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ASTR 600: Cosmology
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PHYS 600 : HW 2

Friedmann Equation II

I mentioned in class that the second Friedmann equation can be derived from the first Friedmann equation and the continuity equation. Work out the details of this derivation.

Cosmological Dimming

Show that the observed (bolometric, see below) surface brightness I_o of a source at redshift z is related to its intrinsic surface brightness I_e by

$$I_{\mathrm{o}} = I_{\mathrm{e}}(1+z)^{-4}$$

Magnitudes and K-corrections

This problem builds on our discussion of luminosity distances, highlighting some standard observational terminology as well as a few subtleties.

Define the apparent magnitude of an object with observed flux f as

$$m = -2.5\log_{10}\left[\frac{f}{f_0}\right]$$

where f_0 is the flux of a standard (traditionally Vega, but more modern systems use "AB-magnitudes"). The absolute magnitude M is defined as the apparent magnitude a source would have if it were at a distance of 10 pc.

- Using these definitions, show that

  $$m = M + DM(z)$$

  where $DM(z)$ is the "distance modulus" defined by

  $$DM(z) = 5\log_{10}\left[\frac{D_L(z)}{10\ \mathrm{pc}}\right]$$

So far, so good. However, these expressions are true for bolometric quantities (i.e. total fluxes integrated over all wavelengths). Astronomical observations are often done in wavelength bandpasses, and this gives rise to the K-correction term. Let us see how this comes about.

Consider the differential flux f_ν and the corresponding differential luminosity L_ν, defined as the flux/luminosity per unit frequency such that

$$f = \int f_\nu \, d\nu$$

and similarly for the luminosity.

- Explain why

$$S_\nu = (1+z) \frac{L_{\nu(1+z)}}{L_\nu} \frac{L_\nu}{4\pi D_L^2}$$

The apparent magnitude formula then gets modified to

$$m = M + DM + K$$

where the K-correction is

$$K = -2.5 \log_{10} \left[(1+z) \frac{L_{\nu(1+z)}}{L_\nu} \right]$$

The paper by Hogg 1999 in the bibliography is very useful here, as is the follow up paper on K-corrections.

A Static Universe

This is a modified version of Huterer, 3.10.

Consider the Friedmann equation where we explicitly write out the contributions from matter ρ_m, the cosmological constant Λ, and curvature $k = \kappa/R_0^2$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$$

Now (and we will discuss this in class further), matter has no pressure, and $P_\Lambda = -\rho_\Lambda$.

- Write out the second Friedmann equation for this model. You will need to recast the cosmological constant term as a density. Do so by making it look like the matter density in the first Friedmann equation. Finally, the density and pressure can be written as a sum of the individual components.
- Find a value of Λ and k such that $\dot{a} = 0$ and $\ddot{a} = 0$. Is this Universe open, closed or flat?

This is, by construction, a static Universe, and was one constructed by Einstein. However, it is also unstable to perturbations. To see this, imagine perturbing the scale factor by

$$a(t) = 1 + \delta a(t)$$

To lowest order, this perturbs the matter density by

$$\rho_m(t) = \rho_m[1 - 3\delta a(t)]$$

The cosmological constant is unaffected (it is a constant after all).

- Substitute this into the second Friedmann equation and derive a differential equation for $\delta a(t)$. Your answer should only depend on Λ.
- Solve for $\delta a(t)$. You may assume initial conditions $\delta a(t) = \delta a_0$ and $d(\delta a)/dt = 0$. Explain why this is an unstable solution.

Redshift Drift

This is a modified version of Problem 2.6 in Huterer, A Course in Cosmology.

Can one directly observe the expansion of the Universe through a changing redshift? To explore this, start from the definition of redshift

$$1 + z = \frac{a(t_0)}{a(t_1)}$$

and show that

$$\frac{dz}{dt_0} = (1 + z)H_0 - H(t_1)$$

To derive this, you may need to derive an expression for dt_1/dt_0 which you can do by eg. relating the frequency of light emitted at t_1 to that observed at t_0. (A similar factor comes into the definition of the luminosity distance).

