Lecture Fys4480, November 23, 2023

Creation and annihilation operator with time

$$H_0 = \sum_k \underbrace{\langle k | h_0 | k \rangle}_{\varepsilon_k} a_k^\dagger a_k$$

$$O_I(t) = e^{iH_0 t/\hbar} O_S e^{-iH_0 t/\hbar}$$

$$i\hbar \frac{\partial}{\partial t} O_I(t) = [O_I(t), \hat{H}_0]$$

$$i\hbar \frac{\partial}{\partial t} a_k(t) =$$

time dependence

$$e^{iH_0 t/\hbar} [a_k, H_0] e^{-iH_0 t/\hbar}$$

no time dependence

$$[a_k, H_0] = \left[a_k, \sum_{k'} \varepsilon_{k'} a^\dagger_{k'} a_{k'}\right]$$

$$= \sum_{k'} \varepsilon_{k'} a_k a^\dagger_{k'} a_{k'} - \sum_{k'} a^\dagger_{k'} a_{k'} a_k \times \varepsilon_{k'}$$

$$= \varepsilon_k a_k$$

$$i\hbar \frac{\partial}{\partial t} a_k(t) = \varepsilon_k \underbrace{e^{iH_0 t/\hbar} a_k e^{-iH_0 t/\hbar}}_{a_k(t)}$$

integrate

$$a_k(t) = a_k e^{-i\varepsilon_k t/\hbar}$$

similarly $\quad a^\dagger_k(t) = a^\dagger_k e^{i\varepsilon_k t/\hbar}$

Wick's theorem for time-dependent operators

$$\langle 0 | a_k(t_1)\, a_{k'}^{+}(t_2) | 0 \rangle$$

$$= e^{-i/\hbar (t_1 - t_2) \varepsilon_k}$$

$$\times \langle 0 | a_k a_{k'}^{+} | 0 \rangle \, \delta_{kk'}$$

$$= e^{-i/\hbar (t_1 - t_2) \varepsilon_k} \delta_{kk'}$$

$t_2 > t_1$, $t_2 = t_1$, $t_1 > t_2$

Breaks causality, we annihilate a particle before it has been created.

In order to respect causality, we need to modify Wick's theorem to its time-dependent

Example

$$T\left[a_1(t_1)\,a_2(t_2)\,a_3^+(t_3)\,a_4(t_4)\right)$$

$$t_1 > t_2 = t_4 > t_3$$

$$= -\,a_1(t_1)\,a_2(t_2)\,a_4(t_4)\,a_3^+(t_3)$$

Previous example with two operators

$$\langle 0 | T\left[a_k(t_1)\,a_{k'}^+(t_2)\right] | 0 \rangle$$

$$=$$

$$\left\{ \begin{array}{l} \langle 0 | a_k(t_1)\, a_{k'}^\dagger(t_2) | 0 \rangle \\ \quad = e^{-i/\hbar (t_1 - t_2)\varepsilon_k} \delta_{kk'} \\ \qquad\qquad t_1 \geq t_2 \end{array} \right.$$

$$- \langle 0 | a_{k'}^\dagger(t_2)\, a_k(t_1) | 0 \rangle = 0$$

$$\text{if} \quad t_1 < t_2$$

Time-dependent Wick's theorem

$$N[a_1(t_1)\, a_2(t_2)\, a_3^\dagger(t_3)\, a_4(t_4)]$$

$$= (-)^2 \; a_3^\dagger(t_3)\, a_1(t_1)\, a_2(t_2)\, a_4(t_4)$$

$$\overbracket{u(t_1)\, v(t_2)} = T[u(t_1)\, v(t_2)] - N[u(t_1)\, v(t_2)]$$

Example

$$\overbracket{a_k(t_1)\, a_{k'}^{+}(t_2)} = e^{-i/\hbar(\varepsilon_k t_1 - \varepsilon_{k'} t_2)} \left\{ a_k a_{k'}^{+} + a_{k'}^{+} a_k \right\}$$

$$\delta_{kk'}$$

for $t_1 \geq t_2$

$$\overline{a_k \, a_{k'}^{\dagger}} = \langle 0 | a_k a_{k'}^{\dagger} | 0 \rangle$$

$$+ N[a_k a_{k'}^{\dagger}]$$

$$= \delta_{kk'} - \underbrace{\langle 0 | a_{k'}^{\dagger} a_k | 0 \rangle}_{= 0}$$

