AnswerLipkin

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1 Solution to exercises week 37 and 38, Lipkin model

Morten Hjorth-Jensen, Department of Physics, University of Oslo

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1.1 Lipkin model

We will study a schematic model (the Lipkin model, see Nuclear Physics **62** (1965) 188), for the interaction among 2 and more fermions that can occupy two different energy levels.

In project 1 we consider a two-fermion case and a four-fermion case.

For four fermions, the case we consider in the examples here, each levels has degeneration $d = 4$, leading to different total spin values. The two levels have quantum numbers $\sigma = \pm 1$, with the upper level having $2\sigma = +1$ and energy $\varepsilon_1 = \varepsilon/2$. The lower level has $2\sigma = -1$ and energy $\varepsilon_2 = -\varepsilon/2$. That is, the lowest single-particle level has negative spin projection (or spin down), while the upper level has spin up. In addition, the substates of each level are characterized by the quantum numbers $p = 1, 2, 3, 4$.

1.2 Four fermion case

We define the single-particle states (for the four fermion case which we will work on here)

$$
|u_{\sigma=-1,p}\rangle = a^{\dagger}_{-p}|0\rangle \qquad |u_{\sigma=1,p}\rangle = a^{\dagger}_{+p}|0\rangle.
$$

The single-particle states span an orthonormal basis.

1.3 Hamiltonian

The Hamiltonian of the system is given by

$$
\begin{aligned}
\hat{H} = &\quad \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \\
\hat{H}_0 = &\quad \frac{1}{2}\varepsilon \textstyle\sum_{\sigma,p} \sigma a^{\dagger}_{\sigma,p} a_{\sigma,p} \\
\hat{H}_1 = &\quad \frac{1}{2}V \textstyle\sum_{\sigma,p,p'} a^{\dagger}_{\sigma,p} a^{\dagger}_{\sigma,p'} a_{-\sigma,p'} a_{-\sigma,p} \\
\hat{H}_2 = &\quad \frac{1}{2}W \textstyle\sum_{\sigma,p,p'} a^{\dagger}_{\sigma,p} a^{\dagger}_{-\sigma,p'} a_{\sigma,p'} a_{-\sigma,p}
\end{aligned}
$$

where V and W are constants. The operator H_1 can move pairs of fermions while H_2 is a spin-exchange term. The latter moves a pair of fermions from a state $(p\sigma, p' - \sigma)$ to a state $(p - \sigma, p'\sigma)$.

1.4 Quasispin operators

We are going to rewrite the above Hamiltonian in terms of so-called quasispin operators

$$
\begin{aligned}
\hat{J}_+ &= \textstyle\sum_p a^\dagger_{p+} a_{p-} \\
\hat{J}_- &= \textstyle\sum_p a^\dagger_{p-} a_{p+} \\
\hat{J}_z &= \tfrac{1}{2} \textstyle\sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} \\
\hat{J}^2 &= J_+ J_- + J_z^2 - J_z
\end{aligned}
$$

We show here that these operators obey the commutation relations for angular momentum.

1.5 Including the number operator

We can in turn express $\hat{H}$ in terms of the above quasispin operators and the number operator

$$
\hat{N} = \sum_{p\sigma} a^\dagger_{p\sigma} a_{p\sigma}.
$$

We have the following quasispin operators

$$
J_\pm = \sum_p a^\dagger_{p\pm} a_{p\mp}, \tag{1}
$$

$$
J_z = \frac{1}{2} \sum_{p,\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma}, \tag{2}
$$

$$
J^2 = J_+ J_- + J_z^2 - J_z, \tag{3}
$$

and we want to compute the commutators

$$
[J_z, J_\pm], \quad [J_+, J_-], \quad [J^2, J_\pm] \quad \text{og} \quad [J^2, J_z].
$$

1.6 Angular momentum magics I

Let us start with the first one and inserting for J_z and $J_\pm$ given by the equations (2) and (1), respectively, we obtain

$$\begin{aligned}
[J_z, J_\pm] &= J_z J_\pm - J_\pm J_z \\
&= \left(\frac{1}{2}\sum_{p,\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma}\right)\left(\sum_{p'} a^\dagger_{p'\pm} a_{p'\mp}\right) - \left(\sum_{p'} a^\dagger_{p'\pm} a_{p'\mp}\right)\left(\frac{1}{2}\sum_{p,\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma}\right) \\
&= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\pm} a_{p'\mp} - a^\dagger_{p'\pm} a_{p'\mp} a^\dagger_{p\sigma} a_{p\sigma}\right).
\end{aligned}$$

