EFT Introduction. Overview

These notes are closely based (i.e., pirated ;)) on the 2013 Nuclear Forces TALENT course designed and taught by Dick Furnstahl and Achim Schwenk. The course website (along with the original notes and video lectures) may be found here.

a. Symmetry constraints on nuclear NN forces

When we come to the discussion of chiral effective field theory (χEFT) in a few lectures, we'll see that symmetry plays a central (if not THE central) role in building EFTs. Let us therefore start with a conventional discussion of general symmetry constraints on the NN Hamiltonian as you would find in nuclear physics textbooks. Indeed, the approach outlined here was how people made progress Consider the NN potential as an operator that acts in the spin, isospin, and position space Hilbert space. That means that our building blocks are the Pauli matrics for spin, $\boldsymbol{\sigma}_i$, and isospin, $\boldsymbol{\tau}_i$, multiplied by functions of the spatial coordinates.

So we can specify the potential operator $\widehat{V}$ by giving all of the matrix elements

$$\langle \mathbf{r}_1' s_1' t_1' \, \mathbf{r}_2' s_2' t_2' | \widehat{V} | \mathbf{r}_1 s_1 t_1 \, \mathbf{r}_2 s_2 t_2 \rangle \tag{1}$$

where $s_i = \pm 1/2$ and $t_i = \pm 1/2$ are spin and isospin projections. Suppressing spin and isospin for the moment, the action of $\widehat{V}$ on the coordinate basis is

$$\widehat{V} |\mathbf{r}_1 \mathbf{r}_2\rangle = \int V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1' \mathbf{r}_2'\rangle d^3 r_1' \, d^3 r_2' \, . \tag{2}$$

The familiar local potential corresponds to the special

$$V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1, \mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_1') \delta(\mathbf{r}_2 - \mathbf{r}_2') \quad \Longrightarrow \quad \widehat{V} |\mathbf{r}_1 \mathbf{r}_2\rangle = V(\mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1 \mathbf{r}_2\rangle \, . \tag{3}$$

This evidently has no dependence on the velocities of the particles, but just their positions. It is not surprising then that we can translate the more general non-locality into a velocity (or momentum dependence). To do so, we first expand in a Taylor series the primed variables about the unprimed ones:

$$\begin{aligned} |\mathbf{r}_1' \mathbf{r}_2'\rangle &= |\mathbf{r}_1 \mathbf{r}_2\rangle + [(\mathbf{r}_1' - \mathbf{r}_1) \cdot \boldsymbol{\nabla}_1 + (\mathbf{r}_2' - \mathbf{r}_2) \cdot \boldsymbol{\nabla}_2] |\mathbf{r}_1 \mathbf{r}_2\rangle + \cdots \\ &= : \exp\left\{ (\mathbf{r}_1' - \mathbf{r}_1) \cdot \boldsymbol{\nabla}_1 + (\mathbf{r}_2' - \mathbf{r}_2) \cdot \boldsymbol{\nabla}_2 \right\} : |\mathbf{r}_1 \mathbf{r}_2\rangle \, , \end{aligned} \tag{4}$$

where the "normal-ordering" notation $: \widehat{O} :$ means here that the derivatives be moved to act only to the right of the coordinates (and not on the coordinates). Thus,

$$\begin{aligned} \widehat{V} |\mathbf{r}_1 \mathbf{r}_2\rangle &= \int V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) \exp\left\{ \frac{i}{\hbar} (\mathbf{r}_1' - \mathbf{r}_1) \cdot \mathbf{p}_1 + \frac{i}{\hbar} (\mathbf{r}_2' - \mathbf{r}_2) \cdot \mathbf{p}_2 \right\} |\mathbf{r}_1 \mathbf{r}_2\rangle d^3 r_1' \, d^3 r_2' \\ &= \widetilde{V}(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2) |\mathbf{r}_1 \mathbf{r}_2\rangle \, . \end{aligned} \tag{5}$$

So the general form is built from these operators, generally restricted to a low order in momentum (quadratic), plus the allowed spin and isospin dependence.

But this is *too* general a formulation: if we allow arbitrary dependence on $\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2$ we will violate symmetries like spatial translation invariance. Similarly with possible structure for spin and isospin. For example, if we let σ_0 be the identity matrix, then since the σ_i, $i = 0, 1, 2, 3$ form a complete basis in the spin space of one particle, a general expression for $\widehat{V}$ is

$$\widehat{V} = \sum_{i,j=0}^{3} V_{ij} (\sigma_1)_i (\sigma_2)_j \ . \tag{6}$$

So there are 16 V_{ij} functions at this point. Similarly for isospin. But not every combination is consistent with the symmetries, so in fact there are fewer. What are they?

a.1 Operator structure of NN forces

Let us step through the constraints on $\widetilde{V}(\mathbf{r}_1, \mathbf{p}_1, \boldsymbol{\sigma}_1, \boldsymbol{\tau}_1, \mathbf{r}_2, \mathbf{p}_2, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_2)$ based on continuous space-time and discrete symmetries (but not yet chiral symmetry). Generically when we use $\widehat{V}(1,2)$, the 1 and 2 refer to all of the indices.

