_2) = T(x_1) + T(x_2) \quad \forall\, x_1, x_2 \in V$

2) $T(\alpha x) = \alpha T(x) \quad \forall\, x \in V \ \& \ \alpha \in \mathbb{C}$

$T(\alpha x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall\, x_1, x_2 \in V \ \& \ \alpha \in \mathbb{C}$

$\because T(0x) = 0T(x) \Rightarrow T(0) = 0$

Transformation:

$V = \mathbb{F}^n \qquad W = \mathbb{F}^m \qquad A \in \mathbb{F}^{m \times n}$

1) $T_A : V \to W$

$Tx = Ax \ , \ x \in V$

2) $T_L x = AX \ , \ X \in V$

$T_R x = XA$

$T_{LR}(X) = PXP^{-1}$ = Similarity transfer

$= P^{-1}XP$

Consider polynomial $\mathbb{F}(x)$:

$V = \mathbb{F}(x)$

If $p \in V$ and linear operator $D \triangleq \frac{d}{dx}$

$D(p) = \frac{dp}{dx}$. It is linear operator

Similarly $D^2 = \frac{d^2 p}{dx^2}$ is also linear operator.

In general $D^n = \frac{d^n}{dx^n}$ is also linear operator

$$\int_a^b (\alpha_1 f_1 + \alpha_2 f_2)\, dx = \alpha_1 \int_a^b f_1\, dx + \alpha_2 \int_a^b f_2\, dx$$

Definite integrals is linear.

$y = mx \rightarrow$ linear system

$y = mx + c \rightarrow$ Affine system

<u>System</u>: Operator on signal space.

Discrete time system

$x : \mathbb{Z} \longrightarrow \mathbb{R}$

$$x(k) = \sum_{m=-\infty}^{\infty} x(m)\, \delta(k-m)$$

Translation invariant: $T(x) = y$

$\Rightarrow T(x(k-m)) = y(k-m)$

where $k, m \in \mathbb{Z}$ and $\forall x \in V$

Convolution:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$y(k) = \sum_{m=-\infty}^{\infty} x(k)\, h(k-m)$$

Impulse response, $h(t)$ is also called point spreading function.

- consider set of all signal : $\mathbb{R}^{\mathbb{Z}}$

i.e $f : \mathbb{Z} \to \mathbb{R}$

i.e discrete signal space, $f(k) \in \mathbb{R}$, $k \in \mathbb{Z}$

$\mathbb{R}^{\mathbb{Z} \times \mathbb{Z}} \Rightarrow f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$

$f(m, n) \in \mathbb{R}$, $(m, n) \in \mathbb{Z}^2$

ADD: $\mathbb{R}^{Z \times Z} \times \mathbb{R}^{Z \times Z} \longrightarrow \mathbb{R}^{Z \times Z}$

Range operation (matrix addition)

Shift: $\mathbb{R}^{Z} \longrightarrow \mathbb{R}^{Z}$

$(\text{Shift } f)(m) = f(m-1)$

Properties of algebraic signal (addition)

1) Associative property: $f+(g+h) = (f+g)+h$

2) Zero signal property: $f+0 = 0+f = f$

3) Negative signal property: $f+(-f) = -f+f = 0$

4) Commutative property: $f+g = g+f$

Signals that satisfy above property belong to abelian group.

Scalar multiplication: $a, b \in \mathbb{R}$, $f \in V$

1) Associative: $(ab)f = a(bf)$

2) unity preservation: $1 \cdot f = f$ $\quad 1 \in \mathbb{R}$

3) distribution property: $a(f+g) = af + fg$

$(a+b)f = af + bf$

$\mathbb{R}^{Z}$ of signals form vector space over field $\mathbb{R}$

Multiplication :

1> Associative : $(fg)h = f(gh)$

2> Distributive : $(f+g)h = fh + gh$

3> commutative : $fg = gf$

4> Unity signal property : $1_2 \cdot f = f \cdot 1_2 = f$
(1_2 is a signal)

$\mathbb{R}^{\mathbb{Z}}$ is commutative ring with identity

NOTE : Field is a group with respect to addition and multiplication.

Systems :

Eigen analysis :

$$\vec{x} \longrightarrow \boxed{A} \longrightarrow A\vec{x} = \lambda\vec{x}$$

Complex exponential is an eigenfunction.
eg : ke^{st} where both s & k can be complex numbers.

NOTE :

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Space of bounded signals

$x \in V$, $\exists\ M_x < \infty$ $\exists\ |x_k| \leq M$ $\forall\ k \in Z$

Convolution $(*)$

$* : \ell_1 \times \ell_1 \longrightarrow \ell_1$

1> Commutative property: $f * g = g * f$

2> Associative property: $f * (g * h) = (f * g) * h$

3> Distributive property: $f * (g+h) = f * g + g * h$
$(f+g) * h = f * h + g * h$

4> Identity (unity) property: $f * \delta = \delta * f = f$

5> Scalar commutativity: $\alpha (f * g) = (\alpha f) * g = f * (\alpha g)$

6> $|f * g|_{\ell_1} \leq |f|_{\ell_1} \cdot |g|_{\ell_1}$

Control systems

1> $y'' + 3y' + 2y = 2x' - x$. Find $H(s)$, if $f(0) = 0$

ZSR : Zero state response
ZIR : Zero initial response

IVP : Initial value problems
BVP : Boundary value problems

Ans $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{2s-1}{s^2+3s+2}$

NOTE: Total response, TR = ZSR + ZIR

$$s^2 Y(s) - s\,y(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s)$$
$$= 2sX(s) - x'(0) - X(s)$$

$$\Rightarrow Y(s)(s^2+3s+2) = X(s)(2s-1)$$
$$\therefore \frac{Y(s)}{X(s)} = \frac{2s-1}{s^2+3s+2}$$

NOTE: $y(t) = \int_{-\infty}^{\infty} x(t) \cdot h(t-\tau)\, d\tau$

if $x(t) = e^{st}$, then :
$y(t) = e^{st} \cdot H(s)$

Laplace Transform

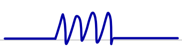
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Inverse laplace transform

$$x(t) = \frac{1}{2\pi j} \oint_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

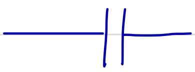
System performance of RLC

Basic elements of electrical system:
Voltage (V), current (I), charge (q), flux (ϕ)

 $\Rightarrow$ $V = IR$

 $\Rightarrow$ $\phi = LI$

$$V = L\frac{dI}{dt} = \frac{d\phi}{dt}$$

 $\Rightarrow$ $q = CV$ $\Rightarrow$ $I = C\frac{dV}{dt}$

State variables: I through inductor, V through capacitor

Q1) $x(t) = \sin\left(t + \frac{\pi}{3}\right) + 2\sin 2t$. Find frequency response

Ans $X(s) = \frac{1}{s^2+1} e^{-\pi/3\, s} + \frac{4}{s^2+4}$

Q2) $x[n] = [2\ \underset{\uparrow}{1}\ 0\ 3\ -1]$. Find z transform.

Ans $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 2z + 1 + \frac{3}{z^2} - \frac{1}{z^3}$

R.O.C ; $0 < |z| < \infty$

If $$y(t) = \int_{-\infty}^{\infty} x(t)\, h(t-\tau)\, d\tau$$

$$L(y(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, e^{-st}\, d\tau\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\, H(s)\, e^{-s\tau}\, d\tau$$

$$= H(s) \cdot \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau$$

$$Y(s) = H(s) \cdot X(s)$$

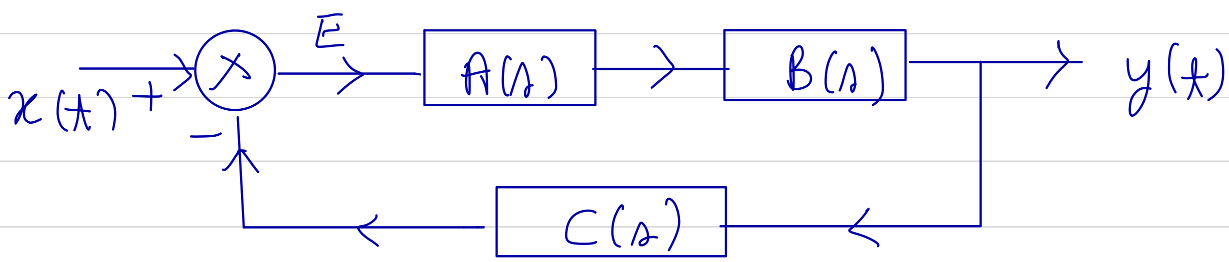
To get $y(n) = x(n) + x(n-1)$

$h(n) = \delta(n) + \delta(n-1)$

$y(n) = x(n) - x(n-1) \Rightarrow h(n) = \delta(n) - \delta(n-1)$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \Rightarrow h(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

Q1) Minimise the following:



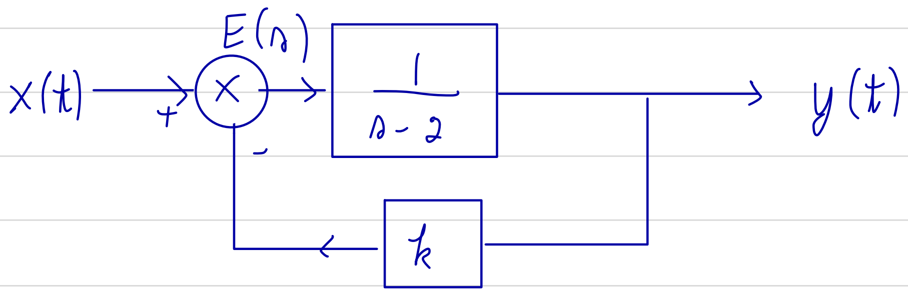
Ans $E(s) = X(s) - C(s) \cdot Y(s)$

$Y(s) = A(s) \cdot B(s) \cdot E(s)$

$$\therefore\ Y(s) = A(s)\,B(s)\,[X(s) - C(s)\,Y(s)]$$

$$\therefore\ \frac{Y(s)}{X(s)} = H(s) = \frac{A(s)\,B(s)}{1 + A(s)\,B(s)\,C(s)} = \frac{AB}{1+AB}$$

ii>



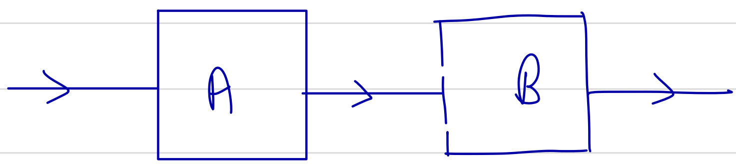
$$E(s) = X(s) - k\,Y(s)$$

$$Y(s) = \frac{E(s)}{s-2} = \frac{1}{s-2}\left(X(s) - k\,Y(s)\right)$$

$$\therefore\ \frac{Y(s)}{X(s)} = \frac{\dfrac{1}{s-2}}{1 + \dfrac{k}{s-2}} = \frac{1}{s-(2-k)} = H(s)$$

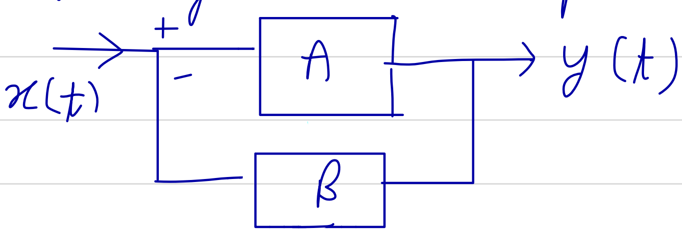
Here k = root locus

NOTE:



→ To prevent loading in this case input impedance of B should be high (ideally ∞) & output impedance of A should be very low, ideally 0.

→ Also a buffer can be connected externally to prevent loading.



$$y(t) = \frac{A}{1+A\beta}\,x(t)$$

If $A \gg 1$, $h(t) = 1/\beta$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Q1> Find $x(t)$ if is $X(S) = \frac{1}{S+3}$

Ans $\mathcal{L}^{-1}(X(S)) = \frac{1}{S+3}$

$$= e^{-3t} \; U(t)$$

$$\therefore x(t) = \begin{cases} e^{-3t} \; U(t) & \text{for } Re(S) > -3 \\ -e^{-3t} \; U(-t) & \text{for } Re(S) < -3 \end{cases}$$

ii> $X(s) = \frac{3s+5}{s^2+3s+2}$ for $-2 < Re(s) < -1$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

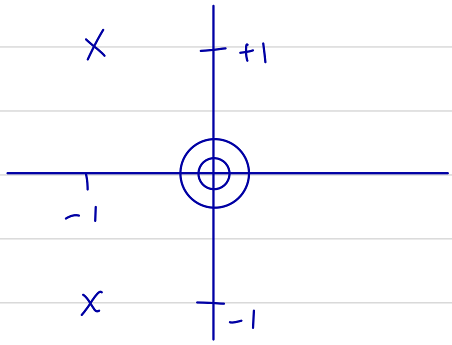
$$= \frac{2}{s+1} + \frac{1}{s+2}$$

$$x(t) = 2e^{-t} \; U(t) + e^{-2t} \; U(t)$$

iii> $X(s) = \frac{1}{s^2+4s+40} = \frac{1}{(s+2)^2+6^2}$

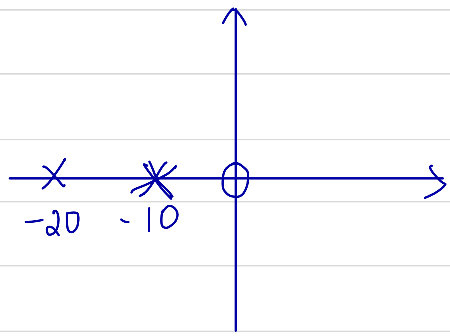
$$x(t) = \frac{1}{6} e^{-2t} \sin 6t \; U(t)$$

Q2> Give $H(s)$ for: i>



Ans
$$H(s) = \frac{k s^2}{(s+1-j)(s+1+j)}$$

ii>



(for $k=1$)

Ans
$$H(s) = \frac{k s}{(s+10)^2 (s+20)}$$

$$= \frac{(-1/5)}{s+20} + \frac{1/5}{s+10} - \frac{1}{(s+10)^2}$$

Zero input response

$$ZIR = \left(\frac{-e^{-20t}}{5} + \frac{e^{-10t}}{5} - t e^{-10t}\right) u(t)$$

Q3> Find ZSR (zero state response) for $y' + 3y = 5x$ when $x(t) = \sin 6t \; u(t)$

Ans

$$H(s) = \frac{5}{s+3}$$

$$X(s) = \frac{6}{s^2+6^2}$$

$$Y(s) = H(S) \cdot X(S)$$

$$= \frac{30}{(s+3)(s^2+6^2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+36}$$

$\therefore y(t) = \left(Ae^{-3t} + B\cos 6t + C\sin 6t\right) u(t)$

ZSR = transient response + steady state response

NOTE: • Decay is a characteristic of system which depends on pole of system.
• when input is $\delta(t)$, it is treated as ZIR.

ZIR = Zero input response
ZSR = Zero state response

Differential equation

General form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = u(t)$$

or $(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y = u(t)$

Homogeneous system: $\{f_1, f_2, f_3, \ldots f_n\}$ is said to be independent if,
$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 0$

Q1) $y'' + 3y' + 2y = u$, $y(0) = 0$, $y'(0) = 1$, $u = 1$

$(D^2 + 3D + 2) = 0 \Rightarrow D = -1$ or $D = -2$

$\therefore e^{-t}$ & e^{-2t}

$W(t) = \alpha_1 e^{-t} + \alpha_2 e^{-2t}$

$$y(t) = \int_0^t w(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_0^t \alpha_1 e^{-(t-\tau)} + \alpha_2 e^{-2(t-\tau)} \quad d\tau$$

$$= \alpha_1 e^{-t} \int_0^t e^{\tau} d\tau + \alpha_2 e^{-2t} \int_0^t e^{2\tau} d\tau$$

$$= \alpha_1 e^{-t} (e^{t} - 1) + \alpha_2 e^{-2t} (e^{2t} - 1)$$

$$= \alpha_1 (1 - e^{-t}) + \alpha_2 (1 - e^{-2t})$$

$y'(0) = 1 \Rightarrow \alpha_1 + 2\alpha_2 = 1$

$\rightarrow y_p(t) = t$

Then $2t = 1 \quad \therefore \quad t = 1/2$

$\therefore \quad y(t) = \alpha_1 e^{-t} + \alpha_2 e^{-2t} + 1/2$

$y(0) = 0 \Rightarrow \alpha_1 + \alpha_2 = -1/2$

$y'(0) = 1 \Rightarrow -\alpha_1 - 2\alpha_2 = 1$

$\therefore \alpha_1 = 0 \qquad \alpha_2 = 1/2$

$\therefore \quad y(t) = \frac{1}{2}(e^{-2t} + 1)$ //

NOTE: Total response = Transient response + steady state response.

Transient response is due to initial state of system. It decays to zero after long time.

→ Steady state response is the response that stays same.

→ Fourier transformation is a signal analysis tool because $j\omega$ is a steady state response.

→ For system we deal with unilateral laplace transform i.e (0 to ∞ instead of −∞ to ∞). So that the transient response can be offered.

→ $$X(\sigma, \omega) = \int_0^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

Converting this to discrete:

$$X(\sigma, \omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-\sigma n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, r^{n}\, e^{-j\omega n}$$

$$z = r e^{j\omega} = |z| e^{j\omega}$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

NOTE: If

$$x(n) \longleftrightarrow X(z)$$

$$x(n-1) \longleftrightarrow z^{-1} \cdot X(z)$$

$$x(n-k) \longleftrightarrow z^{-k} \cdot X(z)$$

Transfer function of unit delay $= \frac{1}{z}$

Delay is a linear operator.

Transfer function, $H(z) = \frac{Y(z)}{X(z)}$

Order of $Y(z)$ can be less than order of $X(z)$ unlike the case in laplace domain.

NOTE: $y' + ay = bu(t)$ has solution:

$$y(t) = y(0)\, e^{-at} + \int_0^t e^{-a(t-\tau)}\, b\, u(\tau)\, d\tau$$

Q> $\frac{dy}{dt} + k y(t) = x(t)$. Discretize this equation.

Ans $$\lim_{T_s \to 0} \frac{y(n) - y(n-1)}{nT_s} + k\,y(n) = x(n)$$

NOTE:

Q) Use laplace transform to solve:

$y'' + 5y' + 6y = u$ with zero initial condition.

Ans $s^2 Y(s) - s y(0) - y'(0) + 5(sY(s) - y(0)] + 6Y(s) = U(s)$

$Y(s)(s^2 - 5s + 6) - s y(0) - y'(0) - 5y(0) = U(s)$

$$Y(s) = \frac{U(s) + (s+5)\,y(0) + y'(0)}{s^2 + 5s + 6}$$

$$\therefore \frac{Y(s)}{U(s)} = H(s) = \frac{1}{s^2 + 5s + 6}$$

Here $\frac{(s+5)\,y(0) + y'(0)}{s^2 + 5s + 6}$ is free response

and $\frac{1}{s^2 + 5s + 6}$ is forced response

Step response : $Y(s) = \frac{1}{s \cdot (s+2) \cdot (s+3)}$

$$Y(s) = \frac{6}{s} + \frac{-2}{s+2} + \frac{3}{s+3}$$

$$\therefore y(t) = \left(6 - 2e^{-2t} + 3e^{3t}\right) U(t)$$

NOTE: • Transfer function is the laplace transform of the output divided by the laplace transform of the input with zero initial with zero initial condition.

• For time limited signal, fourier transform is band unlimited signal

• Similarly for band limited signal inverse fourier transform is time unlimited signal

- Proper transfer function (degree of denominator > degree of numerator) are causal.

Q1> $y'' + y = u$. Calculate step response & $u(t) = \cos t$

Q2> $y'' + y' - 2y = u$. Check BIBO stability. (Bounded input & bounded output)

NOTE
- BIBO $\Leftrightarrow$ pole is in left hand side of s plane.
- If the pole is closer to imaginary axis then it is dominant pole & it decreases stability.

Q1> Ans>

$$y'' + y = u$$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s)$$

$$Y(s)(s^2+1) = U(s)$$

$$\therefore Y(s) = \frac{U(s)}{s(s^2+1)}$$

Step response $Y_1(s) = \dfrac{1}{s(s^2+1)}$

$$Y(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$Y(s) = \frac{1}{s} + \frac{-s}{s^2+1}$$

$$y(t) = (1 - \cos t)\, u(t)$$

$$Y_2(s) = \frac{s}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$$

Q 2)

$$s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) - 2 Y(s) = U(s)$$

$$Y(s)(s^2 + s - 2) = U(s)$$

$$\therefore H(s) = \frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)}$$

Poles are -2 & 1

Since 1 pole lies right of imaginary axis, the system is not BIBO.