1.7 Angular momentum magics II

Using the commutation relations for the creation and annihilation operators

$$\{a_l, a_k\} = 0, \tag{4}$$

$$\{a^\dagger_l, a^\dagger_k\} = 0, \tag{5}$$

$$\{a^\dagger_l, a_k\} = \delta_{lk}, \tag{6}$$

in order to move the operators in the right product to be in the same order as those in the lefthand product

$$\begin{aligned}
[J_z, J_\pm] &= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\pm} a_{p'\mp} - a^\dagger_{p'\pm}\left(\delta_{p'p}\delta_{\mp\sigma} - a^\dagger_{p\sigma} a_{p'\mp}\right) a_{p\sigma}\right) \\
&= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\pm} a_{p'\mp} - a^\dagger_{p'\pm}\delta_{p'p}\delta_{\mp\sigma} a_{p\sigma} + a^\dagger_{p'\pm} a^\dagger_{p\sigma} a_{p'\mp} a_{p\sigma}\right).
\end{aligned}$$

1.8 Angular momentum magics III

It results in

$$\begin{aligned}
[J_z, J_\pm] &= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\pm} a_{p'\mp} - a^\dagger_{p'\pm}\delta_{pp'}\delta_{\mp\sigma} a_{p\sigma} + a^\dagger_{p\sigma} a^\dagger_{p'\pm} a_{p\sigma} a_{p'\mp}\right) \\
&= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\pm} a_{p'\mp} - a^\dagger_{p'\pm}\delta_{pp'}\delta_{\mp\sigma} a_{p\sigma} + a^\dagger_{p\sigma}\left(\delta_{pp'}\delta_{\pm\sigma} - a_{p\sigma} a^\dagger_{p'\pm}\right) a_{p'\mp}\right) \\
&= \frac{1}{2}\sum_{p,p',\sigma} \sigma \left(a^\dagger_{p\sigma}\delta_{pp'}\delta_{\pm\sigma} a_{p'\mp} - a^\dagger_{p'\pm}\delta_{pp'}\delta_{\mp\sigma} a_{p\sigma}\right).
\end{aligned}$$

1.9 Angular momentum magics IV

The last equality leads to

$$\begin{aligned}
[J_z, J_\pm] &= \frac{1}{2}\sum_p \left((\pm 1)a^\dagger_{p\pm}a_{p\mp} - (\mp 1)a^\dagger_{p\pm}a_{p\mp}\right) = \pm\frac{1}{2}\sum_p \left(a^\dagger_{p\pm}a_{p\mp} + (\pm 1)a^\dagger_{p\pm}a_{p\mp}\right) \\
&= \pm\sum_p a^\dagger_{p\pm}a_{p\mp} = \pm J_\pm,
\end{aligned}$$

where the last results follows from comparing with Eq. (1).

1.10 Angular momentum magics V

We can then continue with the next commutation relation, using Eq. (1),

$$\begin{aligned}
[J_+, J_-] &= J_+J_- - J_-J_+ \\
&= \left(\sum_p a^\dagger_{p'+}a_{p-}\right)\left(\sum_{p'} a^\dagger_{p'-}a_{p'+}\right) - \left(\sum_{p'} a^\dagger_{p'-}a_{p'+}\right)\left(\sum_p a^\dagger_{p+}a_{p-}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p'+}a_{p-}a^\dagger_{p'-}a_{p'+} - a^\dagger_{p'-}a_{p'+}a^\dagger_{p+}a_{p-}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p'+}a_{p-}a^\dagger_{p'-}a_{p'+} - a^\dagger_{p'-}\left(\delta_{++}\delta_{pp'} - a^\dagger_{p+}a_{p'+}\right)a_{p-}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p'+}a_{p-}a^\dagger_{p'-}a_{p'+} - a^\dagger_{p'-}\delta_{pp'}a_{p-} + a^\dagger_{p'-}a^\dagger_{p+}a_{p'+}a_{p-}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p'+}a_{p-}a^\dagger_{p'-}a_{p'+} - a^\dagger_{p'-}\delta_{pp'}a_{p-} + a^\dagger_{p+}a^\dagger_{p'-}a_{p-}a_{p'+}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p'+}a_{p-}a^\dagger_{p'-}a_{p'+} - a^\dagger_{p'-}\delta_{pp'}a_{p-} + a^\dagger_{p+}\left(\delta_{--}\delta_{pp'} - a_{p-}a^\dagger_{p'-}\right)a_{p'+}\right) \\
&= \sum_{p,p'} \left(a^\dagger_{p+}\delta_{pp'}a_{p'+} - a^\dagger_{p'-}\delta_{pp'}a_{p-}\right),
\end{aligned}$$