1. $\widehat{H} = \widehat{T} + \widehat{V}$ is hermitian and so is $\widehat{T}$, therefore $\widehat{V}$ is hermitian.
2. For identical particles, invariance under interchange of coordinates: $V(1,2) = V(2,1)$. This is connected to the symmetry of the two-particle wave function $|1\,2\rangle$.
3. Translational invariance in space. The unitary transformation for translations by $\boldsymbol{a}$ is

   $$U = e^{-i\boldsymbol{a}\cdot\boldsymbol{P}} \ , \tag{7}$$

   where $\boldsymbol{P}$ is the total center-of-mass momentum (the generator of translations). It results in

   $$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{a} \ , \quad \mathbf{k}_i' = \mathbf{k}_i \ , \quad \boldsymbol{\sigma}_i' = \boldsymbol{\sigma}_i \ , \quad \boldsymbol{\tau}_i' = \boldsymbol{\tau}_i \ . \tag{8}$$

   Therefore we require

   $$[\boldsymbol{P}, \widehat{V}] = 0 \ . \tag{9}$$

   In the full coordinate-space form (suppressing spin and isospin),

   $$\langle \mathbf{r}_1\ \mathbf{r}_2 | \widehat{V} | \mathbf{r}_1'\ \mathbf{r}_2' \rangle = \langle \mathbf{r}_1 - \mathbf{a}\ \mathbf{r}_2 - \mathbf{a} | \widehat{V} | \mathbf{r}_1' - \mathbf{a}\ \mathbf{r}_2' - \mathbf{a} \rangle \quad \Longrightarrow \quad \langle \mathbf{r}_1 - \mathbf{r}_2 | \widehat{V} | \mathbf{r}_1' - \mathbf{r}_2' \rangle = \langle \mathbf{r} | \widehat{V} | \mathbf{r}' \rangle \ . \tag{10}$$

   For $\widetilde{V}$ this means

   $$V(1,2) = \widetilde{V}(\mathbf{r}, \mathbf{p}_1, \boldsymbol{\sigma}_1, \boldsymbol{\tau}_1, \mathbf{p}_2, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) \ . \tag{11}$$

4. Galilean invariance (more generally Lorentz invariance). Nuclear low-lying states have $|\mathbf{p}| \sim 200\,\mathrm{MeV}$, so $p/m \approx 0.2$ and $(v/c)^2 < 0.1$. This is small but not *negligible*, which suggests we should treat it as a $1/m$ expansion. Galilean invariance says that the physics looks the same

from a moving frame, so $\mathbf{r}_i' = \mathbf{r}_i$, $\mathbf{p}_i' = \mathbf{p}_i - m_i\boldsymbol{u}$, $\boldsymbol{\sigma}_i' = \boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_i' = \boldsymbol{\tau}_i$. The unitary operator is (total mass M and center-of-mass position operator $\boldsymbol{R}$):

$$U = e^{-iM\boldsymbol{u}\cdot\boldsymbol{R}} , \quad \text{where} \quad M = \sum_i m_i \quad \text{and} \quad \boldsymbol{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i . \tag{12}$$

This results in

$$V(1,2) = \widetilde{V}(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\tau}_1, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) . \tag{13}$$

5. Rotational invariance requires $[\boldsymbol{J}, V] = 0$, where is the *total* angular momentum, i.e., $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. There are three independent scalars from $\mathbf{r}$ and $\mathbf{p}$, namely $\mathbf{r}^2$, $\mathbf{p}^2$, and $\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}$. The latter must appear quadratically because of time-reversal invariance (below), and it is more convenient to use $\boldsymbol{L}^2 = (\mathbf{r}\times\mathbf{p})^2$. The only vector to combine with a linear appearance of $\boldsymbol{S}$ is $\boldsymbol{L}$ to make a spin-orbit interaction. There are only limited choices (see Okubo and Marshak, Ann. Phys. **4**, 166 (1968) for full details).

6. Parity P is space reflection, which has the effect:

$$\mathbf{r}_i' = -\mathbf{r}_i , \quad \mathbf{k}_i' = -\mathbf{k} , \quad \boldsymbol{\sigma}_i' = \boldsymbol{\sigma}_i , \quad \boldsymbol{\tau}_i' = \boldsymbol{\tau}_i . \tag{14}$$

If we perform two space reflections we are back where we started, so $P^2 = 1$ and the eigenvalues are ± 1. Parity is conserved by the strong interaction but violated by the weak interactions; however, the parity-violating matrix elements are small (of order $\sim 0.1\,\text{eV}$). This violation was first observed in the beta decay of polarized ^{60}Co by Wu et al. in 1957, where the polarization of the nuclear spin defines a direction and a preferential emission of electrons in the opposite direction of the spin was observed.

7. Time reversal T has the effect:

$$\mathbf{r}_i' = \mathbf{r}_i , \quad \mathbf{k}_i' = -\mathbf{k} , \quad \boldsymbol{\sigma}_i' = -\boldsymbol{\sigma}_i , \quad \boldsymbol{\tau}_i' = \boldsymbol{\tau}_i . \tag{15}$$

T is violated in the standard model [from indirect CP violation in K^0 decay]. C exchanges particles and antiparticles. There are active direct searches for T violation by looking for permanent dipole moments in neutrons, nuclei, and atoms.

8. Baryon and lepton number conservation. B violated by weak interactions. L would be violated if neutrinos are Majorana, which means they are their own antiparticles.

9. Isospin charge symmetry. $p \leftrightarrow n$ and
Charge independence. Scattering length for $^1\text{S}_0$:

$$a_{nn} \approx (a_{pp} - \text{Coulomb}) \approx -18\,\text{fm} \tag{16}$$

which is isospin charge symmetry. But $a_{np} \approx -23.7\,\text{fm}$, which shows that charge independence breaking is stronger. Both are broken by Coulomb and other e/m plus $m_u \neq m_d$ effects.

