FYS4480/9480 lecture September 6, 2024

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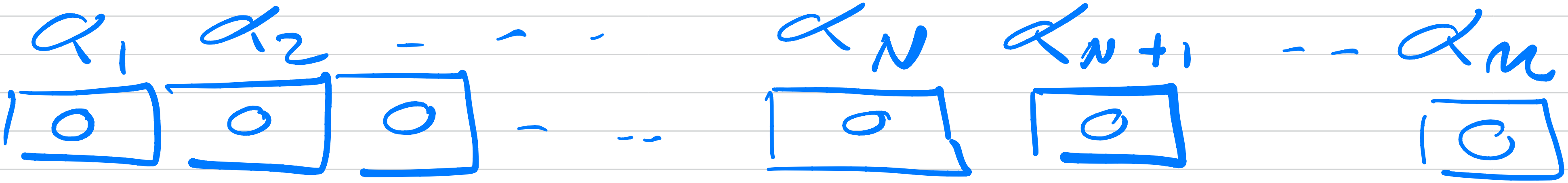
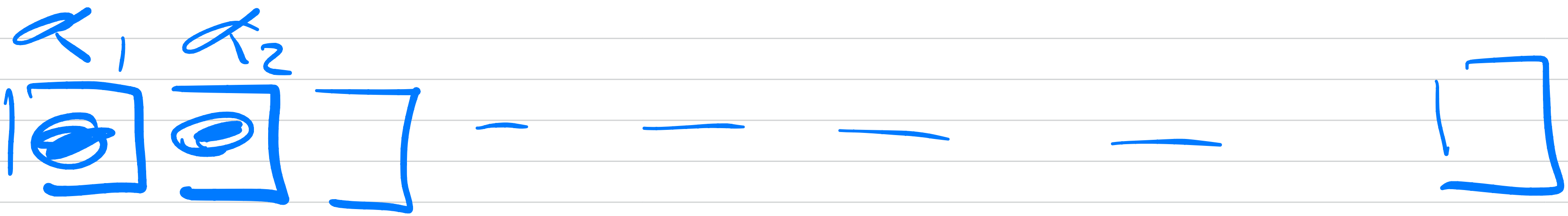
vacuum state $|0\rangle$



$|0\rangle$



$$a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle = |\alpha_1 \alpha_2\rangle$$

$$n = 2$$

$$|\alpha_1\alpha_2\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$$

$$|\alpha_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |10\rangle$$

$$|\alpha_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |01\rangle$$

$$|\alpha_1\rangle \otimes |\alpha_2\rangle = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= |0100\rangle$$

$$|\alpha_1\alpha_2\rangle = |11\rangle$$

with $n = 4 \wedge N = 2$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \boxed{\bullet} & \boxed{\bullet} & \boxed{\circ} & \boxed{\circ} \end{array}$$

$|1100\rangle$

$|1010\rangle$

$|1001\rangle$

$|\alpha_1\rangle \otimes |\alpha_2\rangle$

$|\alpha_1 \alpha_2\rangle$

$$a_\alpha |0\rangle = 0 \qquad a_\alpha |\alpha\rangle = |0\rangle$$

$$\psi_\alpha(x) = \langle x | \alpha \rangle$$

$$SD = |\alpha_1 \alpha_2 \ldots \alpha_N\rangle$$

$$\hat{N} = \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha^\dagger a_\alpha$$

$$\hat{N}^2 = \hat{N} \quad , \quad [\hat{N}_i, \hat{N}_j] = 0$$

$$\hat{N}_i = a_i^\dagger a_i$$

$$[a_i, \hat{N}_j] = \underbrace{\langle i|j\rangle}_{\delta_{ij}} a_i$$

$$[a_i^\dagger, \hat{N}_j] = -\delta_{ij} a_i^\dagger$$

Example $\underline{1}$

$$|\alpha_1 \alpha_2\rangle = a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle \quad \alpha_1 \neq \alpha_2$$

$$\langle \alpha_1 \alpha_2 | \alpha_1 \alpha_2 \rangle =$$

$$\langle 0 | a_{\alpha_2} a_{\alpha_1} a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle =$$

$$\langle 0 | a_{\alpha_2} (\delta_{\alpha_1 \alpha_1} - a^\dagger_{\alpha_1} a_{\alpha_1}) a^\dagger_{\alpha_2} |0\rangle$$