Now, let's do some simple estimates. For a matter dominated flat Universe, we will show that

$$H(z) = H_0(1 + z)^{3/2}$$

Using this, estimate the change in redshift for an object at $z = 1$. Assume $H_0 = 100h\,\mathrm{km/s/Mpc}$

You may find arXiv:0310808 an interesting read on redshifts, and you will find a discussion of this effect towards the end of this article.

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(1.) Friedmann Equation II

Derive Friedmann equation II: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$

Given Friedmann equation I: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{K}{R_0^2 a^2}$
and continuity equation: $\dot{\rho} + 3H(\rho + P) = 0$

Solution:

Beginning with Friedmann I:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{K}{R_0^2 a^2}$$, where we multiply both sides by a^2

$$\dot{a}^2 = \frac{8\pi G}{3}\rho(t)a^2 - \frac{K}{R_0^2}$$, where we take the derivative with respect to time

$$\frac{d}{dt}\dot{a}^2 = \frac{8\pi G}{3}\frac{d}{dt}(\rho(t)a^2) - \cancelto{0}{\frac{d}{dt}\left(\frac{K}{R_0^2}\right)}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2a\dot{a}\rho)$$

$$\dot{a}\ddot{a} = \frac{4\pi G}{3}(\dot{\rho}a^2 + 2a\dot{a}\rho)$$

$$\frac{\dot{a}\ddot{a}}{a\dot{a}} = \frac{4\pi G}{3}(\dot{\rho}a + 2\dot{a}\rho)\frac{1}{\dot{a}}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\dot{\rho}\frac{a}{\dot{a}} + 2\rho\right)$$, where $\frac{a}{\dot{a}} = \frac{1}{H}$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\dot{\rho}\frac{1}{H} + 2\rho\right)$$, where we now incorporate the continuity equation

Continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0$$
$$\dot{\rho} = -3H(\rho + P)$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(-3\cancel{H}(\rho + P)\cancel{\frac{1}{H}} + 2\rho\right)$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(-3\rho - 3P + 2\rho)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\cancel{+3}\rho + 3P \cancel{-2\rho})$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

This matches the second Friedmann equation, so we have derived Friedmann II from Friedmann I and the continuity equation

2. Cosmological Dimming

Show $I_o = I_e(1+z)^{-4}$

$$I_e = \frac{f}{\Omega} = \frac{L}{4\pi d_m^2}\frac{1}{\Omega} = \frac{L}{4\pi d_m^2}\left(\frac{d_m}{D_A}\right)^2 = \frac{L}{4\pi D_A^2}$$

$\hookrightarrow f = \frac{L}{4\pi d_L^2}$, where d_L is luminosity distance

$\hookrightarrow \Omega = \alpha^2 = \left(\frac{D_A}{d_A}\right)^2$, where D_A is size of patch and d_A is angular distance

Substituting for f and Ω:

$$I_o = \frac{f}{\Omega}$$

comoving distance

$$I_o = \frac{\left(\frac{L}{4\pi d_L^2}\right)}{\left(\frac{D_A}{d_A}\right)^2},$$ where luminosity distance $d_L = d_m(1+z)$ and angular distance $d_A = \frac{d_L}{(1+z)^2}$

$$= \frac{\left(\frac{L}{4\pi d_m^2(1+z)^2}\right)}{\left(\frac{D_A^2}{\left(\frac{d_L}{(1+z)^2}\right)^2}\right)}$$

$$= \frac{L}{4\pi \cancel{d_m^2}\cancel{(1+z)^2}}\cdot\frac{1}{D_A^2}\cdot\frac{\cancel{d_L^2}}{(1+z)^4}\cdot\frac{\cancel{d_m^2}\cancel{(1+z)^2}}{\cancel{d_L^2}},$$ where we will now invoke $I_e = \frac{L}{4\pi D_A^2}$ from the first line of the solution

$$= \frac{\cancel{L}}{\cancel{4\pi D_A^2}(1+z)^4}\cdot\frac{I_e\,\cancel{4\pi D_A^2}}{\cancel{L}}$$

$$\boxed{I_o = I_e(1+z)^{-4}}$$

This describes cosmological dimming due to both spreading out of light and Universe expansion.