In the end, the constraints lead to

$$V_{NN} = V_1(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + V_\tau(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2))\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \tag{17}$$

where we have explicitly applied translational and rotational invariance to write these in terms of $\mathbf{r}$ and $\mathbf{p}$ (remember that these are operators). We further classify V_1 and V_τ into central (scalar), vector (spin-orbit), and tensor, with spin structures of rank 0,1,2:

- **central parts:**
$$V_1(\mathbf{r},\mathbf{p}) + V_\sigma(\mathbf{r},\mathbf{p})\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\ , \tag{18}$$
- **vector parts:**
$$V_{LS}(\mathbf{r},\mathbf{p})\boldsymbol{L}\cdot S\ , \tag{19}$$
- **tensor parts:**
$$V_T(\mathbf{r},\mathbf{p})S_{12}(\widehat{\mathbf{r}}) \tag{20}$$
with tensor operator
$$S_{12}(\widehat{\mathbf{r}}) \equiv \widehat{\mathbf{r}}\cdot\boldsymbol{\sigma_1}\widehat{\mathbf{r}}\cdot\boldsymbol{\sigma_2} - \frac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \tag{21}$$
[*Quick question: Why do we subtract off the 2nd term?*]

The full operator form in coordinate space is:
$$\left\{1_{\rm spin},\ \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,\ {\rm S}_{12}(\widehat{\mathbf{r}}),\ {\rm S}_{12}(\widehat{\mathbf{p}}),\ {\rm L}\cdot{\rm S},\ ({\rm L}\cdot{\rm S})^2\right\}\times\left\{\mathbf{1}_{\rm isospin},\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\right\}\ , \tag{22}$$
times scalar operator-like functions of r^2, p^2, and L^2 (rather than $\mathbf{r}\cdot\mathbf{p}$). In momentum space, the full operator form is:
$$\left\{1_{\rm spin},\ \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,\ {\rm S}_{12}(\widehat{\mathbf{q}}),\ {\rm S}_{12}(\widehat{\mathbf{k}}),\ {\rm i}\mathbf{S}\cdot(\mathbf{q}\times\mathbf{k}),\ \boldsymbol{\sigma_1}\cdot(\mathbf{q}\times\mathbf{k})\,\boldsymbol{\sigma_2}\cdot(\mathbf{q}\times\mathbf{k})\right\}\times\left\{\mathbf{1}_{\rm isospin},\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\right\}\ , \tag{23}$$
where $\mathbf{q}\equiv\mathbf{p}'-\mathbf{p}$ and $\mathbf{k}\equiv(\mathbf{p}'+\mathbf{p})/2$, times scalar functions of p^2, p'^2, and $\mathbf{p}\cdot\mathbf{p}'$.

b. Motivation and background for EFT

b.1 Problems with phenomenological potentials

The best potential models (such as AV18) can describe with $\chi^2/{\rm dof}\approx 1$ all of the NN data (about 6000 points) below the pion production threshold. So what more do we need? Some limitations of these potentials:

- They usually have a very strong repulsive short-range part that requires special (non-systematic) treatment in (some types of) many-body calculations of nuclear structure.
- It is difficult to estimate the theoretical error in a calculation and the range of applicability (i.e., where should it fail?).
- Three-nucleon forces (3NF) are largely included as under-constrained and non-systematic models. How can we define *consistent* 3NF's and operators (e.g., including-meson exchange currents)?
- Models are largely unconnected to QCD (e.g., chiral symmetry is only respected in part). They don't connect NN and other strongly interacting processes (e.g., $\pi\pi$ and πN). Lattice QCD will be able to predict NN, 3N observables for high pion masses. How can we extrapolate to physical pion masses?

To address these limitations, we seek a systematic alternative: effective (field) theory. Before getting into the nitty gritty details, let's start with some boilerplate about the general philosophy of low energy effective theories.

b.2 Effective theories: Appeal to authority :)

Here we pull some nice quotes from H. Georgi, Ann. Rev. Nucl. Part. Sci. **43**, 209 (1993). [Note: while Georgi has some nice qualitative discussions in the intro, the bulk of the review article is at a pretty high level and assumes a solid QFT background. Perhaps the best introductory treatment that does not require the heavy machinery of QFT is the lecture notes by Peter Lepage entitled "How to Renormalize the Schrödinger Equation", which can be found on the course website.]

Some relevant quotes from Georgi about effective theories:

- *One of the most astonishing things about the world in which we live is that there seems to be interesting physics at all scales.*
- *To do physics amid this remarkable richness, it is convenient to be able to isolate a set of phenomena from all the rest, so that we can describe it without having to understand everything. Fortunately, this is often possible. We can divide up the parameter space of the world into different regions, in each of which there is a different appropriate description of the important physics. Such an appropriate description of the important physics is an "effective theory."*
- *The common idea is that if there are parameters that are very large or very small compared to the physical quantities (with the same dimension) that we are interested in, we may get a simpler approximate description of the physics by setting the small parameters to zero and the large parameters to infinity. Then the finite effects of the parameters can be included as small perturbations about this simple approximate starting point.*

Any time there is a hierarchy of (separated) energy scales, think EFT! Even if coupling constants are large (i.e., strongly interacting), the existence of a well-separated hierarchy of scales suggests a small parameter to expand in (i.e. ,the ratio of the low- and high-energy scales).

Examples from our physics experiences and this program of such effective theories (you give more!):

- non-relativistic quantum (or classical) mechanics: $c \to \infty$;
- size of a charge distribution $\to 0$ (multipole expansion);
- mass of the proton in a hydrogen atom $\to \infty$;
- asphericity of a cow $\to 0$;
- number of colors in QCD $\to \infty$;
- chiral effective *field* theory (EFT): $m_\pi \to 0$, $M_N \to \infty$.

You add some!

b.3 Principles of low-energy effective theories

Summary of the basic principles:

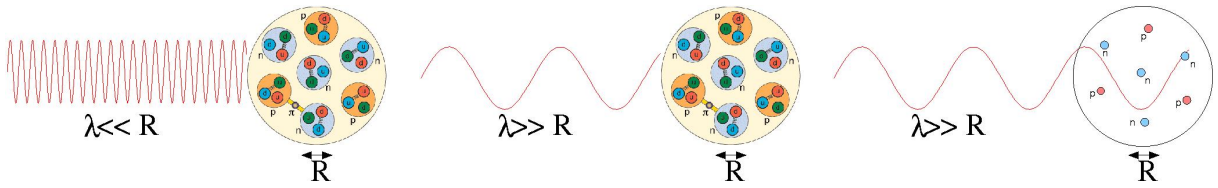


Figure 1: Left: high-resolution, with wavelength of probe short compared to characteristic size of probed structure. Middle: low-resolution probe doesn't resolve the details (because of diffracton). Right: low-energy theory takes advantage and replaces short-distance with long-distance degrees of freedom.