For $t_2 > t_1$

$$\overline{a_k(t_1) \, a_{k'}^{\dagger}}(t_2) = - a_{k'}^{\dagger}(t_2) \, a_k(t_1)$$

$$+ \quad a_{k'}^{\dagger}(t_2) \, a_k(t_1)$$

$$= 0$$

$$T\left[u(t_1)\, v(t_2) \cdots z(t_n)\right] =$$

$$N[\cdots] + \sum_{(1)} N[\quad]$$

$$+ \sum_{(2)} N[\quad] + \cdots$$

$$+ \sum_{\left(\left[\frac{n}{2}\right]\right)} N[\quad \cdots]$$

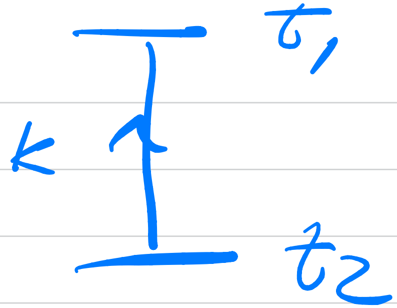
$$\langle 0 | T[u(t_1) v(t_2) \cdots z(t_n)] | 0 \rangle$$

$$= \begin{cases} \sum\limits_{\left(\left[\frac{n}{2}\right]\right)} u(t_1) v(t_2) \cdots z(t_n) & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$$

$$a_k(t_1) a^+_{k'}(t_2) = \delta_{kk'} e^{-i/\hbar\, \varepsilon_k (t_1 - t_2)} \times \epsilon(t_1 - t_2)$$

$t_1 \neq t_2$

$$t_1 > t_2 \qquad = \delta_{kk'}\, g_k(t_1 - t_2)$$

$$g_k(t_1 - t_2) =$$ 

Example

$$\langle 0 | T\left[a^\dagger_{k_1}(t_1)\, a^\dagger_{k_2}(t_1)\, a_{k_3}(t_2)\, a_{k_4}(t_2) \right] \times | 0 \rangle$$

$$= a^\dagger_{k_1}(t_1)\, a^\dagger_{k_2}(t_1)\, a_{k_3}(t_2)\, a_{k_4}(t_2)$$

$$+ \; a^\dagger_{k_1}(t_1)\, a^\dagger_{k_2}(t_1)\, a_{k_3}(t_2)\, a_{k_4}(t_2)$$

$$=$$

$$g_{k_1}(t_2-t_1)\, g_{k_2}(t_2-t_1)$$

$$\times \left[\delta_{k_1 k_4}\, \delta_{k_2 k_3} - \delta_{k_1 k_3}\, \delta_{k_2 k_4} \right]$$

particle-hole operators

$$b_k^+(t) = \begin{cases} e^{+i/\hbar\, \varepsilon_k t}\, a_k^+ & k > F \\ e^{-i/\hbar\, \varepsilon_k t}\, a_k & k \leq F \end{cases}$$

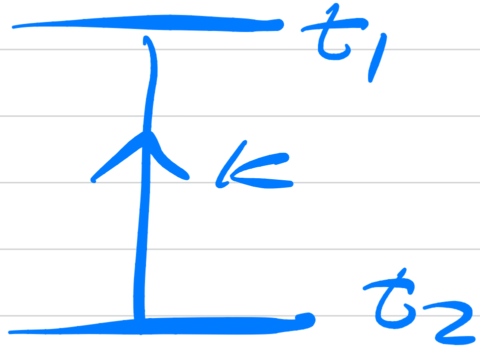
$$b_k(t) = \begin{cases} e^{-i/\hbar\, \varepsilon_k t}\, a_k & k > F \\ e^{i/\hbar\, \varepsilon_k t}\, a_k^+ & k \leq F \end{cases}$$

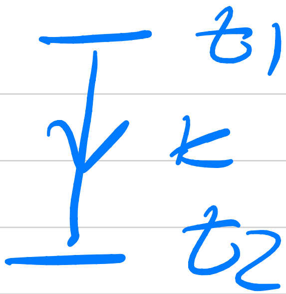
$$b_k(t_1)\, b_{k'}^{+}(t_2) =$$

$$\begin{cases} \delta_{kk'}\, e^{-\frac{i}{\hbar}\varepsilon_k (t_1 - t_2)}\, \theta(t_1 - t_2) = \delta_{kk'}\, g_k^{>}(t_1 - t_2) & k, k' > F \\[2ex] \delta_{kk'}\, e^{+\frac{i}{\hbar}\varepsilon_k (t_1 - t_2)}\, \theta(t_1 - t_2) = \delta_{kk'}\, g_k^{<}(t_1 - t_2) & k, k' \le F \end{cases}$$

