Exercises FYS4480/9480, week 38, September 15-19, 2025

Exercise 1

This exercise is a continuation of the exercises from last week on the so-called Lipkin model. We considered a state with all fermions in the lowest single-particle state

$$|\Phi_{J_z=-2}\rangle = a_{1-}^{\dagger} a_{2-}^{\dagger} a_{3-}^{\dagger} a_{4-}^{\dagger} |0\rangle .$$

This state has $J_z = -2$ and belongs to the set of projections for $J = 2$. We will use the shorthand notation $|J, J_z\rangle$ for states with different spon J and spin projection J_z. The other possible states have $J_z = -1$, $J_z = 0$, $J_z = 1$ and $J_z = 2$.

Use the raising or lowering operators J_+ and J_- in order to construct the states for spin $J_z = -1$ $J_z = 0$, $J_z = 1$ and $J_z = 2$. The action of these two operators on a given state with spin J and projection J_z is given by ($\hbar = 1$) by $J_+ |J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z+1)}\, |J, J_z+1\rangle$ and $J_- |J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z-1)}\, |J, J_z-1\rangle$.

Exercise 2

a) Show that the onebody part of the Hamiltonian

$$\hat{H}_0 = \sum_{pq} \langle p| \hat{h}_0 |q\rangle a_p^{\dagger} a_q$$

can be written, using standard annihilation and creation operators, in normal-ordered form as

$$\begin{aligned}
\hat{H}_0 &= \sum_{pq} \langle p| \hat{h}_0 |q\rangle a_p^{\dagger} a_q \\
&= \sum_{pq} \langle p| \hat{h}_0 |q\rangle \left\{a_p^{\dagger} a_q\right\} + \delta_{pq\in i} \sum_{pq} \langle p| \hat{h}_0 |q\rangle \\
&= \sum_{pq} \langle p| \hat{h}_0 |q\rangle \left\{a_p^{\dagger} a_q\right\} + \sum_{i} \langle i| \hat{h}_0 |i\rangle
\end{aligned}$$

Explain the meaning of the various symbols. Which reference vacuum has been used?

b) Show that the twobody part of the Hamiltonian

$$\hat{H}_I = \frac{1}{4} \sum_{pqrs} \langle pq| \hat{v} |rs\rangle a_p^{\dagger} a_q^{\dagger} a_s a_r$$

can be written, using standard annihilation and creation operators, in normal-ordered form as

$$\begin{aligned}
\hat{H}_I &= \frac{1}{4} \sum_{pqrs} \langle pq| \hat{v} |rs\rangle a_p^{\dagger} a_q^{\dagger} a_s a_r \\
&= \frac{1}{4} \sum_{pqrs} \langle pq| \hat{v} |rs\rangle \left\{a_p^{\dagger} a_q^{\dagger} a_s a_r\right\} + \sum_{pqi} \langle pi| \hat{v} |qi\rangle \left\{a_p^{\dagger} a_q\right\} + \frac{1}{2} \sum_{ij} \langle ij| \hat{v} |ij\rangle
\end{aligned}$$

Explain again the meaning of the various symbols. The two-body matrix elements are anti-symmetrized.