Lecture FYS4480/9480, September 6

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$$\langle 0 | a_\alpha a_\beta^\dagger | 0 \rangle = \underbrace{\delta_{\alpha\beta}}_{\overline{a_\alpha a_\beta^\dagger}} \underbrace{- \langle 0 | a_\beta^\dagger a_\alpha | 0 \rangle}_{N[a_\alpha a_\beta^\dagger]}$$

$$\langle 0 | x y | 0 \rangle = N[xy] + N[\overline{x y}]$$

$$\langle 0 | \overline{x y} | 0 \rangle$$

$$\overbrace{a_\alpha a_\beta^\dagger} = \delta_{\alpha\beta}$$

$$\overbrace{a_\alpha^\dagger a_\beta} = 0 = \langle 0 | a_\alpha^\dagger a_\beta | 0 \rangle$$

$$\overbrace{a_\alpha^\dagger a_\beta^\dagger} = \overbrace{a_\alpha a_\beta} = 0$$

$N = 3$ operators

$$\langle 0 | a_1 a_2 a_3^\dagger | 0 \rangle = N[a_1 a_2 a_3^\dagger]$$

$$+ N[a_1 \underbrace{\overbrace{a_2 a_3^\dagger}}_{\delta_{23}}] + N[\overbrace{a_1 a_2 a_3^\dagger}]$$

$N = 4$

$$\langle 0 | a_1 a_2 a_3^\dagger a_4^\dagger | 0 \rangle =$$

$$\boxed{N[a_1 a_2 a_3^\dagger a_4^\dagger]} + \Big($$

1 contraction

$$N[a_1 a_2 a_3^\dagger a_4^\dagger] + N[a_1 a_2 a_3^\dagger a_4^\dagger]$$

$$+ \; N[a_1 a_2 a_3^\dagger a_4^\dagger] +$$

$$N[a_1 a_2 a_3^\dagger a_4^\dagger] +$$

$$N[a_1 a_2 a_3^\dagger a_4^\dagger] + N[a_1 a_2 a_3^\dagger a_4^\dagger]$$

2 contractions

$$|34\rangle = a_3^\dagger a_4^\dagger |0\rangle$$

$$|12\rangle = a_1^\dagger a_2^\dagger |0\rangle$$

$$\langle 12 | 34 \rangle$$

Wick's theorem

$$\langle 0 | x y z \cdots w | 0 \rangle =$$

$$x y z \cdots w = N[x y z \cdots w]$$

$$+ \sum_{(1)} N[x\, \overbrace{y z \cdots w}]$$

$$+ \sum_{(2)} N[xyz - - w]$$

$$+ \cdots + \sum_{\left(\frac{M}{2}\right)} N[xyz - - w]$$

$$= \sum_{\left(\frac{M}{2}\right)} N([xyz - - w])$$

Example before Proof.

$$H_I = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | v | \gamma\delta \rangle$$
$$\times \; a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$

$$\langle \alpha_1 \alpha_2 | H_I | \alpha_1 \alpha_2 \rangle$$

$$= \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | v | \gamma\delta \rangle$$
$$\times \; \langle 0 | a_{\alpha_2} a_{\alpha_1} a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} | 0 \rangle$$

$$\langle \alpha\beta | v | \gamma\delta \rangle$$

$$a_{\alpha_2} a_{\alpha_1} \; a^\dagger_{\alpha} a^\dagger_{\beta} a_{\delta} a_{\gamma} \; a^\dagger_{\alpha_1} a^\dagger_{\alpha_2}$$

$\delta_{\alpha_2\beta}$ $\delta_{\delta\alpha_2}$ $\delta_{\gamma\alpha_1}$

$$\langle \alpha_1\alpha_2 | v | \alpha_1\alpha_2 \rangle$$

$\delta_{\alpha\alpha_1}$

$$- \langle \alpha_1\alpha_2 | v | \alpha_2\alpha_1 \rangle$$

$$\langle \alpha_2\alpha_1 | v | \alpha_2\alpha_1 \rangle$$

$$- \langle \alpha_2\alpha_1 | v | \alpha_1\alpha_2 \rangle$$

$$= \langle \alpha_1 \alpha_2 | v | \alpha_1 \alpha_2 \rangle$$

$$- \langle \alpha_1 \alpha_2 | v | \alpha_2 \alpha_1 \rangle$$

$$= \langle \alpha_1 \alpha_2 | v | \alpha_1 \alpha_2 \rangle_{AS}$$

Proof of Wick's theorem.