$$= \langle 0 | a_{\alpha_2} a^\dagger_{\alpha_2} |0\rangle \delta_{\alpha_1 \alpha_1}$$

$$- \langle 0 | a_{\alpha_2} a^\dagger_{\alpha_1} a_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle =$$

$$\langle c| a_{\alpha_2} a^\dagger_{\alpha_2} |0\rangle \delta_{\alpha_1\alpha_1}$$

$$- \underbrace{\langle c| a_{\alpha_2} a^\dagger_{\alpha_1} (\delta_{\alpha_1\alpha_2} - a^\dagger_{\alpha_2} a_{\alpha_1}}_{=0}) |0\rangle$$

$$\langle c| (\delta_{\alpha_2\alpha_2} - a^\dagger_{\alpha_2} a_{\alpha_2}) |0\rangle \delta_{\alpha_1\alpha_1}$$

$$= \delta_{\alpha_2\alpha_2} \delta_{\alpha_1\alpha_1}$$

introduce new notations

$$\langle 0| a_\alpha a^\dagger_\beta |0\rangle = \langle 0| \delta_{\alpha\beta} |0\rangle - \langle 0| a^\dagger_\beta a_\alpha |0\rangle =$$

$$\overbrace{a_\alpha a_\beta^\dagger} \qquad + \underbrace{N[a_\alpha a_\beta^\dagger]}_{-\langle 0|a_\beta^\dagger a_\alpha|0\rangle}$$

$$= \langle 0|a_\alpha a_\beta^\dagger|0\rangle$$

$$\delta_{\alpha\beta}$$

$$= \langle \alpha|\beta\rangle$$

Example 2

$$N[a_1 a_2 a_3 a_4^\dagger] = (-)^3 \langle 0|a_4^\dagger a_1 a_2 a_3$$

$$\times |0\rangle$$

$$= 0$$

$$\langle 0| a_1 a_2 a_3^\dagger |0\rangle \qquad 1 \neq 2 \neq 3$$

$$= \langle 0| a_1 (\delta_{32} - a_3^\dagger a_2) |0\rangle$$

$$= \langle 0| a_1 a_2 a_3^\dagger |0\rangle$$

$$- \langle 0| a_1 a_3^\dagger a_2 |0\rangle =$$

$$\langle 0| a_1 a_2 a_3^\dagger |0\rangle$$

$$- \langle 0| (\delta_{13} a_2 - a_3^\dagger a_1 a_2) |0\rangle$$

$$= \langle 0| a_1 a_2 a_3^\dagger |0\rangle - \langle 0| a_1 a_3^\dagger a_2 |0\rangle$$

$$+ \underbrace{\langle 0 | a_3^\dagger a_1 a_2 | 0 \rangle}_{N[a_1 a_2 a_3^\dagger]}$$

$$\langle 0 | a_1 a_2 a_3^\dagger | 0 \rangle$$

$$= \langle 0 | a_1 a_2 a_3^\dagger | 0 \rangle$$

$$+ \langle 0 | a_1 a_2 a_3^\dagger | 0 \rangle$$

$$+ N[a_1 a_2 a_3^\dagger] = 0$$

$$\langle 0 | a_1 a_2 a_3^\dagger a_4^\dagger | 0 \rangle =$$

$$N[a_1 a_2 a_3^\dagger a_4^\dagger] \;{=}\, 0 \;+$$

$$N[a_1 \overbrace{a_2 a_3^\dagger} a_4^\dagger] \;{=}\, 0 \;+ N[a_1 a_2 \overbrace{a_3^\dagger a_4^\dagger}] \;{=}\, 0$$

$$\Big(\overbrace{a_\alpha a_\beta^\dagger} = \delta_{\alpha\beta} \;;\quad \overbrace{a_\alpha a_\beta} = \overbrace{a_\alpha^\dagger a_\beta^\dagger} = \overbrace{a_\beta^\dagger a_\alpha} = 0$$

$$\langle 0 | a_\alpha a_\beta | 0 \rangle = 0$$

$$= 0 \qquad\qquad\qquad = 0$$

$$+ N[\overbrace{a_1 a_2 a_3^\dagger} a_4^\dagger] + N[a_1 \overbrace{a_2 a_3^\dagger a_4^\dagger}]$$