- If system is probed at low energies, fine details are not resolved.
- So we can use low-energy variables for low-energy processes (as they can be easier, more efficient, . . .).
- The short-distance structure can be *replaced* by something simpler (and wrong at short distances!) without distorting low-energy observables. It is important that being wrong at short distances ("incorrect UV behavior") doesn't matter but also cautions us that when it works for long distances, we should not conclude that we know about the UV.
- This is systematically achieved by the effective field theory.

There are some basic physics principles underlying any low-energy effective model or theory. A high-energy, short-wavelength probes sees details. E.g., electron scattering at Jefferson Lab resolves the quark substructure of protons and neutrons in a nucleus. But at lower energies, details are not resolved, and one can replace short distance structure, as in a multipole expansion of a complicated charge or current distribution. So it is not necessary to do full QCD. It is not obvious that this will work in quantum mechanics as it does for pixels or point dots or the classical multipole expansion, because *virtual* states can have high energies that are not, in reality, simple. Renormalization theory says it can be done! (More later!) It doesn't say that we are *insensitive* to all short-distance details, only that there effects at low energies can be accounted for in a simple way. Effective *field* theory is a systematic approach to carrying out this program using a local field theoretical Lagrangian framework.

To begin, we want to apply these ideas to the simplest possible example, for which the resolution is so low we don't resolve *any* interaction between the particles. For the nuclear case, this means that the exchange of pions is considered short-ranged; that is the pion is a heavy degree of freedom that is replaced (as are all other interactions) by contact interactions.

First, we'll lay out the basic ideas of renormalization and power counting for the simplest case: a natural scattering length. After that, the more interesting case for nuclear physics of an unnatural scattering length will be explored in detail.

c. Pionless effective field theory

c.1 Low-energy simplication of boson exchange interactions

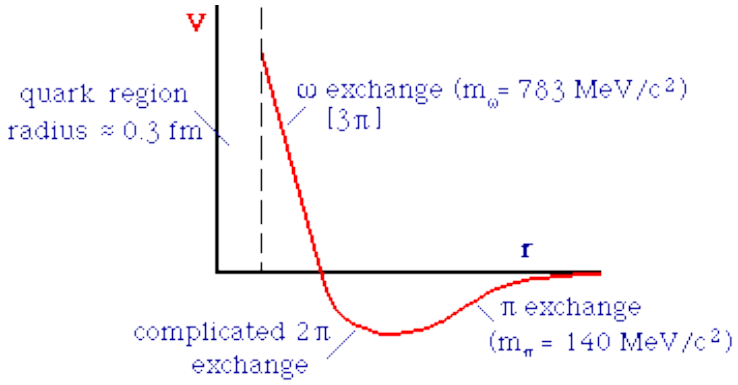


Figure 2: Ranges associated with boson exchange. The intermediate two-pion exchange attraction can be simulated with isoscalar, scalar exchange ("σ").

Let's first get some intuition about what NN forces would look like at low-energies. For this purpose, it is instructive to start with boson exchange potentials. Consider Fourier transforming to momentum space a Yukawa potential with mass μ arising from one of the boson exchanges considered in the last lecture (without paying attention to the overall normalization),

$$g^2 \frac{e^{-\mu|\mathbf{r}|}}{4\pi|\mathbf{r}|} \longleftrightarrow \frac{g^2}{(\mathbf{k}'-\mathbf{k})^2+\mu^2} \ , \tag{24}$$

which manifestly depends on $\mathbf{k}^2$, $\mathbf{k}'^2$, and $\boldsymbol{k}\cdot\boldsymbol{k}'$. (Note: we are ignoring any spin or isospin structure here.) Now suppose we are at very low resolution, which means low momenta $k, k' \ll \mu$ (the deBroglie wavelengths are long). We can Taylor expand in Eq. (24):

$$A_0 + A_2(\mathbf{k}^2+\mathbf{k}'^2) + A_2'\mathbf{k}\cdot\mathbf{k}' + \cdots \tag{25}$$

These terms are called *contact interactions* because when Fourier transformed to coordinate space they become delta functions in the interparticle distance and derivatives of such delta functions:

$$\langle \mathbf{k}|A_0|\mathbf{k}'\rangle \propto A_0 \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}} = A_0(2\pi^3)\delta(\mathbf{x}) \ . \tag{26}$$

So this implies that at low resolution we can replace an extended interaction (in this case, from the exchange of a particle) with a series of contact interactions.

If all we ever did was first-order perturbation theory, then a simple Taylor expansion like this is all we need and we could pick the A_i's to match the corresponding Taylor expansion of the Yukawa term-by-term to reproduce the results of the Yukawa theory. But what if we do second-order perturbation theory (or include the effects of all orders by solving the Schrödinger equation)? Then the sums over intermediate states become integrals over the momenta, which then contain parts where for which $k, k' \geq \mu$ and the expansion does not apply (i.e., it is wrong at high energies). What do we do? Stay tuned!

c.2 Recap of effective range expansion

Because we are considering the low-energy limit of a theory, we expect we will reproduce the physics of the effective range expansion briefly discussed in Morten's lectures. This is generically called the "pionless" EFT, because there are *no* long-range degrees of freedom (i.e., the pion is treated as heavy rather than almost massless!)