1.11 Angular momentum magics VI

Which results in

$$[J_+, J_-] = \sum_p \left(a^\dagger_{p+}a_{p+} - a^\dagger_{p-}a_{p-}\right) = 2J_z,$$

It is straightforward to show that

$$[J^2, J_\pm] = [J_+J_- + J_z^2 - J_z, J_\pm] = [J_+J_-, J_\pm] + [J_z^2, J_\pm] - [J_z, J_\pm].$$

1.12 Angular momentum magics VII

Using the relations

$$[AB, C] = A[B, C] + [A, C]B, \tag{7}$$

$$[A, BC] = [A, B]C + B[A, C], \tag{8}$$

we obtain

$$[J^2, J_\pm] = J_+[J_-, J_\pm] + [J_+, J_\pm]J_- + J_z[J_z, J_\pm] + [J_z, J_\pm]J_z - [J_z, J_\pm].$$

1.13 Angular momentum magics VIII

Finally, from the above it follows that

$$\begin{aligned}
[J^2, J_+] &= -2J_+J_z + J_z[J_z, J_+] + [J_z, J_+]J_z - [J_z, J_+] \\
&= -2J_+J_z + J_zJ_+ + J_+J_z - J_+ \\
&= -2J_+J_z + J_+ + J_+J_z + J_+J_z - J_+ = 0,
\end{aligned}$$

and

$$\begin{aligned}
[J^2, J_-] &= 2J_zJ_- + J_z[J_z, J_-] + [J_z, J_-]J_z - [J_z, J_-] \\
&= 2J_zJ_- - J_zJ_- - J_-J_z + J_- \\
&= J_zJ_- - (J_zJ_- + J_-) + J_- = 0.
\end{aligned}$$

1.14 Angular momentum magics IX

Our last commutator is given by

$$\begin{aligned}
[J^2, J_z] &= [J_+J_- + J_z^2 - J_z, J_z] \\
&= [J_+J_-, J_z] + [J_z^2, J_z] - [J_z, J_z] \\
&= J_+[J_-, J_z] + [J_+, J_z]J_- \\
&= J_+J_- - J_+J_- = 0
\end{aligned}$$

1.15 Angular momentum magics X

Summing up we have

$$[J_z, J_\pm] = \pm J_\pm, \tag{9}$$

$$[J_+, J_-] = 2J_z, \tag{10}$$

$$[J^2, J_\pm] = 0, \tag{11}$$

$$[J^2, J_z] = 0, \tag{12}$$

which are the standard commutation relations for angular (or orbital) momentum $L_\pm$, L_z og L^2.

1.16 Rewriting the Hamiltonian

We wrote the above Hamiltonian as

$$H = H_0 + H_1 + H_2,$$

with

$$H_0 = \frac{1}{2}\varepsilon \sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma},$$

and

$$H_1 = \frac{1}{2}V \sum_{p,p',\sigma} a^\dagger_{p\sigma} a^\dagger_{p'\sigma} a_{p'-\sigma} a_{p-\sigma},$$

and

$$H_2 = \frac{1}{2}W \sum_{p,p',\sigma} a^\dagger_{p\sigma} a^\dagger_{p'-\sigma} a_{p'\sigma} a_{p-\sigma}.$$