$\rightarrow Y(s) = P(s)\ U(s)$ ($P(s)$ - impulse response)

$$y(t) = \int_0^t p(t-\tau)\, u(\tau)\, d\tau$$

$$|u(t)| \leq M < \infty \quad \forall t$$

$$|y(t)| = \left| \int_0^t p(t-\tau)\, u(\tau)\, d\tau \right|$$

$$\leq \int_0^t |p(t-\tau)\, u(\tau)|\, d\tau$$

$$\Rightarrow |y(t)| \leq M \int_0^t |p(t-\tau)|\, d\tau$$

$\rightarrow \lim_{t \to \infty} |p(t)| = 0$ for response to converge & to be stable or else it diverges at $t \to \infty$

$\rightarrow P(s) = \frac{N(s)}{D(s)}$, order $= n$

K distinct poles $\sum_{i=1}^{k} \mu_i = n$

pole $s = s_i$ which repeats μ_i times,
$i = 1, 2, \ldots k$

$$P(s) = \frac{N(s)}{(s+s_1)^{\mu_1}(s+s_2)^{\mu_2}\ldots(s+s_k)^{\mu_k}}$$

$$= \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} \left(\frac{C_{lm}}{(s+s_i)^{m+1}} \right)$$

$$P(t) = \sum_{\langle l \rangle} \sum_{\langle m \rangle} C_{lm}\, t^m e^{s_i t}$$

$$|P(t)| = \left| \sum_{\langle l \rangle} \sum_{\langle m \rangle} C_{lm}\, t^m e^{s_i t} \right|$$

$\lim_{t \to \infty} |p(t)|$ converges when $\sigma_l < 0 \ \forall l$.

So system is stable only when poles lie in left half of the s-plane.

eg 1) $$P(s) = \frac{1}{(s+1)(s+10)} = \frac{1}{9} \frac{(s+10) - (s+1)}{(s+1)(s+10)}$$

$$= \frac{1}{9}\left[\frac{1}{s+1} - \frac{1}{s+10}\right]$$

$\therefore y(t)$ (impulse response) $= \frac{1}{9}\left[e^{-t} - e^{-10t}\right] u(t)$

Eg 2: $P(s) = \dfrac{s+2}{(s+1)(s+10)}$. Find impulse response.

$$= \frac{1}{9}\left(\frac{1}{s+1} + \frac{8}{s+10}\right)$$

$$y(t) = \frac{1}{9}\left(e^{-t} - 8e^{-10t}\right)U(t) \Rightarrow \text{impulse response}$$

Eg 3: $P(s) = \dfrac{s-2}{(s+1)(s+10)}$. Find unit step response

Ans $Y(s) = P(s) \cdot U(s)$

$$= \frac{s-2}{s(s+1)(s+10)}$$

$$Y(s) = \frac{-1}{5s} + \frac{1}{3(s+1)} - \frac{2}{15(s+10)}$$

$\therefore$ $y(t)$ (unit step response) $= \left[\dfrac{-1}{5} + \dfrac{1}{3}e^{-t} - \dfrac{2}{15}e^{-10t}\right]U(t)$

as $t \to \infty$, $y(t) \to -1/5$

NOTE:
- If zeroes are on left side → minimum phase response
- If zeroes are on right side → maximum phase response

Time delay system

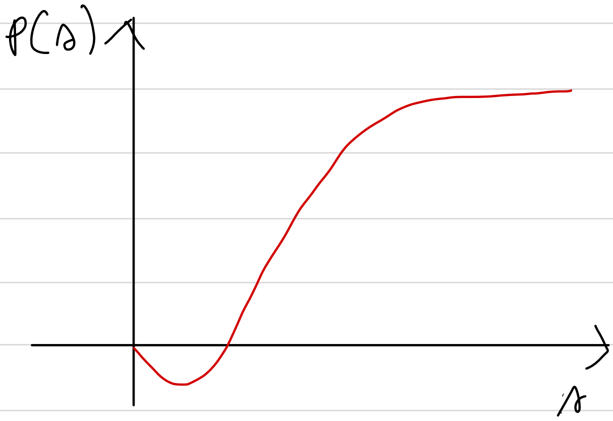
$$y' + y = u(t - T_d)$$

$$P(s) = \frac{e^{-T_d s}}{s+1}$$

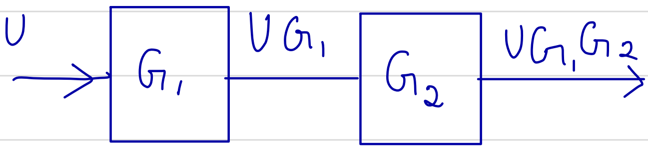
approximation 1; $e^{-T_d s} \approx \dfrac{1}{T_d s + 1}$

approximation 2; $e^{-T_d s} = \dfrac{e^{-\frac{T_d s}{2}}}{e^{\frac{T_d s}{2}}} = \dfrac{2 - T_d s}{2 + T_d s}$

$$\therefore \quad P(s) = \frac{2 - T_d s}{(s+1)(2 + T_d s)}$$

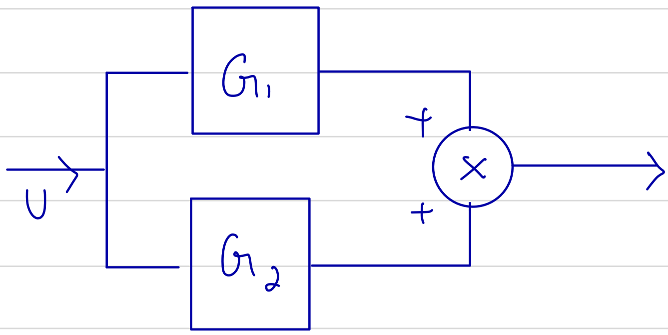


1>



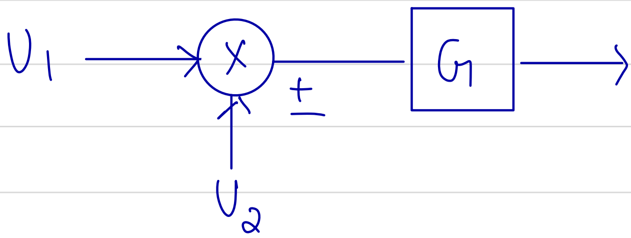
$$\Rightarrow \frac{Y}{U} = G_1 G_2$$

2>



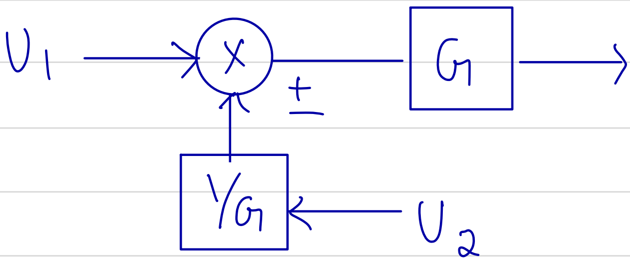
$$Y = U(G_1 + G_2)$$

3>



$$y = G(U_1 \pm U_2)$$

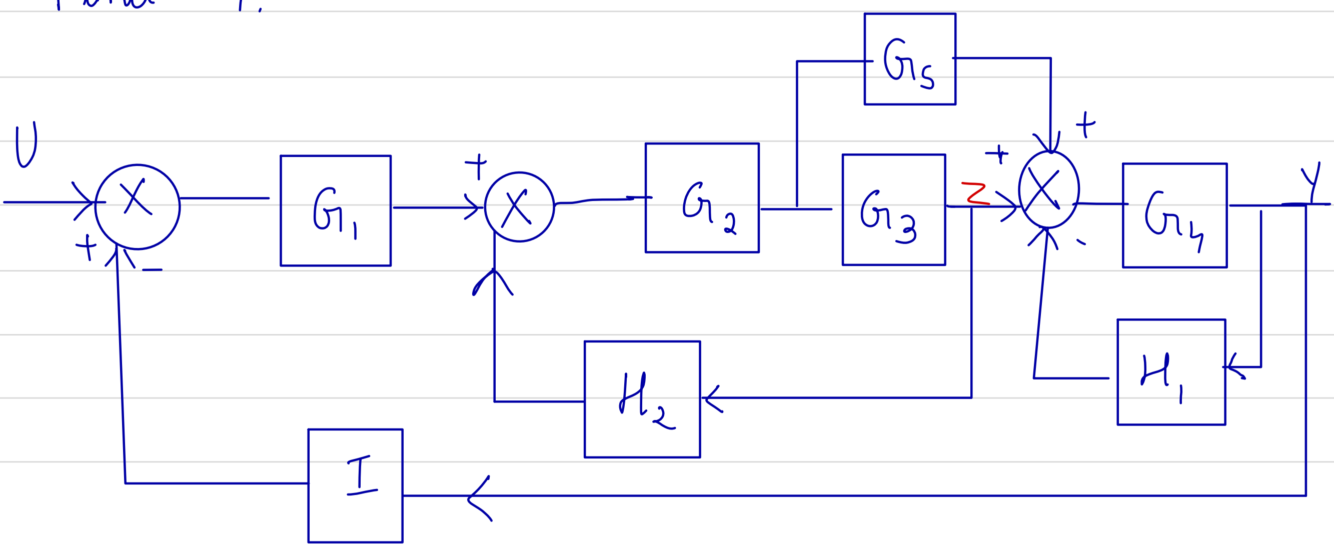
4>



$$y = GU_1 \pm U_2$$

provided $G \neq 0$

Q> Find Y.



Ans $\left(\left(U - IY\right)G_1 - ZH_2\right) G_2 G_3 = Z$

$$(U - IY) G_1 G_2 G_3 = Z(1 + H_2 G_2 G_3)$$

$$\Rightarrow \quad Z = \frac{(U - IY) G_1 G_2 G_3}{1 + H_2 G_2 G_3}$$

$$\left(Z + G_5\left(\left(U - IY\right)G_1 - ZH_2\right) G_2 - YH_1\right) G_4 = Y$$

$$G_4 Z + G_4 G_5 G_2 G_1 U - G_2 H_2 Z - G_1 G_2 G_4 G_5 IY$$
$$- YH_1 G_4 = Y$$

$$\Rightarrow Y(1 + G_1 G_2 G_4 G_5 I + G_4 H_1) = Z(G_4 + G_2 H_2)$$
$$+ UG_1 G_2 G_4 G_5$$

$$\Rightarrow Y(1 + G_1 G_2 G_4 G_5 I + G_4 H_1) = \frac{(U - IY)(G_1 G_2 G_3)(G_4 + G_2 H_2)}{1 + H_2 G_2 G_3}$$
$$+ UG_1 G_2 G_4 G_5$$

$$Y\left((1 + G_2 G_1 G_4 G_5 I + G_4 H_1)(1 + H_2 G_2 G_3) + IG_1 G_2 G_3 (G_4 + G_2 H_2)\right)$$
$$= U\left((G_1 G_2 G_4 G_5)(1 + H_2 G_2 G_3) + (G_1 G_2 G_3)(G_4 + G_2 H_2)\right)$$

$$Y = \frac{U\left((G_1 G_2 G_4 G_5)(1 + H_2 G_2 G_3) + (G_1 G_2 G_3)(G_4 + G_2 H_2)\right)}{\left((1 + G_1 G_2 G_4 G_5 I + G_4 H_1)(1 + H_2 G_2 G_3) + IG_1 G_2 G_3 (G_4 + G_2 H_2)\right)}$$

State Variable method

- $a_0 y + a_1 y^{(1)} + a_2 y^{(2)} + \dots + a_{n-1} y^{(n-1)} + a_n y^{(n)} = U(t)$

n - state variables

$x_1 = y$

$x_2 = \dot{x}_1 = y'$

$x_3 = \dot{x}_2 = y''$

$x_n = \dot{x}_{n-1} = y^{n-1}$

$\dot{x}_n = y^n$

$$y^n = \frac{1}{a_n} U(t) - (a_0 x_1 + a_1 x_2 + a_2 x_3 + \dots + a_{n-1} x_n)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0 \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/a_n \end{bmatrix} U(t)$$

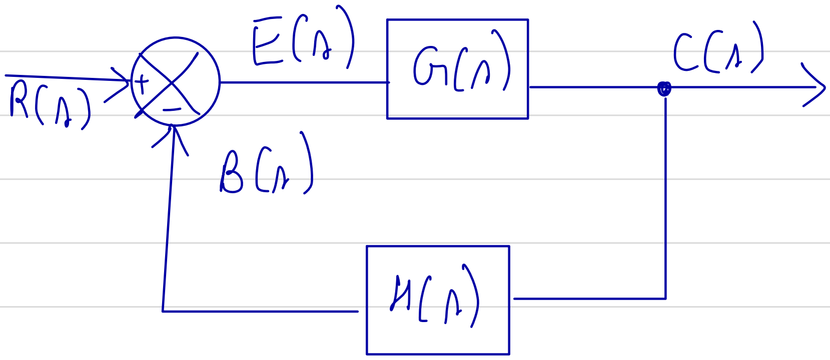
$$\frac{dX}{dt} = Ax + Bu$$

$$x(t) = e^{-At} x(0) + \int_0^t e^{-A(t-\tau)} Bu(\tau)\, d\tau$$

General equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

Q1)



Find transfer function

$R(s)$ = reference i/p
$C(s)$ = o/p signal or controlled variable
$B(s)$ = feedback signal
$E(s)$ = Actuating signal

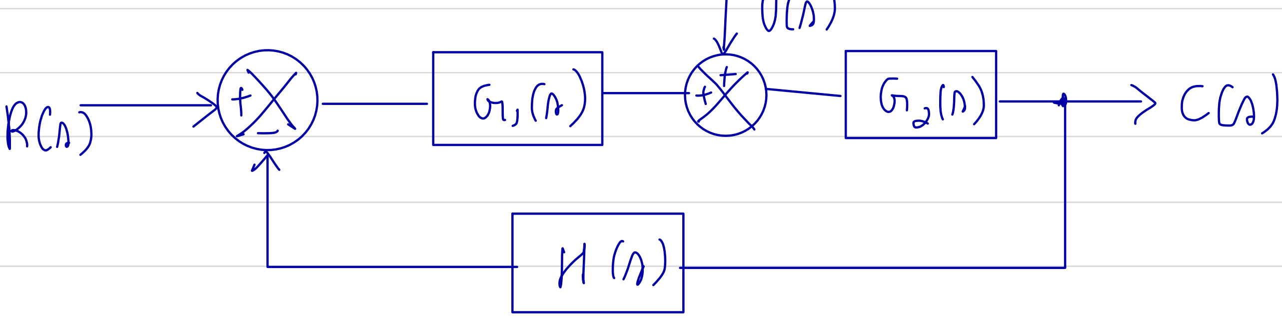
Ans $G(s)\cdot H(s) = \dfrac{C(s)}{E(s)}\cdot H(s) = \dfrac{B(s)}{H(s)}$ → loop transfer function

$T(s) = \dfrac{C(s)}{R(s)}$ → closed loop transfer function

$R(s) - H(s)\cdot C(s) = \dfrac{C(s)}{G(s)} = E(s)$

$R(s)\cdot G(s) - H(s)\cdot G(s)\cdot C(s) = C(s)$
$R(s)\cdot G(s) = C(s)[1 + H(s)\cdot G(s)]$
$\therefore\ T(s) = \dfrac{G(s)}{1+H(s)\cdot G(s)}$

Q2)



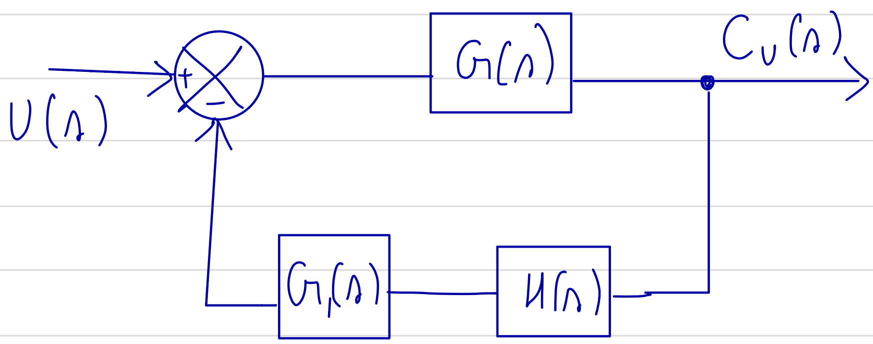
Ans using superposition, make $U(s) = 0$

$$C_R(s) = \frac{G_1(s) \cdot G_2(s)}{1 + G_1(s) \cdot G_2(s) \cdot H(s)} R(s)$$

$R(s) = 0$

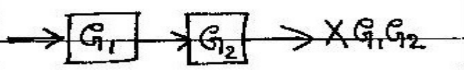
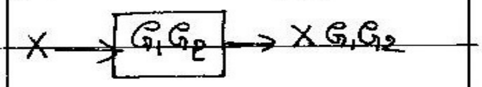
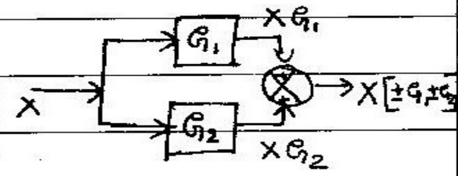
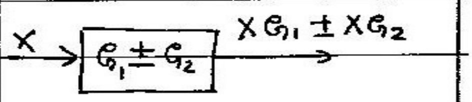
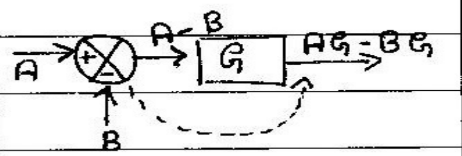
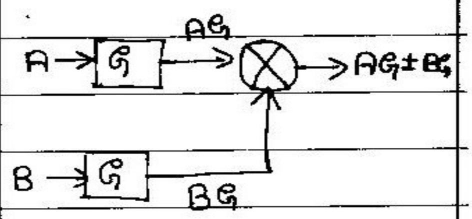
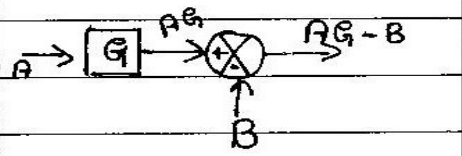
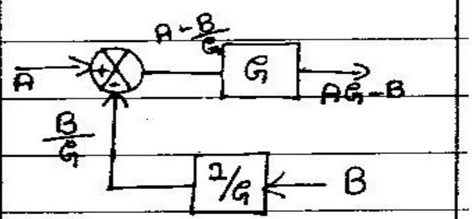
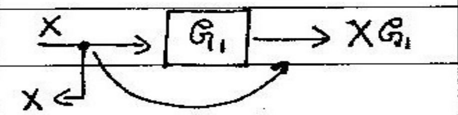
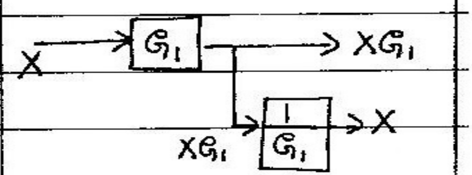
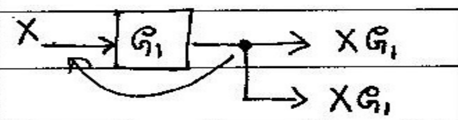
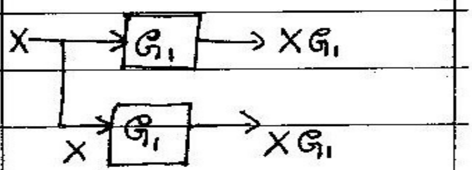
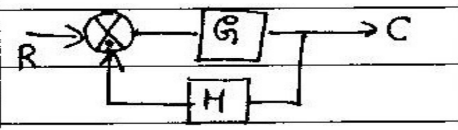
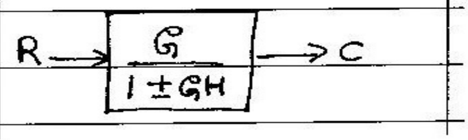
$$C_U(s) = \frac{G_2(s)}{1 + G_1(s) H(s) G_2(s)} U(s)$$

i.e

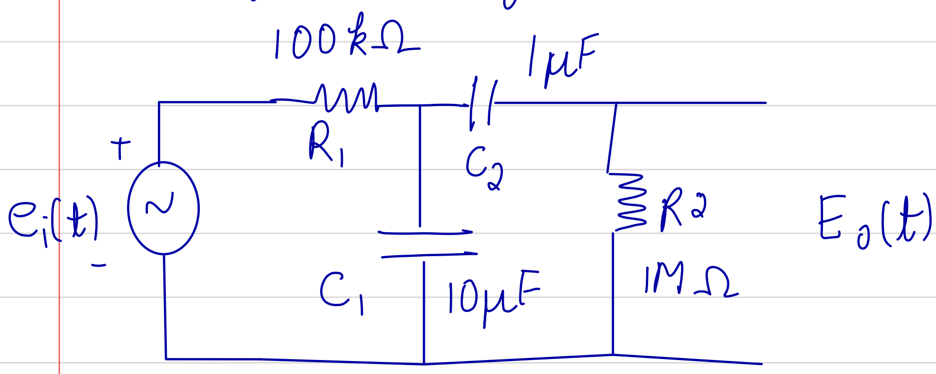


Rules to reduce block diagram

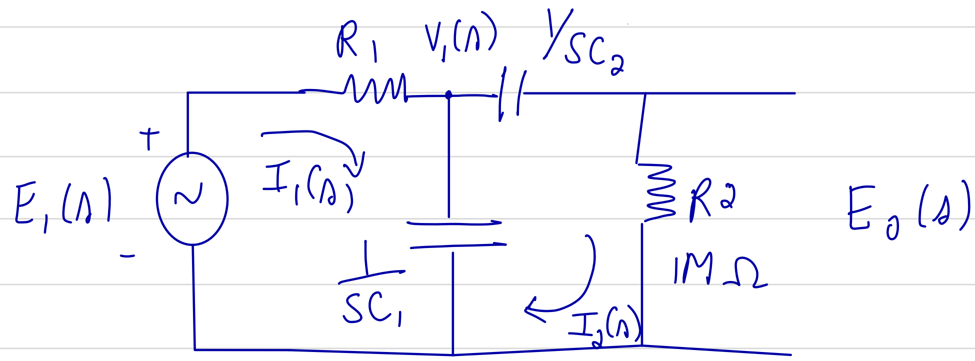
Block Diagram Reduction Rule

	Rules	Original Diagram	Equivalent B.D
1)	Combining blocks in series		
2)	Combining blocks in parallel		
3)	Shifting the Summing eleme-nts after the blocks.		
4)	Shifting the sum elements before blocks		
5)	Shifting the take off point after the block.		
6)	Shifting the take-off point before the block.		
7)	T.F of CLCS		