Recall that the partial wave expansion of the scattering amplitude or the on-shell T-matrix (which is only different by a multiplicative factor) takes the form

$$f(k,\theta) \equiv \frac{m}{4\pi} T(k,\cos\theta) = \sum_{k=0}^{\infty} \frac{2l+1}{k\cot\delta_l(k) - ik} P_l(\cos\theta) \ . \tag{27}$$

As first shown by Schwinger, $k^{2l+1}\cot\delta_l(k)$ has a power series expansion in k^2 (see Newton for more details on proving this). The radius of convergence is dictated by how the potential falls off for large r; for a Yukawa potential of mass m, this radius is $m/2$. For $l=0$ the expansion is

$$k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 - P r_0^3 k^4 + \cdots \ , \tag{28}$$

which defines the S-wave *scattering length* a_0, the S-wave *effective range* r_0 and the S-wave shape parameter P (often these are written a_s and r_s or a and r_e). Note the sign conventions. For $l=1$ the convention is to write

$$k^3\cot\delta_1(k) = -\frac{3}{a_p^3} \ , \tag{29}$$

which defines the P-wave scattering length a_p (with dimensions of a length).

The effective range expansion for hard-sphere scattering (radius R) is (just do the Taylor expansion with $\delta_0(k) = -kR$):

$$k\cot(-kR) = -\frac{1}{R} + \frac{1}{3} R k^2 + \cdots \quad \Longrightarrow \quad a_0 = R \quad r_0 = 2R/3 \tag{30}$$

(note the sign of a_0) so the magnitudes of the effective range parameters are all the order the range of the interaction R. The radius of convergence here is infinite because the potential is identically zero beyond some radius.

For more general cases, we find that while $r_0 \sim R$, the range of the potential, a_0 can be anything:

- if $a_0 \sim R$, it is called "natural";
- $|a_0| \gg R$ (unnatural) is particularly interesting, e.g., for neutrons or cold atoms.

We can associate the sign and size of a_0 with the behavior of the scattering wave function as the energy (or k) goes to zero, since

$$\frac{\sin(kr + \delta_0(k))}{k} \xrightarrow[k\to 0]{} r - a_0 \ , \tag{31}$$

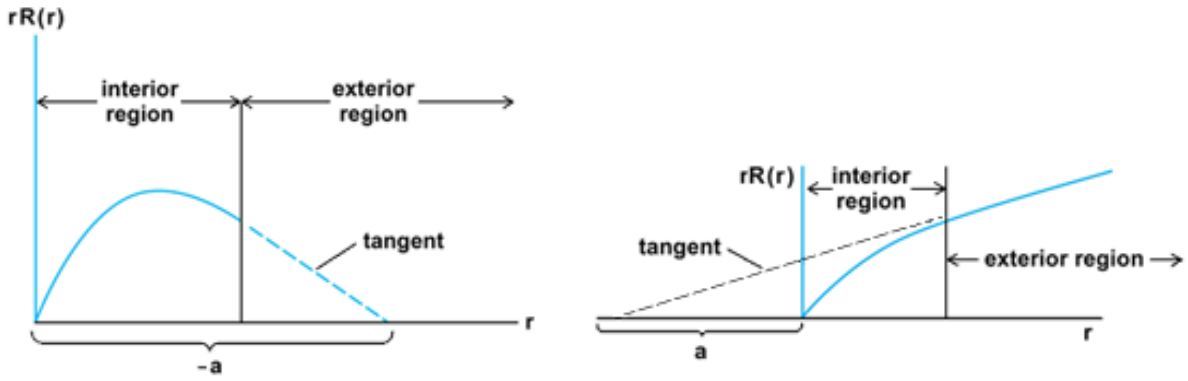


Figure 3: Identifying the S-wave scattering length a_0 by looking at $k \to 0$ wave functions. [credit to ???]

so in the asymptotic region the wave function is a straight line with intercept a_0 (see Fig. 3). We see that a_0 ranges from $-\infty$ to $+\infty$ (are there excluded regions?).

Let us consider the case of low-energy scattering for $l = 0$. Then the S-wave scattering amplitude is (recall Eq. (**??**))

$$f_0(k) = \frac{1}{-1/a_0 - ik} \qquad \Longrightarrow \qquad \sigma(k) = \frac{4\pi}{1/a_0^2 + k^2} \; . \tag{32}$$

In the natural case, $|ka_0| \ll 1$ and $f_0(k) \to -a_0$ at low k and

$$\frac{d\sigma}{d\Omega} = a_0^2 \qquad \Longrightarrow \qquad \sigma = 4\pi a_0^2 \; . \tag{33}$$

Also, $\delta_0(k) \approx -ka_0$ implies that the sign of a corresponds to the sign of V (if strictly attractive or repulsive). Figure 3 tells us that to get large a_0 (unnatural), we need to have close to a zero-energy bound state (so the wave function is close to horizontal in the asymptotic region). If $a_0 > 0$, we have a shallow bound state (i.e., close to zero binding energy) while if $a_0 < 0$ we have a nearly bound (virtual) state. The limit of $|a_0| \to \infty$ is called the unitary limit; the cross section becomes

$$\frac{d\sigma}{d\Omega} \longrightarrow \frac{1}{k^2} \qquad \Longrightarrow \qquad \sigma = \frac{4\pi}{k^2} \; , \tag{34}$$

which is the largest it can be consistent with the constraint of unitarity.

Schwinger first derived the effective range expansion (ERE) back in the 1940's and then Bethe showed an easy way to derive (and understand) it. This is apparently a common pattern with Schwinger's work! The implicit assumption here is that the potential is short-ranged; that is, it falls off sufficiently rapidly with distance. This is certainly satisfied by any potential that actually vanishes beyond a certain distance. Long-range potentials like the Coulomb potential must be treated differently (but a Yukawa potential, which has a finite range although non-vanishing out to $r \to \infty$, is ok).

When first identified, the ERE was a disappointment because it meant that scattering experiments at low energy could not reveal the structure of the potential. For example, it is impossible to invert the expansion at any finite order to find a potential that correctly gives scattering at higher energies. All you determined were a couple of numbers (scattering length and effective range) and any potential whose ERE was fit to these would reproduce experimental cross sections.

