01NAEX - Lecture 11
Nested and Split-Plot Design

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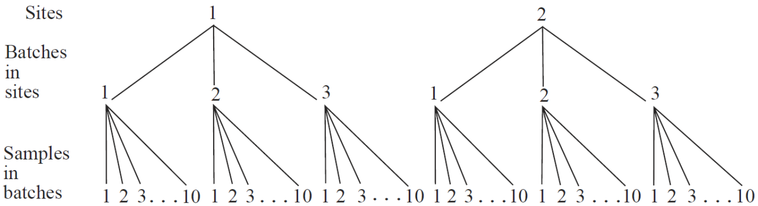
Nested Design

When factor *B* is nested in levels of factor *A*:

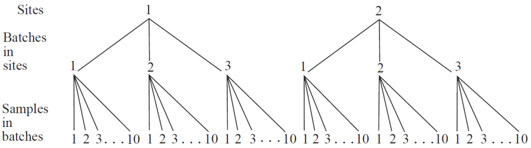
- it is possible to distinguish nested design from a crossed factorial design.
- levels of the nested factor *B* don't have exactly the same meaning under each level of the main factor *A*.
- levels of factor *B* are not identical to each other at different levels of factor *A*.



Two stage design with batches nested in sites

The Two-Stage Nested Design

Let us assume an experiment with sites (factor A) and batches (factor B).



Two stage design with batches nested in sites

- Three batches taken from site 1 are different from the three batches taken from site 2.
- Factor B (batches) is said to be nested in factor A (sites).
- It is impossible to evaluate the effect of the interaction of factor B with factor A, because each level of factor B does not appear with each level of factor A.

The linear statistical model for the two-stage nested design is:

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{(ij)k} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

There is no interaction term $\tau_i \beta_{ij}$ and the term for factor B, $\beta_{j(i)}$, is subscripted to denote the jth level of factor B is nested in the ith level of factor A.

The Two-Stage Nested Design

When B is a random factor, whether factor A is a fixed or random factor the error term for testing the hypothesis about A is based on the mean squares due to B nested in A.

				EMS		
Source	**SS**	**df**	**MS**	**Fixed**	**Mixed (A Fixed)**	**Random**
A	SSA	$a-1$	MSA	$\sigma_\varepsilon^2 + bn\theta_A$	$\sigma_\varepsilon^2 + n\sigma_\beta^2 + bn\theta_A$	$\sigma_\varepsilon^2 + n\sigma_\beta^2 + bn\sigma_\alpha^2$
$B(A)$	SSB(A)	$a(b-1)$	MSB(A)	$\sigma_\varepsilon^2 + n\theta_B$	$\sigma_\varepsilon^2 + n\sigma_\beta^2$	$\sigma_\varepsilon^2 + n\sigma_\beta^2$
Error	SSE	$ab(n-1)$	MSE	σ_ε^2	σ_ε^2	σ_ε^2
Total	TSS	$abn-1$				

Expected mean squares in the two-stage nested design for different combinations of factor A and B being fixed or random.

θ_A is sum of effects divided by appropriate degree of freedoms, i.e. $\frac{\sum_i^a \tau_i}{a-1}$.

The Two-Stage Nested Design

The total corrected sum of squares can be partioned into a sum of squares due to factor A, a sum of squares due to factor B under the leves of A, and a sum of squares due to error:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2$$

Symbolically, it can be written as $SS_T = SS_A + SS_{B(A)} + SS_E$.

The appropriate statistics for testing the effects of factor A and B depend on whether A and B are fixed or random.

The Two-Stage Nested Design

The expected mean squares:

$E(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$E(MS_A)$	$\sigma^2 + \frac{bn \sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn \sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$E(MS_{B(A)})$	$\sigma^2 + \frac{n \sum \sum \beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$E(MS_E)$	σ^2	σ^2	σ^2

The analysis of variance table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bn \sum (\bar{y}_{i..} - \bar{y}_{...})^2$	$a-1$	MS_A
B within A	$n \sum \sum (\bar{y}_{ij.} - \bar{y}_{i..})^2$	$a(b-1)$	$MS_{B(A)}$
Error	$\sum \sum \sum (y_{ijk} - \bar{y}_{ij.})^2$	$ab(n-1)$	MS_E
Total	$\sum \sum \sum (y_{ijk} - \bar{y}_{...})^2$	$abn-1$	

Question: Is the purity of the material the same across suppliers?
Model: The batches are random samples from each supplier, i.e. suppliers are fixed, the batches are random, and the observations are random.

