

## Lecture 1: Approximate Counting, Distinct Elements

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# 1 Approximate counting

The problem is to maintain a counter that supports following operations:

reset(), [ $n \leftarrow 0$ ]  
 inc(), [ $n \leftarrow n + 1$ ]  
 query(), [output  $n$ ]

Simple lower bound of  $\log(n)$  bits for exact (information-theoretic lower bound).

**Goal:** algorithm that queried outputs  $n'$  such that  $\Pr(|n - n'| > \varepsilon n) < \delta$ .

## 1.1 Morris' algorithm [Sr.78]

**Local state:**  $X$  [int], represents  $n \sim 2^X$ . The crucial part of algorithm is to design how we increase  $X$ .

**Inc:**  $X \leftarrow X + 1$  with some small probability ( $\sim 2^{-X}$ ), with intuition being that the ppb of  $n$  being exactly  $2^{X+1} - 1$  is  $2^{-X}$ .

Let us analyze increment probability  $= 2^{-X}$ . Let  $X_n$  be random variable denoting state of algorithm after  $n$  increases.

**Theorem 1.**

$$\mathbf{E}[2^{X_0}] = 2^{X_0} = 1 \tag{1}$$

$$\mathbf{E}[2^{X_n}] = n + 1 \text{ by induction} \tag{2}$$

*Proof.*

$$\begin{aligned} \mathbf{E}2^{X_{n+1}} &= \sum_{j=0}^{\infty} \Pr(X_n = j) \cdot \mathbf{E}(2^{X_{n+1}} | X_n = j) \\ &= \sum_{j=0}^{\infty} \Pr(X_n = j) \cdot \left( 2^j \left( 1 - \frac{1}{2^j} \right) + \frac{1}{2^j} \cdot 2^{j+1} \right) \\ &= \sum_{j=0}^{\infty} \Pr(X_n = j) 2^j + \sum_j \Pr(X_n = j) \\ &= \mathbf{E}2^{X_n} + 1 \\ &= (n + 1) + 1 \end{aligned}$$

□

Morris algorithm output:  $Z = 2^{X_n} - 1 \leftarrow$ , which is an unbiased estimator of  $n$  (that is  $\mathbf{E}[Z] = n$ ).

### 1.1.1 Analysis of variance to extract guarantees:

**Theorem 2.** We show inductively that  $\mathbf{E}[2^{2X_n}] = 3/2n^2 + 3/2n + 1$ .

*Proof.* see exercise □

Since

$$\begin{aligned}\mathbf{Var}[Z] &= \mathbf{Var}[2^{X_n}] \\ &= \mathbf{E}[2^{2X_n}] - (\mathbf{E}[2^{X_n}])^2 \\ &= \frac{3}{2}n^2 + 3/2n + 1 - (n+1)^2 \\ &= \frac{1}{2}n^2 - \frac{1}{2}n,\end{aligned}$$

by Chebyshev's inequality  $\Pr(|Z - n| > \varepsilon n) \leq 1/(2\varepsilon^2)$ .

This only gives failure probability  $\delta < \frac{1}{2}$  for  $\varepsilon > 1$ , which is not very informative: (large) constant approximation with constant probability. But that was to be expected: our algorithm only outputs powers of two, so it cannot do much better job.

## 1.2 Morris+

Repeat  $k$  times independently, take average of estimations. Since variance is additive:  $\mathbf{Var}(Z') = \frac{1}{k^2}(\mathbf{Var}(Z_1) + \mathbf{Var}(Z_2) + \dots + \mathbf{Var}(Z_k)) = 1/k \mathbf{Var}(Z)$  so number of iterations necessary becomes:  $k = \mathcal{O}(\frac{1}{\varepsilon^2\delta})$  (ok for 9/10 ppb of correctness, bad for whp correctness).

