Exercises FYS4480, week 37, September 9-13, 2024

Exercise 1

We will study a schematic model (the Lipkin model, Nucl. Phys. **62** (1965) 188) for the interaction among 4 fermions that can occupy two different energy levels. Each levels has degeneration $d = 4$. The two levels have quantum numbers $\sigma = \pm 1$, with the upper level having $\sigma = +1$ and energy $\varepsilon_1 = \varepsilon/2$. The lower level has $\sigma = -1$ and energy $\varepsilon_2 = -\varepsilon/2$. In addition, the substates of each level are characterized by the quantum numbers $p = 1, 2, 3, 4$.

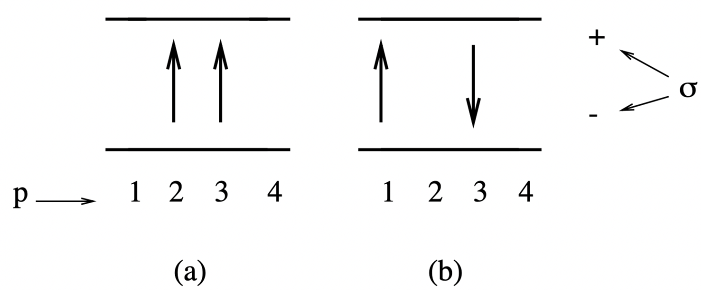
We define the single-particle states

$$|u_{\sigma=-1,p}\rangle = a^\dagger_{-p}|0\rangle \qquad |u_{\sigma=1,p}\rangle = a^\dagger_{+p}|0\rangle .$$

The single-particle states span an orthonormal basis. The Hamiltonian of the system is given by

$$\begin{aligned}
\hat{H} =& \ \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \\
\hat{H}_0 =& \ \tfrac{1}{2}\varepsilon \textstyle\sum_{\sigma,p} \sigma a^\dagger_{\sigma,p} a_{\sigma,p} \\
\hat{H}_1 =& \ \tfrac{1}{2}V \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{\sigma,p'} a_{-\sigma,p'} a_{-\sigma,p} \\
\hat{H}_2 =& \ \tfrac{1}{2}W \textstyle\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{-\sigma,p'} a_{\sigma,p'} a_{-\sigma,p}
\end{aligned}$$

where V and W are constants. The operator H_1 can move pairs of fermions as shown in the left part of the fugure (a). while H_2 is a spin-exchange term. As shown in (b), H_2 moves a pair of fermions from a state $(p\sigma, p' - \sigma)$ to a state $(p - \sigma, p'\sigma)$.



We will encounter this model again in our analysis of mean field methods like the Hartree-fock method and full configuration interaction theory. It is a model which has been used widely in many-body physics and recently also in quantum computing, see for example `https://journals.aps.org/prc/abstract/10.1103/PhysRevC.104.024305`.

a. Quasispin operators Introduce the quasispin operators

$$\begin{aligned}
\hat{J}_+ =& \ \textstyle\sum_p a^\dagger_{p+} a_{p-} \\
\hat{J}_- =& \ \textstyle\sum_p a^\dagger_{p-} a_{p+} \\
\hat{J}_z =& \ \tfrac{1}{2}\textstyle\sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} \\
\hat{J}^2 =& \ J_+J_- + J_z^2 - J_z
\end{aligned}$$

Show that these operators obey the commutation relations for angular momentum.

b. Number operator Express $\hat{H}$ in terms of the above quasispin operators and the number operator

$$\hat{N} = \sum_{p\sigma} a^\dagger_{p\sigma} a_{p\sigma}.$$

c. Commutation relations Show that $\hat{H}$ commutes with J^2, viz., J is a good quantum number. Does it commute with J_z?

d. Wick's theorem Consider thereafter a state with all four fermions in the lowest level (see the above figure). We can write this state as

$$|\Phi_0\rangle = |\Phi_{J_z=-2}\rangle = a^\dagger_{1-} a^\dagger_{2-} a^\dagger_{3-} a^\dagger_{4-} |0\rangle .$$

This state has $J_z = -2$ (convince yourself about this) and belongs to the set of possible projections of $J = 2$. We introduce the shorthand notation $|J, J_z\rangle$ for states with different values of spin J and its projection J_z. We can think of this as our computational basis for $J = 2$ and all five projections J_z. We will also assume that the state Φ_0 can be considered as an ansatz for the ground state of the system.

Use Wick's theorem to calculate the expectation values of

$$\langle \Phi_0 | \hat{N} | \Phi_0 \rangle ,$$

and

$$\langle \Phi_0 | \hat{H} | \Phi_0 \rangle .$$

Comment your results.

e. Using quasispin operators Show that you can obtain the same result for

$$\langle \Phi_0 | \hat{H} | \Phi_0 \rangle .$$

using the quasispin representation of the Hamiltonian (plus the number operator). Comment your results.