The Two-Stage Nested Design

F-test for mentioned three of the more common situations are following:

1. The F test for factor B is always

$$F = \frac{\text{MSB(A)}}{\text{MSE}}$$

2. The F test for factor A in the fixed-effects model is

$$F = \frac{\text{MSA}}{\text{MSE}}$$

For the random- and mixed-effects model, however, the corresponding test for factor A is

$$F = \frac{\text{MSA}}{\text{MSB(A)}}$$

3. When $n = 1$, there is no test for factor B, but we can test for factor A in the random- and mixed-effects model using

$$F = \frac{\text{MSA}}{\text{MSB(A)}}$$

The Two-Stage Nested Design - example

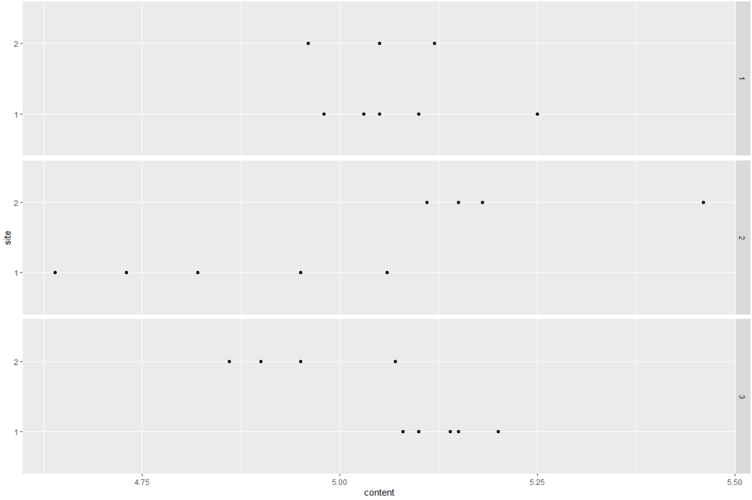
Researchers conducted an experiment to determinthe content uniformity of film-coated tablets produced for a cardiovascular drug used to lower blood pressure.
They obtained a random sample of three batches from each of two blending sites; within each batch they assayed a random sample of five tablets.

Site	1			2		
Batches within each site	1	2	3	1	2	3
Tablets within each batch	5.03	4.64	5.10	5.05	5.46	4.90
	5.10	4.73	5.15	4.96	5.15	4.95
	5.25	4.82	5.20	5.12	5.18	4.86
	4.98	4.95	5.08	5.12	5.18	4.86
	5.05	5.06	5.14	5.05	5.11	5.07

The Two-Stage Nested Design - example

Use ggplot2 to visualize the data set:

```
ggplot(tablets, aes(y = site, x = content)) +
  geom_point() + facet_grid(batch ~ .)
```



The Two-Stage Nested Design - example

Wrong way - full cross-classified fixed effect model

```
> summary(aov(content ~ site*batch , data=tablets))
            Df Sum Sq Mean Sq  F value   Pr(>F)
site         1 0.0183 0.01825   1.510    0.231
batch        2 0.0115 0.00576   0.477    0.627
site:batch   2 0.4425 0.22124  18.297 1.49e-05
Residuals   24 0.2902 0.01209
```

The Two-Stage Nested Design - example

Better way - nested fixed effect model (two possibilities)

```
> summary(aov(content ~ site + batch%in%site, data=tablets))
             Df Sum Sq Mean Sq   F value Pr(>F)
site          1 0.0183 0.01825   1.510   0.231115
site:batch    4 0.4540 0.11350   9.387   0.000103
Residuals    24 0.2902 0.01209
---
```

```
> summary(aov(content ~ site/batch, data=tablets) )
           Df Sum Sq Mean Sq   F value   Pr(>F)
site        1 0.0183 0.01825   1.510     0.231115
site:batch  4 0.4540 0.11350   9.387     0.000103
Residuals  24 0.2902 0.01209
```

The analysis of variance table contains the correct test for the factor `batch`.