Q> Draw a block diagram for the given electric circuit shown & calculate transfer function $E_o(s) / E_i(s)$

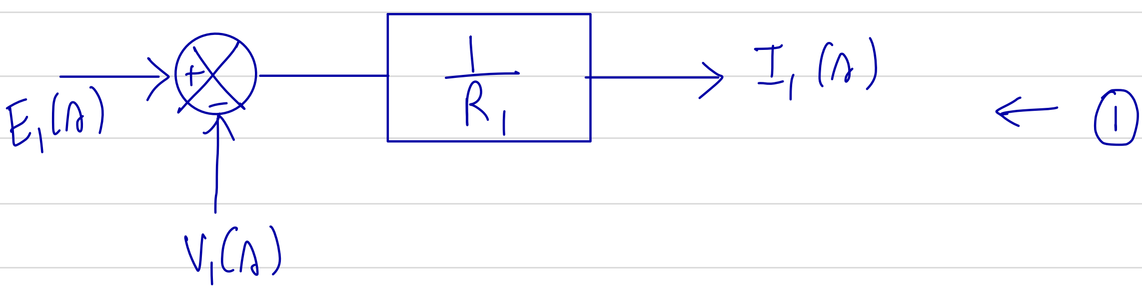


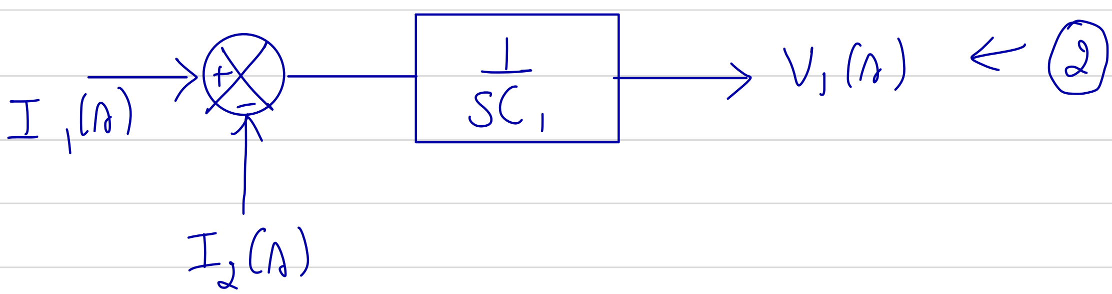
Ans



$$I_1(s) = \frac{E_1(s) - V_1(s)}{R_1} \rightarrow (1)$$

$$V_1(s) = \frac{1}{sC_1}\left[I_1(s) - I_2(s)\right] \rightarrow (2)$$

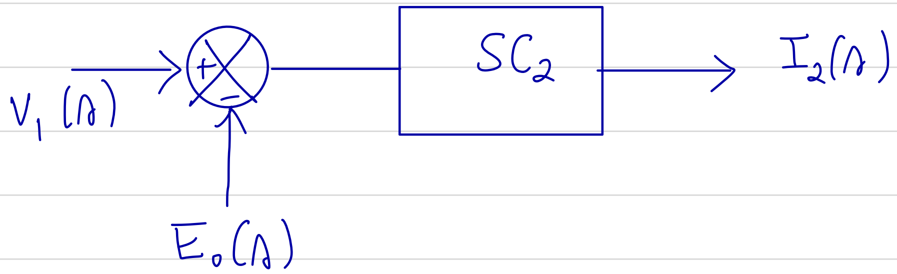




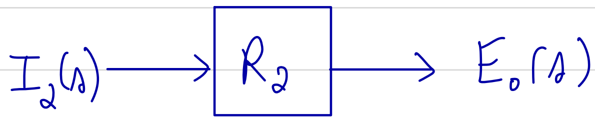
$$I_2(s) = [V_1(s) - E_o(s)]\, sC_2 \rightarrow (3)$$

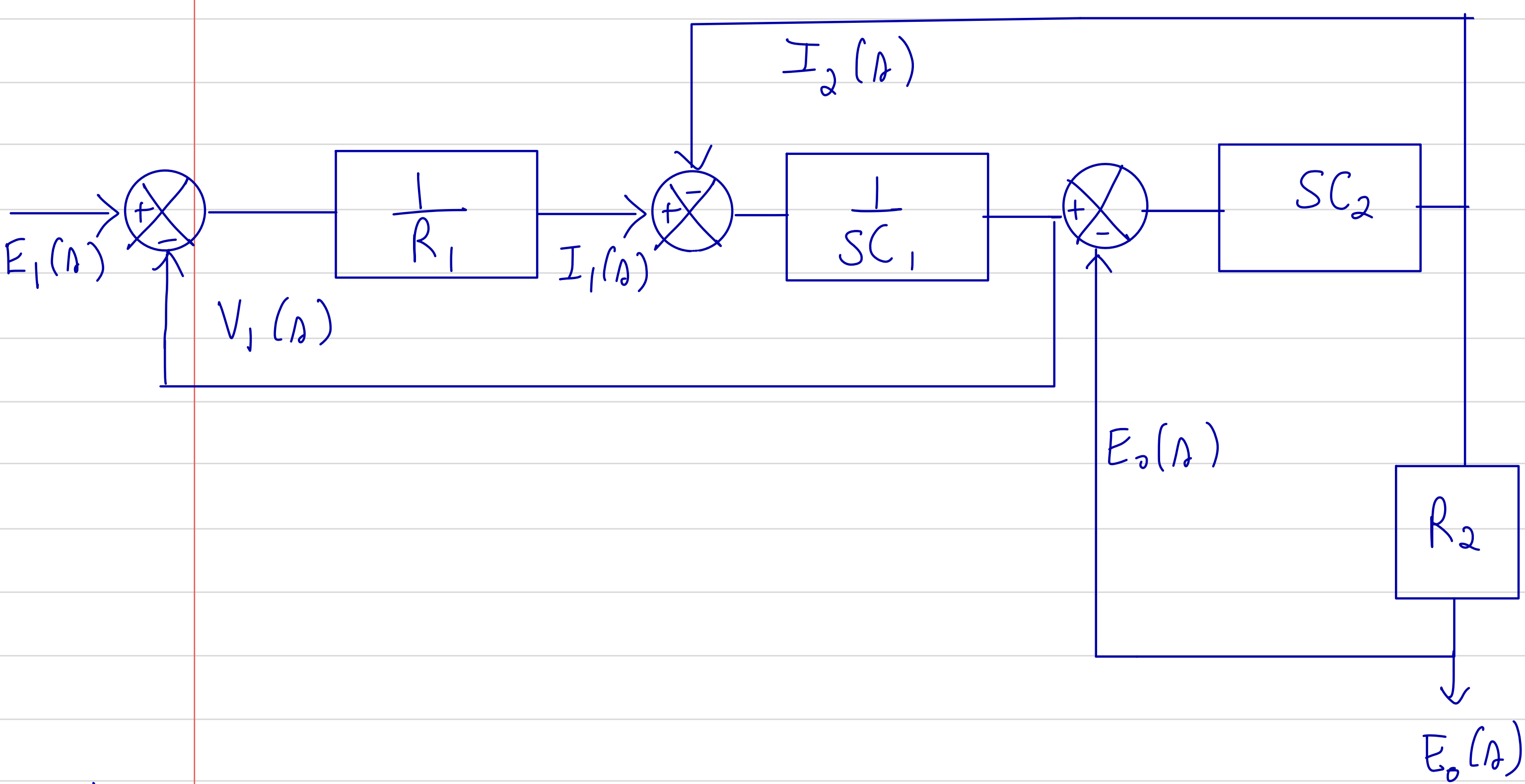
$$E_o(s) = R_2 I_2(s) \rightarrow (4)$$

(3) $\Rightarrow$

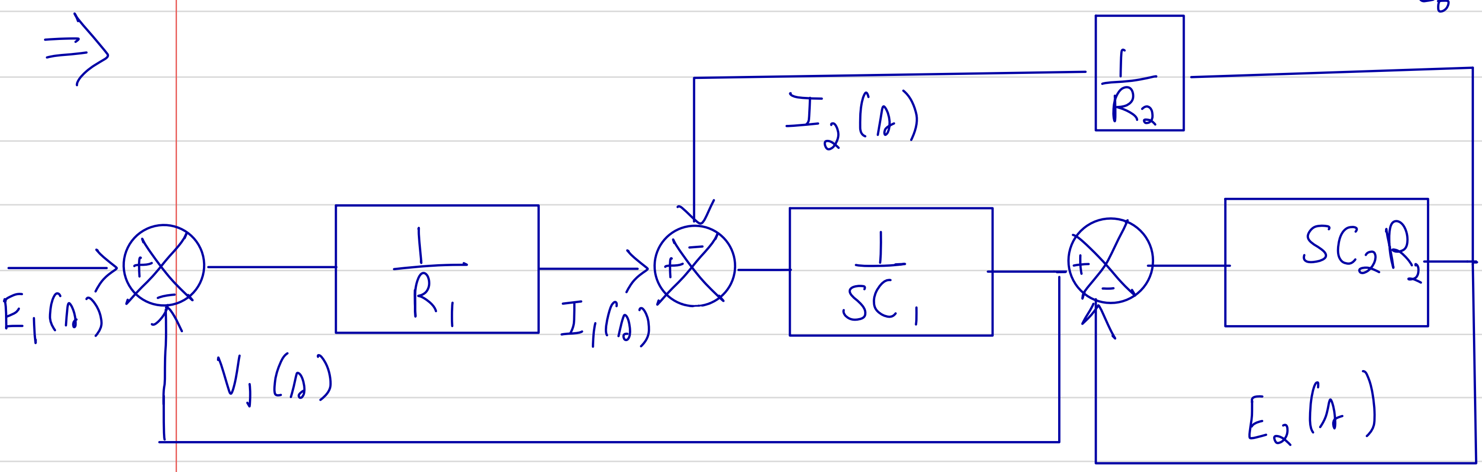


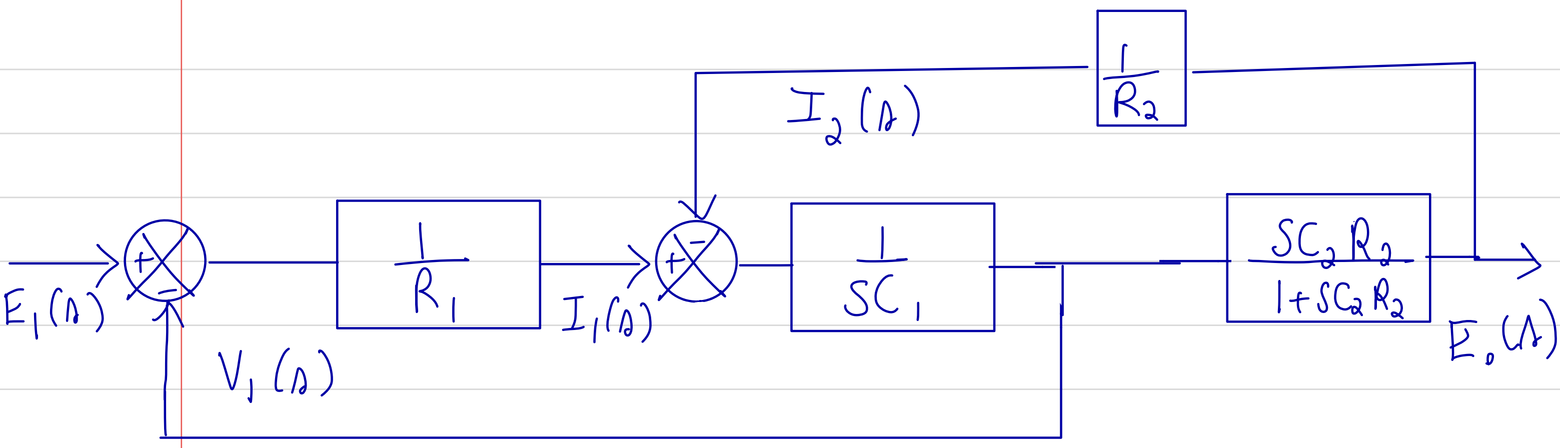
(4) $\Rightarrow$



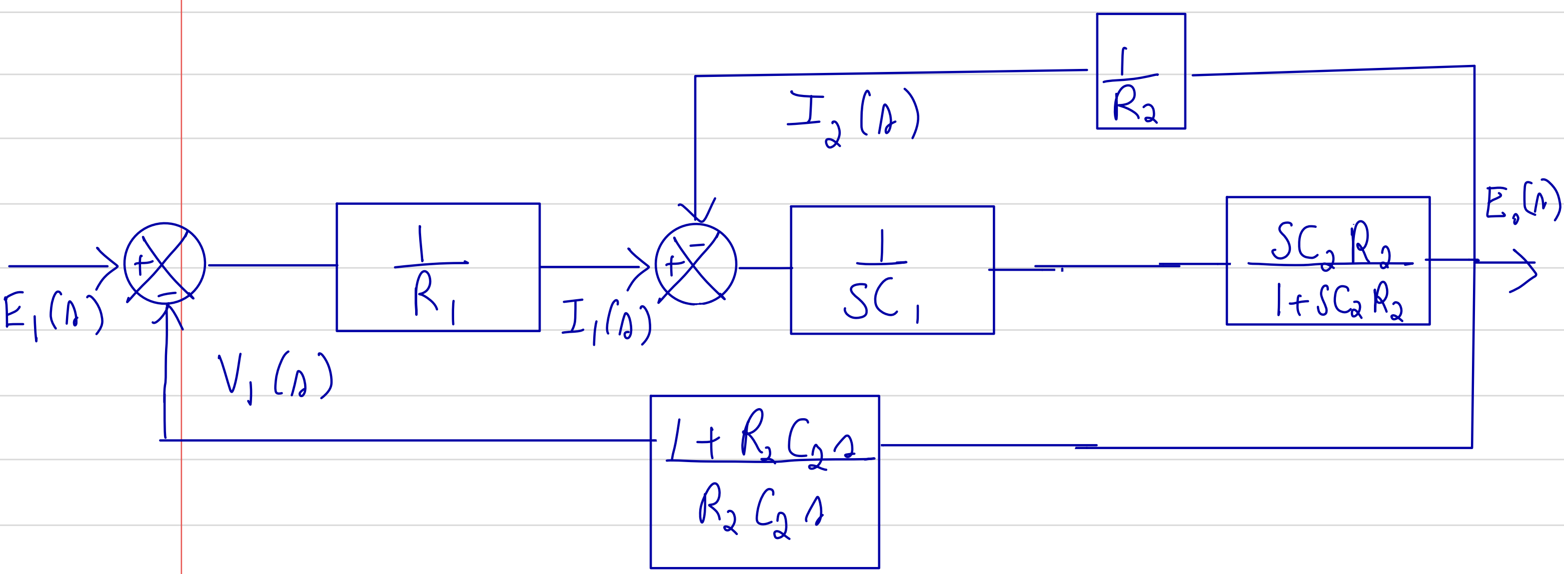


$\Rightarrow$





$E_1(s)$
+
−
$\frac{1}{R_1}$
$V_1(s)$
$I_1(s)$
+
−
$\frac{1}{sC_1}$
$I_2(s)$
$\frac{1}{R_2}$
$\frac{sC_2R_2}{1+sC_2R_2}$
$E_o(s)$

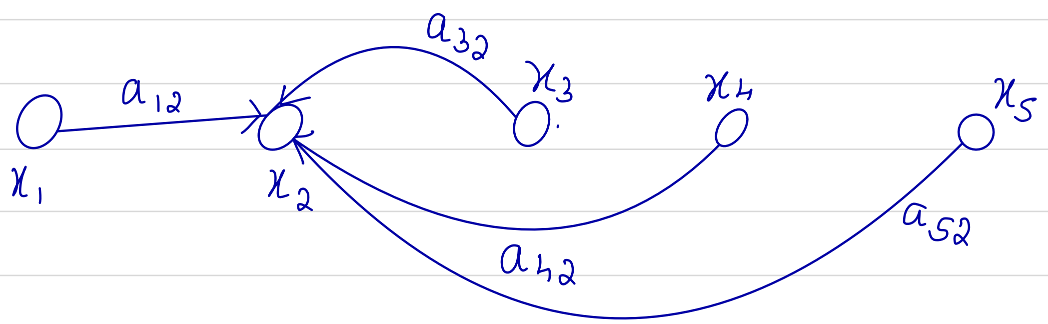


$E_1(s)$
+
−
$\frac{1}{R_1}$
$V_1(s)$
$I_1(s)$
+
−
$\frac{1}{sC_1}$
$I_2(s)$
$\frac{1}{R_2}$
$\frac{sC_2R_2}{1+sC_2R_2}$
$E_o(s)$
$\frac{1+R_2C_2s}{R_2C_2s}$

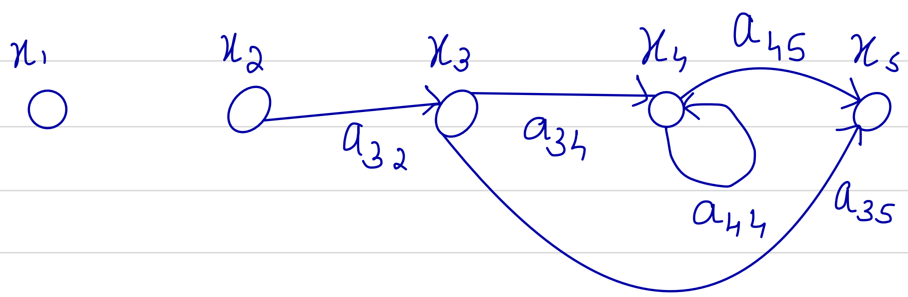
Q> Construction of signal flow graph:

$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5 \rightarrow (1)$$
$$x_3 = a_{32}x_2 \rightarrow (2)$$
$$x_4 = a_{34}x_3 + a_{44}x_4 \rightarrow (3)$$
$$x_5 = a_{35}x_3 + a_{45}x_4 \rightarrow (4)$$

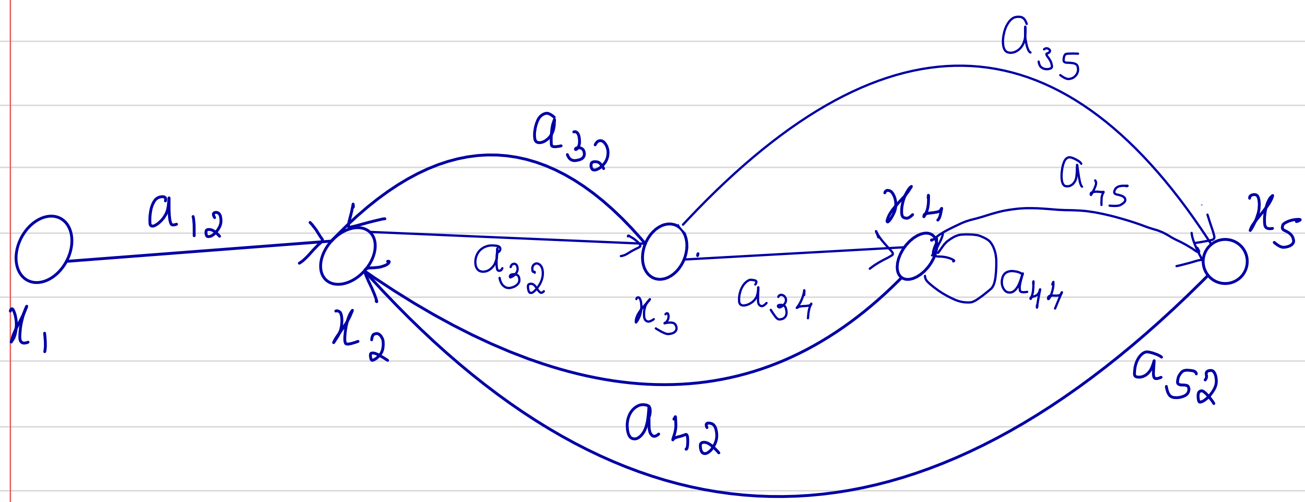
Ans (1) $\Rightarrow$



(2), (3), (4)



Finally



Mason's gain formula:

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

P_k = Path gain of k^{th} forward Path

Δ = Determinant of graph

Δ = 1 − (Sum of loop gains of all individual loops) + (Sum of gain products of all possible combinations of 2 non touching loops) − (Sum of gain products of all possible combinations of 3 non touching loops) + ...

$$\Delta = 1 - \sum_m P_{m_1} + \sum_m P_{m_2} - \sum P_{m_3} + \sum_m P_{m_4} - \cdots$$

Δ_k = The value of Δ for the part of the graph not touching k^{th} forward path.

