

Algorithms for Big Data

Spring Semester 2022

Exercise Set 6

Definition 1 (Hadamard matrix) We define $H_1 = [1]$ and $H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$. We write $F = \frac{1}{\sqrt{n}}H_n$, dropping n from the index (and assuming n is a power of two).

Exercise 1:

Show that $\|Fx\|_2 = \|x\|_2$ for any $x \in \mathbb{R}^n$.

Exercise 2:

Show that $F \times F = I$.

Exercise 3:

Show algorithm that given $x \in \mathbb{R}^n$ computes Fx in time $\mathcal{O}(n \log n)$.

Definition 2 (Fourier matrix) Let $\omega = (-1)^{1/n} = \cos(2\pi/n) + i \sin(2\pi/n)$. We define W

as a $n \times n$ matrix such that $W = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}$.

Exercise 4:

Show that $\|Wx\|_2 = \|x\|_2$ for any $x \in \mathbb{C}^n$.

Exercise 5:

Show that $W \times \overline{W} = I$.

Exercise 6:

Show algorithm that given $x \in \mathbb{C}^n$ computes Wx in time $\mathcal{O}(n \log n)$.