The Two-Stage Nested Design - example

To get the correct test for `site`, you have to specify the appropriate error term for the factor `site`:

```
> summary(aov(content ~ site + Error(batch %in% site),
data=tablets))
Error: batch:site
          Df Sum Sq Mean Sq   F value  Pr(>F)
site       1 0.0183 0.01825   0.161    0.709
Residuals  4 0.4540 0.11350

Error: Within
          Df Sum    Sq Mean Sq F value Pr(>F)
Residuals 24 0.2902 0.01209
```

The Two-Stage Nested Design - example

Good way - nested random effect model:

```
> summary(lme (content ~ 1,    random = ~1|site/batch, data = tablets)
Linear mixed-effects model fit by REML
Data: tablets
AIC       BIC    logLik
-24.06435 -18.59516 16.03217

Random effects:
Formula: ~1 | site
        (Intercept)
StdDev: 3.236734e-06

Formula: ~1 | batch %in% site
          (Intercept)  Residual
StdDev:   0.1283446    0.1099621

Fixed effects: content ~ 1
            Value    Std.Error DF  t-value p-value
(Intercept) 5.043333  0.056111 24 89.88136       0

Number of Observations: 30
Number of Groups:
site batch %in% site
2               6
```

The Two-Stage Nested Design - example

Good way - nested random effect model:

```
> summary(lmer(content ~ 1 + (1|site/batch), data = tablets))
Linear mixed model fit by REML
t-tests use  Satterthwaite approximations to degrees of freedom
Formula: content ~ 1 + (1 | site/batch)
Data: tablets

REML criterion at convergence: -32.1

Random effects:
Groups     Name        Variance Std.Dev.
batch:site (Intercept) 0.01647  0.1283
site       (Intercept) 0.00000  0.0000
Residual               0.01209  0.1100
Number of obs: 30, groups:  batch:site, 6; site, 2

Fixed effects:
            Estimate   Std. Error  df        t value Pr(>|t|)
(Intercept)  5.04333    0.05611     5.00000   89.88   3.23e-09
```

Nested Design vs. Hierarchical models

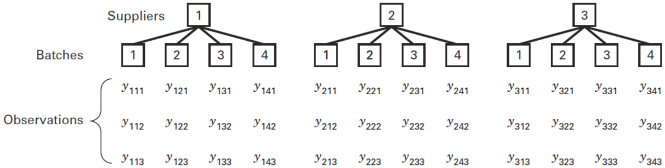
Sometimes is better to avoid Nested design and number the data set in another way!

Models with optimal way of numbering (origin number to each "nested" measurements) are called **Hierarchical models**.

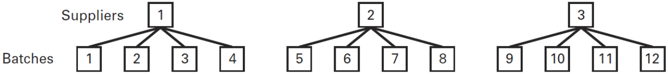
R can cope with both - Nested models numbering (setting) and with Hierarchical models numbering too.

The Two-Stage Nested Design - another simple example

Let us consider a company that purchase its raw material from three different suppliers, there are four batches of this raw material available from each supplier, and three determinations of purity are to be taken from each batch.



Note: if the levels of the factor can be renumbered arbitrarily, then the factor is nested. An alternate layout for the two stage nested design is



Because not every level of factor B appears with every level of factor A, there can be no interaction between A and B.

Let us denote this notation by Batches2 in the code.

The Two-Stage Nested Design - example

Wrong ways:

```
> str(data_purity)
'data.frame': 36 obs. of  4 variables:
$ Suppliers: Factor 3 levels "Supplier 1","Suplier 2",..: 1 1
$ Batches  : Factor 4 levels "1","2","3","4": 1 1 1 2 2 2 3 3
$ Purity   : int 1 -1 0 -2 -3 -4 -2 0 1 1 ...
$ Batches2 : Factor 12 levels "1","2","3","4",..: 1 1 1 2 2 2

# crossed fixed effect model
anova(lm(Purity ~ Suppliers+Batches, data = data_purity))

# mixed model
anova(lme(Purity ~ Suppliers, random = ~1|Batches, data = data_
```

The Two-Stage Nested Design - example

Nested fixed effect model, three possibilities, same results

```
summary(aov(Purity~Suppliers + Batches%in%Suppliers))
summary(aov(Purity~Suppliers/Batches, data=data_purity))
summary(aov(Purity~Suppliers + Batches2, data=data_purity))
```

```
                  Df  Sum Sq Mean Sq F value Pr(>F)
Suppliers          2   15.06   7.528   2.853 0.0774 .
Suppliers:Batches 9   69.92   7.769   2.944 0.0167 *
Residuals         24   63.33   2.639
```

Recall: F-test for the factor B is always the same (B random or fixed).

