University of Wrocław: Algorithms for Big Data (Fall'19) 13/01/2020

Lecture 12: Coresets

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1 Coresets [BHI02]

Setup: given set of $P \subseteq \mathbb{R}^d$, compute $C_P() : \mathbb{R}^d \to \mathbb{R}$. For example:

- MEB, a minimal enclosing ball: $C_P(o)$ is a radius of minimal enclosing ball for set P centered at o.

$|P| = n$ is large, so storing it explicitly is out of the question. Observe that for fixed o, a single $p \in P$ is enough, that is $\forall_o \exists_{p \in P} C_P(o) = C_{\{p\}}(o)$. Can we generalize this observation so it works for all $o \in \mathbb{R}^2$ at once?

Definition 1. *We say that $S \subseteq P$ is a c-coreset for P if for any o and any $T \subseteq \mathbb{R}^d$ there is:*

$$C_{S \cup T}(o) \leq C_{P \cup T}(o) \leq c \cdot C_{S \cup T}(o)$$

Note: this is a stronger definition than just requiring that $f_S(o)$ is a c-approximation to $C_P(o)$ (take $T = \emptyset$).

Observe that in case of MEB we have for $A \subseteq B$: $C_A(o) \leq C_B(o)$, so the first inequality is "for free".

Lemma 2 (Merge property)**.** *If S is a c-coreset for P, and S' is a c-coreset for P', then $S \cup S'$ is a c^2-coreset for $P \cup P'$.*

Lemma 3 (Reduce property)**.** *If S is a c-coreset for P and P is a c-coreset for Q, then S is a c^2-coreset for Q.*

We sometimes want a stronger property (which for example MEB satisfies)

Property 4 (Disjoint merge)**.** *If S is a c-coreset for P and S' is a c-coreset for P' and $P \cap P' = \emptyset$, then $S \cup S'$ is a c-coreset for $P \cup P'$.*

Exercise: proof for MEB.

Theorem 5. *Assume that a problem is supported by a $(1+\alpha)$-coreset of size $f(\alpha)$, computable in linear space, with disjoint merge property. Then there is a streaming algorithm with $1+\varepsilon$ guarantee, with space $\mathcal{O}(f(\varepsilon/\log n)\log n)$.*

Proof. Sketch of a proof: put stream of n elements into binary tree. Each node stores coreset for the range below it.

For two sibling nodes N_1, N_2 covering sets A_1, A_2, at level i, and parent node N at level $i+1$, there is:

- N_1 is $(1+\alpha)^i$-coreset for A_1 (and the same for N_2, A_2)

- $N_1 \cup N_2$ is $(1+\alpha)^i$-coreset for $A_1 \cup A_2$
- N is constructed as $(1+\alpha)$-coreset for $N_1 \cup N_2$
- thus N is $(1+\alpha)^{i+1}$-coreset for $A_1 \cup A_2$

As a end-result, we have in the root $(1+\alpha)^{\log n}$-coreset for whole input. Selecting $\alpha = \mathcal{O}(\varepsilon/\log n)$ is enough. □

1.1 Coreset for MEB

Construction goes as follow. Choose dense set of directions $\{v_i\}_{i=1}^m$, such that for any other direction u, there is always some v_i such that angle$(u, v_i) \le \alpha$: this is $\sim \alpha$-net on unit-ball (up to trigonometry). We can choose such set of $m = (1/\alpha)^{\mathcal{O}(d)}$ directions.

Claim 6. *For any direction v_i, pick $p_i \in P$ that is extremal in that direction. Set $S = \{p_i\}_{i=1}^m$ is a $(1+\mathcal{O}(\alpha^2))$-coreset for P.*

Proof. Pick arbitrary T ($T = \emptyset$ w.l.o.g. in this proof), and P and constructed set S. Fix o. Pick furthest point $x \in P$, and close direction v_i, and maximal in this direction point $p_i \in S$. The angle x-o-p_i is small (at most α), so the stretch is upper bounded by $\frac{1}{\cos\alpha} = 1 + \mathcal{O}(\alpha^2)$. □

1.2 Coreset for median

Approximate median: given sequence of numbers A of number $a_1, \ldots, a_n$, return a such that $(1/2 \pm \varepsilon)n$ elements in A are smaller/larger than a.

Alternative formulation: find a that minimizes $C_A(a) = \max(|\{i : a_i \ge a\}|, |\{i : a_i \le a\}|)$.

Coreset of size $1/\varepsilon$: pick every εn element from sorted A. Easy to see that

$$C_{A\cup T}(a) \le C_{A\cup S}(a) \le (1+\varepsilon) C_{A\cup T}(a)$$

Plugging into the theorem, we obtain streaming median computation in space $\mathcal{O}(\log^2 n/\varepsilon)$. In fact this works for any quantile computation (but the error is additive). Improvement: pre-filter and keep only $1/\varepsilon^2$ elements (randomly), so space becomes $\mathcal{O}(\log^2(1/\varepsilon)/\varepsilon)$.

2 Graph algorithms

2.1 Certificates

Graph-theoretic approach, for decision problems.