In the modern view we are delighted with this: the low-energy theory is determined by a small number of constants that encapsulate the limited high-energy features that affect the low-energy physics (in many ways like a multipole expansion). We can devise an effective theory (called the "pionless effective theory") to reproduce the ERE and consistently extend it to include the coupling to external probes (and other physics). Here we first consider how to reproduce this result in the natural case $a_0 \sim R$; the more interesting (and relevant for nuclear physics!) unnaturally large scattering length case $|a_0| >> R$ will be looked at after that.

c.3 In search of a perturbative expansion

If a_0 is natural, then low-energy scattering simplifies further. Let's take the hard sphere potential as a concrete example. The ERE in this case is indeed natural (recall that $\delta(k) = -kR$):

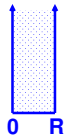
$$k\cot(-kR) = -\frac{1}{a_0} + \frac{1}{2}r_0k^2 + \cdots = -\frac{1}{R}\Big[1 - \frac{(kR)^2}{3} - \frac{(kR)^4}{45} + \cdots\Big]\,, \tag{35}$$

and so if kR is small, we have a convergent expansion. This means that the scattering amplitude itself also has a perturbative expansion in k:

$$\begin{aligned} f_0(k) = \frac{1}{k\cot\delta(k) - ik} &= \frac{1}{-\frac{1}{a_0} + \frac{1}{2}r_0k^2 + \cdots - ik} = \frac{-a_0}{1 - \frac{1}{2}a_0r_0k^2 + \cdots + ia_0k} \\ &\longrightarrow -a_0\big[1 - ia_0k - (a_0^2 - a_0r_0/2)k^2 + \mathcal{O}(k^3a_0^3)\big] \\ &\longrightarrow -R[1 - ikR - 2k^2R^2/3 + \mathcal{O}(k^3R^3)]\,, \end{aligned} \tag{36}$$

where in the last line we substituted the particular case of hard-sphere scattering. That is, for scattering at momentum $k \ll 1/R$, we should recover a perturbative expansion in kR for the scattering amplitude (because a_0 and r_0 are of order R), or more generally, where we leave the coefficients of the expansion free. Note that the perturbative expansion is *not* an expansion in the strength of the potential.

Can we reproduce this simple expansion for the hard-sphere potential? As just noted, ordinary perturbation theory in the original hard-sphere potential manifestly won't work:



$$\Longrightarrow \quad \langle \mathbf{k}|V|\mathbf{k}'\rangle \propto \int d\mathbf{x}\; e^{i\mathbf{k}\cdot\mathbf{x}}\, V(\mathbf{x})\, e^{-i\mathbf{k}'\cdot\mathbf{x}} \longrightarrow \infty\,, \tag{37}$$

so even first-order perturbation theory fails and higher orders just get worse. The standard solution is to solve the scattering problem *nonperturbatively*, then expand in kR. For our example, this is

easy: just use $\delta_0(k) = -kR$ and give it to Mathematica to find the last line in Eq. (36). Now this is easy to do for 2–2 scattering, but not for the many-body problem! So even for this simple system we would like a more systematic approach. In anticipation of this more interesting application to finite density, we consider the EFT approach: the condition $k \ll 1/R$ means that we probe at low resolution (long wavelenths), so we can replace the potential with a simpler but general interaction.

c.4 EFT for a "natural" short-range interaction

As we motivated when considering boson exchange, a general low-energy expansion in momentum space is:

$$\langle \mathbf{k}|V_{\rm eft}|\mathbf{k}'\rangle = C_0 + \frac{1}{2}C_2(\mathbf{k}^2 + \mathbf{k}'^2) + C_2'\mathbf{k}\cdot\mathbf{k}' + \cdots \,. \tag{38}$$

Recall the conditions we set on a nuclear interaction: hermiticity, invariance under particle 1 $\longleftrightarrow$ particle 2, translational, Galilean, and rotational invariance, parity and time-reversal invariance, conservation of baryon number, isospin invariance. These tell us that this expansion really is the most general to this order (in the exercises you can explore the constraints on the k^4 terms). That is, despite our motivation, we are not restricted to the model that the interaction comes from boson exchange, but rather we have a complete expansion that accommodates *any* underlying physics. That is why we call this expansion *model independent*. At this point we are not considering spin, but we'll return briefly to this question below.

In field theory language, we can write the EFT Lagrangian density $\mathcal{L}_{\rm eft}$ with the most general local (contact) interactions (again, not including spin-dependent interactions):

$$\begin{aligned}\mathcal{L}_{\rm eft} = \psi^\dagger\Big[i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^2}{2m}\Big]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\big[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{ h.c.}\big] \\ + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots\end{aligned} \tag{39}$$

where "h.c." stands for hermitian conjugate and $\overleftrightarrow{\nabla}$ is the Galilean invariant derivative:

$$\overleftrightarrow{\nabla} \equiv \overleftarrow{\nabla} - \overrightarrow{\nabla} \,. \tag{40}$$

Using this derivative ensures that the Lagrangian is unchanged if all of the particle momenta are boosted by $\boldsymbol{v}$: $\mathbf{p} \to \mathbf{p} + m\mathbf{v}$.

[When this Lagrangian is used in the literature, it often is accompanied by words like: "this is a general but not unique form of the Lagrangian for short-range spin-independent interactions." This may sound contradictory, but what it means is that there are other choices that can be reached by "field redefinitions," which are basically changes of variable for ψ. These other choices lead to different but physically equivalent forms (e.g., with more time derivatives). Physically equivalent means that you will get the same result in any calculation of measurable quantities.]

