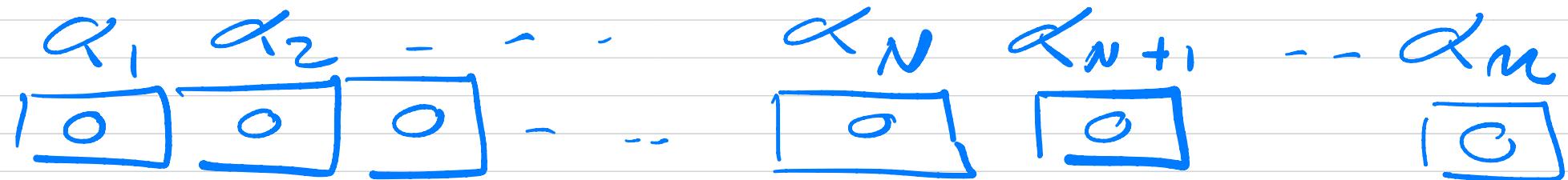


**FYS4480/9480**  
**lecture September**  
**6, 2024**

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vacuum state  $|0\rangle$



$|0\rangle$

A diagram showing the creation of a state from the vacuum  $|0\rangle$ . On the left, there is a diagram of the vacuum state  $|0\rangle$  (from the previous diagram). To its right is a bracket with a minus sign under it, followed by a plus sign. Next is another bracket with a plus sign under it. To the right of these brackets is the expression  $a_1^+ a_2^+ |0\rangle$ . This is followed by an equals sign and the final state  $|a_1 a_2\rangle$ .

$$|0\rangle - + a_1^+ a_2^+ |0\rangle = |a_1 a_2\rangle$$

$$n = 2$$

$$|\alpha_1, \alpha_2\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$$

$$|\alpha_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |10\rangle$$

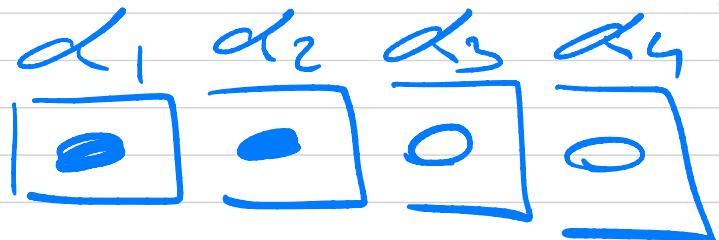
$$|\alpha_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |01\rangle$$

$$|\alpha_1\rangle \otimes |\alpha_2\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= |0100\rangle$$

$$|\alpha_1, \alpha_2\rangle = |11\rangle$$

with  $m = 4 \wedge N = 2$



$|d_1\rangle \otimes |d_2\rangle$

$|d_1, d_2\rangle$

$|1100\rangle$

$|1010\rangle$

$|1001\rangle$

$$a_d |0\rangle = 0$$

$$a_d |1\rangle = |0\rangle$$

$$c_d(x) = \langle x | d \rangle$$

$$SD = |d_1, d_2, \dots, d_N\rangle$$

$$\hat{N} = \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha^\dagger a_\alpha$$

$$\hat{N}^2 = \hat{N}, \quad [\hat{N}_i, \hat{N}_j] = 0$$

$$\hat{N}_i = q_i^\dagger q_i$$

$$[a_i, \hat{N}_j] = \underbrace{\langle i | j \rangle}_{S_{ij}} a_i$$

$$[a_i^\dagger, \hat{N}_j] = S_{ij} - S_{ij} q_i^\dagger$$

Example 1

$$\langle d_1, d_2 \rangle = q_{d_1}^+, q_{d_2}^+ |0\rangle \quad d_1 \neq d_2$$

$$\langle d_1, d_2 | d_1, d_2 \rangle =$$

$$\langle 0 | q_{d_2} q_{d_1}^+, q_{d_1}^+, q_{d_2}^+ | 0 \rangle =$$

$$\langle 0 | q_{d_2} (S_{d_1, d_1} - q_{d_1}^+ q_{d_1}) q_{d_2}^+ | 0 \rangle$$

$$= \langle 0 | q_{d_2} q_{d_2}^+ | 0 \rangle S_{d_1, d_1}$$

$$- \langle 0 | q_{d_2} q_{d_1}^+ q_{d_1}^+ q_{d_2}^+ | 0 \rangle =$$

$$\langle c | \alpha_2 \alpha_2^\dagger | 0 \rangle S_{\alpha_1 \alpha_1}$$

$$- \underbrace{\langle c | \alpha_2 \alpha_1^\dagger (S_{\alpha_1 \alpha_2} - \alpha_2^\dagger \alpha_1) | 0 \rangle}_{= 0}$$

