University of Wrocław: Algorithms for Big Data (Fall'19)	20/01/2020

Lecture 13: MPC

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1 Massively Parallel Computing

Modern distributed computing model (captures map-reduce, hadoop, spark, etc.).

- input of size n
- k machines, each of space S, $S = n^{1-\delta}$, $k \cdot S = \mathcal{O}(n)$.
- output (might be too large, e.g. size n that does not fit on a single machine)

Computation:

- computation happens over R rounds
- each machine: near linear computation per round, so total computation cost $\mathcal{O}(n^{1+o(1)}R)$
- each machine communicates $\sim S$ bits per round, so total communication cost $\mathcal{O}(nR)$

goal: minimize R

1.1 Sorting (Tera-Sort)

Intuition: if we partition input onto machines, so each machine receives contiguous fragment of size S, then we are done in a single round (each machine sorts and outputs its own part, output == concatenation of outputs).

Idea:

- each machine receives input
- each machine samples randomly from its input
- each sample is sent to single machine (1 round)
- 1 machine gathers all the samples, sorts locally, and sents back to everyone approximate histogram
- machines use approximate histogram to decide how to partition locally their input and sent it to proper receivers
- then everyone sorts their parts

Total sample size $s = \mathcal{O}(S)$, and whp histogram is with $\pm\varepsilon n$ error, where $\frac{\log n}{\varepsilon^2} \leq s$. We are ok if error is $\mathcal{O}(S)$. This is satisfied when $S^{3/2} = \Omega(n\sqrt{\log n})$, or $S = \widetilde{\Omega}(n^{2/3})$, or $k = \widetilde{\mathcal{O}}(n^{1/3}) \implies$ sorting in $\mathcal{O}(1)$ rounds

2 Connectivity

Input: list of edges, partitioned (arbitrarily), size N output: each vertex labeled with id of connected component

Notation: id of vertex $\pi(v)$, label of vertex $\ell(v)$, $\Gamma(v)$ denotes neighbourhood Approach:

- let L_v be the set of vertices $\{u : \ell(u) = \ell(v)\}$ – always a subset of connected component of v
- initially $\ell(v) \leftarrow \pi(v)$
- some vertcies are called *active*
- every L_v will have exactly one active vertex (e.g. one with smallest ℓ), that makes decisions wrt whole L_v, assume cannonically its v
- in each round:
  - each active vertex becomes a *leader* with ppb 1/2
  - if v is a leader, mark whole L_v as a leaders
  - every active non-leader v, find $w \in \Gamma(L_v)$ that is a leader and minimizes $\pi(w)$
  - whole L_v joins L_w

Observation: each round decreases *in expectation* number of components by a factor of $1/4$, so $\mathcal{O}(\log n)$ rounds are enough. Implementation details:

- edges are stored locally
- split vertices of large degree into smaller vertices of degree $\mathcal{O}(S)$ (this is already non-trivial and requires e.g. sorting)
- ditributed data structure implementing vertex info required (implemented e.g. via sorting, emulates sending messages over the edges etc.)

3 MST [PRS16]

Input: graph of $N = n^{1+\delta}$ edges. Output: some approximation of a maximal matching. Output fits into single machine (one edge per vertex). Assume $S = n^{1+\varepsilon}$.

- Split output randomly into machines.
- Each machine i computes MST of its input T_i
- Recurse on $\bigcup_i T_i$

Filtering step reduces total number of edges from $n^{1+\delta}$ to $n^{1+\delta-\varepsilon}$, thus the number of rounds is $\mathcal{O}(\delta/\varepsilon)$.

4 Maximal matching [BHH19]

(Setting the same as in previous algorithm)

- Sample E', set of edges of size $\mathcal{O}(S)$.
- Send E' to one machine. Compute M', the maximal matching of E'.
- Remove all vertices from $V[E']$ from graph. Recurse on remaining graph. Call M the result of recursion.
- Return $M \cup M'$.

Correctness: simulated greedy.

Lemma 1. *If E' is a set of edges picked by sampling with probability p. Let I be set of vertices not adjacent to E'. With very high probability, $|E[I]| \leq 2n/p$.*

Proof. Follows from concentration bounds, since if we pick set E_1 of size $2n/p$, then expected intersection size $\mathbb{E}|E[I] \cap E'| = 2n$, so with exponentially small probability its not empty. We then take union bound over all possible subsets. □

Theorem 2. *Number of rounds is $\mathcal{O}(\delta/\varepsilon)$.*

Proof. Sampling probability needs to be $p = S/m = n^{\varepsilon-\delta}$. So in each step number of edges is reduced from $n^{1+\delta}$ to $2n^{1+\delta-\varepsilon}$. □

5 Maximal matching, approach 2

- Partition edges randomly into machines.
- Each machine receives G_i.
- Each machine computes M_i, maximal matching of G_i.
- Everyone sends M_i to a coordinator machine.
- Coordinator outputs M, maximal matching from $\bigcup_i M_i$.

The limiting factor is that all of the maximal-matchings need to be aggregated on a single machine, so $S \geq n \cdot \frac{N}{S}$, or $S \geq \sqrt{n \cdot N}$, which for dense graphs is $n^{3/2}$.

Claim: algorithm outputs maximal matching of size $\Theta(n)$, so its $\Theta(1)$-approximation of MM.

Proof. Consider greedy algorithm going through edges in order of $M_1, \ldots, M_k$. W.l.o.g. maximal matching is of size $\mathcal{O}(n)$. Consider step i (processing M_i).

Lemma 3. *Assume we have already selected $o(n)$ edges from $M_1 \cup \ldots \cup M_{i-1}$, as otherwise we are done.*
E_{old} – all edges in G_i that are adjacent to already selected vertex
μ_{old} – size of a maximum matching in G_i using only edges in E_{old}.
There exists matching in $G_1 \cup \ldots \cup G_i$ of size $\mu_{old} + \Omega(n/k)$.

Proof. G contains a matching of size $\Theta(n) - 2\mu_{\text{old}} = \Theta(n)$ that is not adjacent to any of the vertices from already selected maching. By random partitioning, $\Theta(n/k)$ of those edges land in G_i, so G_i contains matching of size $\mu_{\text{old}} + \Omega(n/k)$. $\square$

So any maximal matching in G_i needs to be of size at least $\mu_{\text{old}} + \Omega(n/k)$. At most μ_{old} of edges in any maximal matching can be adjacent to vertices already blocked in previous rounds, so from maximal matching we are getting $\Omega(n/k)$ new edges. $\square$

References

[BHH19] Soheil Behnezhad, MohammadTaghi Hajiaghayi, and David G. Harris. Exponentially faster massively parallel maximal matching. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1637–1649. IEEE Computer Society, 2019.

[PRS16] Gopal Pandurangan, Peter Robinson, and Michele Scquizzato. Fast distributed algorithms for connectivity and MST in large graphs. In Christian Scheideler and Seth Gilbert, editors, *Proceedings of the 28th ACM Symposium on Parallelism in Algorithms and Architectures, SPAA 2016, Asilomar State Beach/Pacific Grove, CA, USA, July 11-13, 2016*, pages 429–438. ACM, 2016.