Lemma:

a chain of operators

$N[xyz \cdots w]$ and add

a new operator Ω

$$N[xyz\ldots w]\Omega =$$

$$N[xyz\ldots w\Omega] +$$

$$\sum_{(1)\Omega} N[xyz\overline{\ldots w\Omega}]$$

(i) valid immediately if Ω is an annihilation operator since then $N[xyz\ldots w]\Omega$ is already normal-ordered and

$$\overbrace{a_n^{+}\, a_j^{+}} = 0 \qquad \overbrace{a_n\, a_j} = 0$$

(ii) we assume that the sequence $xyz \ldots w$ is normal-ordered

(iii) if x is a creation operator we need to prove the lemma if all $xyz \ldots w$ are annihilation operators since all $\overbrace{a^{+}a^{+}} = 0$

(IV) in (iii) we anticommute Ω through all $xyz \ldots w$ operator get the first term and all the anticommutations which are produced give the second term of the lemma

Examples of (iii) and (iv)

$$N[a_1 a_2 \dots a_N]\, a_e^+$$

$$N[a_1 a_2 \dots a_N]\, a_e$$

<u>$N = 1$</u>

$$N[a_1]\, a_e^+ = N[a_1 a_e^+]$$

$$+ N[\overline{a_1 a_e}^{\,+}]$$

$$= - \langle 0 | a_e^+ a_1 | 0 \rangle$$

$$+ \overline{a_1 a_e}^{\,+}$$

This is easy to extend to $N > 1$ and

$$N[a_1 a_2 \dots a_N]\, a_e^\dagger$$

Wick's generalized theorem

an arbitrary product of creation and annihilation operators in which some operators are already in a normal-ordered form is given by

$$N[A_1 A_2 \ldots]\, N[B_1 B_2 \ldots]\, N[C_1 C_2 \ldots]$$

$$\langle \alpha_1 \alpha_2 | = \langle 0 | a_{\alpha_2} a_{\alpha_1} \qquad a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma \qquad a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} |0\rangle$$

Normal order

$$= N[A_1 A_2 \ldots B_1 B_2 \ldots C_1 C_2 \ldots]$$

$$+ \sum_{\substack{\text{ALL} \\ \text{contractions}}} N[A_1 A_2 \ldots B_1 B_2 \ldots C_1 C_2 \ldots]$$

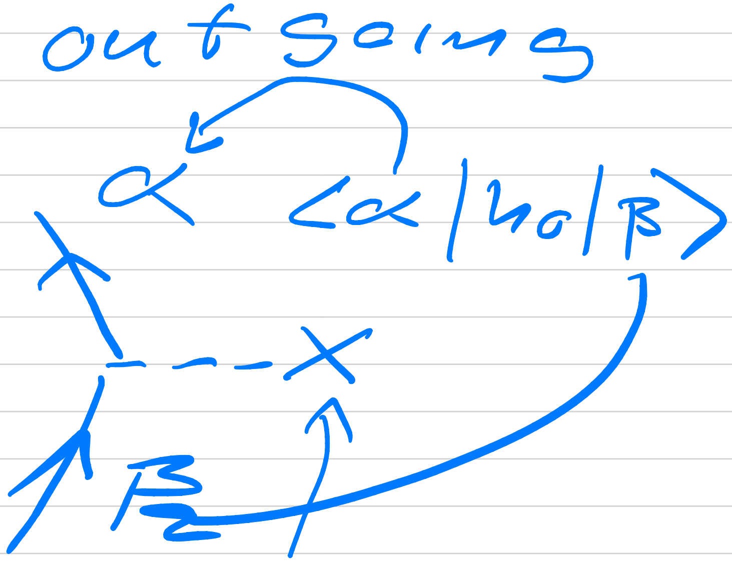
Diagrammatic notation

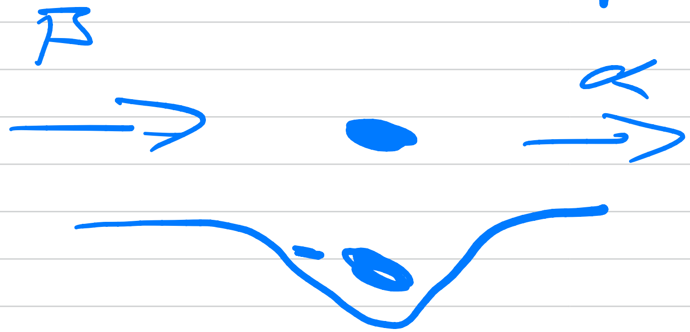
(i) $\quad \overline{a_\alpha a_\alpha^\dagger}$

arrow pointing up represents a particle state on top of $|0\rangle$, all states are particle states