The Feynman rules for these vertices are given in Fig. 4. Note how the factors introduced in the Lagrangian are canceled in the rules and how we end up with relative momenta (with factors

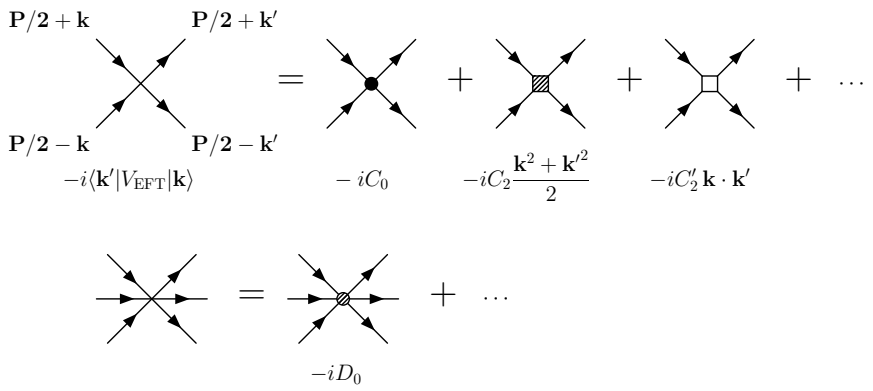


Figure 4: Feynman rules for spinless interaction. The leading two-body and three-body vertices are given.

of i) from the derivatives. Those of you with some field theory experience might try deriving these rules, for example from a path integral representation (e.g., the i in front of each comes from the i in e^{iS}). We can understand the factors of relative momentum if we imagine substituting second quantized field expansions for the ψ's and $\psi^\dagger$'s:

$$\psi \longrightarrow \widehat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}\lambda} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \eta_\lambda a_{\mathbf{p}\lambda} \qquad \psi^\dagger \longrightarrow \widehat{\psi}^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}\lambda} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \eta_\lambda^\dagger a_{\mathbf{p}\lambda}^\dagger \,, \tag{41}$$

where we are in volume V, λ is the spin projection with two-component spinor η_λ and $a^\dagger$ and a are creation and destruction operators, respectively. Each of the operators is associated with one leg of the scattering; the derivatives ensure that the total momentum $\mathbf{P}$'s are cancelled and the hermiticity ensures that the potential is symmetric in $\mathbf{k}$ and $\mathbf{k}'$. There are also combinatoric factors because of the freedom of assigning operators to the legs (this gives factors of 2 to each of the two-body terms and a $3! = 6$ factor to the three-body term).

Let's reproduce the $l = 0$ term in $T(k, \cos\theta)$ (which we'll call T_0) in perturbation theory, which is the same as calculating the Born series for the Lippmann-Schwinger equation,

$$T_0(E) = V_{l=0} + V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} + V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} + \ldots \tag{42}$$

to reproduce the effective range expansion:

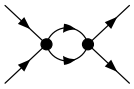
$$T_0(k) = -\frac{4\pi a_0}{m} \left[1 - i a_0 k - (a_0^2 - a_0 r_0/2) k^2 + \mathcal{O}(k^3 a_0^3)\right] \,. \tag{43}$$

Again, at very low momentum (energy), the scattering is described to good accuracy by specifying just the scattering length a_0. The higher corrections require r_0 and a_p, and so on. We've pulled out a factor of $-4\pi a_0/m$ to make the perturbative expansion manifest.

Consider the leading potential $V_{\rm EFT}^{(0)}(\mathbf{x}) = C_0 \delta(\mathbf{x})$ or

$$\langle \mathbf{k}|V_{\rm eft}^{(0)}|\mathbf{k}'\rangle \implies \implies C_0 \tag{44}$$

Choosing $C_0^{(0)} = -4\pi a_0/m$ gets the first term. (We've added a superscript (0) in anticipation that this value of C_0 will be modified order-by-order in the expansion.) The next term is $\langle \mathbf{k}|VG_0V|\mathbf{k}'\rangle$ (where $G_0 \equiv 1/(E - H_0)$):



$$\implies C_0^{(0)} m \int \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} C_0^{(0)} \longrightarrow \infty! \tag{45}$$

We can get this result by direct substitution and inserting a complete set of momentum $|\mathbf{q}\rangle$ states (with a different normalization than we've used in earlier lectures) or from the field theory perspective one includes a propagator for each internal line and then integrates over the frequency. [Note: I'll include the details of the latter calculation eventually.]

But the result is infinite! This infinite result is called a "linear divergence" because when we put an upper limit on q of Λ_c, then the high-q part of the integral in Eq. (45) is proportional to one power of Λ_c: $\int_c^\Lambda dq\, q^2/q^2 \propto \Lambda_c$. We can extract the leading dependence in a power series in k without actually evaluating the integral:

$$\begin{aligned} I_0(k,\Lambda_c) &\equiv m \int^{\Lambda_c} \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} = -\frac{m}{2\pi^2} \int_0^{\Lambda_c} dq + \frac{mk^2}{2\pi^2} \int_0^{\Lambda_c} \frac{dq}{k^2 - q^2 + i\epsilon} \\ &= -\frac{m}{2\pi^2}\Lambda_c + \frac{mk^2}{2\pi^2} \mathcal{P} \int_0^{\Lambda_c} dq\, \frac{1}{k^2 - q^2} - i\pi \frac{mk^2}{2\pi^2} \int_0^{\Lambda_c} dq\, \delta(k^2 - q^2) \\ &= -\frac{m}{2\pi^2}\Lambda_c \left(1 + \mathcal{O}(\frac{k^2}{\Lambda_c}\right) - \frac{ikm}{4\pi} + \end{aligned} \tag{46}$$

where we wrote $q^2 = (q^2 - k^2) + k^2$ to get the first equality, used

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \tag{47}$$

to get the second quality, and used

$$\int_0^{\Lambda_c} dq\, \delta(k^2 - q^2) = \frac{1}{2k} \int_0^{\Lambda_c} dq\, \delta(k - q) = \frac{1}{k}\,, \tag{48}$$

to get the final result.