$$\langle c | (S_{\alpha_2 \alpha_2} - \alpha_2^\dagger \alpha_2) | 0 \rangle S_{\alpha_1 \alpha_1}$$

$$= S_{\alpha_2 \alpha_2} S_{\alpha_1 \alpha_1}$$

introduce new notations

$$\langle 0 | \alpha \alpha^\dagger | 0 \rangle = \langle 0 | S_{\alpha \beta} | 0 \rangle$$
$$- \langle 0 | \alpha_\beta^\dagger \alpha_\alpha | 0 \rangle =$$

$$[a_{\alpha} a_{\beta}^+] + N [a_{\alpha} a_{\beta}^+]$$

$$= \langle 0 | a_{\alpha} a_{\beta}^+ | 0 \rangle$$

$S_{\alpha\beta}$

$$\underbrace{- \langle 0 | a_{\beta}^+ a_{\alpha} | 0 \rangle}$$

$$= \langle \alpha | \beta \rangle$$

Example

$$N [a_1 a_2 a_3 a_4^+] = (-)^3 \langle 0 | a_4 a_1 a_2 a_3$$

$$x | 0 \rangle$$

$$= \textcircled{O}$$

$$\langle c | a_1 a_2 q_3^+ | 10 \rangle \quad | \neq 2 \neq 3$$

$$= \langle c | a_1 (\delta_{32} - a_3^+ q_2) | 10 \rangle$$

$$= \langle c | a_1 a_2 q_3^+ | 10 \rangle$$

$$- \langle c | a_1 a_3^+ a_2 | 10 \rangle =$$

$$\langle c | a_1 q_2 q_3^+ | 10 \rangle$$

$$- \langle c | (\delta_{13} a_2 - a_3^+ a_1 a_2) | 10 \rangle$$

$$= \langle c | a_1 a_2 q_3^+ | 10 \rangle - \langle c | a_1 q_3^+ q_2 | 10 \rangle$$

$$+ \underbrace{\langle c | q_3^+ q_1 q_2 | c \rangle}_{N [q_1 q_2 q_3^+]}$$

$$\langle c | q_1 q_2 q_3^+ | c \rangle$$

$$= \langle c | q_1 \overline{q_2} \overline{q_3^+} | c \rangle$$

$$+ \langle c | \overline{q_1} \overline{q_2} \overline{q_3^+} | c \rangle$$

$$+ N [q_1 q_2 q_3^+] = 0$$

$$\langle 0 | a_1 a_2 a_3^\dagger a_4^\dagger | 0 \rangle =$$

$$N [ a_1 a_2 a_3^\dagger a_4^\dagger ] \stackrel{=0}{=} +$$

$$N [ a_1 a_2 a_3^\dagger a_4^\dagger ] \stackrel{=0}{=} + N [ a_1 a_2 a_3^\dagger a_4^\dagger ] \stackrel{=0}{=}$$

$$( a_\alpha^\dagger a_\beta^\dagger = \delta_{\alpha\beta} ; a_\alpha a_\beta^\dagger = a_\alpha^\dagger a_\beta^\dagger > a_\beta^\dagger a_\alpha = 0 )$$

$$\langle 0 | a_\alpha a_\beta | 0 \rangle = 0$$

$$+ N [ \underbrace{a_1 a_2 a_3^\dagger a_4^\dagger}_{=0} ] + N [ \overbrace{a_1 a_2 a_3^\dagger a_4^\dagger}^{\text{FL}} ]$$