1.17 Hamiltonian and angular momentum magics I

We will now rewrite the Hamiltonian in terms of the above quasi-spin operators and the number operator

$$N = \sum_{p,\sigma} a^\dagger_{p\sigma} a_{p\sigma}. \tag{13}$$

Going through each term of the Hamiltonian and using the expressions for the quasi-spin operators we obtain

$$H_0 = \varepsilon J_z. \tag{14}$$

1.18 Hamiltonian and angular momentum magics II

Moving over to H_1 and using the anti-commutation relations (4) through (6) we obtain

$$\begin{aligned}
H_1 &= \frac{1}{2} V \sum_{p,p',\sigma} a^\dagger_{p\sigma} a^\dagger_{p'\sigma} a_{p'-\sigma} a_{p-\sigma} \\
&= \frac{1}{2} V \sum_{p,p',\sigma} -a^\dagger_{p\sigma} a^\dagger_{p'\sigma} a_{p-\sigma} a_{p'-\sigma} \\
&= \frac{1}{2} V \sum_{p,p',\sigma} -a^\dagger_{p\sigma} \left(\delta_{pp'} \delta_{\sigma-\sigma} - a_{p-\sigma} a^\dagger_{p'\sigma} \right) a_{p'-\sigma} \\
&= \frac{1}{2} V \sum_{p,p',\sigma} a^\dagger_{p\sigma} a_{p-\sigma} a^\dagger_{p'\sigma} a_{p'-\sigma}
\end{aligned}$$

1.19 Hamiltonian and angular momentum magics III

Rewriting the sum over σ we arrive at

$$\begin{aligned}
H_1 &= \frac{1}{2} V \sum_{p,p'} a^\dagger_{p+} a_{p-} a^\dagger_{p'+} a_{p'-} + a^\dagger_{p-} a_{p+} a^\dagger_{p'-} a_{p'+} \\
&= \frac{1}{2} V \left[\sum_p \left(a^\dagger_{p+} a_{p-} \right) \sum_{p'} \left(a^\dagger_{p'+} a_{p'-} \right) + \sum_p \left(a^\dagger_{p-} a_{p+} \right) \sum_{p'} \left(a^\dagger_{p'-} a_{p'+} \right) \right] \\
&= \frac{1}{2} V \left[J_+ J_+ + J_- J_- \right] = \frac{1}{2} V \left[J_+^2 + J_-^2 \right],
\end{aligned}$$

which leads to

$$H_1 = \frac{1}{2} V \left(J_+^2 + J_-^2 \right). \tag{15}$$

1.20 Hamiltonian and angular momentum magics IV

Finally, we rewrite the last term

$$
\begin{aligned}
H_2 &= \frac{1}{2}W\sum_{p,p',\sigma} a^\dagger_{p\sigma}a^\dagger_{p'-\sigma}a_{p'\sigma}a_{p-\sigma}\\
&= \frac{1}{2}W\sum_{p,p',\sigma} -a^\dagger_{p\sigma}a^\dagger_{p'-\sigma}a_{p-\sigma}a_{p'\sigma}\\
&= \frac{1}{2}W\sum_{p,p',\sigma} -a^\dagger_{p\sigma}\left(\delta_{pp'}\delta_{-\sigma-\sigma} - a_{p-\sigma}a^\dagger_{p'-\sigma}\right)a_{p'\sigma}\\
&= \frac{1}{2}W\sum_{p,p',\sigma} -a^\dagger_{p\sigma}\delta_{pp'}a_{p'\sigma} + a^\dagger_{p\sigma}a_{p-\sigma}a^\dagger_{p'-\sigma}a_{p'\sigma}\\
&= \frac{1}{2}W\left(-\sum_{p,\sigma} a^\dagger_{p\sigma}a_{p\sigma} + \sum_{p,p',\sigma} a^\dagger_{p\sigma}a_{p-\sigma}a^\dagger_{p'-\sigma}a_{p'\sigma}\right)
\end{aligned}
$$