## 1.3 Morris++

Run  $t$  copies of Morris+ algorithm, each with  $\delta = \frac{1}{3}$  and take median of estimations as a final estimation. Each estimation is ok with probability  $\geq \frac{2}{3}$ , so for the median to fail at least  $\frac{1}{6}$  fraction of estimations need to fail (all too large or all too small) Chernoff bound gives us:

$$\Pr\left(\sum_{i=1}^t Y_i \leq \frac{t}{2}\right) \leq \Pr\left(\left|\sum_{i=1}^t Y_i - \mathbf{E}\sum_{i=1}^t Y_i\right| \geq \frac{t}{6}\right) \leq 2e^{-t/3} < \delta \quad (3)$$

for  $t = \Theta(\log(1/\delta))$ . Final **bit** complexity  $\mathcal{O}(\log \log(n/(\varepsilon\delta))^{\frac{1}{\varepsilon^2}} \log(\frac{1}{\delta}))$ .

**Lower bound:**  $\Omega(\log \log_{1+\varepsilon} n) = \Omega(\log(1/\varepsilon) + \log \log n)$  (for  $\delta = 0$ , its trickier to prove lowerbound involving  $\delta$ )

## 2 Distinct elements

**Input:** Stream of values  $i_1, i_2, \dots, i_m$  from  $[n]$  query()  $\leftarrow$  number of distinct elements

**Trivial solution:** remember the stream, bitvector

## 2.1 Flajolet Martin [FM85]

Pick a hash function  $h : [n] \rightarrow [0, 1]$  (for a moment let us assume ideal real numbers, and perfectly random hash function).

1. initially  $Z = 1$
2. input  $X$ :  $Z = \min(Z, h(X))$
3. estimator:  $Y = 1/Z - 1$

**Observation 3.** *Repeats do not affect  $Z$ .*

If  $t$  is the number of distinct elements, then  $Z = \min(r_1, r_2, \dots, r_t)$  where  $r_i$  are all independent and from  $[0, 1]$ .

**Lemma 4.**

$$\mathbf{E}[Z] = \frac{1}{t+1} \quad (4)$$

*Proof.* Pick fresh  $A$  at random from  $[0, 1]$ . By symmetry,

$$\mathbf{E}[Z] = \mathbf{Pr}[A < Z] = \mathbf{Pr}[A \text{ is minimal among } A, r_1, \dots, r_t] = \frac{1}{(t+1)}.$$

□

**Lemma 5.**

$$\mathbf{E}[Z^2] \leq \frac{2}{(t+1)(t+2)} \quad (5)$$

*Proof.* Pick fresh  $A, B$  at random from  $[0, 1]$ . By symmetry,  $\mathbf{E}[Z^2] = \mathbf{Pr}[A < Z \wedge B < Z] = \frac{2}{(t+1)(t+2)}$  □

*Alternative proof.*

$$\begin{aligned} \mathbf{E}[Z^2] &= \int_0^\infty \mathbf{Pr}(Z^2 > \lambda) d\lambda \\ &= \int_0^\infty \mathbf{Pr}(Z > \sqrt{\lambda}) d\lambda \\ &= \int_0^1 (1 - \sqrt{\lambda})^t d\lambda \\ &= 2 \int_0^1 u^t (1 - u) du \quad [u = 1 - \sqrt{\lambda}] = \frac{2}{(t+1)(t+2)} \end{aligned}$$

□

$$\mathbf{Var}[Z] = \frac{2}{(t+1)(t+2)} - \frac{1}{(t+1)^2} = \frac{t}{(t+1)^2(t+2)} < (\mathbf{E}[Z])^2 \quad (6)$$

**Remark 6.** *Applying Chebyshev's inequality  $\rightarrow$  results in a guarantee of a (large) constant approximation with lets say  $\frac{9}{10}$  probability.*

**Issue:**  $\mathbb{E}[\frac{1}{Z}] \neq \frac{1}{\mathbb{E}[Z]}$ , but concentrating  $Z$  with  $1 + \varepsilon$  multiplicative error will give  $1 + \varepsilon$  multiplicative error for  $\frac{1}{Z}$ .