$g_k^{>}(t_1 - t_2)$

$k > F$





$k \le F \qquad g_k^{<}(t_1 - t_2)$

Adiabatic hypothesis

$\Delta E = E_0 - W_0 \qquad H_0|\Phi_0\rangle = W_0|\Phi_0\rangle$

$$= \frac{\langle \Phi_0 | H_1 | \Psi_0 \rangle}{\langle \Phi_0 | \Psi_0 \rangle}$$

Linking $\qquad |\Psi_0\rangle = u(t, t_0)|\Psi_0(t_0)\rangle$

$|\Psi_0(t_0)\rangle = |\Phi_0\rangle$

$t_0 \to t = -\infty$

$$H(t) = H_0 + H_I\, e^{\alpha t}$$

as time $t \to -\infty$, $e^{\alpha t} \to 0$

$H_I\, e^{\alpha t} \to 0$ when $t \to -\infty$

$$i\hbar \frac{\partial}{\partial t} |\psi_I(t)\rangle = H_I(t) |\psi_I(t)\rangle$$

$$\hbar = 1$$

$$H_I(t) = \exp(iH_0 \cdot t^{+}) H_I \exp(\alpha t)$$
$$\times \exp(-iH_0 t)$$

$$|\psi_I(t)\rangle = u(t, t') |\psi_I(t')\rangle$$

$$|\Psi_I(t)\rangle = U(t,-\infty)|\Psi_I(-\infty)\rangle$$

$$|\Psi_I(-\infty)\rangle = |\Phi_0\rangle$$

$$|\Psi_I(t)\rangle = U(t,-\infty)|\Phi_0\rangle$$

$$|\Psi_I(t)\rangle = \exp(iH_0t)|\Psi_S(t)\rangle$$

$$|\Psi_I(0)\rangle = |\Psi_S(0)\rangle$$

$$|\Psi_I(0)\rangle = |\Psi_0\rangle$$

$$|\Psi_0\rangle = U(0,-\infty)|\Phi_0\rangle$$

$$\Delta E = \lim_{\alpha\to 0} \frac{\langle\Phi_0|H_I e^{\alpha t} U(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|U(0,-\infty)|\Phi_0\rangle}$$

$$\Delta E = \frac{\langle \Phi_0 | H_I \, U(0, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | \underbrace{U(0, -\infty) | \Phi_0 \rangle}_{|\Psi_0\rangle}}$$

$$\frac{\partial}{\partial t} \langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle$$

$$= \langle \Phi_0 | \frac{\partial}{\partial t} U_\alpha(t, -\infty) | \Phi_0 \rangle$$

$$U_\alpha(t, -\infty) \; : \; \lim_{\alpha \to 0} U(t, -\infty)$$

$$\frac{\partial}{\partial t} \langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle$$

$$= -i \langle \Phi_0 | H_1(t) U_\alpha(t, -\infty) | \Phi_0 \rangle$$

$$\frac{\partial \log}{\partial t} \left(\langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle \right)$$

$$= - i \frac{\langle \Phi_0 | H_1(t) U_\alpha(t, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle}$$

$$H_I(0) = H_I$$

$$\Delta E = \lim_{\alpha \to 0} i \left[\frac{\partial}{\partial t} \log \langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle \right]$$

taken at $t=0$.

Can we link this with time independent RS pert. theory?

$$U_\alpha(t,-\infty) = 1 + \sum_{n=1}^{\infty} U_n$$

$$U_n = (-i)^n \int_{-\infty}^{t} H_1(t_1)\,dt_1 \int_{-\infty}^{t_1} H_1(t_2)\,dt_2 \cdots \int_{-\infty}^{t_{n-1}} H_1(t_n)\,dt_n$$

$$\langle \Phi_0 | U_\alpha(t,-\infty) | \Phi_0 \rangle =$$

$$= \langle \Phi_0 | 1 + \sum_{n=1}^{\infty} u_n | \Phi_0 \rangle$$

$$= 1 + \sum_{n=1}^{n} A_n$$

$$A_n = \langle \Phi_0 | u_n | \Phi_0 \rangle$$

(result of correlation operator)

$$\log \langle \Phi_0 | U_\alpha(t, -\infty) | \Phi_0 \rangle =$$

$$\log\left(1 + \sum_n A_n\right)$$

$$= \sum_n A_n - \frac{1}{2}\left(\sum A_n\right)^2 + \frac{1}{3}\left(\sum A_n\right)^3$$

$$+ \cdots$$