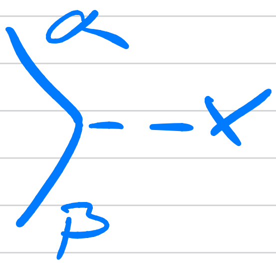
(ii) one body operator

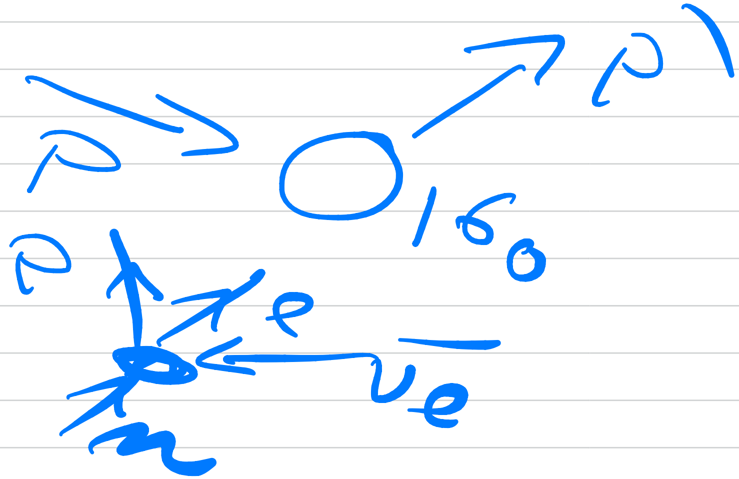
Example $H_0 = \sum_{\alpha\beta} \langle \alpha | h_0 | \beta \rangle \times a_\alpha^\dagger a_\beta$





incoming operator h_0 ($O^{(1)}$)



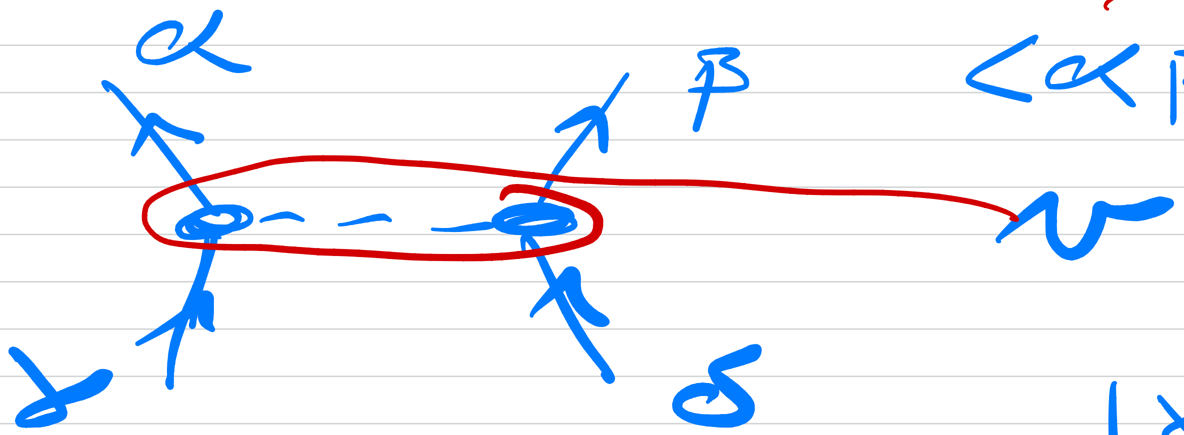


outgoing label α represents

$\langle \alpha |$

incoming β $\longrightarrow$ $| \beta \rangle$

(iii) Two-body



left outgoing

$\langle \alpha \beta |$

Right

$| \gamma \delta \rangle$

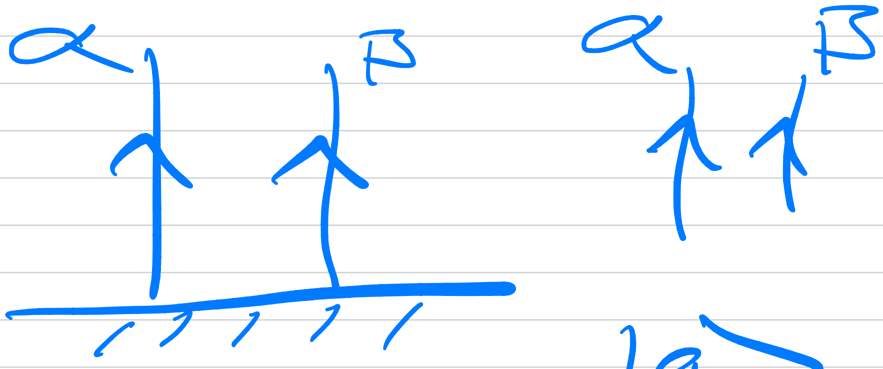
right

incoming left

$$\langle \alpha \beta | v | \gamma \delta \rangle$$

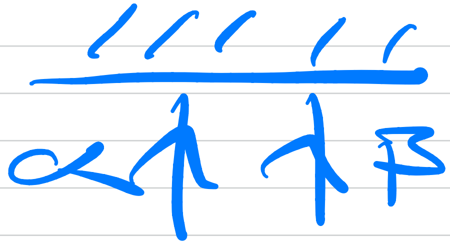
(iv) Specific state

$|\alpha\beta\rangle$



$|0\rangle$

$\langle\alpha\beta|$ $\langle 0|$



(v) $\langle \alpha_1 \alpha_2 | H_1 | \alpha_1 \alpha_2 \rangle$