## 2.2 FM+

To reach better approximation guarantee, we need to concentrate our output around expected value.

**Approach 1** copy approach from Morris' algorithm - "repeat  $k$  times and take average" to improve variance, set  $k = \mathcal{O}(\frac{1}{\varepsilon^2})$  for  $\frac{9}{10}$  probability of  $1 + \varepsilon$  approximation.

**Approach 2** replace "take minimum" with "take  $k$ -th smallest value" (to be analyzed  $\rightarrow$  exercise).

## 2.3 FM++

To improve probability of success, repeat FM+ algorithm  $t = \mathcal{O}(\log \delta^{-1})$  times, and take median of answers. This boosts probability of success to  $1 - \delta$ .

Total memory complexity is

$\mathcal{O}(\log n \frac{1}{\varepsilon^2} \log \delta^{-1})$  of **words** (each word is  $\log n$  bits).

## 2.4 Issues

**Recall** "for a moment let us assume ideal real numbers".

We only care about relative order of hashes, and use actual value as an estimator. Using hash-functions of form  $h : [n] \rightarrow \{\frac{0}{M}, \frac{1}{M}, \dots, \frac{M-1}{M}, \frac{M}{M}\}$  for some  $M = n^3$ , as it only introduces small relative error (whp each hash is  $\geq \frac{1}{n}$  thus relative error introduced is at most  $(1 + \frac{1}{n})$ , and wlog  $\varepsilon > \frac{1}{n}$ ), and whp there are no collisions of hashes.

**Recall** "and perfectly random hash function".

Randomness vs. pseudorandomness  $\rightarrow$  c.f. exercises

## 3 Further reading

- hyperloglog algorithm, which very efficient in theory and practice, but has extremely nontrivial analysis [DF03] [HNH13]
- [Bla18] - optimal  $\Theta(\log n + \frac{\log \delta^{-1}}{\varepsilon^2})$  bits.

## References

- [Bla18] Jaroslaw Blasiok. Optimal streaming and tracking distinct elements with high probability. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 2432–2448, 2018.
- [DF03] Marianne Durand and Philippe Flajolet. Loglog counting of large cardinalities (extended abstract). In *Algorithms - ESA 2003, 11th Annual European Symposium, Budapest, Hungary, September 16-19, 2003, Proceedings*, pages 605–617, 2003.

- [FM85] Philippe Flajolet and G. Nigel Martin. Probabilistic counting algorithms for data base applications. *J. Comput. Syst. Sci.*, 31(2):182–209, 1985.
- [HNH13] Stefan Heule, Marc Nunkesser, and Alexander Hall. Hyperloglog in practice: algorithmic engineering of a state of the art cardinality estimation algorithm. In *Joint 2013 EDBT/ICDT Conferences, EDBT '13 Proceedings, Genoa, Italy, March 18-22, 2013*, pages 683–692, 2013.
- [Sr.78] Robert H. Morris Sr. Counting large numbers of events in small registers. *Commun. ACM*, 21(10):840–842, 1978.

## A Probability recap

**Definition 7.** 1. The empty set is an event,  $\emptyset \in \mathcal{F}$

2. Given a countable set of events  $A_1, A_2, \dots$ , its union is also an event,  $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

3. if  $A$  is an event, then so is the complementary set  $A^c$

**Definition 8.** 1.  $\Pr(\emptyset) = 0, \Pr(\Omega) = 1$

2. if  $A_1, A_2, \dots$  are mutually excluding events, then  $\Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$

A  $\Pr : \mathcal{F} \mapsto [0, 1]$  satisfying these is called a probability.

The triple  $(\Omega, \mathcal{F}, \Pr)$  is called a probability space.

