FYS4480/9480 lecture September 5, 2025

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$M = 2$

$$\langle 0 | AB | 0 \rangle = \langle 0 | a_i a_j^\dagger | 0 \rangle$$

$$= \overbrace{a_i a_j^\dagger} + N[a_i a_j^\dagger]$$

$$\left(a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} \right)$$

$$N[a_i a_j^\dagger] = - \langle 0 | a_j^\dagger a_i | 0 \rangle$$

$$\overbrace{a_i a_j^\dagger} = \langle 0 | a_i a_j^\dagger | 0 \rangle$$

$$\overbrace{a_i a_j} = \overbrace{a_i^\dagger a_j^\dagger} = \overbrace{a_j^\dagger a_i} = 0$$

$$\overbrace{a_i a_j^\dagger} = \delta_{ij}$$

$$M = 4$$

$$\langle 0 | \hat{A} \hat{B} \hat{C} \hat{D} | 0 \rangle =$$

$$\hat{A} \hat{B} \hat{C} \hat{D} = a_i a_j a_k^\dagger a_l^\dagger$$

$$\langle 0 | a_i a_j a_k^\dagger a_l^\dagger | 0 \rangle = a_i a_j a_k^\dagger a_l^\dagger$$

$$= N\left[a_i a_j a_k^\dagger a_l^\dagger \right] +$$

$$\sum_{(1)} N\left[a_i\, \overbrace{a_j\, a_k^{\dagger}}\, a_l^{\dagger}\right]$$

$\langle 0 | a_j a_k^{\dagger} | 0 \rangle$

$$\left(\delta_{kj} \cdot N\left[a_i\, a_l^{\dagger}\right] \right)$$

$$\underbrace{- \langle 0 | a_l^{\dagger} a_i | 0 \rangle}_{= 0}$$

all possible pairs contracted

$$+ \sum_{(2)} N\left[\overbrace{a_i\, \overbrace{a_j\, a_k^{\dagger}}\, a_l^{\dagger}}\right]$$

$$\delta_{kj}\ \delta_{il}$$

$\langle 0 | a_j a_k^{\dagger} | 0 \rangle$ $\qquad$ $\langle 0 | a_i a_l^{\dagger} | 0 \rangle$

assume that we have a chain of normal ordered operators

$$[XYZ \ldots W]$$

we need a lemma,

<u>L1</u>: add a new operator to right of the previous

$$N[XYZ \ldots W]\Omega =$$

$$N[XYZ \ldots W\Omega] +$$

$$\sum_{(i)\Omega} N[xyz \ldots w\Omega]$$

(i) is valid immediately if Ω is an annihilation operator since then

$N[xyz \ldots w]\Omega$ is already normal-ordered

$$a^\dagger \underset{\Omega}{a} = 0$$

(we have assumed that the sequence $xyz \ldots w$ is normal-ordered)

(ii) if all $xyz \ldots w$ are annihilation operators, and Ω is an annihilation operator, then (2.1) is trivially satisfied

(iii) if Ω is a creation operator we need to move $\underline{\langle 1}$ if all $xyz\ldots w$ are annihilation operators since $\overline{a_i^{+}\,a_j^{+}} = 0$

(iv) we anticommute Ω through all $xyz\ldots w$ to get $N[xyz\ldots w\Omega]$ an the contractions which are produced, give the second term,

$$N[xyz \cdots wz]$$

Example

$$M = 1$$

$$N[a_1]\, a_e^\dagger = N[a_1 a_e^\dagger] + N[a_1 a_e^\dagger]$$

$$= -\langle 0| a_e^\dagger a_1 |0\rangle \qquad \delta_{1e}$$

$$+ \langle 0| a_1 a_e^\dagger |0\rangle$$

$$a_0 N[a_1] a_e^{+} = (N[a_0 a_1] a_e^{+}$$

$$= a_0 \left(N[a_1 a_e^{+}] + N[\overbrace{a_1 a_e^{+}}] \right)$$

$$= N[a_0 a_1 a_e^{+}]$$

$$+ N[a_0 \overbrace{a_1 a_e^{+}}] +$$

$$N[\overbrace{a_0 a_1 a_e^{+}}]$$

$$(-)^{M=2} \langle 0 | a_e^{+} a_0 a_1 | 0 \rangle +$$

$$N[a_0] \cdot \delta_{1e} + \delta_{0e} N[a_1])$$

generally:

