

1 Approximate counting

The problem is to maintain a counter that supports following operations:

reset(), [$n \leftarrow 0$]
 inc(), [$n \leftarrow n + 1$]
 query(), [output n]

Simple lower bound of $\log(n)$ bits for exact (information-theoretic lower bound).

Goal: algorithm that queried outputs n' such that $\Pr(|n - n'| > \varepsilon n) < \delta$.

1.1 Morris' algorithm [Sr.78]

Local state: X [int], represents $n \sim 2^X$. The crucial part of algorithm is to design how we increase X .

Inc: $X \leftarrow X + 1$ with some small probability ($\sim 2^{-X}$), with intuition being that the ppb of n being exactly $2^{X+1} - 1$ is 2^{-X} .

Let us analyze increment probability = 2^{-X} . Let X_n be random variable denoting state of algorithm after n increases.

Theorem 1.

$$\mathbf{E}[2^{X_0}] = 2^{X_0} = 1 \tag{1}$$

$$\mathbf{E}[2^{X_n}] = n + 1 \text{ by induction} \tag{2}$$

Proof.

$$\begin{aligned} \mathbf{E}2^{X_{n+1}} &= \sum_{j=0}^{\infty} \Pr(X_n = j) \cdot \mathbf{E}(2^{X_{n+1}} | X_n = j) \\ &= \sum_{j=0}^{\infty} \Pr(X_n = j) \cdot \left(2^j \left(1 - \frac{1}{2^j} \right) + \frac{1}{2^j} \cdot 2^{j+1} \right) \\ &= \sum_{j=0}^{\infty} \Pr(X_n = j) 2^j + \sum_j \Pr(X_n = j) \\ &= \mathbf{E}2^{X_n} + 1 \\ &= (n + 1) + 1 \end{aligned}$$

□

Morris algorithm output: $Z = 2^{X_n} - 1 \leftarrow$, which is an unbiased estimator of n (that is $\mathbf{E}[Z] = n$).

1.1.1 Analysis of variance to extract guarantees:

Theorem 2. We show inductively that $\mathbf{E}[2^{2X_n}] = 3/2n^2 + 3/2n + 1$.

Proof. see exercise □

Since

$$\begin{aligned}\mathbf{Var}[Z] &= \mathbf{Var}[2^{X_n}] \\ &= \mathbf{E}[2^{2X_n}] - (\mathbf{E}[2^{X_n}])^2 \\ &= \frac{3}{2}n^2 + 3/2n + 1 - (n+1)^2 \\ &= \frac{1}{2}n^2 - \frac{1}{2}n,\end{aligned}$$

by Chebyshev's inequality $\Pr(|Z - n| > \varepsilon n) \leq 1/(2\varepsilon^2)$.

This only gives failure probability $\delta < \frac{1}{2}$ for $\varepsilon > 1$, which is not very informative: (large) constant approximation with constant probability. But that was to be expected: our algorithm only outputs powers of two, so it cannot do much better job.

1.2 Morris+

Repeat k times independently, take average of estimations. Since variance is additive: $\mathbf{Var}(Z') = \frac{1}{k^2}(\mathbf{Var}(Z_1) + \mathbf{Var}(Z_2) + \dots + \mathbf{Var}(Z_k)) = 1/k \mathbf{Var}(Z)$ so number of iterations necessary becomes: $k = \mathcal{O}(\frac{1}{\varepsilon^2 \delta})$ (ok for 9/10 ppb of correctness, bad for whp correctness).

1.3 Morris++

Run t copies of Morris+ algorithm, each with $\delta = \frac{1}{3}$ and take median of estimations as a final estimation. Each estimation is ok with probability $\geq \frac{2}{3}$, so for the median to fail at least $\frac{1}{6}$ fraction of estimations need to fail (all too large or all too small) Chernoff bound gives us:

$$\Pr\left(\sum_{i=1}^t Y_i \leq \frac{t}{2}\right) \leq \Pr\left(\left|\sum_{i=1}^t Y_i - \mathbf{E}\sum_{i=1}^t Y_i\right| \geq \frac{t}{6}\right) \leq 2e^{-t/3} < \delta \quad (3)$$

for $t = \Theta(\log(1/\delta))$. Final **bit** complexity $\mathcal{O}(\log \log(n/(\varepsilon\delta))^{\frac{1}{\varepsilon^2}} \log(\frac{1}{\delta}))$.

Lower bound: $\Omega(\log \log_{1+\varepsilon} n) = \Omega(\log(1/\varepsilon) + \log \log n)$ (for $\delta = 0$, its trickier to prove lowerbound involving δ)

2 Distinct elements

Input: Stream of values i_1, i_2, \dots, i_m from $[n]$ query() \leftarrow number of distinct elements

Trivial solution: remember the stream, bitvector

2.1 Flajolet Martin [FM85]

Pick a hash function $h : [n] \rightarrow [0, 1]$ (for a moment let us assume ideal real numbers, and perfectly random hash function).