T = Total gain.

Here 2 forward paths:

$x1 \to x2 \to x3 \to x4 \to x5 \quad \to 1$

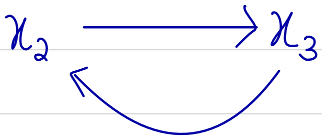
$x1 \to x2 \to x3 \to x5 \quad \to 2$

(Forward path = Starts at source, ends at sink, nodes are traversed only once).

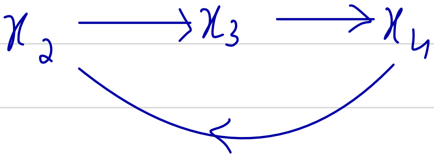
$P_1 = a_{12}\, a_{23}\, a_{34}\, a_{45}$

$P_2 = a_{12}\, a_{23}\, a_{35}$

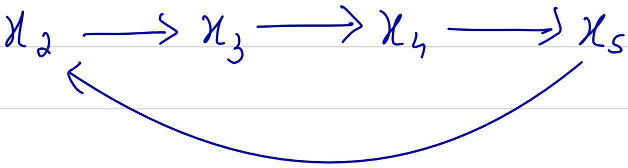
Individual loops : 5



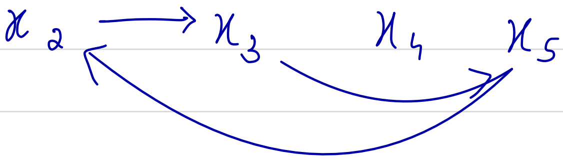
: gain = $a_{23}\,a_{32} = P_{11}$



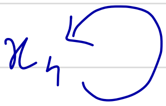
gain = $a_{23}\,a_{34}\,a_{42} = P_{21}$



gain = $a_{23}\,a_{34}\,a_{45}\,a_{52}$
$= P_{31}$

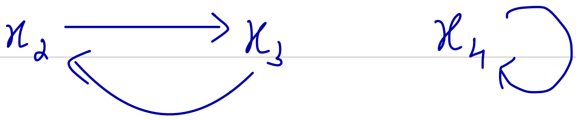


gain = $a_{23}\,a_{35}\,a_{52} = P_{41}$

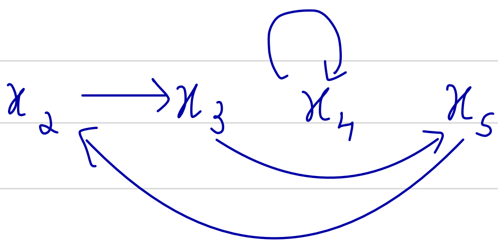


gain = $a_{44} = P_{51}$

Non touching loops : 2



$P_{12} = a_{23}\,a_{32}\,a_{44}$



$P_{22} = a_{23}\,a_{44}\,a_{35}\,a_{52}$

$$\therefore \Delta = 1 - (a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52} + a_{44}) + (a_{23}a_{32}a_{44} + a_{23}a_{44}a_{35}a_{52})$$

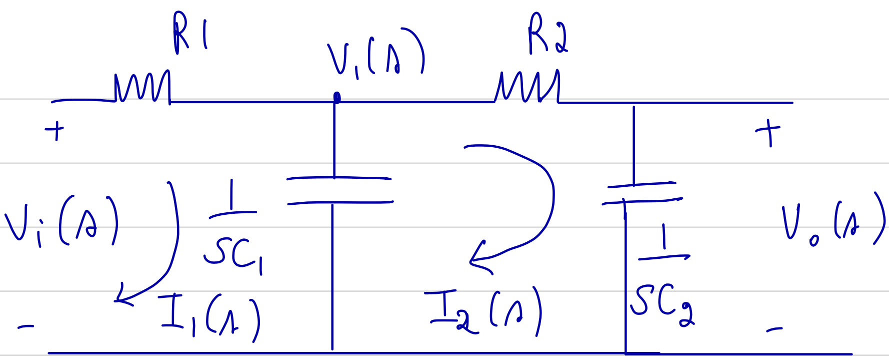
$\Delta_1 = 1$

$\Delta_2 = 1 - a_{44}$

$$T = \frac{x_5}{x_1} = \frac{1}{\Delta} \sum_{k=0}^{2} P_k \Delta_k$$

$$T = \frac{a_{12} a_{23} a_{34} a_{45} + a_{12} a_{23} a_{35} (1 - a_{44})}{1 - (a_{23} a_{32} + a_{23} a_{34} a_{42} + a_{23} a_{34} a_{45} a_{52} + a_{23} a_{35} a_{52} + a_{44}) + a_{23} a_{35} a_{52} a_{44} + a_{23} a_{32} a_{44}}$$

Q)



Find transfer function:

Ans

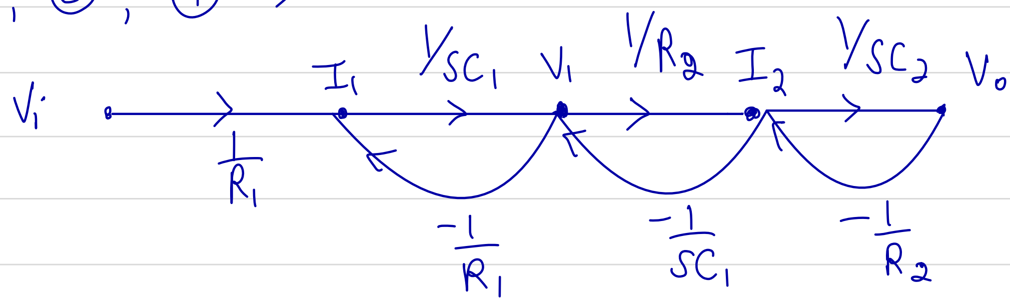
$$I_1(s) = \frac{V_i(s) - V_1(s)}{R_1} \rightarrow (1)$$

$$V_1(s) = \frac{I_1(s) - I_2(s)}{SC_1} \rightarrow (2)$$

$$I_2(s) = \frac{V_1(s) - V_o(s)}{R_2} \rightarrow (3)$$

$$V_o(s) = \frac{I_2(s)}{SC_2} \rightarrow (4)$$

(1), (2), (3), (4) $\Rightarrow$



Forward Path : $V_i \rightarrow I_1 \rightarrow V_1 \rightarrow I_2 \rightarrow V_o \Rightarrow k = 1$

$$P_1 = \frac{1}{s^2 R_1 R_2 C_1 C_2}$$

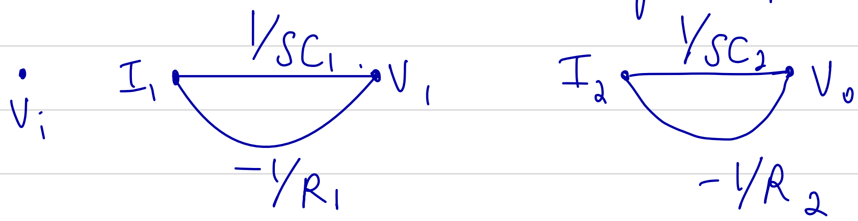
Individual loops : 3

1) $I_1 \rightarrow V_1 \rightarrow I_1 \Rightarrow$ $P_{11} = -1/R_1SC_1$

2) $V_1 \rightarrow I_2 \rightarrow V_1 \Rightarrow$ $P_{21} = -1/R_2SC_1$

3) $I_2 \rightarrow V_o \rightarrow I_2 \Rightarrow$ $P_{31} = -1/R_2SC_2$

Pair of non touching loops:



$$P_{21} = \frac{+1}{S^2 R_1 R_2 C_1 C_2}$$

$$\Delta_1 = 1 - 0$$

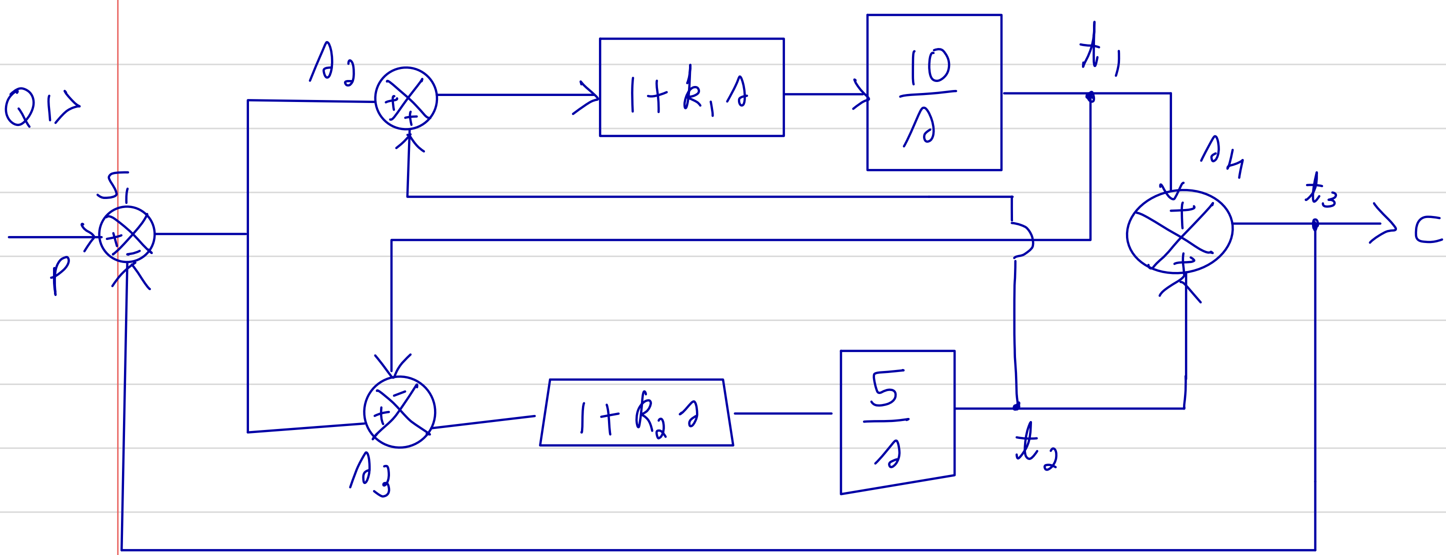
$$T = \frac{1}{\Delta} \sum P_k \Delta_k$$

$$\Delta = 1 + \frac{1}{S}\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2}\right) + \frac{1}{s^2 R_1 R_2 C_1 C_2}$$

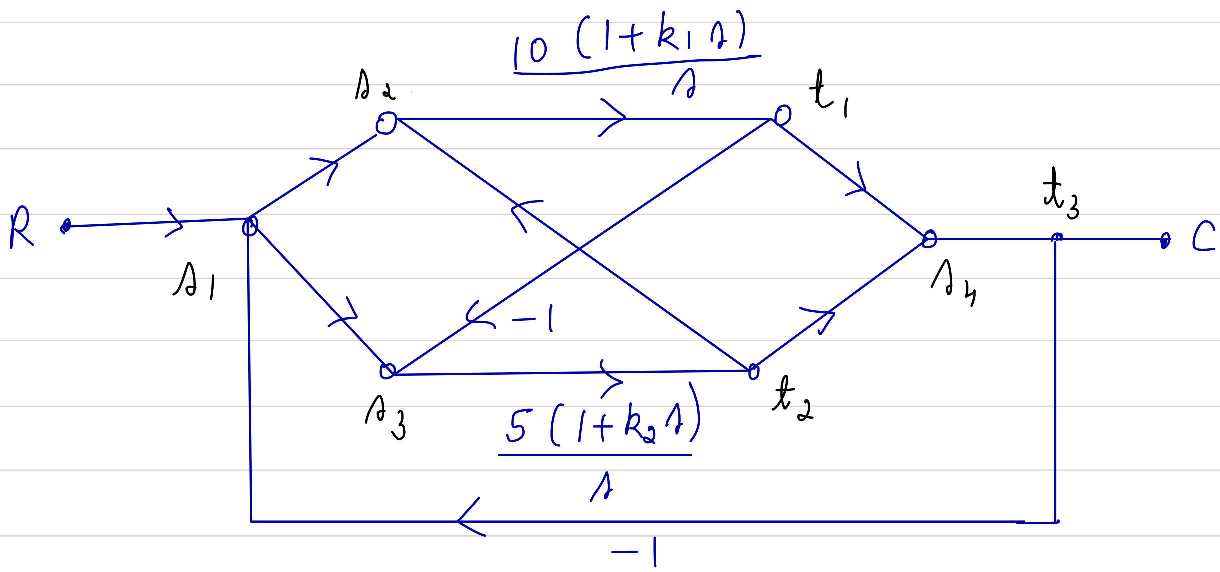
$$= \frac{s^2 R_1 R_2 C_1 C_2 + s(R_2 C_2 + R_1 C_2 + R_1 C_1) + 1}{s^2 R_1 R_2 C_1 C_2}$$

$$T = \sum P_1 \Delta_1$$

Q1)

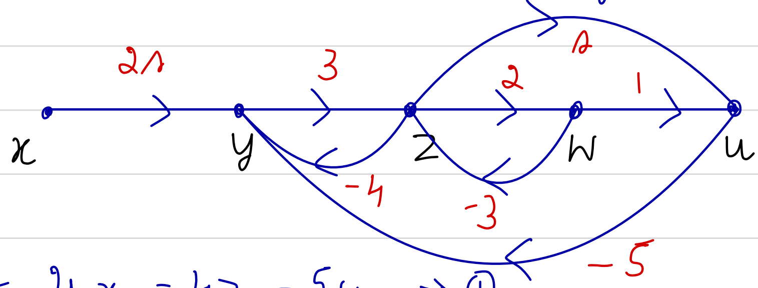


Ans



Forward path: 4
Loop : 5

Q) Obtain the block diagram for:



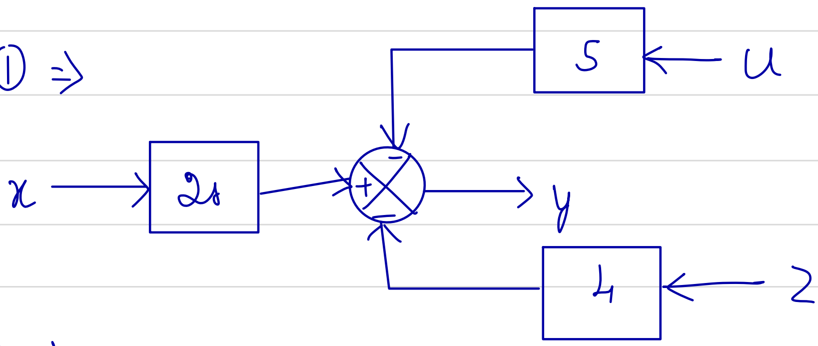
Ans $y = 2s\,x - 4z - 5u \rightarrow (1)$

$z = 3(y - w) \rightarrow (2)$

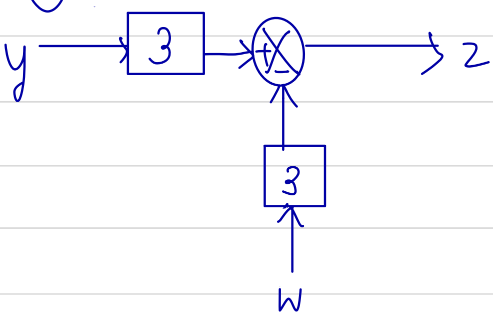
$w = 2z \rightarrow (3)$

$u = w + sz \rightarrow (4)$

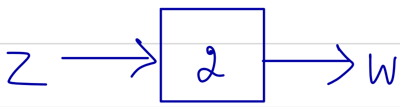
(1) ⇒



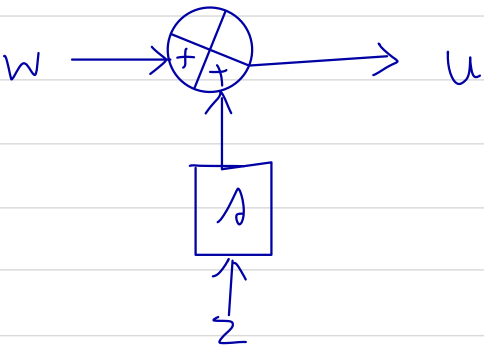
(2) ⇒



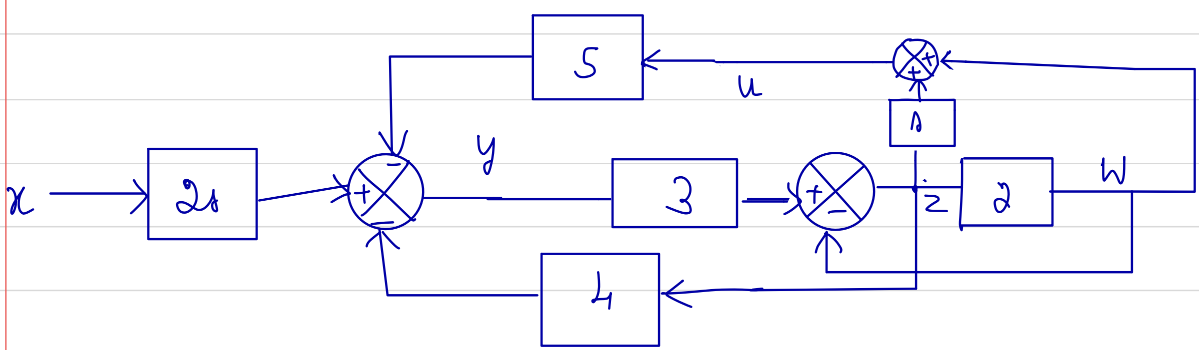
(3)



(4) ⇒



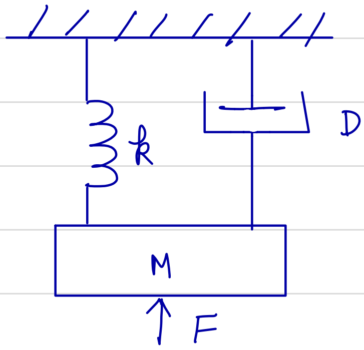
Finally



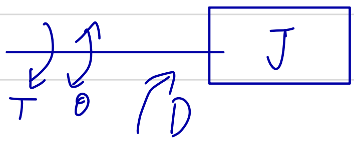
Mechanical systems

Consider the following mechanical system:

$$F = kx + Dv + M\frac{d^2x}{dt^2}$$



Q) Determine the mathematical model.

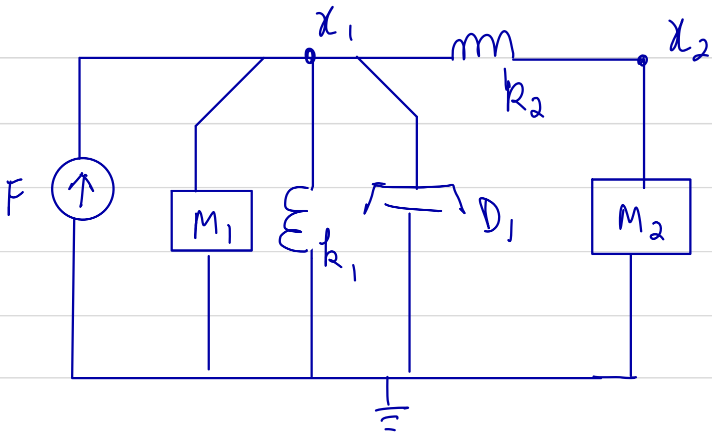


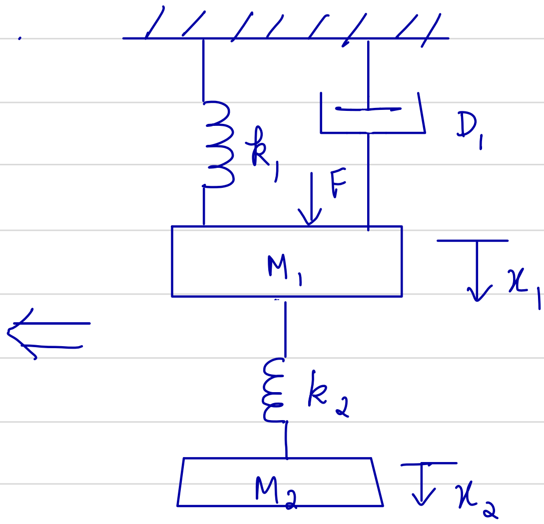
Ans

$$T = D\frac{d\theta}{dt} + J\frac{d^2\theta}{dt^2}$$

Q) Find mathematical model.