3. Magnitudes + K-Corrections

a. Show that $m = M + DM(z)$

Given $m = -2.5\log\left(\frac{f}{f_0}\right)$

$DM(z) = 5\log\left(\frac{D_L(z)}{10pc}\right)$

Solution:

$m = M + DM(z)$, where M is m at a distance of 10 pc

$$-2.5\log\left(\frac{f}{f_0}\right) = -2.5\log\left(\frac{f_{10}}{f_0}\right) + 5\log\left(\frac{D_L(z)}{10pc}\right)$$

$$-2.5(\log f - \log f_0) = -2.5(\log f_{10} - \log f_0) + 5(\log D_L(z) - \log(10pc))$$

$$-2.5\log f + \cancel{2.5\log f_0} = -2.5\log f_{10} + \cancel{2.5\log f_0} + 5\log D_L(z) - 5\log(10pc)$$

$$-2.5\log f = -2.5\log f_{10} + 5\log D_L(z) - 5\log(10pc)$$

$$-2.5\log\left(\frac{f_{10}}{f}\right) = 5\log\left(\frac{d_L}{10}\right), \text{ where } f = \frac{L}{4\pi d_L^2}$$

$$-2.5\log\left(\frac{\frac{\cancel{L}}{\cancel{4\pi}(10)^2}}{\frac{\cancel{L}}{\cancel{4\pi}d_L^2}}\right) = 5\log\left(\frac{d_L}{10}\right)$$

$$-2.5\log\left(\frac{10^2}{d_L^2}\right) = 5\log\left(\frac{d_L}{10}\right)$$

$$-2.5\log\left(\frac{10}{d_L}\right)^2 = 5\log\left(\frac{d_L}{10}\right)$$

$$-5\log\left(\frac{10}{d_L}\right) = 5\log\left(\frac{d_L}{10}\right)$$

$$5\log\left(\frac{d_L}{10}\right) = 5\log\left(\frac{d_L}{10}\right)$$

$$1 = 1 \checkmark$$

So we have confirmed (without assuming any expression for M) that $m = M + DM(z)$.

b. Explain why $S_\nu = (1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}$

where $m = M + DM + K$

where $K = -2.5\log_{10}\left[(1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right]$

Solution:

When we consider frequency-specific fluxes as opposed to bolometric, we need to account for the impact of the distance on the flux. For this, we use a 'K-correction'.

I interpret the question to be more about the S_ν equation, though, so here's my understanding of that.

- $S_\nu = f_\nu$ and refers to observed flux in a certain wavelength.
- $(1+z)$ is a required factor for converting from emitted to observed fluxes due to redshift.
- The luminosity ratio simplifies to $\frac{L_{\nu(1+z)}}{4\pi D_L^2}$ and this is the expression for flux $f = \frac{L}{4\pi D_L^2}$, so overall it converts L to f (S) in the ν band and ensures we're considering the observed flux S_ν as intended.

4. Static Universe

Friedmann I

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$k = \frac{K}{R_0^2}$$

$$P_m = 0, \quad P_\Lambda = -\rho_\Lambda$$

a. Write Friedmann II

Goal is to get this format:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{k}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \underbrace{\frac{8\pi G}{3}\rho_\Lambda}_{} - \frac{k}{a^2} \leftarrow \text{Friedmann I}$$

$$\frac{8\pi G}{3}\rho_\Lambda = \frac{\Lambda}{3} \quad \text{in our case}$$

$$\rho_\Lambda = \frac{3}{8\pi G}\frac{\Lambda}{3}$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

Now find Friedmann II:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

$$= -\frac{4\pi G}{3}(\rho_m + \rho_\Lambda + 3P_m + 3P_\Lambda), \text{ where } P_m = 0, P_\Lambda = -\rho_\Lambda$$

$$= -\frac{4\pi G}{3}(\rho_m + \rho_\Lambda + \cancel{3(0)} + 3(-\rho_\Lambda)), \text{ where } \rho_\Lambda = \frac{\Lambda}{8\pi G}$$

$$= -\frac{4\pi G}{3}\left(\rho_m + \frac{\Lambda}{8\pi G} + 3\left(-\frac{\Lambda}{8\pi G}\right)\right)$$

$$= -\frac{4\pi G}{3}\left(\rho_m - 2\frac{\Lambda}{8\pi G}\right)$$

$$= -\frac{4\pi G}{3}\left(\rho_m - \frac{\Lambda}{4\pi G}\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}}$$