1. initially $Z = 1$
2. input X : $Z = \min(Z, h(X))$
3. estimator: $Y = 1/Z - 1$

Observation 3. *Repeats do not affect Z .*

If t is the number of distinct elements, then $Z = \min(r_1, r_2, \dots, r_t)$ where r_i are all independent and from $[0, 1]$.

Lemma 4.

$$\mathbf{E}[Z] = \frac{1}{t+1} \quad (4)$$

Proof. Pick fresh A at random from $[0, 1]$. By symmetry,

$$\mathbf{E}[Z] = \mathbf{Pr}[A < Z] = \mathbf{Pr}[A \text{ is minimal among } A, r_1, \dots, r_t] = \frac{1}{(t+1)}.$$

□

Lemma 5.

$$\mathbf{E}[Z^2] \leq \frac{2}{(t+1)(t+2)} \quad (5)$$

Proof. Pick fresh A, B at random from $[0, 1]$. By symmetry, $\mathbf{E}[Z^2] = \mathbf{Pr}[A < Z \wedge B < Z] = \frac{2}{(t+1)(t+2)}$ □

Alternative proof.

$$\begin{aligned} \mathbf{E}[Z^2] &= \int_0^\infty \mathbf{Pr}(Z^2 > \lambda) d\lambda \\ &= \int_0^\infty \mathbf{Pr}(Z > \sqrt{\lambda}) d\lambda \\ &= \int_0^1 (1 - \sqrt{\lambda})^t d\lambda \\ &= 2 \int_0^1 u^t (1 - u) du \quad [u = 1 - \sqrt{\lambda}] = \frac{2}{(t+1)(t+2)} \end{aligned}$$

□

$$\mathbf{Var}[Z] = \frac{2}{(t+1)(t+2)} - \frac{1}{(t+1)^2} = \frac{t}{(t+1)^2(t+2)} < (\mathbf{E}[Z])^2 \quad (6)$$

Remark 6. *Applying Chebyshev's inequality \rightarrow results in a guarantee of a (large) constant approximation with lets say $\frac{9}{10}$ probability.*

Issue: $\mathbb{E}[\frac{1}{Z}] \neq \frac{1}{\mathbb{E}[Z]}$, but concentrating Z with $1 + \varepsilon$ multiplicative error will give $1 + \varepsilon$ multiplicative error for $\frac{1}{Z}$.

2.2 FM+

To reach better approximation guarantee, we need to concentrate our output around expected value.

Approach 1 copy approach from Morris' algorithm - "repeat k times and take average" to improve variance, set $k = \mathcal{O}(\frac{1}{\varepsilon^2})$ for $\frac{9}{10}$ probability of $1 + \varepsilon$ approximation.

Approach 2 replace "take minimum" with "take k -th smallest value" (to be analyzed \rightarrow exercise).

2.3 FM++

To improve probability of success, repeat FM+ algorithm $t = \mathcal{O}(\log \delta^{-1})$ times, and take median of answers. This boosts probability of success to $1 - \delta$.

Total memory complexity is

$\mathcal{O}(\log n \frac{1}{\varepsilon^2} \log \delta^{-1})$ of **words** (each word is $\log n$ bits).

2.4 Issues

Recall "for a moment let us assume ideal real numbers".

We only care about relative order of hashes, and use actual value as an estimator. Using hash-functions of form $h : [n] \rightarrow \{\frac{0}{M}, \frac{1}{M}, \dots, \frac{M-1}{M}, \frac{M}{M}\}$ for some $M = n^3$, as it only introduces small relative error (whp each hash is $\geq \frac{1}{n}$ thus relative error introduced is at most $(1 + \frac{1}{n})$, and wlog $\varepsilon > \frac{1}{n}$), and whp there are no collisions of hashes.

Recall "and perfectly random hash function".

Randomness vs. pseudorandomness \rightarrow c.f. exercises

3 Further reading

- hyperloglog algorithm, which very efficient in theory and practice, but has extremely nontrivial analysis [DF03] [HNH13]
- [Bla18] - optimal $\Theta(\log n + \frac{\log \delta^{-1}}{\varepsilon^2})$ bits.

References

- [Bla18] Jaroslaw Blasiok. Optimal streaming and tracking distinct elements with high probability. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 2432–2448, 2018.
- [DF03] Marianne Durand and Philippe Flajolet. Loglog counting of large cardinalities (extended abstract). In *Algorithms - ESA 2003, 11th Annual European Symposium, Budapest, Hungary, September 16-19, 2003, Proceedings*, pages 605–617, 2003.