1.21 Hamiltonian and angular momentum magics V

Using the expression for the number operator we obtain

$$
\begin{aligned}
\sum_{p,p',\sigma} a^\dagger_{p\sigma}a_{p-\sigma}a^\dagger_{p'-\sigma}a_{p'\sigma} &= \sum_{p,p'} a^\dagger_{p+}a_{p-}a^\dagger_{p'-}a_{p'+} + a^\dagger_{p-}a_{p+}a^\dagger_{p'+}a_{p'-}\\
&= \sum_{p}\left(a^\dagger_{p+}a_{p-}\right)\sum_{p'}\left(a^\dagger_{p'-}a_{p'+}\right) + \sum_{p}\left(a^\dagger_{p-}a_{p+}\right)\sum_{p'}\left(a^\dagger_{p'+}a_{p'-}\right)\\
&= J_+J_- + J_-J_+,
\end{aligned}
$$

resulting in

$$
H_2 = \frac{1}{2}W\left(-N + J_+J_- + J_-J_+\right). \tag{16}
$$

We have thus expressed the Hamiltonian in term of the quasi-spin operators. Below, we will show how we can rewrite these expressions in terms of Pauli X, Y and Z matrices.

1.22 Commutation relations for the Hamiltonian

The above expressions can in turn be used to show that the Hamiltonian commutes with the various quasi-spin operators. This leads to quantum numbers which are conserved. Let us first show that $[H, J^2] = 0$, which means that J is a so-called *good* quantum number and that the total spin is a conserved quantum number.

We have

$$
\begin{aligned}
[H, J^2] &= [H_0 + H_1 + H_2, J^2]\\
&= [H_0, J^2] + [H_1, J^2] + [H_2, J^2]\\
&= \varepsilon[J_z, J^2] + \frac{1}{2}V[J_+^2 + J_-^2, J^2] + \frac{1}{2}W[-N + J_+J_- + J_-J_+, J^2].
\end{aligned}
$$

1.23 Hamiltonian and commutators

We have previously shown that

$$[H, J^2] = \frac{1}{2}V\left([J_+^2, J^2] + [J_-^2, J^2]\right) + \frac{1}{2}W\left(-[N, J^2] + [J_+J_-, J^2] + [J_-J_+, J^2]\right)$$

Using that $[J_\pm, J^2] = 0$, it follows that $[J_\pm^2, J^2] = 0$. We can then see that $[J_+J_-, J^2] = 0$ and $[J_-J_+, J^2] = 0$ which leads to

$$\begin{aligned}
[H, J^2] &= -\frac{1}{2}W[N, J^2] \\
&= \frac{1}{2}W\left(-[N, J_+J_-] - [N, J_z^2] + [N, J_z]\right) \\
&= \frac{1}{2}W\left(-[N, J_+]J_- - J_+[N, J_-] - [N, J_z]J_z - J_z[N, J_z] + [N, J_z]\right).
\end{aligned}$$

1.24 Including the number operator

Combining with the number operator we have

$$\begin{aligned}
[N, J_\pm] &= NJ_\pm - J_\pm N \\
&= \left(\sum_{p,\sigma} a_{p\sigma}^\dagger a_{p\sigma}\right)\left(\sum_{p'} a_{p'\pm}^\dagger a_{p'\mp}\right) - \left(\sum_{p'} a_{p'\pm}^\dagger a_{p'\mp}\right)\left(\sum_{p,\sigma} a_{p\sigma}^\dagger a_{p\sigma}\right) \\
&= \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p\sigma} a_{p'\pm}^\dagger a_{p'\mp} - a_{p'\pm}^\dagger a_{p'\mp} a_{p\sigma}^\dagger a_{p\sigma} \\
&= \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p\sigma} a_{p'\pm}^\dagger a_{p'\mp} - a_{p'\pm}^\dagger\left(\delta_{\mp\sigma}\delta_{pp'} - a_{p\sigma}^\dagger a_{p'\mp}\right)a_{p\sigma} \\
&= \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p\sigma} a_{p'\pm}^\dagger a_{p'\mp} - a_{p'\pm}^\dagger \delta_{\mp\sigma}\delta_{pp'} a_{p\sigma} + a_{p'\pm}^\dagger a_{p\sigma}^\dagger a_{p'\mp} a_{p\sigma} \\
&= \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p\sigma} a_{p'\pm}^\dagger a_{p'\mp} + a_{p\sigma}^\dagger a_{p'\pm}^\dagger a_{p\sigma} a_{p'\mp} - \sum_p a_{p\pm}^\dagger a_{p\mp} \\
&= \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p\sigma} a_{p'\pm}^\dagger a_{p'\mp} + a_{p\sigma}^\dagger\left(\delta_{pp'}\delta_{\pm\sigma} - a_{p\sigma} a_{p'\pm}^\dagger\right)a_{p'\mp} - \sum_p a_{p\pm}^\dagger a_{p\mp} \\
&= \sum_p a_{p\pm}^\dagger a_{p\mp} - \sum_p a_{p\pm}^\dagger a_{p\mp} = 0.
\end{aligned}$$