Ans





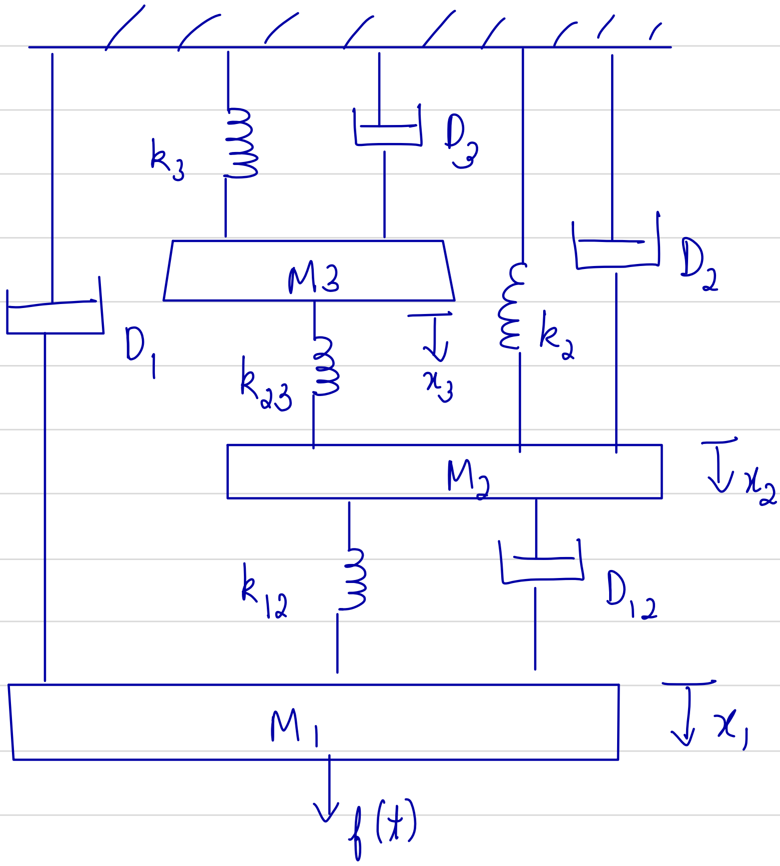
At node x1:

$$F(t) = M_1\frac{d^2x}{dt^2} + k_1x_1 + D_1\frac{dx_1}{dt} + k_2(x_2 - x_1)$$

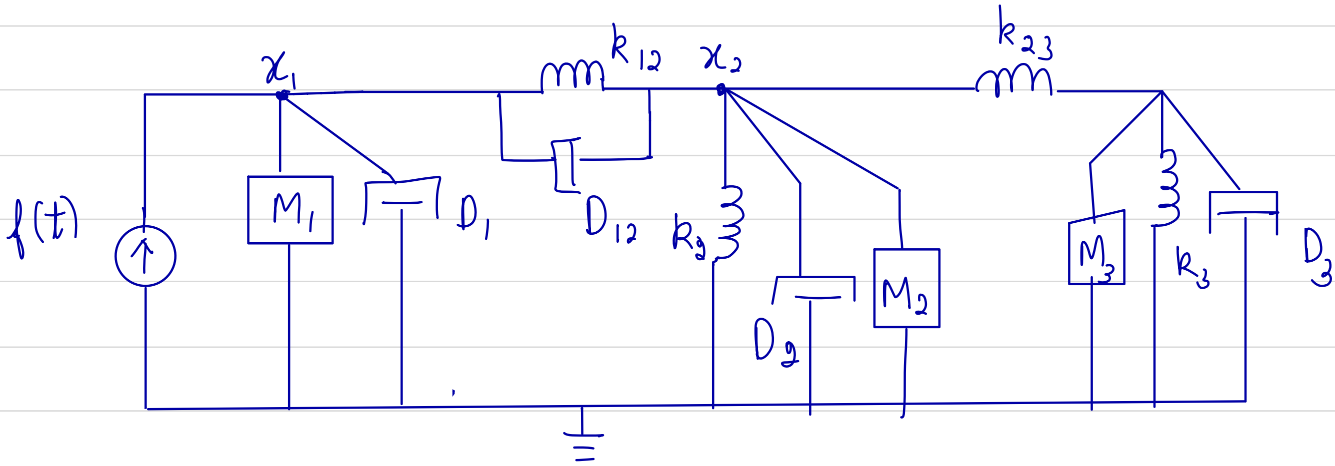
At node x2:

$$k_2(x_2 - x_1) + M_2\frac{dx^2}{dt^2} = 0$$

Q> Find nodal equations.



Ans



At $x1$:

$$f(t) = M_1 \frac{d^2x_1}{dt^2} + D_1 \frac{dx_1}{dt} + k_{12}(x_1 - x_2) + D_{12} \frac{d}{dt}(x_1 - x_2)$$

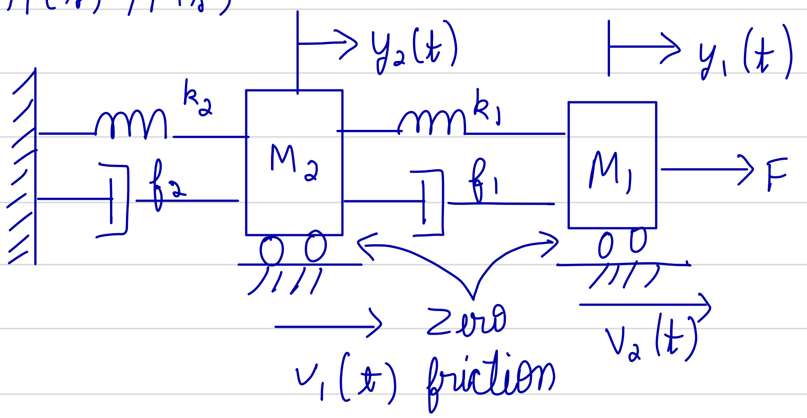
At $x2$;

$$k_{12}(x_1 - x_2) + D_{12}(x_1 - x_2) = k_2 x_2 + D_2 \frac{dx_2}{dt} + M \frac{d^2x_2}{dt}$$

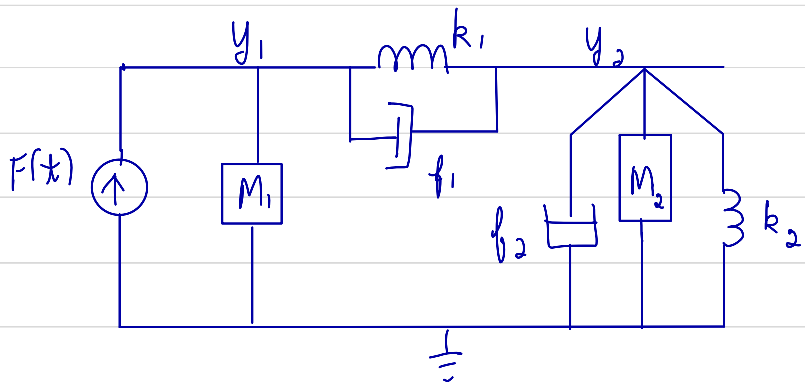
At x_3:

$$k_{13}(x_2 - x_3) = D_3 \frac{dx_3}{dt} + M \frac{d^2 x_3}{dt^2} + k_3 x_3$$

Q2> For mechanical system shown below, find $Y_1(s)/F(s)$



Ans



At y_1:

$$F = M_1 \frac{d^2 y_1}{dt^2} + k_1 (y_1 - y_2) + f_1 \frac{d}{dt}(y_1 - y_2)$$

At y_2:

$$k_1 (y_2 - y_1) + f_1 \frac{d}{dt}(y_2 - y_1) + M_2 \frac{d^2 y_2}{dt^2} + k_2 y_2 + f_2 \frac{dy_2}{dt}$$

Time response of 1st order system

Q> If $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$, find $C(s)$ for $r(t) = u(t)$

Ans $C(s) = \frac{1}{s}\left(\frac{1}{sT+1}\right) = \frac{1}{s}\left(\frac{1/T}{s+1/T}\right) = \frac{1}{s} - \frac{1}{s+1/T}$

$\therefore c(t) = 1 - e^{-t/T}$

Error, $\varepsilon(t) = r(t) - c(t) = e^{-t/T}$

Steady state error, $\varepsilon_{ss} = \lim_{t\to\infty} \varepsilon(t) = 0$

$\left.\frac{d(c(t))}{dt}\right|_{t=0} = \frac{1}{T}$, T = Time constant.

Consider 2 values of T, : T_1 & T_2.
If $T_1 > T_2$, 2nd system reaches steady state faster.

At $t = T$
$c(t) = 1 - e^{-1} = 1 - 0.368 = 0.632$

NOTE: Rise time, t_r: Time required for $c(t)$ to rise from 0.1 to 0.9.

$c(t) = 0.1 \Rightarrow 1 - e^{-t/T} = 0.1$

$e^{-t/T} = 0.9$

$1.111 = e^{t/T}$

$t = T \ln(1.11) = 0.104T$

then , $1 - e^{-t/T} = 0.9$

$$\Rightarrow \quad e^{-t/T} = 0.1$$

$$e^{t/T} = 10$$

$$t = T \ln 10 = 2.303T$$

$$\therefore \; t_r = T(2.303 - 0.104)$$

$$t_r \cong 2.2T$$

NOTE: Settling time: (t_s) Time for response to reach and stay within 2% of final value.

It need not be 2%, it can range from 2 - 5.

$$c(t_s) = 0.98$$

$$e^{-t/T} = 0.02$$

$$t_s = T \ln 50$$

$$\therefore \; t_s \cong 4T$$

Time constant : τ : Time required to reach 63% of the final value

Q> Find τ if final value is 0.72.

Ans $0.63 \times 0.72 = 0.45 = e^{-t/\tau} = T = 0.13$

If $c(s) = \dfrac{K/a}{(s+a)}$

$$c(t) = \frac{K}{a} e^{-at}$$

$$a = 1/\tau = 7.7$$

$$\therefore \; \frac{K}{a} = 0.72 \Rightarrow K = 7.7 \times 0.72 = 5.54$$

Time Response of 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ where $\zeta > 0$, $\omega_n > 0$

ω_n = undamped natural frequency
ζ = Damping factor.

NOTE: Denominator = 0 $\Rightarrow$ characteristic equation.

$$s^2 + 2\zeta s\omega_n + \omega_n^2 = 0$$

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_1, s_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$ //

Case 1: $\zeta = 0$
$s_1, s_2 = \pm j\omega_n$ $\rightarrow$ undamped

Case 2: $\zeta = 1$
$s_1, s_2 = -\omega_n$ $\rightarrow$ critically damped

case 3: $0 < \zeta < 1$ $\rightarrow$ underdamped

$$s_1, s_2 = -\omega_n\left(\zeta \pm j\sqrt{1 - \zeta^2}\right)$$

$$= -\zeta\omega_n \pm j\,\omega_d$$

ω_d = damped frequency of oscillation.

case 4: $\zeta > 1$ $\rightarrow$ overdamped
$$s_1, s_2 = -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

$\zeta = \dfrac{\text{exponential decay frequency}}{\text{Natural frequency}}$

Q) Consider $G(s) = \dfrac{b}{s^2+as+b}$, find ζ

Ans To find w_n, we find $s^2 + 0 + b = 0$

$\therefore s = \pm j\sqrt{b}$ $\therefore$ $w_n = \sqrt{b}$

$2\zeta w_n = a$

$\therefore \zeta = \dfrac{a}{2\sqrt{b}}$ //

Note: ζw_n = exponential decay frequency $= \dfrac{a}{2}$ //

Step response of 2nd order system

If $\dfrac{C(s)}{R(s)} = \dfrac{w_n^2}{s^2+2\zeta w_n s + w_n^2}$ and

if $R(s) = \dfrac{1}{s}$

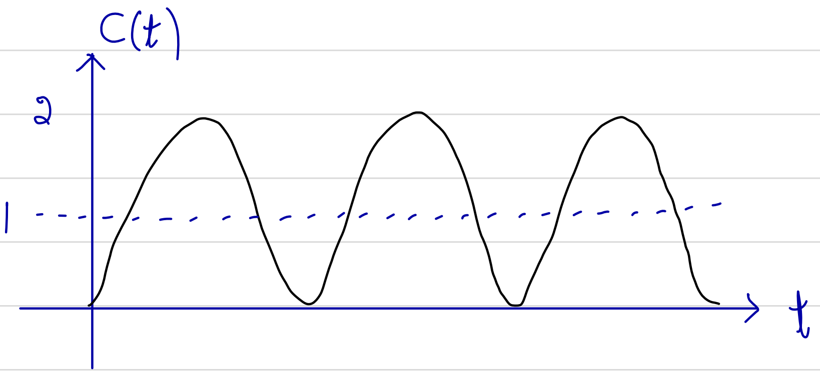
$$C(s) = \frac{w_n^2}{s(s^2+2\zeta w_n s + w_n^2)}$$

case 1: $\zeta = 0$

$$C(s) = \left(\frac{1}{s}\right)\frac{w_n^2}{s^2+w_n^2}$$

$$= \frac{1}{s} - \frac{s}{s^2+w_n^2}$$

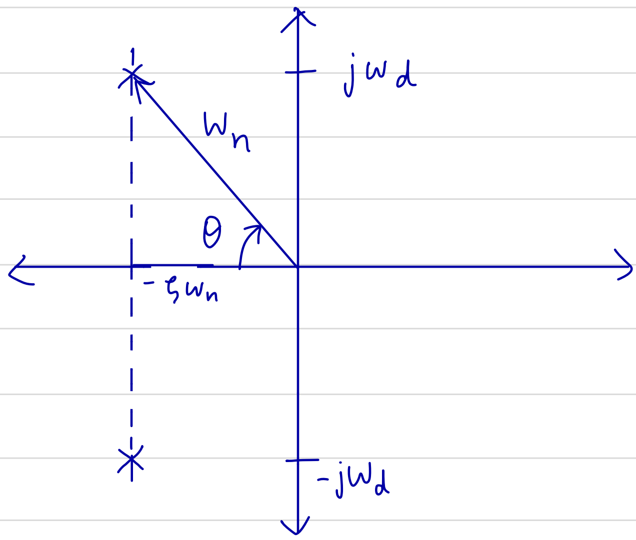
$$c(t) = (1 - \cos w_n t)\, u(t)$$ //



case 2 ; $0 < \zeta < 1$

$$s_1, s_2 = \omega_n\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)$$

S- plane:



$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$A = 1$

Also,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

$\Rightarrow \quad B + 1 = 0$

$\Rightarrow B = -1$

Put $s = 1$

$$\frac{\omega_n^2}{\omega_n^2 + 2\zeta\omega_n + 1} = 1 + \frac{C - 1}{\omega_n^2 + 2\zeta\omega_n + 1}$$

$$\omega_n^2 = \omega_n^2 + 2\zeta\omega_n + 1 + C - 1$$

$$C = -2\zeta\omega_n$$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\omega_n^2 + 2\zeta\omega_n s + s^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s + 2\zeta\omega_n s + (\zeta\omega_n)^2 - (\zeta\omega_n)^2 + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

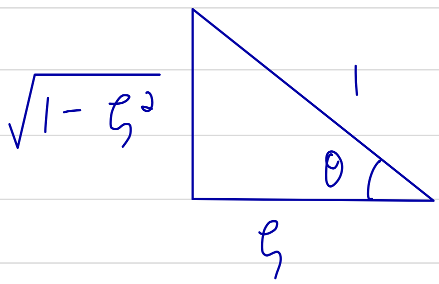
$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \zeta\frac{\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t - \zeta\frac{\omega_n}{\omega_d}\sin\omega_d t\right)$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t - \zeta\frac{\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t\right)$$

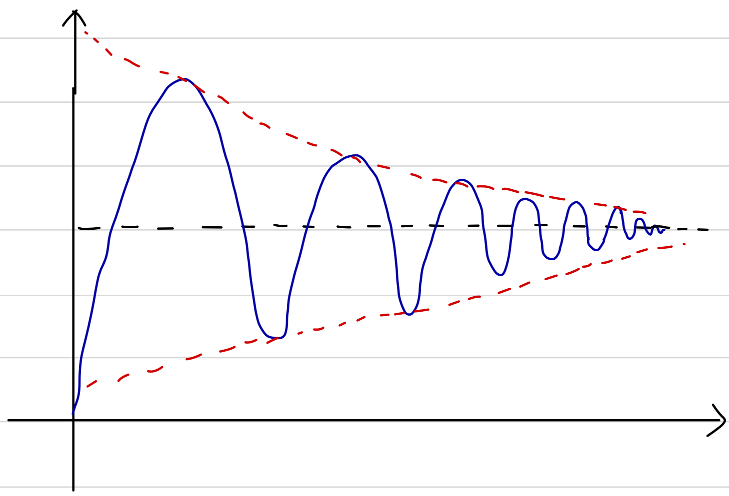
Triangle:



$$\therefore \cos\theta = \zeta \qquad \sin\theta = \sqrt{1-\zeta^2}$$

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore\ c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$



NOTE: ζ is closer to 1, there will more damping and less oscillation.

Q> If $T(s) = \dfrac{1}{Ms^2 + Ds + K}$. Find ζ & ω_n

Ans $T(s) = \dfrac{1/M}{s^2 + \frac{D}{M}s + \frac{K}{M}}$

$$\omega_n = \sqrt{\frac{K}{M}}$$

$$2\zeta\omega_n = \frac{D}{M}$$

$$\zeta = \frac{D}{M} \times \sqrt{\frac{M}{K}} \times \frac{1}{2} = \frac{D}{2\sqrt{MK}}$$

Q> For each of the transfer function given below, find ζ, w_n. Also characterise nature of system.

i> $T(s) = \dfrac{400}{s^2 + 12s + 400}$

Ans $w_n^2 = 400 \Rightarrow w_n = 20$

$2\zeta w_n = 12$

$\zeta = \dfrac{6}{20} = 0.3$ $\Rightarrow$ underdamped

ii> $T(s) = \dfrac{225}{s^2 + 30s + 225}$

Ans $w_n^2 = 225 \Rightarrow w_n = 15$

$2\zeta w_n = 30$

$\zeta = \dfrac{15}{15} = 1 \Rightarrow$ critically damped

<u>case 3</u>: $\zeta = 1$; critically damped:

$$C(s) = \frac{w_n^2}{s(s^2 + 2w_n s + w_n^2)}$$

$$= \frac{A}{s} + \frac{B}{(s+w_n)^2} + \frac{C}{s+w_n}$$

$A = 1$

$B = -w_n$

Put $s = 1$

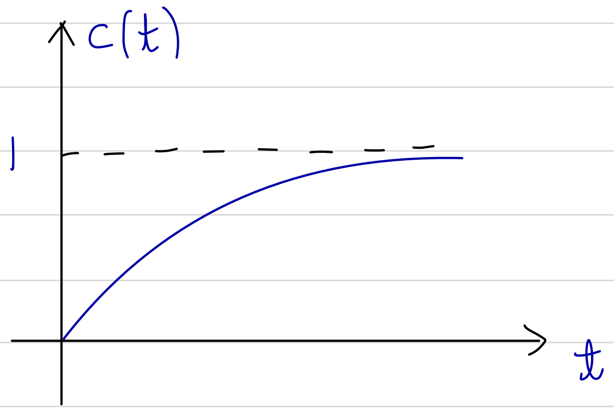
$$\frac{w_n^2}{(1+w_n)^2} = 1 - \frac{w_n}{(1+w_n)^2} + \frac{C}{1+w_n}$$

$$w_n^2 = 1 + w_n^2 + 2w_n - w_n + C(1+w_n)$$

$\Rightarrow C + 1 = 0 \qquad \Rightarrow C = -1$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$$\therefore c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

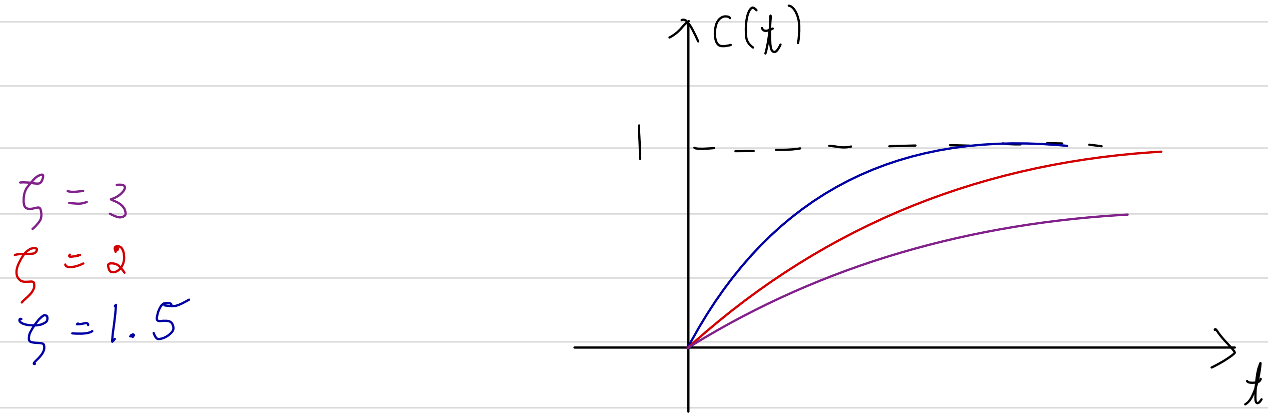


Case 4: $\zeta > 1$

$$S_1, S_2 = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$C(s) = \frac{1}{S} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{S_1}\left(\frac{1}{S+S_1}\right) + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{S_2}\left(\frac{1}{S+S_2}\right)$$

$$\therefore c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]$$



Q> Measurements conducted on a servo mechanism show the system response to be $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$, when subjected to a unit step function. Obtain the closed loop transfer function. Also determine the undamped natural frequency & damping ratio.