$$a_0 \, N[a_1 a_2 \dots a_M] \, a_e^+$$

$$= \sum_{(i)} N[a_0 a_1 \dots a_M a_e^+]$$

$$+ \; N[a_0 a_1 \dots a_M a_e^+]$$

Example

$$a_0 [a_1 a_2] a_e^+$$

$$= N[a_0 a_1 a_2 a_e^+] + N[a_0 a_1 a_2 a_e^+] + N[a_0 a_1 a_2 a_e^+]$$

$$+ \; N[a_0 a_1 a_2 a_\ell^\dagger]$$

Wick's theorem

$$\langle c| x y z \cdots w \Omega |c\rangle$$

$$(x y z \cdots w)\Omega =$$

$$N[x y z \cdots w \Omega]$$

$$+ \sum_{(1)\neq\Omega} N[x y z \cdots w \Omega]$$

$$+ \sum_{(1)\Omega} N[x y z \cdots w \Omega]$$

$$+$$

$$\sum_{(2)\neq\Omega} N[xyz \cdots w\Omega]$$

$$+ \sum_{(2)\Omega} N[xyz \cdots w\Omega]$$

$$+ \cdots$$

$$\sum_{\left[\frac{M}{2}\right]\neq\Omega} N[xyz \cdots w\Omega]$$

$$+ \sum_{\left[\frac{M}{2}\right]\Omega} N[xyz \cdots w\Omega]$$

$$+ \sum_{\left[\frac{M+1}{2}\right] \neq \Omega} N[- \ - \ -] + \sum_{\left[\frac{M+1}{2}\right] \Omega} N[- \ -]$$

$M+1$ gives an odd operator, then we will always have an uncontracted operator

$$\langle c | a | c \rangle = c$$

$$\langle c | a^\dagger | c \rangle = 0$$

$$\sum_{[m] \neq \Omega} + \sum_{[m] \Omega} = \sum_{[m]}$$

Wick's theorem:

$$\langle 0| XYZ \cdots W |0\rangle$$

$$= XYZ \cdots W =$$

$$\sum_{\left[\frac{M}{2}\right]} N[XYZ \cdots W]$$

only non-zero,

Wick's generalized theorem

Example

$$|ij\rangle = a_i^+ a_j^+ |0\rangle$$

Normal ordered

$$\langle ij| = \langle 0| a_j a_i$$

$$\mathcal{H}_I = \frac{1}{2} \sum_{pqrs} \langle pq|v|rs\rangle \times a_p^+ a_q^+ a_s a_r$$

Normal ordered

$$\langle ij | \mathcal{H}_I | ij \rangle$$

$$\propto \quad \langle c | \boxed{a_j a_i} \boxed{a_p^\dagger a_q^\dagger a_s a_r} \boxed{a_i^\dagger a_j^\dagger} | 0 \rangle$$

$$N[A_1 A_2 \; - -]$$

$$N[B_1 B_2 \; - -]$$

$$N[C_1 C_2 \; - -]$$

Wick's generalized theorem: states that for an arbitrary product of creation and annihilation operators in which some of the operators are already in a normal-ordered form, we can write Wick's generalized theorem as

$$N[A_1 A_2 \cdots]\, N[B_1 B_2 \cdots]\, N[C_1 C_2 \cdots]$$

$$= N[A_1 A_2 \cdots B_1 B_2 \cdots C_1 C_2 \cdots]$$

$$+ \sum_{\text{All}} N[A_1 A_2 \cdots B_1 B_2 \cdots C_1 C_2 \cdots]$$

where contractions are summed over all contractions from different Normal-ordered groups of operators