The Two-Stage Nested Design - example

Nested mixed effect model (Suppliers fixed, Batches random), three possibilities, same results

```
summary(aov(Purity~Suppliers+Error(Batches %in% Suppliers)))
summary(aov(Purity~Suppliers+Error(Suppliers/Batches)))
summary(aov(Purity~Suppliers+Error(Batches2)))
Error: Batches:Suppliers
          Df Sum Sq Mean Sq F value Pr(>F)
Suppliers  2  15.06   7.528   0.969  0.416
Residuals  9  69.92   7.769
Error: Within
         Df Sum Sq  Mean Sq F value Pr(>F)
Residuals 24  63.33   2.639
```

If Suppliers (A) are fixed end Batches (B) random, then $H_0 : \tau_i = 0$ is tested by $MS_A/MS_{B(A)}$ and $H_0 : \sigma^2_\beta = 0$ is tested by $MS_{B(A)}/MS_E$.
$\hat{\sigma}^2_\beta = \frac{MS_{B(A)} - MS_E}{n} = \frac{7.769 - 2.639}{3} = 1.71$, $\hat{\sigma}^2_\tau = \frac{MS_A - MS_{B(A)}}{n}$

The Two-Stage Nested Design - example

Nested mixed effect model - analysis by Mixed effects models

```
library(nlme)
purity_lme<-lme(Purity~Suppliers, random=~1|Suppliers/Batches)
purity_lme<-lme(Purity~Suppliers, random=~1|Batches2)
summary(purity_lme), intervals(purity_lme)
> anova(purity_m2_lme)
               numDF denDF  F-value    p-value
(Intercept)     1    24     0.6042908  0.4445
Suppliers       2     9     0.9690107  0.4158
> VarCorr(purity_m2_lme)
Batches2 = pdLogChol(1)
Variance StdDev
(Intercept) 1.709877 1.307622
Residual    2.638889 1.624466

library(lme4)
purity_lmer <- lmer(Purity ~ Suppliers + (1|Batches2))
summary(purity_lmer)
anova(purity_lmer)
```

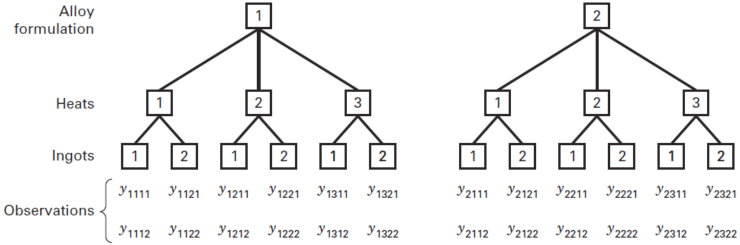
R doesn’t use F-distribution to test the significance of random component.

The General *m*-Stage Nested Design

Results from the Two-Stage Nested Design can be extended to the case of *m* completely nested factors. The model for the general three stage nested design is

$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_{k(ij)} + \epsilon_{(ijk)l} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

As an example, suppose a foundry wishes to investigate the hardness of two different metal alloy formulations. Three heats of each alloy formulations are prepared, two ingots are selected at random from each heat, and two hardness measurement are made on each ingot.