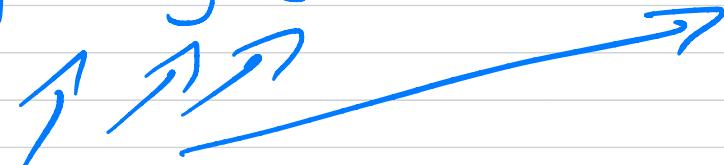
$$- S_{13} N [ a_2 a_4^\dagger ] \stackrel{=0}{=}$$

$$+ N \overline{[q_1 q_2 q_3]} \overline{q_4}^+$$

$$+ N \overline{[q_1 q_2 q_3]} \overline{q_4}^+$$

Wick's theorem

$$\langle c | x y z \dots w | 0 \rangle =$$



string of annihilation  
and creation operators

$$N[xyz \dots w]$$

$$+ \sum N \overbrace{[xyz \dots w]}^{\text{(1 set of permutations)}}$$

$$+ \sum_{\substack{\text{(2)} \\ \dots}} N \overbrace{[xyz \dots w]}^{\text{+}} + \sum_{\substack{\text{M} \\ \text{2 set}}} N \overbrace{[xyz \dots w]}^{\text{# of operators}}$$

$$\hat{N} : \sum_{\alpha} a_{\alpha}^+ a_{\alpha}$$

$$\sum_{\alpha} \underbrace{\langle \alpha_1 \alpha_2 | a_{\alpha}^+ a_{\alpha} | \alpha_1 \alpha_2 \rangle}_{\delta_{\alpha}}$$

$$\langle c | a_{\alpha_2} a_{\alpha_1} a_{\alpha}^+ a_{\alpha} a_{\alpha_1}^+ a_{\alpha_2} | o \rangle$$

$$\langle c | a_{\alpha_2} a_{\alpha_1} a_{\alpha}^+ a_{\alpha} \underbrace{a_{\alpha_1}^+ a_{\alpha_1}}_{\delta_{\alpha_1}} a_{\alpha_2} | o \rangle = \delta_{\alpha_1}$$

$$\langle c | a_{\alpha_2} a_{\alpha_1} a_{\alpha}^+ a_{\alpha} a_{\alpha_1}^+ a_{\alpha_2} | o \rangle = \delta_{\alpha_2}$$

$$\hat{H}_0 = \sum_{\alpha\beta} \langle \alpha | h_0 | \beta \rangle a_\alpha^\dagger a_\beta$$

(one-body - operation)

$$h_0 | \beta \rangle = \epsilon_\beta | \beta \rangle \quad \langle \alpha | \beta \rangle = \delta_{\alpha\beta}$$

$$\hat{H}_0 = \sum_{\alpha} \epsilon_\alpha a_\alpha^\dagger a_\alpha$$

$$\hat{N} = \sum_{\alpha} a_\alpha^\dagger a_\alpha$$

$$\langle \alpha_1, \alpha_2 | \hat{H}_0 | \alpha_1, \alpha_2 \rangle =$$

$$\sum_{\alpha} \sum_{\alpha} \langle c | a_{\alpha_2} a_{\alpha_1} a_{\alpha}^+ a_{\alpha} a_{\alpha_1}^+ a_{\alpha_2}^+ | c \rangle$$

$$= \sum_{\alpha} \sum_{\alpha} (S_{\alpha \alpha_1} + S_{\alpha \alpha_2})$$

$$= \varepsilon_{\alpha_1} + \varepsilon_{\alpha_2}$$

spectral decomposition of  
operators

Define an ONB

$$|i\rangle = \{ |0\rangle, |1\rangle, \dots, |n-1\rangle \}$$

$\hat{A}$  has  $|i\rangle$  as an eigen basis

Expand

$$|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle \quad \left| \sum_{i=0}^{n-1} |\alpha_i|^2 = 1 \right.$$

$$\hat{A}|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i \underbrace{\hat{A}|i\rangle}_{\lambda_i|i\rangle}$$

Define a projection operator

$$P_j = |j\rangle\langle j|$$

$$\hat{P}_j | \psi \rangle = | j \rangle \langle j | \left( \sum_{i=0}^{n-1} a_i | i \rangle \right)$$

$$= \sum_{i=0}^{n-1} a_i | j \rangle \langle j | i \rangle = a_j | j \rangle$$

$$\hat{A} | \psi \rangle = \sum_{i=0}^{n-1} a_i x_i | i \rangle$$

$$= \left( \sum_{i=0}^{n-1} \lambda_i \hat{P}_i \right) | \psi \rangle \Rightarrow$$

$$\hat{A} = \sum_{i=0}^{n-1} \lambda_i P_i$$