Ans $C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$

$R(s) = \frac{1}{s}$

$T(s) = \frac{C(s)}{R(s)} = 1 + \frac{0.2s}{s+60} - \frac{1.2s}{s+10}$

characteristic equation: $(s+60)(s+10)$
$= s^2 + 70s + 600$

$W_n^2 = 600$
$W_n = 10\sqrt{6} = 24.49$ rad/s //

$2 \zeta W_n = 70$
$\zeta = \frac{35}{25.49} \approxeq 1.43$ //

NOTE:- Rise time: t_r:

$\xi > 1$, $t_r \rightarrow 10\% - 90\%$

$\xi = 1$, $t_r \rightarrow 5\% - 95\%$

$\xi < 1$, $t_r \rightarrow 0\% - 100\%$

For underdamped system:

$$c(t) = 1 \Rightarrow 1 = 1 - \frac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}} \sin(w_d t + \theta)$$

$$\Rightarrow \sin(w_d t + \theta) = \pi$$

$$\Rightarrow \left(w_n\sqrt{1-\xi^2}\right) t_r = -\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) + \pi$$

$$t_r = \pi - \frac{1}{w_n\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) //$$

(π - first time raise)

Peak time, t_p:

$$\frac{d(c(t))}{dt} = 0$$

$$\frac{-1}{\sqrt{1-\xi^2}}\left(-\xi w_n e^{-\xi w_n t} \sin(w_d t + \theta) + w_d \cos(w_d t + \theta) e^{-\xi w_n t}\right) = 0$$

$$\tan(w_d t_p + \theta) = \frac{W_d}{\xi w_n}$$

$$\tan(W_d t + \theta) = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\omega_d t_p + \theta = \pi + \theta$$

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Peak overshoot: (M_p)

$$M_p = c(t_p)$$

$$c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$= 1 + \frac{e^{-\pi\cot\theta}}{\sqrt{1-\zeta^2}} \sin\theta$$

$$= 1 + \frac{e^{-\pi\cot\theta}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2}$$

$$\therefore M_p = 1 + e^{-\pi\cot\theta}$$

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\pi\cot\theta} - 1}{1} \times 100$$

$$\therefore \% M_p = e^{-\pi\cot\theta} \times 100$$

NOTE
- Ideal value of ζ : 0.4 - 0.8
- Also $\zeta - \uparrow \Rightarrow M_p \downarrow$ & $t_r \uparrow$

Settling time (t_s)

$$\frac{e^{-\xi \omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02 \qquad (2\% \text{ tolerance})$$

$$\Rightarrow \quad e^{-\xi \omega_n t_s} \cong 0.02$$

$$\Rightarrow \quad e^{\xi \omega_n t_s} = 50$$

$$\xi \omega_n t_s = 3.91$$

$$\therefore t_s = \frac{3.91}{\omega_n \xi} \cong \frac{4}{\xi \omega_n}$$

let $T = \frac{1}{\xi \omega_n}$

$$\therefore t_s = \frac{4}{T}$$

For other tolerance,

$$t_s = \frac{-\ln(\% \text{ tolerance})}{\xi \omega_n}$$

Q) A closed loop servo system is represented by differential equation: $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$

where c is the displacement of output shaft, and r is the displacement of input shaft and $e = r - c$. Determine damping ratio, damped natural frequency, % Mp for unit step input.

Ans $$\frac{d^2C}{dt^2} + 8\frac{dC}{dt} = 64(r-c)$$

$$s^2 C(s) - s\,c(0) - c'(0) + 8s\,C(s) - c(0) = 64(R-C)$$

$$s^2 C(s) + 8s\,C(s) + 64\,C(s) = 64\,R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{64}{s^2 + 8s + 64}$$

Also $\quad W_n^2 = 64 \quad \Rightarrow \quad W_n = 8$

$2\zeta w_n = 8$

$\zeta = 1/2 = 0.5$

$$W_d = W_n\sqrt{1-\zeta^2} = \frac{8\sqrt{3}}{2} = 4\sqrt{3}$$

$$C(s) = \frac{1}{s} \cdot \left(\frac{64}{s^2 + 8s + 64}\right)$$

$\zeta < 1 \Rightarrow$ undamped.

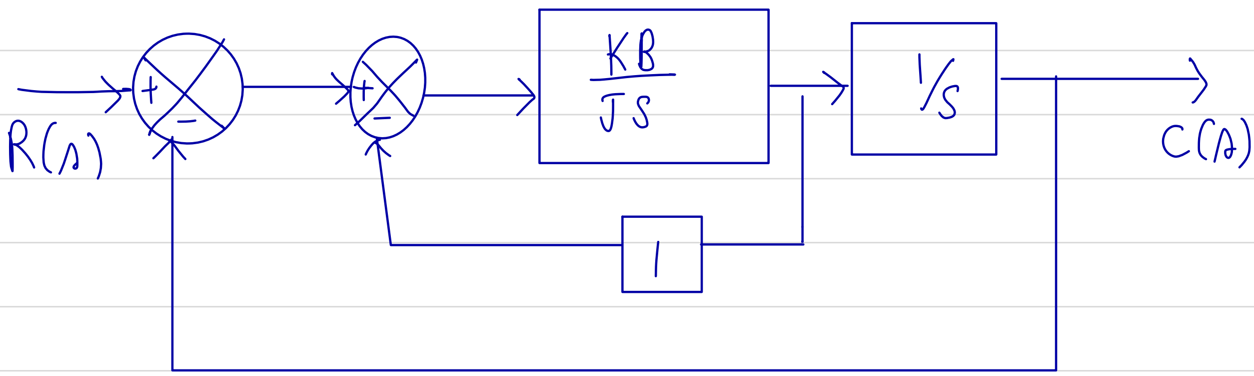
$$\%\,M_p = e^{-\pi \cot\theta} \times 100$$

$$\cot\theta = \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{0.5}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

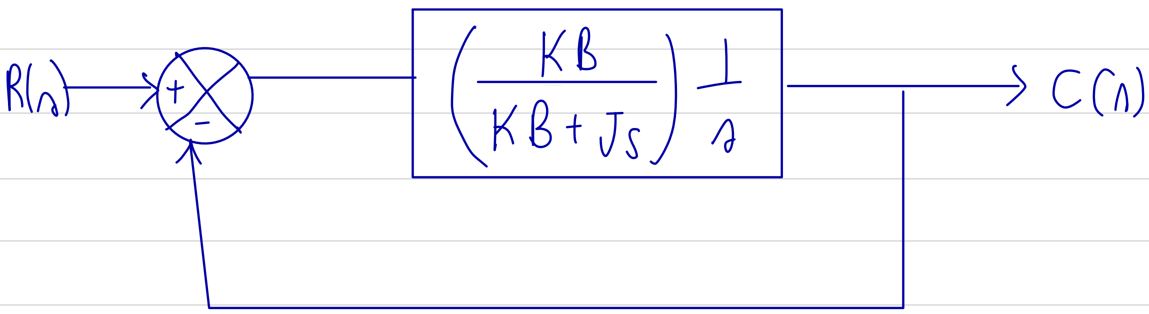
$$\therefore \%\,M_p = \left(e^{-\pi/\sqrt{3}}\right) \times 100$$

Q> For closed loop system shown in figure determine the value of K & B, for unit step input with $\%\,M_p = 20\%$ and peak time = 2.5 seconds

Take $J = 1\,\text{Kgm}^2$



Ans



$$\frac{C(s)}{R(s)} = \frac{KB}{s(KB+Js)+KB} = \frac{KB}{s^2+KBs+KB}$$

$(\because J=1)$

$$\therefore \quad \omega_n^2 = KB \quad \Rightarrow \quad \omega_n = \sqrt{KB}$$

$$2\xi\omega_n = KB$$

$$\xi = \sqrt{KB}/2$$

$$\% M_p = e^{-\pi\cot\theta} \times 100$$

$$\cot\theta = \frac{\xi}{\sqrt{1-\xi^2}} = \frac{\sqrt{KB}/2}{\sqrt{1-\frac{KB}{4}}}$$

$$\Rightarrow e^{-\pi\cot\theta} = \frac{20}{100} = 0.2$$

$$\pi\cot\theta = \ln 5$$

$$\therefore \cot\theta = 0.512$$

$$\frac{\frac{KB}{4}}{1-\frac{KB}{4}} = 0.2621$$

$$\Rightarrow \frac{x}{1-x} = 0.2621 \qquad \text{let } \frac{KB}{4} = x$$

$$1.262\,x = 0.2621$$

$$\therefore \frac{KB}{4} = 0.207$$

$$\therefore KB = 0.8306$$

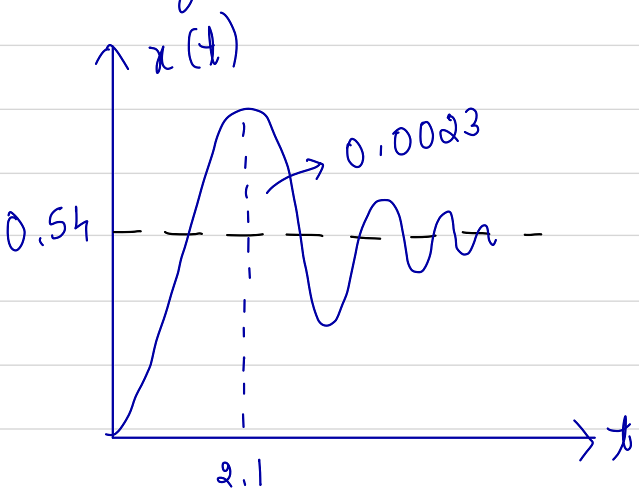
Let $K = 2$

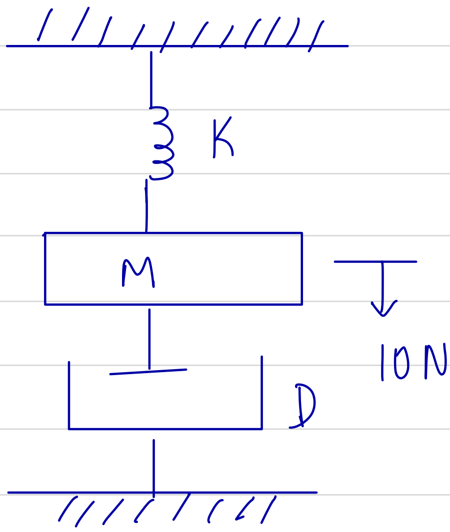
$$B = 0.4153$$

$$\zeta = \sqrt{\frac{KB}{2}} = 0.4556$$

$$w_n = 0.91137$$

Q) Calculate the value of M, D & K for the system shown below. The system is initially at rest.





Ans $$X(s) = \frac{10/M}{s\left(s^2 + \frac{D}{M}s + \frac{K}{M}\right)}$$

$$\lim_{s \to 0} sX(s) = 0.54$$

$$\Rightarrow \lim_{s \to 0} \frac{10/M}{s^2 + \frac{Ds}{M} + \frac{K}{M}} = 0.54$$

$$\frac{10}{K} = 0.54$$

$$\therefore \quad K = 18.51$$

$$W_n^2 = \frac{K}{M} \qquad W_n = \sqrt{\frac{K}{M}}$$

$$2\zeta W_n = \frac{D}{M}$$

$$\zeta = \frac{D}{M} \times \sqrt{\frac{M}{K}} \times \frac{1}{2} = \frac{D}{2\sqrt{MK}}$$

$$t_p = \frac{\pi}{W_d} = \frac{\pi}{\sqrt{\frac{K}{M}}\left(\sqrt{1 - \frac{D^2}{4MK}}\right)} \qquad M_p = 0.00426$$

$$0.00426 = \frac{e^{-\zeta\pi}}{\sqrt{1-\zeta^2}}$$

$$\zeta = 0.8668$$

$$2.1 = \frac{\pi}{W_n \times 0.498}$$

$$W_n \approx 3$$

$$\therefore \quad M = \frac{K}{9} = \quad = 2.0566 \text{ Kg}$$

$$0.8668 = \frac{D}{2\sqrt{18.51 \times 2.056}}$$

$$D = 10.69 \; Ns/m$$

HW: A system has $G(s) = \frac{K}{s(1+as)}$ with unity feedback. Here k & a are constants. Determine the factor by which k should be multiplied to reduce overshoot from 80% to 45%.

Q) Find steady state error e_{ss} for unit step input.

Ans

$$e_{ss} = \lim_{s\to 0} \frac{R(s)\, s}{1+G(s)H(s)}$$

$$\Rightarrow \lim_{s\to 0} \frac{s \cdot 1/s}{1+G(s)H(s)} = \frac{1}{1+k_p}$$

k_p is defined as velocity error constant with $k_p = \lim_{s\to 0} G(s)\cdot H(s)$

<u>For ramp input:</u>

$$e_{ss} = \lim_{s\to 0} \frac{(1/s^2)\,s}{1+G(s)H(s)} = \lim_{s\to 0} \frac{1}{s+sGH}$$

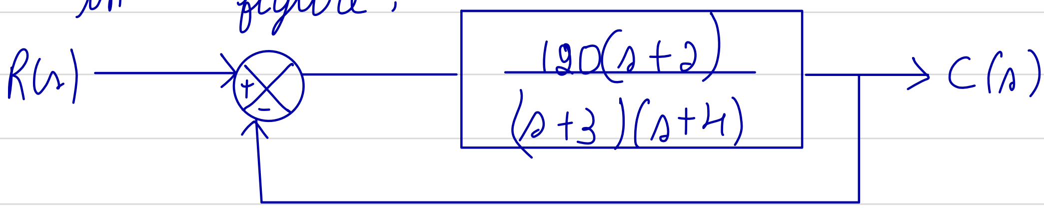
$$= \frac{1}{1+k_v} \quad \text{with } k_v = \lim_{s\to 0} sG(s)\cdot H(s)$$

<u>For parabolic input:</u>

$$e_{ss} = \lim_{s\to 0} \frac{(s)\ 1/s^2}{1+G(s)H(s)} = \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$= \frac{1}{1+k_a} \quad \text{with} \quad k_a = \lim_{s\to 0} s^2 G(s)H(s)$$

Q> Find steady state errors for input $5u(t)$, $5tu(t)$, $5t^2u(t)$ to the stable systems shown in figure.



Ans

$$e_{ss_1} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} \frac{s\times(5/s)}{1+\frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{1+\frac{120\times 2}{12}} = \frac{5}{21}$$

$$e_{ss_2} = \lim_{s\to 0} \frac{s(5/s^2)}{1+\frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{s+\frac{120(s+2)s}{(s+3)(s+4)}} = \infty$$

$$e_{ss_3} = \infty$$

Q> For a system $P(s) = 1/(s-1)$. $H(s) = 1$
$C(s) = k_p$ (controller). Find possible values of k_p.

Ans $G(s) = \dfrac{k_p}{(s-1)}$

$\therefore T(s) = \dfrac{k_p}{s+(k_p-1)}$ $\therefore$ pole $\Rightarrow s = 1-k_p$

For system to be stable, $1-k_p < 0$

$\therefore k_p > 1$

Routh Criterion

$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots$

s^n	a_0	a_2	a_4	$\ldots$
s^{n-1}	a_1	a_3	a_5	$\ldots$
s^{n-2}	b_1	b_2	b_3	$\ldots$
s^{n-3}	c_1	c_2	c_3	$\ldots$
$\vdots$	d_1	d_2	d_3	$\ldots$
s^2	$\vdots$	a_n		
s^1	$\vdots$			
s^0	a_n			

$b_1 = \dfrac{a_1 a_2 - a_0 a_3}{a_1}$ $b_2 = \dfrac{a_1 a_4 - a_0 a_5}{a_1}$

$b_3 = \dfrac{a_1 a_6 - a_0 a_7}{a_1}$

$c_1 = \dfrac{b_1 a_3 - a_1 b_2}{b_1}$

Q> Find stability of $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Ans

s^4	1	18	5
s^3	8	16	0
s^2	16	5	0
s^1	13.5	0	
s^0	5		

$\Rightarrow$ Stable

NOTE: • If all roots in first column are +ve then it is stable.

• No of sign changes in first column tells no. of roots on right side

Q> Determine whether the system is stable or not: $3s^4 + 10s^3 + 5s^2 + 5s + 2 = 0$

Ans

s^4	3	5	2
s^3	10/5	5/5	0
s^2	3.5	2	0
s^1	$-1/7$	0	
s^0	2		

$\Rightarrow$ unstable

Q> Determine whether the system is stable or not: $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

Ans

s^5	1	2	3	0
s^4	1	2	5	0
s^3	$\cancel{0} \to \varepsilon$	-2	0	
s^2	$2(\varepsilon+1)/\varepsilon$	5	0	
s^1	-2	0		
s^0	5			

Replace 0 with ε and use $\lim_{\varepsilon \to 0}$

$\lim_{\varepsilon \to 0} \frac{2\varepsilon + 2}{\varepsilon}$ is +ve

$\therefore$ There are 2 sign changes $\Rightarrow$ 2 roots.
System is marginally stable

NOTE: If a there is a 0 in first column then system is marginally stable.

• Substitute $s = 1/z$ & multiply equation with z^5

$$1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5 = 0$$

z^5	5	2	1	0
z^4	3	2	1	0
z^3	$-4/3$	$-2/3$	0	
z^2	$1/2$	1	0	
z^1	2	0		
z^0	1			

$\Rightarrow$ 2 roots

Q> $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

Ans

s^6	1	8	20	16	0
s^5	$2/2$	$12/2$	$16/2$	0	
s^4	$2/2$	$12/2$	$16/2$	0	
s^3	0	0	0		
s^3	$4/4$	$12/4$	0		
s^2	3	8			
s^1	$1/3$	0			
$6\times s^0$	$8/3$	0			
s^0	16				

NOTE The row above the row in which there are all zeroes is called auxialiary polynomial.

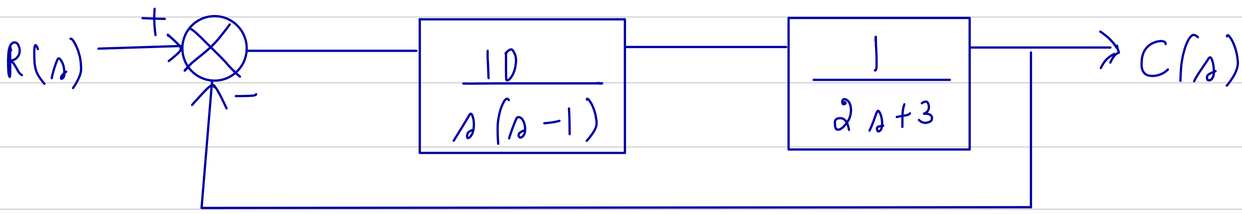
$$A(s) = s^4 + 6s^2 + 8$$

$$\frac{dA}{ds} = 4s^3 + 12s$$

→ Replace the zero row with coefficients of dA/ds

$s^4 + 6s^2 + 8 = 0 \quad \Rightarrow \quad s = \pm j\sqrt{2}$

Q> Is this system stable or not? If it is unstable determine the no of roots on the right half of s plane



$$T(s) = \frac{\left(\frac{10}{s(s-1)}\right)\left(\frac{1}{2s+3}\right)}{1 + \frac{10}{s(s-1)(2s+3)}} = \frac{10}{s(s-1)(2s+3) + 10}$$

$$= \frac{10}{2s^3 + s^2 - 3s + 10}$$

Characteristic equation : $2s^3 + s^2 - 3s + 10 = 0$

s^3	2	−3	0
s^2	1	10	0
s	−23	0	
s^0	10		

⇒ unstable

⇒ 2 sign changes ⇒ 2 roots on right half.

Q> $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

Ans

s^5	1	8	7	0
s^4	4	8	4	0
s^4	1	2	1	0
s^3	6	6	0	
s^3	1	1	0	
s^2	1	1	0	
s	0	0		
s	2	0		
$2\times$ s^0	2	0		
s^0	4			

$A(s) = s^2 + 1$

$\frac{dA}{ds} = 2s$

$s = \pm j$

$\Rightarrow$ Marginally stable.

Q> Check if $s^3 + 5s^2 + 25s + 30$ have roots whose real part is more negative than -1.