b. $\dot{a} = 0$ and $\ddot{a} = 0$

Friedmann II

$$0 = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}$$

$$\boxed{\Lambda = 4\pi G\rho_m} \longrightarrow$$

Friedmann I

$$0 = \frac{8\pi G}{3}\rho_m + \frac{\overset{4\pi G\rho_m}{\uparrow}}{\Lambda}{}\!\;\;\;\;\;\;\;\;\;\;\;\frac{\Lambda}{3} - \frac{k}{a^2}$$

$$= 2\frac{4\pi G}{3}\rho_m + \frac{4\pi G}{3}\rho_m - \frac{k}{a^2}$$

$$= 4\pi G\rho_m - \frac{k}{a^2}$$

$$\boxed{k = 4\pi G a^2\rho_m} = \Lambda a^2$$

This is a positive curvature (open):
- $4\pi G$ is always positive
- Density (ρ_m) cannot be negative
- a^2 cannot be negative (squared)

Overall a positive value for curvature k

c. Perturbation $a(t) = 1 + \delta a(t)$ means $\rho_m(t) = \rho_m(1 - 3\delta a(t))$

Friedmann II

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}$$

$$\frac{1}{a}\frac{d^2}{dt^2}a = -\frac{4\pi G}{3}\rho_m(1 - 3\delta a(t)) + \frac{\Lambda}{3}$$

$$\frac{1}{\cancel{1+\delta a(t)}}\frac{d^2}{dt^2}(1 + \delta a(t)) = -\frac{4\pi G}{3}\rho_m + 4\pi G\rho_m\delta a(t) + \frac{\Lambda}{3}, \text{ where } \rho_m = \frac{\Lambda}{4\pi G}$$

$$1 \cdot \delta\ddot{a}(t) = -\frac{\cancel{4\pi G}}{3}\left(\frac{\Lambda}{\cancel{4\pi G}}\right) + \cancel{4\pi G}\left(\frac{\Lambda}{\cancel{4\pi G}}\right)\delta a(t) + \frac{\Lambda}{3}$$

$$= -\frac{\Lambda}{3} + \Lambda\delta a(t) + \frac{\Lambda}{3}$$

$$\boxed{\delta\ddot{a}(t) = \Lambda\delta a(t)}$$

d. Solve for $\delta a(t)$ in initial conditions.

$$\underbrace{\frac{d}{dt}\frac{d}{dt}}_{\downarrow}\delta a(t) = \Lambda\underset{\downarrow}{\delta a(t)}$$

0, as given in initial conditions $\frac{d(\delta a)}{dt} = 0$

$\delta a(t) = \delta a_0$

* I'm assuming this meant to solve for $\delta a(t)$ at the initial conditions. So that's what I did.

$$0 = \Lambda\delta a_0$$

$\underline{\delta a_0 = 0}$ because Λ shouldn't be 0. But this means there's no perturbation in the first place, so something's wrong. Let's discuss why!

Why unstable?

Our differential equation is almost the spring equation: $\ddot{x} = -kx$. Because our result is positive (we know this because Λ cannot be negative because $\Lambda = 4\pi G\rho_m$ and density ρ_m cannot be negative), the scenario is not oscillation but instead any perturbation to the system will amplify exponentially, resulting in an unstable system.

⑤ Redshift Drift

Huterer problem 2.6 is similar

Show $\frac{dz}{dt_0} = (1+z)H_0 - H(t_1)$

Given $1+z = \frac{a(t_0)}{a(t_1)}$ ← definition of redshift z

Solution

(a) $1+z = \frac{a(t_0)}{a(t_1)}$

$$\frac{d}{dt_0}(1+z) = \frac{d}{dt_0}\left(\frac{a(t_0)}{a(t_1)}\right)$$

$$\cancel{\frac{d}{dt_0}(1)}^{0} + \frac{d}{dt_0}(z) = \frac{d}{dt_0}\left(\frac{a(t_0)}{a(t_1)}\right)$$

$$\frac{dz}{dt_0} = \frac{d}{dt_0}\left(\frac{\overbrace{a(t_0)}^{f(x)}}{\underbrace{a(t_1)}_{g(x)}}\right),$$ where we use the quotient rule

