

Exercises FYS4480/9480, week 37, September 8-12, 2025

Exercise 1

We will study a schematic model (the Lipkin model, Nucl. Phys. **62** (1965) 188) for the interaction among 4 fermions that can occupy two different energy levels. Each level has degeneration $d = 4$. The two levels have quantum numbers $\sigma = \pm 1$, with the upper level having $\sigma = +1$ and energy $\varepsilon_1 = \varepsilon/2$. The lower level has $\sigma = -1$ and energy $\varepsilon_2 = -\varepsilon/2$. In addition, the substates of each level are characterized by the quantum numbers $p = 1, 2, 3, 4$.

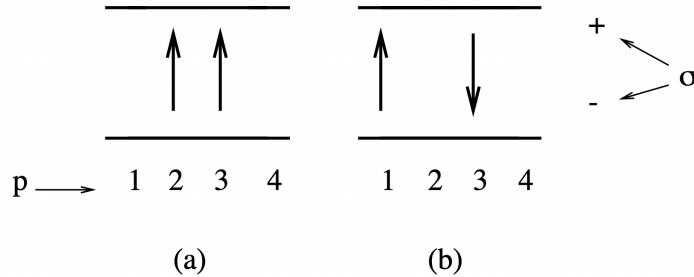
We define the single-particle states

$$|u_{\sigma=-1,p}\rangle = a_{-p}^\dagger |0\rangle \quad |u_{\sigma=1,p}\rangle = a_{+p}^\dagger |0\rangle.$$

The single-particle states span an orthonormal basis. The Hamiltonian of the system is given by

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \\ \hat{H}_0 &= \frac{1}{2}\varepsilon \sum_{\sigma,p} \sigma a_{\sigma,p}^\dagger a_{\sigma,p} \\ \hat{H}_1 &= \frac{1}{2}V \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{\sigma,p'}^\dagger a_{-\sigma,p'} a_{-\sigma,p} \\ \hat{H}_2 &= \frac{1}{2}W \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{-\sigma,p'}^\dagger a_{\sigma,p'} a_{-\sigma,p} \end{aligned}$$

where V and W are constants. The operator H_1 can move pairs of fermions as shown in the left part of the figure (a). while H_2 is a spin-exchange term. As shown in (b), H_2 moves a pair of fermions from a state $(p\sigma, p' - \sigma)$ to a state $(p - \sigma, p'\sigma)$.



We will encounter this model again in our analysis of mean field methods like the Hartree-fock method and full configuration interaction theory. It is a model which has been used widely in many-body physics and recently also in quantum computing, see for example <https://journals.aps.org/prc/abstract/10.1103/PhysRevC.104.024305>.

a. *Quasispin operators* Introduce the quasispin operators

$$\begin{aligned} \hat{J}_+ &= \sum_p a_{p+}^\dagger a_{p-} \\ \hat{J}_- &= \sum_p a_{p-}^\dagger a_{p+} \\ \hat{J}_z &= \frac{1}{2} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \\ \hat{J}^2 &= J_+ J_- + J_z^2 - J_z \end{aligned}$$

Show that these operators obey the commutation relations for angular momentum.

b. Number operator Express \hat{H} in terms of the above quasispin operators and the number operator

$$\hat{N} = \sum_{p\sigma} a_{p\sigma}^\dagger a_{p\sigma}.$$

c. Commutation relations Show that \hat{H} commutes with J^2 , viz., J is a good quantum number. Does it commute with J_z ?

d. Wick's theorem Consider thereafter a state with all four fermions in the lowest level (see the above figure). We can write this state as

$$|\Phi_0\rangle = |\Phi_{J_z=-2}\rangle = a_{1-}^\dagger a_{2-}^\dagger a_{3-}^\dagger a_{4-}^\dagger |0\rangle.$$

This state has $J_z = -2$ (convince yourself about this) and belongs to the set of possible projections of $J = 2$. We introduce the shorthand notation $|J, J_z\rangle$ for states with different values of spin J and its projection J_z . We can think of this as our computational basis for $J = 2$ and all five projections J_z . We will also assume that the state Φ_0 can be considered as an ansatz for the ground state of the system.

Use Wick's theorem to calculate the expectation values of

$$\langle\Phi_0|\hat{N}|\Phi_0\rangle,$$

and

$$\langle\Phi_0|\hat{H}|\Phi_0\rangle.$$

Comment your results.

e. Using quasispin operators Show that you can obtain the same result for

$$\langle\Phi_0|\hat{H}|\Phi_0\rangle.$$

using the quasispin representation of the Hamiltonian (plus the number operator). Comment your results.