Definition 7. *For property $\mathcal{P}$, and a graph G, we say that G' is a strong certificate for G if: for any H, $G \cup H$ is in $\mathcal{P}$ iff $G' \cup H$ is in $\mathcal{P}$.*

Examples:

- connectivity: any spanning forest
- non-bipartiteness: any spanning forest + single odd-cycle inducing edge
- edge/vertex connectivity

2.2 Spanners [ADD+93]

Subgraph approximately preserving distances:

Definition 8. *H, an edge subgraph of G, is a t-spanner of G, if for any u, v there is*

$$d_H(u,v) \le t \cdot d_G(u,v)$$

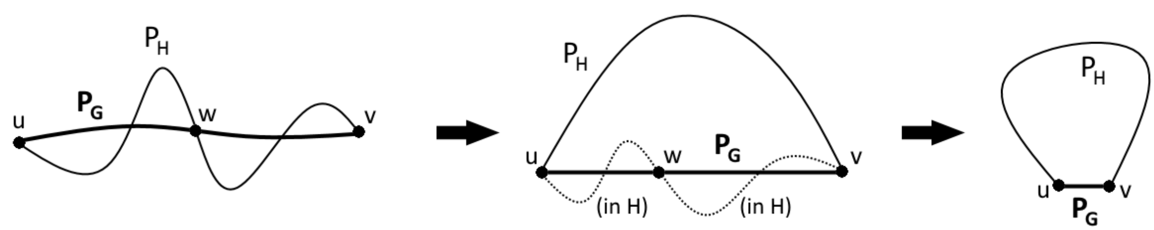
Theorem 9. *Any unweighted G contains t-spanner with $\mathcal{O}(n^{1+2/(t-1)})$ edges, and it can be computed in one pass.*

Proof. Maintain subgraph. Process edges one-by-one. If an edge causes a cycle of length $\le t+1$, do not insert it.

1. This constructs a t-spanner. (Nice easy exercise)

Proof. Let's denote a spanner by H as previously.

- (by contradiction) Assume that H maintained in that way is not a t-spanner. So, there exist a pair of vertices u, v (counterexample) s.t. $d_H(u,v) > td_G(u,v)$. Let's choose some shortest path between u, v in G, P_G and in H, P_H. The proof will have three stages.



- Now we argument that there also exists a counterexample s.t. P_G and P_H don't cross - only common vertices between them are u, v. This follows, because if there is a vertex w s.t. $w \ne u, w \ne v, w \in P_G, w \in P_H$, then we can substitute one of u, v (wlog, u) with w in the counterexample. That's because P_G, P_H are shortest paths between u, v in G, H, so the equalities below are true (if they wouldn't be, there would exist some shorter path):

$$d_H(u,w) + d_H(w,v) = d_H(u,v) > td_G(u,v) = td_G(u,w) + td_G(w,v)$$

  so either $d_H(u,w) > td_G(u,w)$ or $d_H(w,v) > td_G(w,v)$ (in other case contradiction by summing sides of opposite ($\le$) inequalities). By substituting $u \leftarrow w$ as many times as needed we have $P_G \cap P_H = \{u, v\}$.

- We have that P_G, P_H only have u, v common and now we argument that there also exists a counterexample s.t. $|P_G| = 1$. Similarly as previously, if there exists some w on P_G between u, v, we can substitute one of u, v (wlog, u) with it in the counterexample. This is because from triangle inequality in H we have (please note that there is no longer an equality in the left part because w is not on P_H and we need paths for $d_H(u,w), d_H(w,v)$ that contain some edges from outside of P_H, but the equality in the right part which we need is still true as $w \in P_G$):

$$d_H(u,w) + d_H(w,v) \ge d_H(u,v) > td_G(u,v) = td_G(u,w) + td_G(w,v)$$

then by same argument as previously, either $d_H(u,w) > td_G(u,w)$ or $d_H(w,v) > td_G(w,v)$. By substituting $u \leftarrow w$ as many times as needed, we finally have that u, v are neighbours in G (please note that by such substitution, we can again have intersections between P_G, P_H and we may need to move to the previous step, but $|P_G|$ constantly decreases).

- As we now have a counterexample where $|P_G| = 1$ and $|P_H| > t|P_G| = t$, we arrive at a contradiction: when we were processing the only edge on P_G, we should have added it to H as it didn't cause a cycle of length $\leq t+1$ - it doesn't cause that now as $d_H(u,v) = |P_H| \geq t+1$ and in the time of processing of that edge it also didn't as H at that time was a subgraph of the final H.

$\square$

2. The graph is sparse enough:

- Let $d = 2m/n$ be average degree.
- There is subgraph H of G with minimum degree $d' = d/2$: keep removing vertices with degree smaller than d'. We cannot end with no vertices, since then we removed less than m edges.
- H has every vertex of degree at least m/n, and no cycle of length $t+1$ or less.
- BFS tree of depth $t/2$ has no cycles in H
- $(m/n - 1)^{t/2} \leq |H| \leq n$, which implies bound on m

$\square$

References

[ADD+93] Ingo Althöfer, Gautam Das, David P. Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. *Discrete & Computational Geometry*, 9:81–100, 1993.

[BHI02] Mihai Badoiu, Sariel Har-Peled, and Piotr Indyk. Approximate clustering via core-sets. In John H. Reif, editor, *Proceedings on 34th Annual ACM Symposium on Theory of Computing, May 19-21, 2002, Montréal, Québec, Canada*, pages 250–257. ACM, 2002.