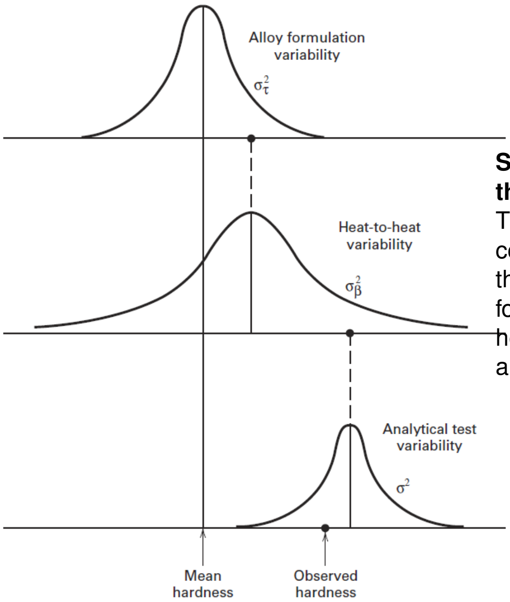


The 3-Stage Nested Design

The Analysis of variance table for three-stage Nested design is

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bcn \sum_i (\bar{y}_{i...} - \bar{y}_{....})^2$	$a - 1$	MS_A
B (within A)	$cn \sum_i \sum_j (\bar{y}_{ij..} - \bar{y}_{i...})^2$	$a(b - 1)$	$MS_{B(A)}$
C (within B)	$n \sum_i \sum_j \sum_k (\bar{y}_{ijk.} - \bar{y}_{ij..})^2$	$ab(c - 1)$	$MS_{C(B)}$
Error	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{ijk.})^2$	$abc(n - 1)$	MS_E
Total	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{....})^2$	$abcn - 1$	

The 3-Stage Nested Design



Sources of variation in the three-stage nested design:
The overall variability in hardness consists of three components: one that results from alloy formulations, one that results from heats, and one that results from analytical test error.

The Split-Plot Design

The split plot design is used when it is not able to completaly randomize the order of the runs.
Not all factors can be easily changed (economical or physical reasons, ...)

An example: experiment on the tensile strength of paper:

The Experiment on the Tensile Strength of Paper

	Replicate 1			Replicate 2			Replicate 3		
Pulp Preparation Method	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**
Temperature (°F)									
200	30	34	29	28	31	31	31	35	32
225	35	41	26	32	36	30	37	40	34
250	37	38	33	40	42	32	41	39	39
275	36	42	36	41	40	40	40	44	45

The experimenter didn't collect data by completely randomized design (require 36 batches of pulp), but produced only 9 batches of pulp, three by each method of interest (require only 9 batches of pulp).

The Split-Plot Design

In the example, we have 9 whole plots, and the preparation method is called the whole plot treatments (Taguchi: Inner Array).
Each whole plot is divided into four parts called subplots (or Split-Plots, Outer Array), and one temperature is assigned to each, temperature is called the subplot treatment.

Because the whole-plot treatments are confounded with the whole-plots, it is best to assign the factor we are most interest into the subplots.

The linear model for split-plot design is:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, a \\ k = 1, 2, \ldots, b \end{cases}$$

The Split-Plot Design

The Expected Mean Squares for Split-Plot Design

	Model Term	Expected Mean Square
Whole plot	τ_i	$\sigma^2 + ab\sigma_\tau^2$
	β_j	$\sigma^2 + b\sigma_{\tau\beta}^2 + \frac{rb\sum \beta_j^2}{a-1}$
	$(\tau\beta)_{ij}$	$\sigma^2 + b\sigma_{\tau\beta}^2$
Subplot	γ_k	$\sigma^2 + a\sigma_{\tau\gamma}^2 + \frac{ra\sum \gamma_k^2}{(b-1)}$
	$(\tau\gamma)_{ik}$	$\sigma^2 + a\sigma_{\tau\gamma}^2$
	$(\beta\gamma)_{jk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + \frac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
	$(\tau\beta\gamma)_{ijk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
	$\epsilon_{(ijk)h}$	σ^2 (not estimable)

The main factor A in the whole plot is tested against the whole-plot error, whereas the subtreatment B is tested against the (replicates X subtreatment) interactions. The AB interaction is tested against the subplot error.

Notice: there are no tests for the replicates effect A.

The Split-Plot Design

Results for Tensile Strength Data by Montgomery

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Replicates	77.55	2	38.78		
Preparation method (A)	128.39	2	64.20	7.08	0.05
Whole plot error (replicates $\times$ A)	36.28	4	9.07		
Temperature (B)	434.08	3	144.69	41.94	<0.01
Replicates $\times$ B	20.67	6	3.45		
AB	75.17	6	12.53	2.96	0.05
Subplot error (replicates $\times$ AB)	50.83	12	4.24		
Total	822.97	35			