$$- \delta_{13} N[a_2 a_4^\dagger] = 0$$

$$+ N[a_1 a_2 a_3^{\dagger} a_4^{\dagger}]$$

$$+ N[a_1 a_2 a_3^{\dagger} a_4^{\dagger}]$$

Wick's theorem

$$\langle 0 | x y z \cdots w | 0 \rangle =$$

string of annihilation and creation operator

$$N[x\,y\,z\,\cdots\,w]$$

$$+\sum_{\substack{(1\ \text{set of}\\ \text{permutations})}} N[x\,\overbrace{y\,z\,\cdots}\,\cdots\,w]$$

$$+\sum_{(2)} N[x\,\overbrace{y\,z\,\cdots}\,\overbrace{\cdots\,w}]$$

$$+\cdots+\sum_{\substack{\frac{M}{2}\\ \text{set}}} N[x\,y\,z\,\cdots\,w]$$

M # of operators

$$\hat{N} : \sum_{\alpha} a^{+}_{\alpha} a_{\alpha}$$

$$\sum_{\alpha} \underbrace{\langle \alpha_1 \alpha_2 | a^{+}_{\alpha} a_{\alpha} | \alpha_1 \alpha_2 \rangle}$$

$$\langle 0 | a_{\alpha_2} a_{\alpha_1} a^{+}_{\alpha} a_{\alpha} a^{+}_{\alpha_1} a^{+}_{\alpha_2} | 0 \rangle$$

$$\langle 0 | a_{\alpha_2} a_{\alpha_1} a^{+}_{\alpha} \underbrace{a_{\alpha} a^{+}_{\alpha_1}}_{\delta_{\alpha\alpha_1}} a^{+}_{\alpha_2} | 0 \rangle = \delta_{\alpha\alpha_1}$$

$$\langle 0 | a_{\alpha_2} a_{\alpha_1} a^{+}_{\alpha} a_{\alpha} a^{+}_{\alpha_1} a^{+}_{\alpha_2} | 0 \rangle = \delta_{\alpha\alpha_2}$$

$$\hat{H}_0 = \sum_{\alpha\beta} \langle \alpha | h_0 | \beta \rangle a_\alpha^\dagger a_\beta$$

(one-body-operator)

$$h_0 |\beta\rangle = \varepsilon_\beta |\beta\rangle \qquad \langle \alpha | \beta \rangle = \delta_{\alpha\beta}$$

$$\hat{H}_0 = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha$$

$$\hat{N} = \sum_\alpha a_\alpha^\dagger a_\alpha$$

$$\langle \alpha_1 \alpha_2 | \hat{H}_0 | \alpha_1 \alpha_2 \rangle =$$

$$\sum_{\alpha} \varepsilon_{\alpha} \langle c | a_{\alpha_2} a_{\alpha_1} a^{+}_{\alpha} a_{\alpha} a^{+}_{\alpha_1} a^{+}_{\alpha_2} | c \rangle$$

$$= \sum_{\alpha} \varepsilon_{\alpha} \left(\delta_{\alpha\alpha_1} + \delta_{\alpha\alpha_2} \right)$$

$$= \varepsilon_{\alpha_1} + \varepsilon_{\alpha_2}$$

Spectral decomposition of operators

Define an ONB

$$|i\rangle = \{ |0\rangle, |1\rangle, \ldots, |n-1\rangle \}$$

$\hat{A}$ has $|i\rangle$ as an eigen basis

Expand

$$|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle \quad \Big| \quad \sum_{i=0}^{n-1} |\alpha_i|^2 = 1$$

$$A|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i \underbrace{A|i\rangle}_{\lambda_i |i\rangle}$$

Define a projection operator $P_j = |j\rangle\langle j|$

$$\hat{P}_j |\psi\rangle = |j\rangle\langle j| \left(\sum_{i=0}^{n-1} \alpha_i |i\rangle \right)$$

$$= \sum_{i=0}^{n-1} \alpha_i |j\rangle\langle j|i\rangle = \alpha_j |j\rangle$$

$$\hat{A} |\psi\rangle = \sum_{i=0}^{n-1} \alpha_i \lambda_i |i\rangle$$

$$= \left(\sum_{i=0}^{n-1} \lambda_i \hat{P}_i \right) |\psi\rangle \Rightarrow$$

$$\hat{A} = \sum_{i=0}^{n-1} \lambda_i P_i$$