Ans Substitute $s = (z-1)$

$(z-1)^3 + 5(z-1)^2 + 25(z-1) + 30 = 0$

$z^3 - 1 - 3z^2 + 3z + 5z^2 + 5 - 10z + 25z - 25 + 30 = 0$

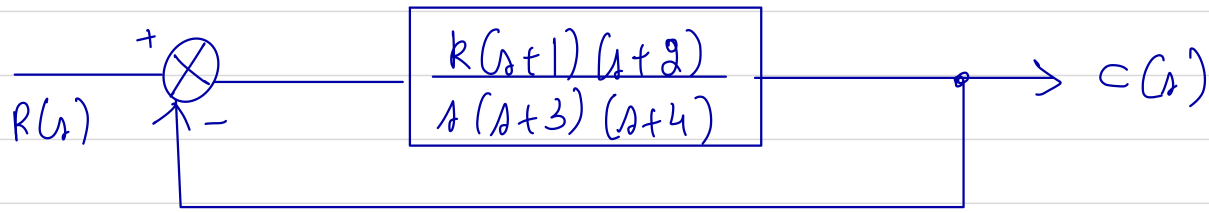
$z^3 + 2z^2 + 18z + 9 = 0$

z^3	1	18	0
z^2	2	9	0
z	$27/2$	0	
1	9	0	

$\Rightarrow$ Stable

All roots have real part less than -1

Q>

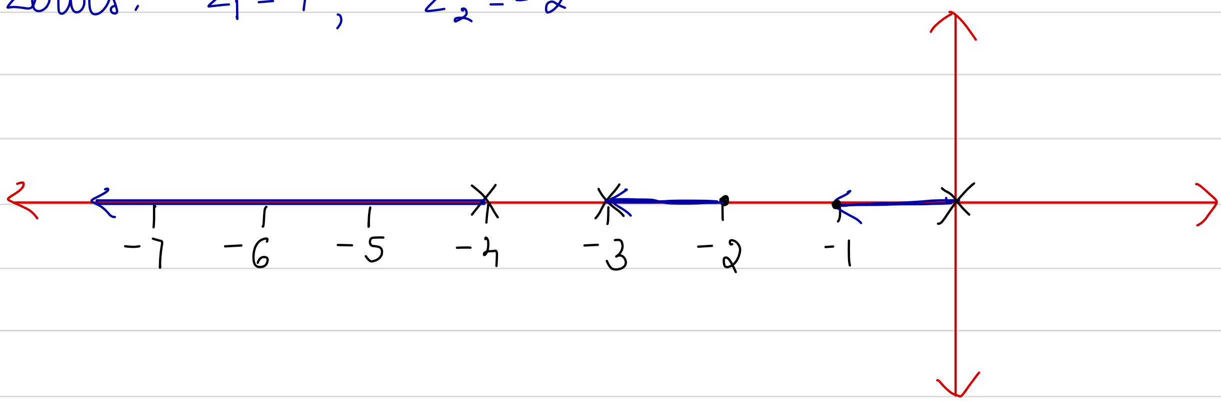


Using root locus method solve:

Ans open loop TF = $G(s)\cdot H(s) = \dfrac{K(s+1)(s+2)}{s(s+3)(s+4)}$

Poles: $p_1 = 0$, $p_2 = -3$, $p_3 = -4$

zeros: $z_1 = -1$, $z_2 = -2$



NOTE: $\phi_A = \dfrac{(2q+1)\,180^\circ}{n-m}$

$$-\sigma_A = \frac{\Sigma \text{ real part of poles} - \Sigma \text{ real parts of zeroes}}{\text{number of poles} - \text{number of zeroes}}$$

Root locus:

1) Draw the root locus plot for the feedback system with the characteristic equation:

$$1 + \frac{k}{s(s+1)(s+2)} = 0$$

Ans Step 1: Locate poles & zeros:

Poles : $s=0$, $s=-1$, $s=-2$

Step 2: Angle of asymptotes

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}$$

n = no. of poles = 3

m = no. of zeros = 0

$$\phi_A = (2q+1)60^\circ$$

$q=0 \Rightarrow \phi_A = 60^\circ$

$q=1 \Rightarrow \phi_A = 180^\circ$

$q=2 \Rightarrow \phi_A = 300^\circ$

Step 3: Find centroid.

$$-\sigma_A = \frac{-1-2}{3} = -1$$

Step 4: Find breakaway point:

$$\frac{dk}{ds} = 0$$

$$k = -s(s+1)(s+2)$$

$$\therefore \frac{dk}{ds} = -[(s+1)(s+2) + s(s+2) + s(s+1)] = 0$$

$\therefore\ s_1 = -0.423$

$s_2 = -1.577 \rightarrow$ not included as it does not lie in root locus

Step 5: Use routh criteria to find point of interaction with $j\omega$ axis.

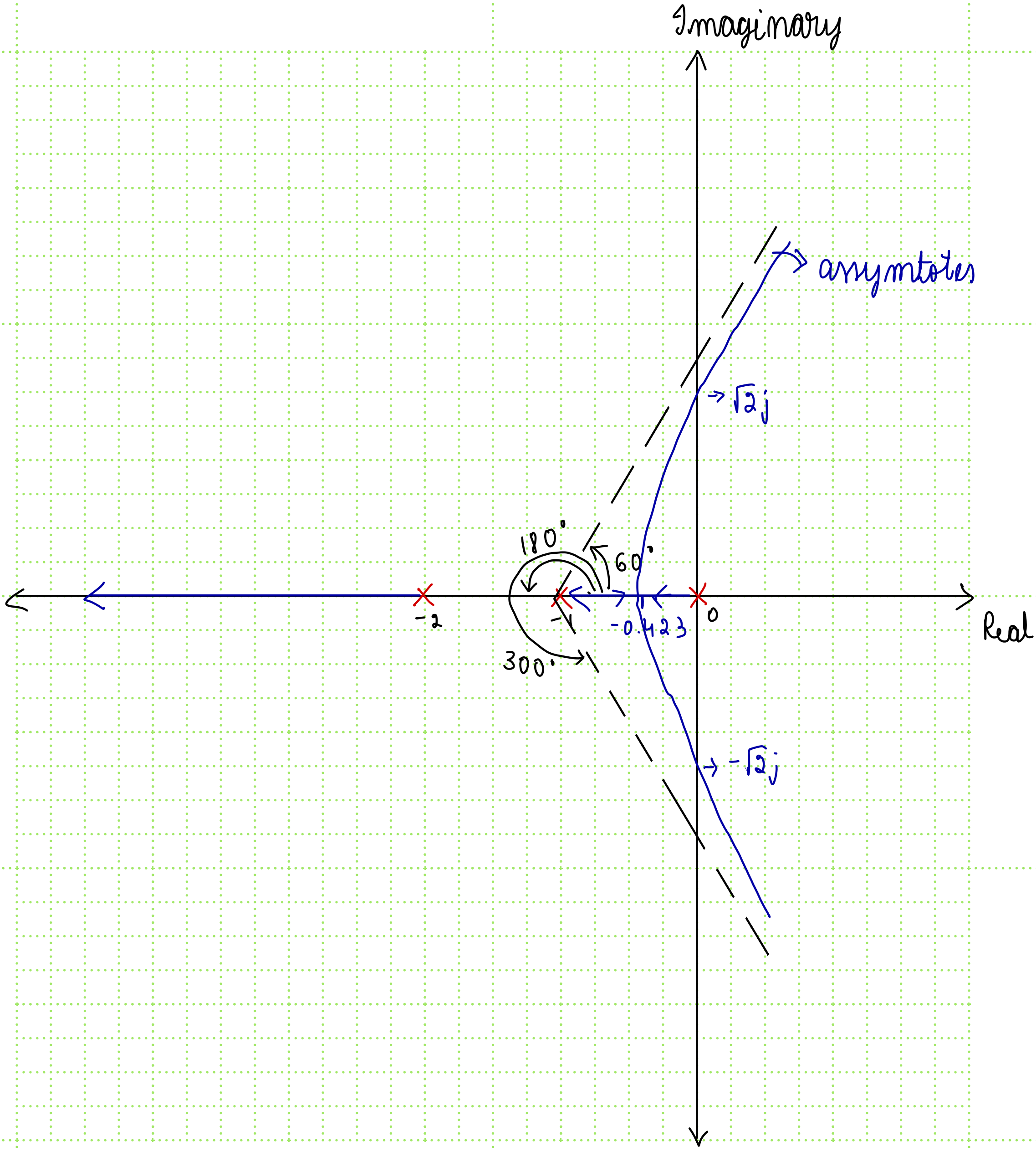
$s(s+1)(s+2)+k = s(s^2+3s+2)+k = s^3+3s^2+2s+k$

s^3	1	2	0
s^2	3	k	0
s	0	0	
1			

For $k=6$, $s \rightarrow 0$

Auxiliary equation $3s^2+6=0$

$s = \pm j\sqrt{2}$



Imaginary
asymtotes
√2j
180°
60°
-2
-1
-0.423
0
Real
300°
-√2j

Q> Draw the root locus for open loop TF : $G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$

Ans characteristic equation:

$$1 + \frac{K}{s(s+4)(s^2+4s+20)}$$

1> Poles : $0, -4, -2+4j, -2-4j$

2> $\phi_A \Rightarrow q = 0 \Rightarrow 45^\circ$

$q = 1 \Rightarrow 135^\circ$

$q = 2 \Rightarrow 225^\circ$

$q = 3 \Rightarrow 315^\circ$

$$\phi_A = \frac{(2q+1)180^\circ}{4}$$

3> $-\sigma_A = \frac{-4 \quad 2-2}{4} = -2$ //

4> $K = -s(s+4)(s^2+4s+20$

$$\frac{dK}{ds} = -\left[(s+4)(s^2+4s+20) + s(s^2+4s+20) + (2s+4)s(s+4)\right] = 0$$

$s_1 = -2 \qquad s_2 = -2 + 2.45j \qquad s_3 = -2 - 2.45j$

5> Angle of departure $= 180 - (63 + 117 + 90)$
$= -90^\circ$ //

6> Point of intersection with $j\omega$ axis :

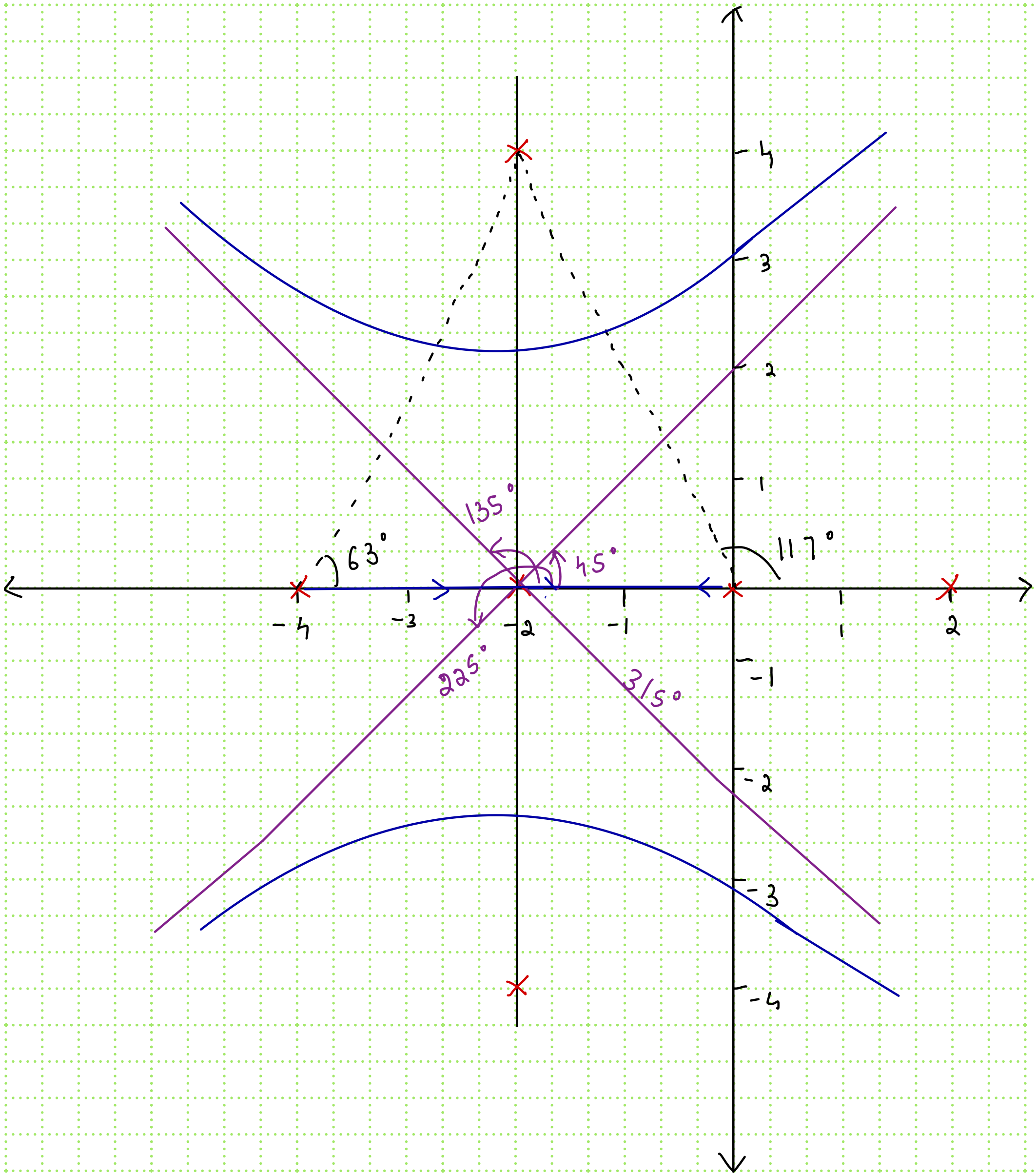
$$s(s+4)(s^2+4s+20)+k = s(s^3+4s^2+20s+4s^2+16s+80)+k$$
$$= s^4+8s^3+36s^2+80s+k$$

s^4	1	36	k
s^3	1	10	0
s^2	26	k	0
s	0	0	
1			

$s \to 0$ for $k = 260$

$26s^2 + 260 = 0$

$\Rightarrow s = \pm j\sqrt{10}$



4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
63°
117°
135°
45°
225°
315°

Q) A feedback control system has an open loop transfer function:

$$G(s)H(s) = \frac{k}{s(s+3)(s^2+2s+2)}$$

Find root locus as k is varied from 0 to ∞

Ans i) Poles: $0, -3, -1+j, -1-j$

$$s^2 + 2s + 2 = 0 \Rightarrow s_1, s_2 = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$s_1, s_2 = -1 \pm j$$

ii) $\phi_A = \frac{(2q+1)180^\circ}{n-m}$ $\quad n = 4$, $m = 0$

$\phi_A = 45^\circ$ when $q = 0$
$= 135^\circ$ when $q = 1$
$= 225^\circ$ when $q = 2$
$= 315^\circ$ when $q = 3$

iii) $-\sigma_A = \frac{-3-1-1}{4} = -1.25$

iv) $\frac{K}{s(s+3)(s^2+2s+2)} + 1 = 0$

$$K = -s(s+3)(s^2+2s+2)$$

$$K = -s(s^3 + 2s^2 + 2s + 3s^2 + 6s + 6)$$

$$K = -(s^4 + 5s^3 + 8s^2 + 6s)$$

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 16s + 6) = 0$$

$s_1 = -2.28$ $\qquad s_2, s_3 = -0.73 \pm 0.35j$

iv> Angle of departure $= 180^\circ - (90^\circ + 28^\circ + 133^\circ)$

$$\phi_p = -71^\circ$$

v> Cutoff points :

$$s^4 + 5s^3 + 8s^2 + 6s + K = 0$$

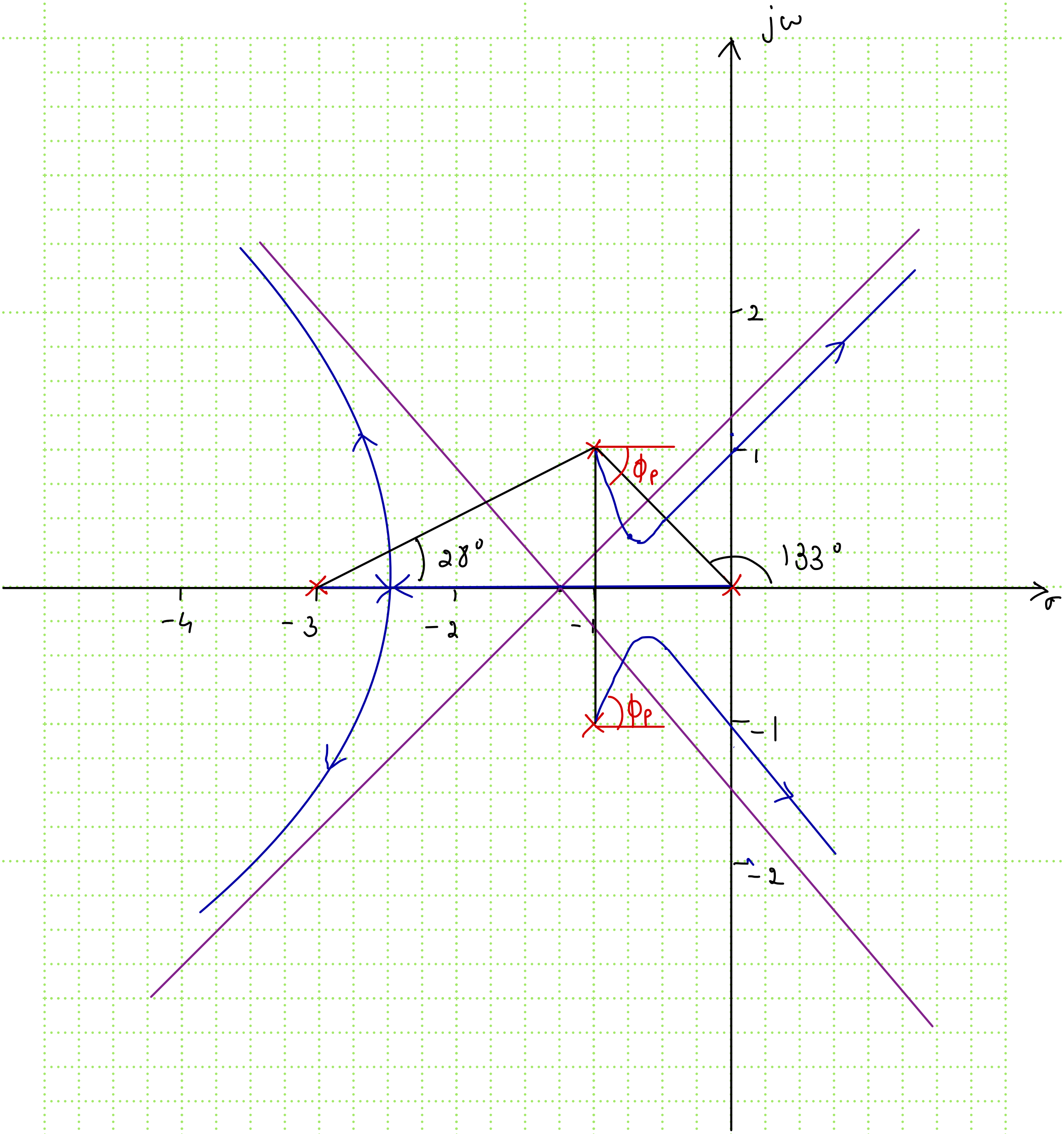
s^4	1	8	K
s^3	5	6	0
s^2	6.8	K	
s	0	0	

$$6.8 \times 6 = 5K$$

$$K = 8.16$$

Auxiliary equation: $6.8s^2 + 8.16 = 0$

$$s = \pm j\sqrt{1.2}$$



jω
σ
2
1
-1
-2
-4
-3
-2
-1
28°
133°
ϕ_p
ϕ_p

Q> Plot root locus for, $G(s)H(s) = \frac{k(s^2+6s+10)}{s^2+2s+10}$

Ans 1> Poles : $s^2+2s+10=0$

$s = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3j$ $\quad n=2$

Zeroes : $s^2+6s+10=0$, $s = -3 \pm j$ $\quad m=2$

2> No assymptotes, $n=m$

3> Breakaway points ;

$k = -\frac{s^2+2s+10}{s^2+6s+10} \rightarrow (1)$

$\frac{dk}{ds} = \frac{(2s+2)(s^2+6s+10) - (2s+6)(s^2+2s+10)}{(s^2+6s+10)^2} = 0$

$(s+1)(s^2+6s+10) - (s+3)(s^2+2s+10) = 0$

$\Rightarrow s^3 + 6s^2 + 10s + s^2 + 6s + 10 - s^3 - 2s^2 - 10s - 3s^2 - 6s - 30 = 0$

$\Rightarrow 2s^2 - 20 = 0$ $\quad s = \pm\sqrt{10}$

When $s = \pm\sqrt{10}$ is substituted in (1), k is negative. But $k > 0$, so no breakaway points.