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{a(t_1)\frac{da(t_0)}{dt_0} - a(t_0)\frac{da(t_1)}{dt_0}}{a^2(t_1)},$$ where I want to state that $\frac{da(t_1)}{dt_0} = \frac{da(t_1)}{dt_1}\cdot\frac{dt_1}{dt_0}$

$$= \frac{a(t_1)\frac{da(t_0)}{dt_0} - a(t_0)\frac{da(t_1)}{dt_1}\cdot\frac{dt_1}{dt_0}}{a^2(t_1)},$$ where $\frac{dt_1}{dt_0} = \frac{V_0}{V_1} = \frac{1}{1+z}$ because $\frac{V_{emit}}{V_{obs}} = 1+z$

emit / obs / obs / emit

$$= \frac{a(t_1)\frac{da(t_0)}{dt_0} - a(t_0)\frac{da(t_1)}{dt_1}\cdot\frac{1}{1+z}}{a^2(t_1)}$$

$$= \underbrace{\frac{1}{a(t_1)}}\frac{da(t_0)}{dt_0} - \underbrace{\frac{a(t_0)}{a^2(t_1)}}\frac{da(t_1)}{dt_1}\left(\frac{1}{1+z}\right),$$ where $\underbrace{\frac{a(t_0)}{a(t_1)} = 1+z}$ and so $\underbrace{\frac{1}{a(t_1)} = \frac{1+z}{a(t_0)}}$

$$= \underbrace{\frac{1+z}{a(t_0)}\frac{da(t_0)}{dt_0}}_{H_0} - \underbrace{\frac{\cancel{1+z}}{a(t_1)}\frac{da(t_1)}{dt_1}}_{H_1}\left(\frac{1}{\cancel{1+z}}\right),$$ where $\underbrace{H_0 = \frac{\dot{a}(t_0)}{a_0(t_0)} = \frac{da(t_0)}{dt_0}\cdot\frac{1}{a(t_0)}}$ and $\underbrace{H_1 = \frac{\dot{a}(t_1)}{a(t_1)} = \frac{da(t_1)}{dt_1}\cdot\frac{1}{a(t_1)}}$

$$\boxed{\frac{dz}{dt_0} = (1+z)H_0 - H(t_1)}$$

we would use the product rule for $\frac{da(t_1)}{dt_0}$ because t_1 is a function of t_0, but I ended up avoiding it

$$\frac{d}{dx}f(g(x)) = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dt_0}a(t_1(t_0)) = a(t_0)t_1'(t_0) + t_1(t_0)\,a'(t_0)$$

$$\frac{da(t_1(t_0))}{dt_0} = a\frac{dt_1}{dt_0} + t_1\frac{da}{dt_0}$$

(b) Use $H(z) = H_0(1+z)^{3/2}$ (for matter-dominated flat Universe) to estimate redshift change for object at $z=1$. (Assume $H_0 = 100h\ \frac{km}{s\,Mpc}$)

From earlier in the question: $\frac{dz}{dt_0} = (1+z)H_0 - H(t_1)$

$$H(z) = H_0(1+z)^{3/2}$$

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{\frac{3}{2}}$$

using $z=1$ as instructed

$$= H_0(1+1) - H_0(1+1)^{\frac{3}{2}}$$

$$= 2H_0 - 2.828\,H_0$$

$$= -0.828\,H_0$$

$$= -0.828\left(100h\ \frac{km}{s\,Mpc}\right)$$

Use $dt_0 = 100$ yrs

$$\frac{dz}{dt_0} = -82.8\underset{0.7}{h}\ \frac{km}{s\,Mpc}$$ ← convert

$$\frac{dz}{100yr} = -82.8h\ \frac{km}{s\,Mpc}$$

$$dz = \frac{-82.8 \mid 0.7 \mid 100yr \mid km \mid \pi\times10^7\,s \mid 1\,Mpc}{\quad\mid\quad\mid\quad\mid s\,Mpc \mid yr \mid 3.086\times10^{19}\,km}$$

$$\boxed{dz = -5.9\times10^{-9}}$$ over 100 yr

A miniscule change on human timescales.