1.25 Hamiltonian and angular momentum commutators

We obtain then

$$
\begin{aligned}
[N, J_z] &= NJ_z - J_zN \\
&= \left(\sum_{p,\sigma} a^\dagger_{p\sigma} a_{p\sigma}\right)\left(\frac{1}{2}\sum_{p',\sigma} \sigma a^\dagger_{p'\sigma} a_{p'\sigma}\right) - \left(\frac{1}{2}\sum_{p',\sigma} \sigma a^\dagger_{p'\sigma} a_{p'\sigma}\right)\left(\sum_{p,\sigma} a^\dagger_{p\sigma} a_{p\sigma}\right) \\
&= \sum_{p,p',\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} a^\dagger_{p'\sigma} a_{p'\sigma} - \sigma a^\dagger_{p'\sigma} a_{p'\sigma} a^\dagger_{p\sigma} a_{p\sigma} = 0,
\end{aligned}
$$

which leads to

$$
[H, J^2] = 0, \tag{17}
$$

and J is a good quantum number.

1.26 Constructing the Hamiltonian matrix for $J = 2$

We start with the state (unique) where all spins point down

$$
|2, -2\rangle = a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4-} |0\rangle \tag{18}
$$

which is a state with $J_z = -2$ and $J = 2$. (we label the states as $|J, J_z\rangle$). For $J = 2$ we have the spin projections $J_z = -2, -1, 0, 1, 2$. We can use the lowering and raising operators for spin in order to define the other states

$$
J_+|J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z+1)}|J, J_z + 1\rangle, \tag{19}
$$

$$
J_-|J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z-1)}|J, J_z - 1\rangle. \tag{20}
$$

1.27 Constructing the Hamiltonian matrix for the other states

We can then construct all other states with $J = 2$ using the raising operator J_+ on $|2, -2\rangle$

$$
J_+|2, -2\rangle = \sqrt{2(2+1) - (-2)(-2+1)}|2, -2+1\rangle = \sqrt{6-2}|2, -1\rangle = 2|2, -1\rangle,
$$

which gives

$$
|2, -1\rangle = \frac{1}{2} J_+|2, -2\rangle \tag{21}
$$

$$= \frac{1}{2} \sum_p a^\dagger_{p+} a_{p-} a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4-} |0\rangle \tag{22}$$

$$= \frac{1}{2} \left(a^\dagger_{1+} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4-} + a^\dagger_{1-} a^\dagger_{2+} a^\dagger_{3-} a^\dagger_{4-} + a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3+} a^\dagger_{4-} + a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4+} \right) |0\rangle. \tag{23}$$

1.28 Constructing the Hamiltonian matrix

We can construct all the other states in the same way. That is

$$J_+|2,-1\rangle = \sqrt{2(2+1)-(-1)(-1+1)}|2,-1+1\rangle = \sqrt{6}|2,0\rangle,$$

which results in

$$|2,0\rangle = \frac{1}{\sqrt{6}} \left(a^\dagger_{1+} a^\dagger_{2+} a^\dagger_{3-} a^\dagger_{4-} + a^\dagger_{1+} a^\dagger_{2-} a^\dagger_{3+} a^\dagger_{4-} + a^\dagger_{1+} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4+} + a^\dagger_{1-} a^\dagger_{2+} a^\dagger_{3+} a^\dagger_{4-} + \right.$$
$$\left. a^\dagger_{1-} a^\dagger_{2+} a^\dagger_{3-} a^\dagger_{4+} + a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3+} a^\dagger_{4+} \right) |0\rangle \tag{24}$$