Relation between time & frequency domain

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta j\omega + \omega_n^2}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

let $u = \frac{\omega}{\omega_n}$ → normalised frequency

$u_r = \frac{\omega_r}{\omega_n}$ = resonant frequency

$$T(j\omega) = \frac{1}{1 - u^2 + j2\zeta u}$$

$$= \frac{1}{\sqrt{(1-u^2) + 2\zeta u}} \left(- \tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)\right)$$

Magnitude, $M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$

Phase, $\phi = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$

$\frac{dM}{du} = 0$ (to find peak magnitude)

$u^2 = 1 - 2\zeta^2$

$\therefore$ Resonant frequency, $u_r = \sqrt{1 - 2\zeta^2}$

Magnitude $M(u_r) = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

Phase, $\phi(u_r) = -\tan^{-1}\left(\dfrac{2\zeta\sqrt{1-2\zeta^2}}{1-(1-2\zeta^2)}\right)$

$$= -\tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

Also when $u = 0$, $M = 1$, $\phi = 0$

$u = \infty$, $M \to 0$, $\phi = -\pi$

$u = 1$, $M = \dfrac{1}{2\zeta}$, $\phi = -\dfrac{\pi}{2}$

Q> Draw the bode plote for:

$$G(s) = \frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$$

Ans is TF in time constant form:

$$G(s) = \frac{64 \times 2\left(\frac{s}{2}+1\right)}{0.5s\left(\frac{s}{0.5}+1\right)64\left(\frac{s^2}{64}+\frac{s}{20}+1\right)}$$

$$G(j\omega) = \frac{4(0.5j\omega+1)}{j\omega(2j\omega+1)((0.125j\omega)^2+0.05j\omega+1)}$$

Term	cutoff	Slope	change
$\frac{4}{j\omega}$	$\omega_{c_1} = 0$	-20	-20
$1+\frac{j\omega}{0.5}$	$\omega_{c_2} = 0.5$	-20	$-20-20=-40$
$1+\frac{j\omega}{2}$	$\omega_{c_2} = 2$	$+20$	$-40+20=-20$
$\left(\frac{j\omega}{8}\right)^2+0.4\frac{j\omega}{8}+1$	$\omega_{c_3} = 8$	-40 (2nd order)	$-20-40=-60$

Initial gain:

$$G(0.4j) = \frac{4(0.5\times0.4j+1)}{j\times0.4(1+0.8j)(-(0.125\times0.4)^2+1+0.02j)}$$

$$|G(0.4j)| =$$

$$|G(j0.4)| = \frac{10 \times 1.019}{1.28 \times 0.9977}$$

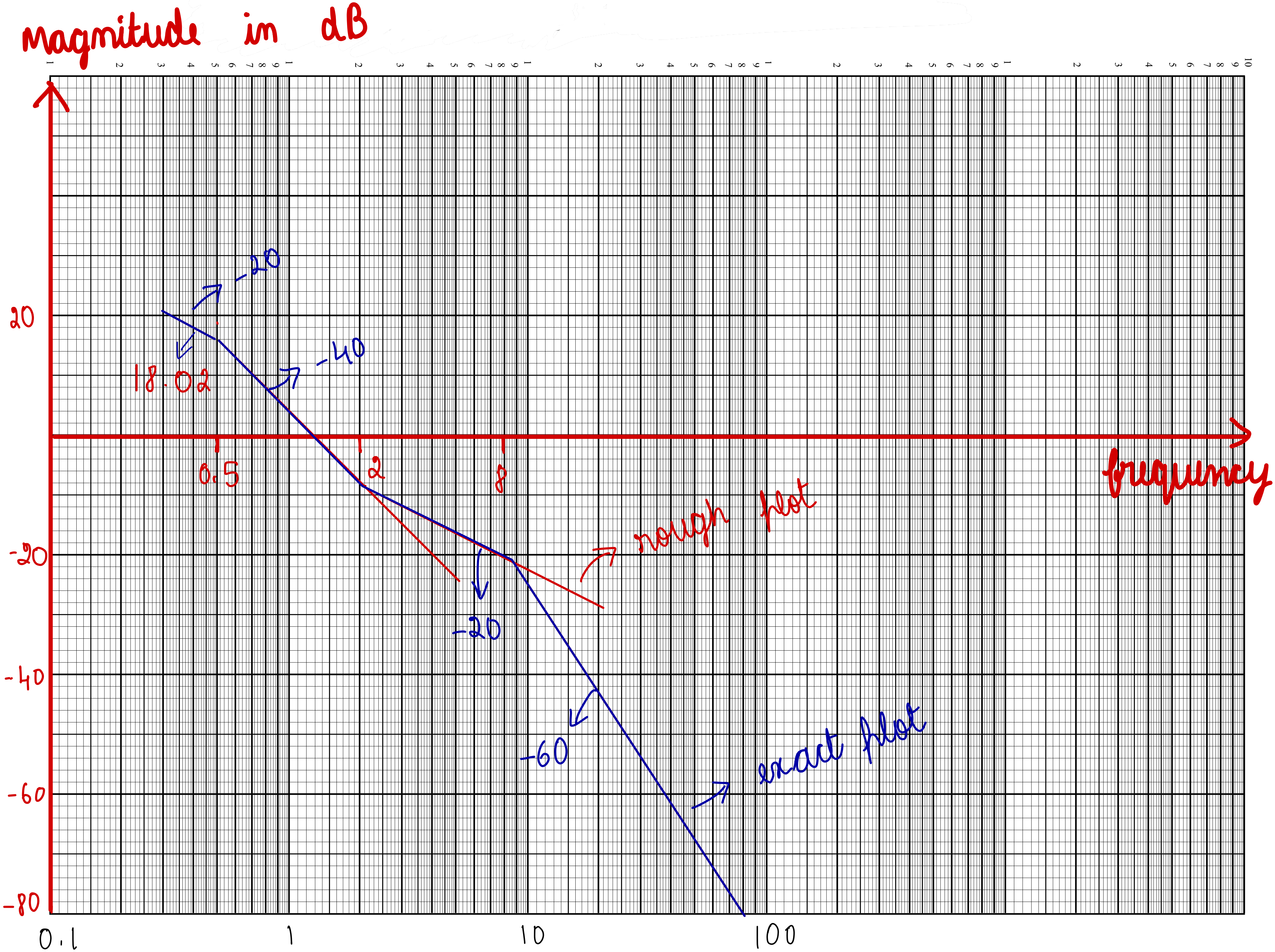
$$= 7.97$$

in dB, $20 \log (7.97) = 18.029$

For phase plot:

$$\phi = \tan^{-1}\left(\frac{\omega}{2}\right) - 90^\circ - \tan^{-1}(2\omega) - \tan^{-1}\left(\frac{3.2\omega}{64-\omega^2}\right)$$

To find phase plot: Put 5-6 values of ω and then find ϕ.


magnitude in dB
20
-20
-40
-60
-80
18.02
-20
-40
-20
-60
0.5
2
8
frequency
rough plot
exact plot
0.1
1
10
100

Q2> Sketch Bode plot for $G(s) = \dfrac{75(1+0.2s)}{s(s^2+16s+100)}$

Ans $G(s) = \dfrac{75\ (1+s/5)}{100s\left(\left(\frac{s}{10}\right)^2 + 1.6\frac{s}{10} + 1\right)}$

$= \left(\dfrac{3}{4s}\right) \dfrac{(1+s/5)}{\left(\left(\frac{s}{10}\right)^2 + 1.6\frac{s}{10} + 1\right)}$

Term	cutoff	Slope	change
$(3/4s)$	$\omega_{c_1} = 0$	-20	-20
$(1+s/5)$	$\omega_{c_2} = 5$	-20	0
$\left(\frac{s}{10}\right)^2 + 1.6\frac{s}{10} + 1$	$\omega_{c_3} = 10$	-40	-40

Initially at $\omega = 5$ approx value:
$20\log(0.75) - 20\log(5) = -16.45$

Angles:

ω	$-90°$	$\tan^{-1}(0.2\omega)$	$-\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$	Net
0.5	$-90°$	5.7	-4.6	-88.9
1	$-90°$	11.3	-9.2	-87.9
5	$-90°$	43	-46.8	-91.8
10	$-90°$	63.43	90	-116.6
50	$-90°$	84.3	161.6	-167.3
100	$-90°$	87.1	170.8	-173.7

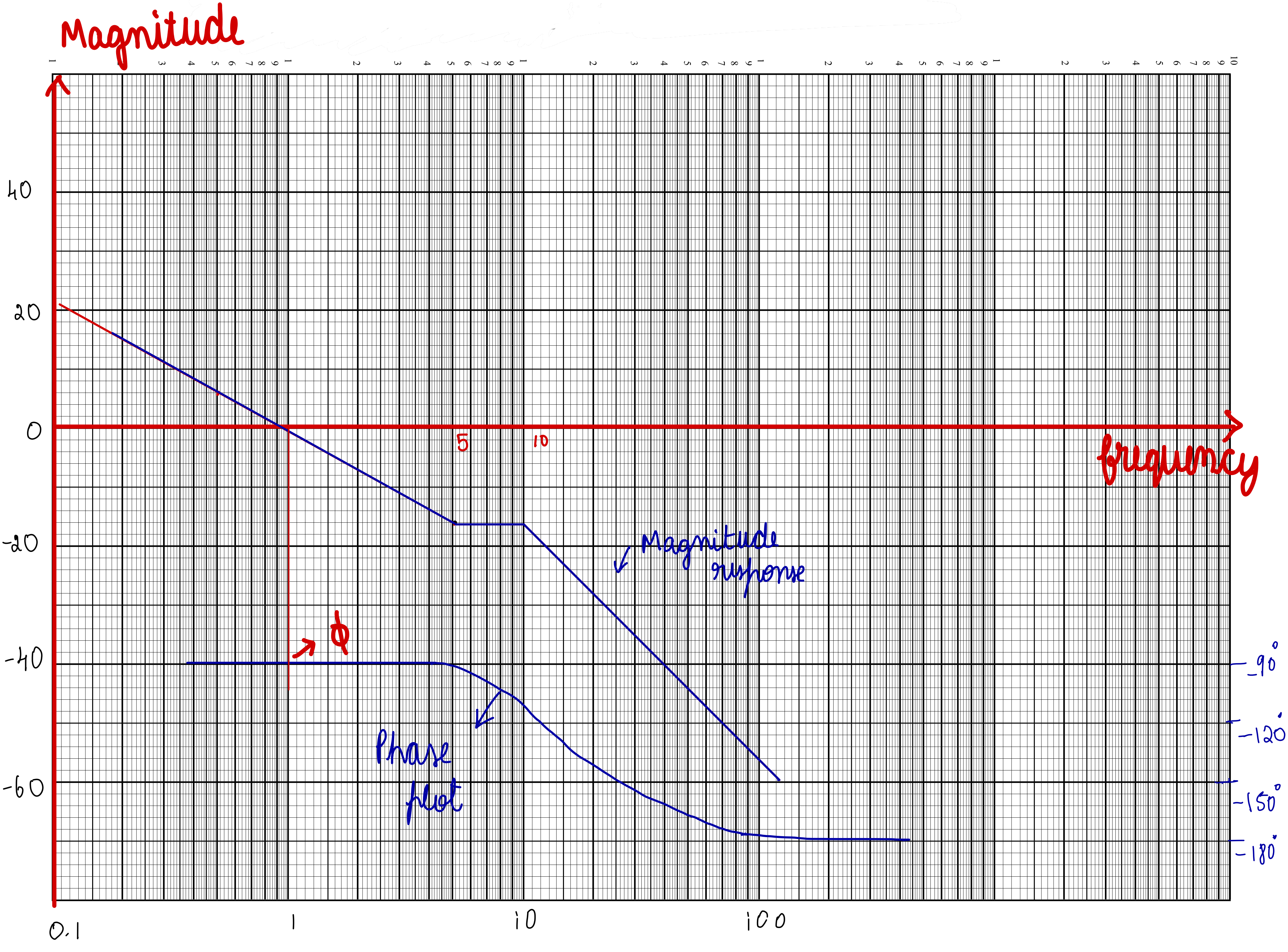
Phase margin $= \phi + 180°$

ϕ = Phase when magnitude = 0 dB

$\phi = -88°$

$\therefore$ Phase margin $= 92°$

Gain margin = Gain at which phase $= 180°$



Magnitude
40
20
0
-20
-40
-60
5
10
frequency
Magnitude response
ϕ
Phase plot
-90°
-120°
-150°
-180°
0.1
1
10
100

Q3) Find Bode plot for $G(s) = \frac{k}{s(s+2)(s+10)}$ for unit feedback system.

Find k for marginally stability

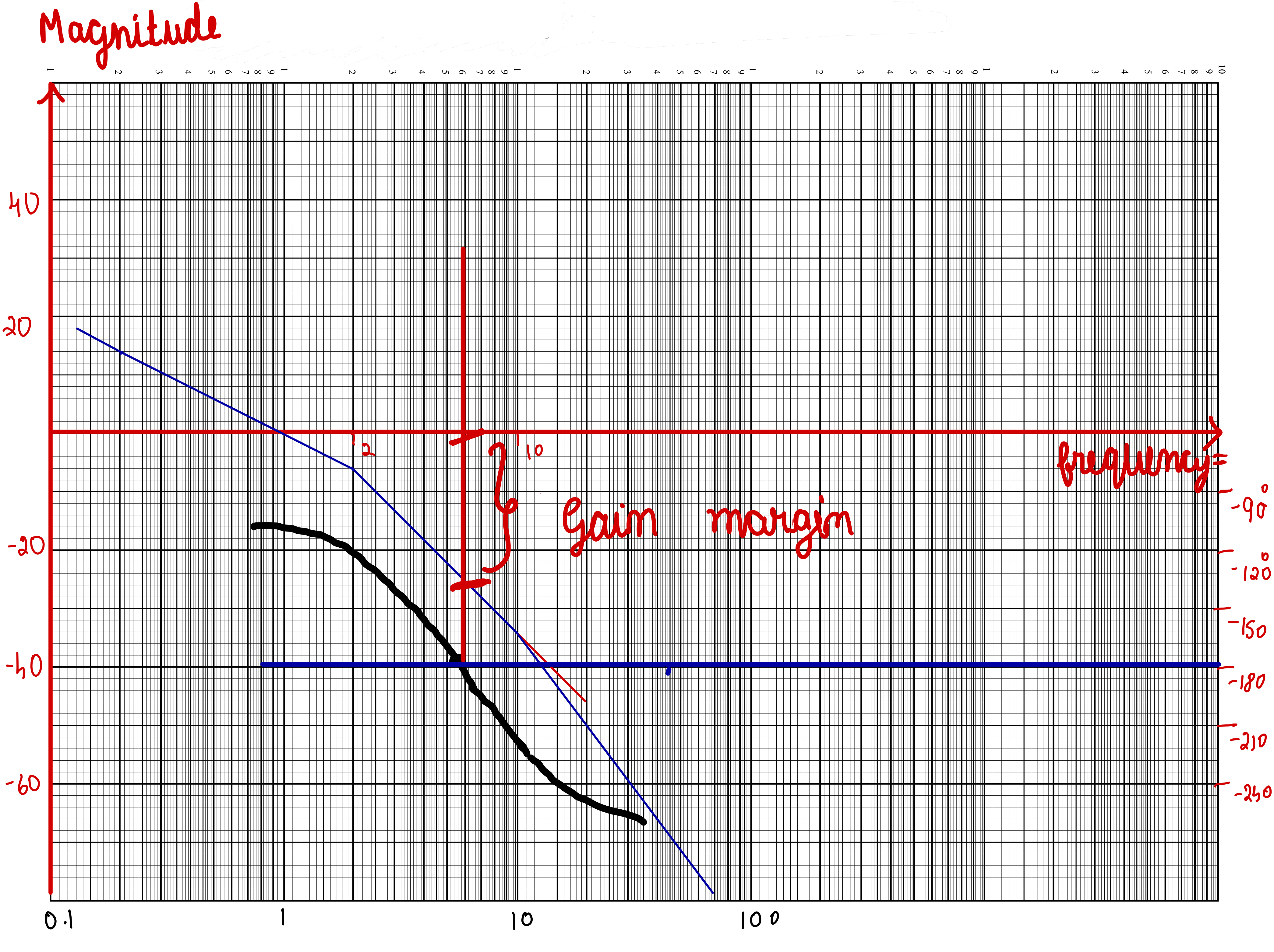
Ans $G(s) = \frac{k}{20s\,(1+s/2)(1+s/10)}$

Let $k' = \frac{k}{20} = 1$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega/2)(1+j\omega/10)}$$

Term	cutoff	Slope	Change
$1/j\omega$	$\omega_{c_1} = 0$	-20	-20
$1+j\omega/2$	$\omega_{c_2} = 2$	-20	-40
$1+j\omega/10$	ω_{c_3} 10	-20	-60

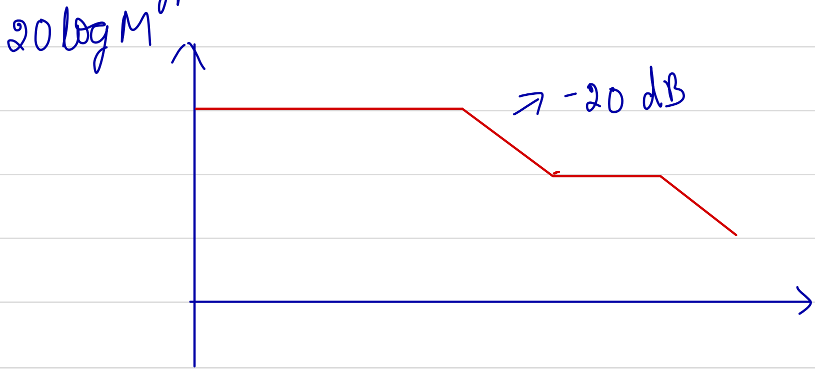
Gain margin $= 20 \log k'$



Magnitude
40
20
-20
-40
-60
2
10
frequency
Gain margin
-90°
-120°
-150
-180
-210
-240
0.1
1
10
100

Q) For each of the bode plot shown, find the type & constant.

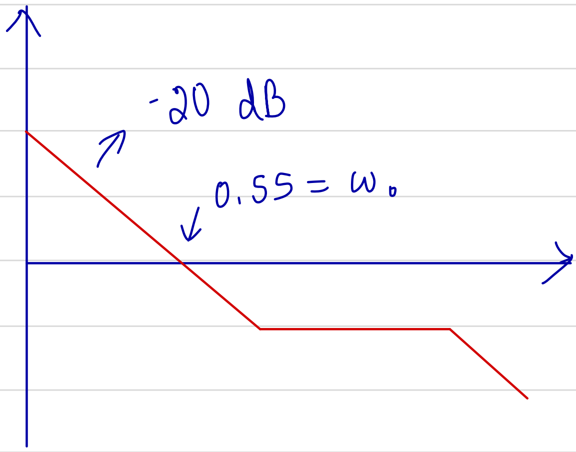
i)



Ans Type 0: $20 \log K_p = 25$

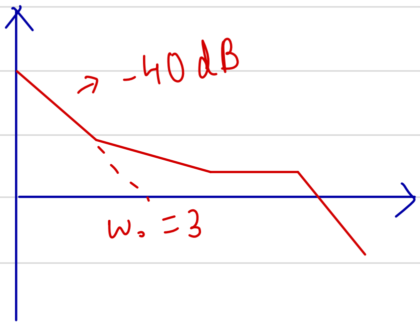
$K_p = 17.78$

ii)



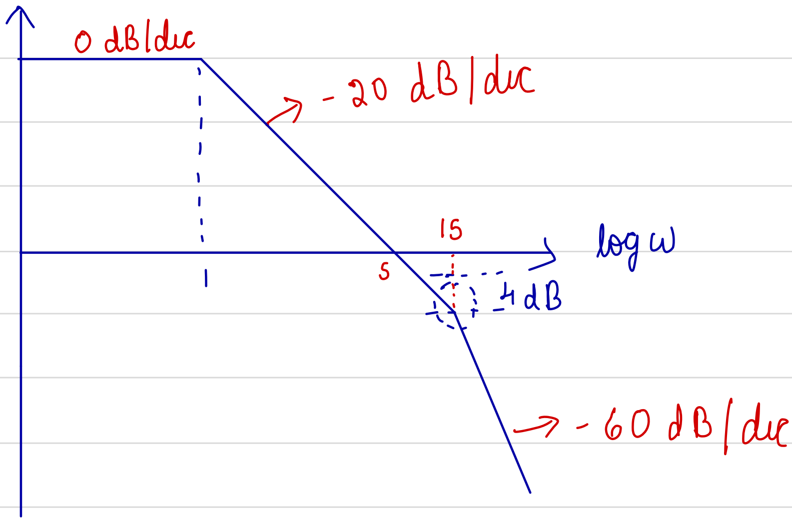
Ans Type 1: $K_v = \omega_0 = 0.55$

iii)



Ans Type 2: $\sqrt{K_a} = \omega_0 \Rightarrow K_a = \omega_0^2 = 9$

Q> Obtain the expression for open loop transfer function for a system with unity feedback whose log |magnitude| plot is shown in figure below.



Ans This is type 0 system:

$$G(j\omega) = \frac{k}{(1+j\omega)\left(1 + j2\xi\frac{\omega}{15} + \left(\frac{j\omega}{15}\right)^2\right)}$$

(15 = cutoff)

$$-20 = \frac{y_2 - 0}{\log 1 - \log 5}$$

$$\Rightarrow 20 \log 5 = y_2 = 14\,dB$$

$$20 \log K = 14$$

$$K = 5$$

NOTE

$$-10 \log (4\xi^2) = \text{error}$$

$$-10 \log (4\xi^2) = 4$$

$$4\xi^2 = 0.398$$

$$\xi = \sqrt{0.099}$$

$$\therefore \xi = 0.315$$

$$\therefore\ G(j\omega) = \frac{5}{(1+j\omega)\left(1+0.042j\omega - \frac{\omega^2}{15}\right)}$$

NOTE: 0 dB ⇒ k

-20 dB/dec ⇒ 1st order term

4 dB error = 2nd order term

3 dB error or error not mentioned ⇒ Square of same 1st order term.