The Split-Plot Design

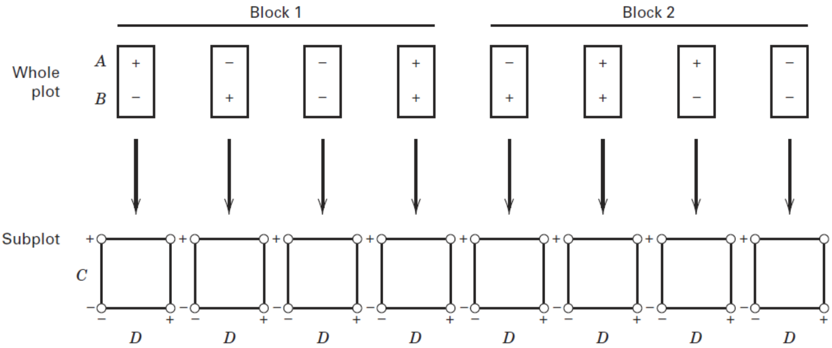
Results for Tensile Strength Data from R

```
> aov_paper <- aov(Strength ~ Temp*Method +Replicates:Temp +
                Error(Replicates/Method), data = paper)
> summary(aov_paper)

Error: Replicates
Df Sum Sq Mean Sq
Temp:Replicates  2  77.56   38.78

Error: Replicates:Method
          Df Sum Sq Mean Sq F value Pr(>F)
Method     2 128.39   64.19   7.078 0.0485 *
Residuals  4  36.28    9.07
-----------------------------------------------------
Error: Within
                Df Sum Sq Mean Sq F value   Pr(>F)
Temp             3  434.1  144.69  34.157 3.71e-06 ***
Temp:Method      6   75.2   12.53   2.957    0.052 .
Temp:Replicates  6   20.7    3.44   0.813    0.580
Residuals       12   50.8    4.24
```

The Split-Plot Design with more than two factors

A split-plot design with 4 design factors, 2 in the whole plot and 2 in the subplot.

$$
\begin{aligned}
y_{ijklm} = \mu &+ \tau_i + \beta_j + \gamma_k + (\beta\gamma)_{jk} + \theta_{ijk} + \delta_l + \lambda_m + (\delta\lambda)_{lm} \\
&+ (\beta\delta)_{jl} + (\beta\lambda)_{jm} + (\gamma\delta)_{kl} + (\delta\lambda)_{lm} + (\beta\gamma\delta)_{jkl} + (\beta\gamma\lambda)_{jkm} \\
&+ (\beta\delta\lambda)_{jlm} + (\gamma\delta\lambda)_{klm} + (\beta\gamma\delta\lambda)_{jklm} + \epsilon_{ijklm} \quad \begin{cases} i = 1, 2 \\ j = 1, 2 \\ k = 1, 2 \\ l = 1, 2 \\ m = 1, 2 \end{cases}
\end{aligned}
$$



Today Exercises

Solve problem 14.5 from Montgomery DaAoE:

Consider the three-stage nested to investigate alloy hardness. Using the data that follow, analyze the design, assuming that alloy chemistry and heats are fixed factors and ingots are random. Use the restricted form of the mixed model.

	Alloy Chemistry 1					
Heats	1		2		3	
Ingots	1	2	1	2	1	2
	40	27	95	69	65	78
	63	30	67	47	54	45
	Alloy Chemistry 2					
Heats	1		2		3	
Ingots	1	2	1	2	1	2
	22	23	83	75	61	35
	10	39	62	64	77	42

Today Exercises

Solve problem 14.20, 14.21 from Montgomery DaAoE:

An experiment is designed to study pigment dispersion in paint. Four different mixes of a particular pigment are studied. The procedure consists of preparing a particular mix and then applying that mix to a panel by three application methods (brushing, spraying, and rolling). The response measured is the percentage reflectance of pigment. Three days are required to run the experiment, and the data obtained follow. Analyze the data and draw conclusions, assuming that mixes and application methods are fixed.
Repeat Problem 14.20, assuming that the mixes are random and the application methods are fixed.

Day	Application Method	Mix 1	Mix 2	Mix 3	Mix 4
1	1	64.5	66.3	74.1	66.5
	2	68.3	69.5	73.8	70.0
	3	70.3	73.1	78.0	72.3
2	1	65.2	65.0	73.8	64.8
	2	69.2	70.3	74.5	68.3
	3	71.2	72.8	79.1	71.5
3	1	66.2	66.5	72.3	67.7
	2	69.0	69.0	75.4	68.6
	3	70.8	74.2	80.1	72.4