Example

$$|ij\rangle = a_i^+ a_j^+ |0\rangle$$

$$\mathcal{H}_I = \frac{1}{2} \sum_{pqrs} \langle pq|v|rs\rangle \times a_p^+ a_q^+ a_s a_r$$

$$\langle ij|\mathcal{H}_I|ij\rangle$$

$$\frac{1}{2} \sum_{pqrs} \langle pq|v|rs\rangle \times \langle 0| \boxed{a_j a_i} \boxed{a_p^+ a_q^+ a_s a_r} \boxed{a_i^+ a_j^+} |0\rangle$$

$$a_j \, a_i \, a_p^\dagger a_q^\dagger a_s \, a_r \, a_i^\dagger a_j^\dagger$$

$$-\delta_{ip} \qquad \delta_{ri}$$

$$\delta_{jq} \qquad \delta_{sj}$$

$$\langle ij|v|ij\rangle$$

$$-\langle ji|v|ij\rangle$$

$$\langle ji|v|ji\rangle$$

$$= \langle ij|v|ij\rangle$$

$$-\langle ij|v|ji\rangle$$

One-body operator

$$\hat{O}^{(1)} = \sum_{pq} \langle p|\hat{o}^{(1)}|q\rangle\, a_p^\dagger a_q$$

$$\langle ij|\hat{O}^{(1)}|kl\rangle$$

$$= \sum_{pq} \langle p|\hat{o}^{(1)}|q\rangle \langle 0|\boxed{a_j a_i}\,\boxed{a_p^\dagger a_q}\,\boxed{a_k^\dagger a_l^\dagger}|0\rangle$$

$$a_j a_i a_p^\dagger a_q a_k^\dagger a_l^\dagger$$

$$\delta_{jl} \langle i|\hat{o}^{(1)}|k\rangle$$

$$\langle SD_1 | O^{(1)} | SD_2 \rangle = 0$$

$|1010\rangle$ $|1100\rangle$

they differ by one or more than single particle state (not acted upon by $O^{(1)}$)

$$\langle 12 | \hat{O}^{(1)} | 13 \rangle$$

$$\langle 1 | \hat{o}^{(1)} | 1 \rangle \delta_{23}$$

$$a_j \, a_i \, a_p^{\dagger} \, a_q \, a_k^{\dagger} \, a_e^{\dagger}$$

$$a_j \, a_i \, a_p^{\dagger} \, a_q \, a_k^{\dagger} \, a_e^{\dagger}$$

$$\delta_{jp} \, \delta_{ik} \, \delta_{qe}$$

$$\langle j | \Theta^{(1)} | e \rangle \, \delta_{ik}$$

$$\langle \underline{i} j | O | \underline{k} e \rangle$$

$$\langle ij | O^{(2)} | kl \rangle$$

$$a_j \, a_i \quad a_p^{+} a_q^{+} a_s \, a_r \, a_k^{+} a_l^{+}$$

$$\langle ij | O^{(2)} | kl \rangle$$

$$-\langle ji | O^{(2)} | kl \rangle$$

$$\langle ji | O^{(2)} | lk \rangle$$

$$-\langle ij | O^{(2)} | lk \rangle$$

$$|123\rangle = a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle$$

$$|456\rangle = a_4^\dagger a_5^\dagger a_6^\dagger |0\rangle$$

$$\langle 123 | \hat{O}^{(2)} | 456 \rangle$$

$$a_3 \, a_2 a_1 \, a_p^\dagger a_q^\dagger \, a_s a_r \, a_4^\dagger a_5^\dagger \, a_6^\dagger$$

$$\delta_{36}$$

$$\langle 12 | O^{(2)} | 45 \rangle$$

$$- \langle 12 | O^{(2)} | 54 \rangle$$

if $|SD\rangle_1$ and $|SD\rangle_2$

$${}_1\langle SD|O^{(2)}|SD\rangle_2 = 0$$

if $|SD\rangle_1$ and $|SD\rangle_2$

differ by more than

two single particle states

$${}_1\langle SD|O^{(2)}|SD\rangle_1$$

$$= \frac{1}{2}\sum_{ij}^{N}\Big(\langle ij|O^{(2)}|ij\rangle - \langle ji|O^{(2)}|ij\rangle\Big)$$