- [FM85] Philippe Flajolet and G. Nigel Martin. Probabilistic counting algorithms for data base applications. *J. Comput. Syst. Sci.*, 31(2):182–209, 1985.
- [HNH13] Stefan Heule, Marc Nunkesser, and Alexander Hall. Hyperloglog in practice: algorithmic engineering of a state of the art cardinality estimation algorithm. In *Joint 2013 EDBT/ICDT Conferences, EDBT '13 Proceedings, Genoa, Italy, March 18-22, 2013*, pages 683–692, 2013.
- [Sr.78] Robert H. Morris Sr. Counting large numbers of events in small registers. *Commun. ACM*, 21(10):840–842, 1978.

A Probability recap

Definition 7. 1. The empty set is an event, $\emptyset \in \mathcal{F}$

2. Given a countable set of events A_1, A_2, \dots , its union is also an event, $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

3. if A is an event, then so is the complementary set A^c

Definition 8. 1. $\Pr(\emptyset) = 0, \Pr(\Omega) = 1$

2. if A_1, A_2, \dots are mutually excluding events, then $\Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$

A $\Pr : \mathcal{F} \mapsto [0, 1]$ satisfying these is called a probability.

The triple $(\Omega, \mathcal{F}, \Pr)$ is called a probability space.

Definition 9. We define conditional probability as

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Theorem 10. Let B_1, \dots, B_n be a partition of Ω , then

$$\Pr(A) = \sum_{i=1}^n \Pr(A|B_i) \Pr(B_i) \tag{7}$$

Definition 11. Events A and B are called independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B). \tag{8}$$

When $0 < \Pr(B) < 1$, this is the same as

$$\Pr(A|B) = \Pr(A) = \Pr(A|B^c) \tag{9}$$

A family $\{A_i : i \in I\}$ of events is called independent if

$$\Pr(\bigcap_{i \in J} A_i) = \prod_{i \in J} \Pr(A_i) \tag{10}$$

for any finite subset J of I .

Definition 12. A random variable is Informally: A quantity which is assigned by a random experiment. Formally: A mapping $X : \Omega \rightarrow \mathbf{R}$.

Definition 13. The cumulated distribution function(cdf) is:

$$F(x) = \Pr(X \leq x) \quad (11)$$

If satisfies following properties:

1. $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow +\infty} F(x) = 1$
2. $x < y \Rightarrow F(x) \leq F(y)$
3. F is right-continuous, ie. $F(x+h) \rightarrow F(x)$ as $h \downarrow 0$

Definition 14. The mean of a stochastic variable is

$$\mathbf{E}X = \sum_{i \in \mathbb{Z}} i \Pr(X = i)$$

in the discrete case, and

$$\mathbf{E}X = \int_{-\infty}^{+\infty} f(x) dx$$

in the continuous case. In both cases we assume that the sum/integral exists absolutely. The variance of X is

$$\mathbf{Var}X = \mathbf{E}(X - \mathbf{E}x)^2 = \mathbf{E}X^2 - (\mathbf{E}X)^2$$

Definition 15. The conditional expectation is the mean in the conditional distribution

$$\mathbf{E}(Y|X = x) = \sum_y y f_{Y|X}(y|x) \quad (12)$$

It can be seen as a stochastic variable: Let $\psi(x) = \mathbf{E}(Y|X = x)$, then $\psi(X)$ is the conditional expectation of Y given X

$$\psi(X) = \mathbf{E}(Y|X) \quad (13)$$

We have

$$\mathbf{E}(\mathbf{E}(Y|X)) = \mathbf{E}Y \quad (14)$$

Definition 16. Conditional variance $\mathbf{Var}(Y|X)$ is the variance in the conditional distribution.

$$\mathbf{Var}(Y|X = x) = \sum_y (y - \psi(x))^2 f_{Y|X}(y|x) \quad (15)$$

This can also be written as

$$\mathbf{Var}(Y|X) = \mathbf{E}(Y^2|X) - (\mathbf{E}(Y|X))^2$$

and can be manipulated into

$$\mathbf{Var} = \mathbf{E}\mathbf{Var}(Y|X) + \mathbf{Var}\mathbf{E}(Y|X)$$

which partitions the variance of Y .

Theorem 17 (Markov's inequality). Let $X \geq 0$ be a random variable. Then for any $k \geq 1$:

$$\Pr(X \geq k \cdot \mathbf{E}[X]) \leq \frac{1}{k} \quad (16)$$

Theorem 18 (Chebyshev's inequality). *Let X be a random variable. For any $k > 0$:*

$$\Pr(|X - \mathbf{E}[X]| \geq k \cdot \sqrt{\mathbf{Var}[X]}) \leq \frac{1}{k^2} \quad (17)$$

Theorem 19. *Hoeffding bound/ Let $X_1, X_2, \dots, X_n \in \{0, 1\}$ be fully independent random variables. Let $X = \sum_i X_i$. Then:*

$$\Pr(|X - \mathbf{E}[X]| \geq t) \leq 2 \exp\left(-\frac{t^2}{n}\right) \quad (18)$$