**Definition 9.** We define conditional probability as

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

**Theorem 10.** Let  $B_1, \dots, B_n$  be a partition of  $\Omega$ , then

$$\Pr(A) = \sum_{i=1}^n \Pr(A|B_i) \Pr(B_i) \tag{7}$$

**Definition 11.** Events  $A$  and  $B$  are called independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B). \tag{8}$$

When  $0 < \Pr(B) < 1$ , this is the same as

$$\Pr(A|B) = \Pr(A) = \Pr(A|B^c) \tag{9}$$

A family  $\{A_i : i \in I\}$  of events is called independent if

$$\Pr(\bigcap_{i \in J} A_i) = \prod_{i \in J} \Pr(A_i) \tag{10}$$

for any finite subset  $J$  of  $I$ .

**Definition 12.** A random variable is Informally: A quantity which is assigned by a random experiment. Formally: A mapping  $X : \Omega \rightarrow \mathbf{R}$ .

**Definition 13.** The cumulated distribution function(cdf) is:

$$F(x) = \Pr(X \leq x) \quad (11)$$

If satisfies following properties:

1.  $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow +\infty} F(x) = 1$
2.  $x < y \Rightarrow F(x) \leq F(y)$
3.  $F$  is right-continuous, ie.  $F(x+h) \rightarrow F(x)$  as  $h \downarrow 0$

**Definition 14.** The mean of a stochastic variable is

$$\mathbf{E}X = \sum_{i \in \mathbb{Z}} i \Pr(X = i)$$

in the discrete case, and

$$\mathbf{E}X = \int_{-\infty}^{+\infty} f(x) dx$$

in the continuous case. In both cases we assume that the sum/integral exists absolutely. The variance of  $X$  is

$$\mathbf{Var}X = \mathbf{E}(X - \mathbf{E}x)^2 = \mathbf{E}X^2 - (\mathbf{E}X)^2$$

**Definition 15.** The conditional expectation is the mean in the conditional distribution

$$\mathbf{E}(Y|X = x) = \sum_y y f_{Y|X}(y|x) \quad (12)$$

It can be seen as a stochastic variable: Let  $\psi(x) = \mathbf{E}(Y|X = x)$ , then  $\psi(X)$  is the conditional expectation of  $Y$  given  $X$

$$\psi(X) = \mathbf{E}(Y|X) \quad (13)$$

We have

$$\mathbf{E}(\mathbf{E}(Y|X)) = \mathbf{E}Y \quad (14)$$

**Definition 16.** Conditional variance  $\mathbf{Var}(Y|X)$  is the variance in the conditional distribution.

$$\mathbf{Var}(Y|X = x) = \sum_y (y - \psi(x))^2 f_{Y|X}(y|x) \quad (15)$$

This can also be written as

$$\mathbf{Var}(Y|X) = \mathbf{E}(Y^2|X) - (\mathbf{E}(Y|X))^2$$

and can be manipulated into

$$\mathbf{Var} = \mathbf{E}\mathbf{Var}(Y|X) + \mathbf{Var}\mathbf{E}(Y|X)$$

which partitions the variance of  $Y$ .

**Theorem 17** (Markov's inequality). Let  $X \geq 0$  be a random variable. Then for any  $k \geq 1$ :

$$\Pr(X \geq k \cdot \mathbf{E}[X]) \leq \frac{1}{k} \quad (16)$$

**Theorem 18** (Chebyshev's inequality). *Let  $X$  be a random variable. For any  $k > 0$ :*

$$\Pr(|X - \mathbf{E}[X]| \geq k \cdot \sqrt{\mathbf{Var}[X]}) \leq \frac{1}{k^2} \quad (17)$$

**Theorem 19** (Hoeffding bound). *Let  $X_1, X_2, \dots, X_n \in \{0, 1\}$  be fully independent random variables. Let  $X = \sum_i X_i$ . Then:*

$$\Pr(|X - \mathbf{E}[X]| \geq t) \leq 2 \exp\left(-\frac{t^2}{n}\right) \quad (18)$$