1.29 Constructing the Hamiltonian matrix, last two states

The two remaining states are

$$|2,1\rangle = \frac{1}{2} \left(a^\dagger_{1+} a^\dagger_{2+} a^\dagger_{3+} a^\dagger_{4-} + a^\dagger_{1+} a^\dagger_{2+} a^\dagger_{3-} a^\dagger_{4+} + a^\dagger_{1+} a^\dagger_{2-} a^\dagger_{3+} a^\dagger_{4+} + a^\dagger_{1-} a^\dagger_{2+} a^\dagger_{3+} a^\dagger_{4+} \right). \tag{25}$$

and

$$|2,2\rangle = a^\dagger_{1+} a^\dagger_{2+} a^\dagger_{3+} a^\dagger_{4+} |0\rangle. \tag{26}$$

1.30 Final Hamiltonian matrix

These five states can in turn be used as computational basis states in order to define the Hamiltonian matrix to be diagonalized. The matrix elements are given by $\langle J, J_z|H|J', J'_z\rangle$. The Hamiltonian is hermitian and we obtain after all this labor of ours

$$H_{J=2} = \begin{bmatrix} -2\varepsilon & 0 & \sqrt{6}V & 0 & 0 \\ 0 & -\varepsilon+3W & 0 & 3V & 0 \\ \sqrt{6}V & 0 & 4W & 0 & \sqrt{6}V \\ 0 & 3V & 0 & \varepsilon+3W & 0 \\ 0 & 0 & \sqrt{6}V & 0 & 2\varepsilon \end{bmatrix} \tag{27}$$

1.31 Comparing with standard diagonalization

We can now select a set of parameters and diagonalize the above matrix. We select $\epsilon = 2$, $V = -1/3$, $W = -1/4$ and our matrix becoes

$$H^{(1)}_{J=2} = \begin{bmatrix} -4 & 0 & -\sqrt{6}/3 & 0 & 0 \\ 0 & -2-3/4 & 0 & -1 & 0 \\ -\sqrt{6}/3 & 0 & -1 & 0 & -\sqrt{6}/3 \\ 0 & -1 & 0 & 2+-3/4 & 0 \\ 0 & 0 & -\sqrt{6}/3 & 0 & 4 \end{bmatrix},$$

which gives the eigenvalue

$$D = \begin{bmatrix} -4.21288 & 0 & 0 & 0 & 0 \\ 0 & -2.98607 & 0 & 0 & 0 \\ 0 & 0 & -0.91914 & 0 & 0 \\ 0 & 0 & 0 & 1.48607 & 0 \\ 0 & 0 & 0 & 0 & 4.13201 \end{bmatrix}.$$

The lowest state has an admixture of basis states given by

$$|\psi_0\rangle = 0.96735|2,-2\rangle + 0.25221|2,0\rangle + 0.02507|2,2\rangle,$$

with energy $E_0 = -4.21288$.

1.32 Comparing with standard diagonalization, other parameters

We can now change the parameters to $\varepsilon = 2$, $V = -4/3$, $W = -1$. Our matrix reads then

$$H^{(2)}_{J=2} = \begin{bmatrix} -4 & 0 & -4\sqrt{6}/3 & 0 & 0 \\ 0 & -5 & 0 & -4 & 0 \\ -4\sqrt{6}/3 & 0 & -4 & 0 & -4\sqrt{6}/3 \\ 0 & -4 & 0 & -1 & 0 \\ 0 & 0 & -4\sqrt{6}/3 & 0 & 4 \end{bmatrix},$$

with the following eigenvalues

$$D = \begin{bmatrix} -7.75122 & 0 & 0 & 0 & 0 \\ 0 & -7.47214 & 0 & 0 & 0 \\ 0 & 0 & -1.55581 & 0 & 0 \\ 0 & 0 & 0 & 1.47214 & 0 \\ 0 & 0 & 0 & 0 & 5.30704 \end{bmatrix}.$$

The new ground state (lowest state) has the following admixture of computational basis states

$$|\psi_0\rangle = 0.64268|2,-2\rangle + 0.73816|2,0\rangle + 0.20515|2,2\rangle,$$

with energy $E_0 = -7.75122$.

1.33 Analysis

For the first set of parameters, the likelihood for observing the system in the computational basis state $|2,-2\rangle$ is rather large. This is expected since the interaction matrix elements are smaller than the single-particle energies. For the second case, with larger matrix elements, we see a much stronger mixing of the other states, again as expected due to the ratio of the interaction matrix elements and the single-particle energies.