Now we can absorb the linear Λ_c dependence by adjusting the value of C_0; this is renormalization! In particular, take

$$C_0^{(0)} + C_0^1 = \frac{4\pi a_0}{m} + (C_0^0)^2 \frac{m}{2\pi^2}\Lambda_c = \frac{4\pi a_0}{m}\left(1 + \frac{2a_0\Lambda_c}{\pi} + \cdots\right) \tag{49}$$

and the linear Λ_c dependence is removed as the contribution from $C_0^{(1)}$ in the contact term cancels the leading Λ_c part of the loop integral.

To summarize, when we sum over intermediate states, the momentum of the states get arbitarily high. The vertices are certainly not correct for that momentum. But we can fix the problem by noting that these vertices *are* correct for low momentum, and for high momentum the intermediate state is at high energy, so it is *highly virtual*. Therefore it can only last a short time and the two vertices are not far apart, so the high-momentum part acts like a local vertx. We can "fix" the incorrect part by just adjusting ("renormalizing") the values of the constants C_0, C_2, and so on.

c.5 (Time Permitting) Dimensional regularization with minimal subtraction

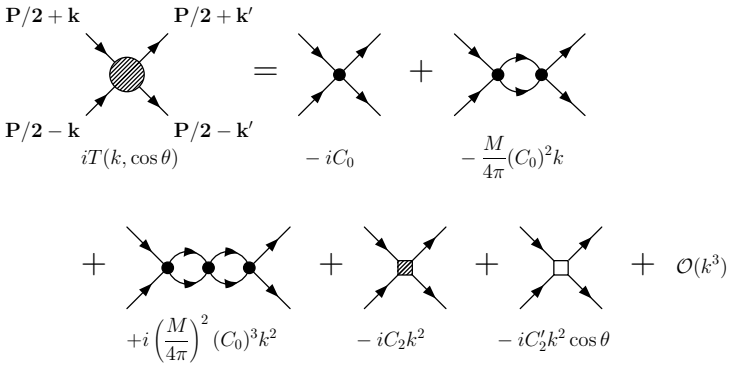
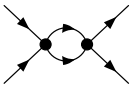


Figure 5: Effective range expansion for a natural scattering length.

Dimensional regularization with minimal subtraction is a cleaner way to regulate and renormalize this EFT than using a cutoff because there will be only one power of k per diagram! We won't be able to use it in more general cases (as when the pion is a long-range degree of freedom) but it is instructive to see how it works here (without filling in too many details). It is based on the integral:



$$\Longrightarrow\ I_0(k) \equiv m\int \frac{d^Dq}{(2\pi)^3}\frac{1}{k^2-q^2+i\epsilon} \underset{D\to 3}{\longrightarrow} -\frac{ik}{4\pi}m\ , \tag{50}$$

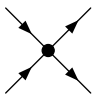
which we present without derivation. This leads to very simple *power counting*, which in this context means an association with each diagram of a definite power of the expansion parameter following these rules:

- Each propagator contributes M/k^2 because of the energy denominator;
- each d^4k loop: k^5/M;
- every n-body vertex with $2i$ derivatives: $k^{2i}R^{2i+3n-5}/M$. The result is that a diagram with E external lines and V_{2i}^n vertices scales as k^ν with

$$\nu = 5 - \frac{3}{2}E + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i+3n-5)V_{2i}^n\ . \tag{51}$$

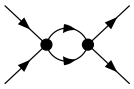
The application of these rules to the leading diagrams is in Fig. 5.

Let's verify this formula for a few cases. Start with one two-body vertex with no derivatives, so $V_0^2 = 1$ ($n = 2, i = 0$); four external lines so $E = 4$, yielding



$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)1 = 0 \quad\Longrightarrow\quad k^0 \tag{52}$$

Try a diagram with a loop: two 2-body vertices with no derivatives, so $V_0^2 = 2$ ($n = 2, i = 0$), one loop, so $L = 1$, 4 external lines



$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)2 = 1 \quad \Longrightarrow \quad k^1 \tag{53}$$

Finally, try a vertex with two derivatives, so $V_2^2 = 1$ ($n = 2$,$i = 1$), yielding

$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 1 + 3\cdot 2 - 5)1 = 2 \quad \Longrightarrow \quad k^2 \,. \tag{54}$$

So the formula seems to work (try the 3-body vertex!).

Matching (to the underlying theory *or* to data) yields:

$$C_0 = \frac{4\pi}{m}a_0 = \frac{4\pi}{m}R \,, \quad C_2 = \frac{4\pi}{m}\frac{a_0^2 r_0}{2} = \frac{4\pi}{m}\frac{R^3}{3} \,, \quad C_2' = \frac{4\pi}{m}a_p^3 \quad \cdots \tag{55}$$

In general, we see that dimensional analysis applied to either Eq. (38) or Eq. (39) tells us that

$$C_{2i} \sim \frac{4\pi}{m}R^{2i+1} \,, \quad D_{2i} \sim \frac{4\pi}{m}R^{2i+4} \tag{56}$$

Every term comes with the same $1/m$ factor (this is from Galilean invariance; relativistic corrections would give higher powers of $1/m$). The factor of 4π doesn't come from dimensional analysis (since it is dimensionless!) but is manifest from connecting the scattering amplitude to the T-matrix.

c.6 Recipe for an effective field theory

We can summarize the steps that we've applied for a particular simple EFT that will also apply more generally:

1. Use the most general Lagrangian with low-energy degrees of freedom consistent with global and local symmetries of underlying theory

   - For purely short-distance interactions, the Lagrangian

   $$\mathcal{L}_{\rm eft} = \psi^\dagger\big[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M}\big]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots \,, \tag{57}$$

   is general (but not unique).

2. Declaration of a regularization and renormalization scheme

   - We need to add a regulator that makes the theory well behaved (but wrong!) at high energies (high momenta in our loop integars, which are sums over intermediate states).
   - For a natural-sized scattering length a_0, dimensional regularization and minimal subtraction are the most efficient but one can use a sharp cutoff as well as other functions that are smoother (exerise!).

3. Identify a well-defined power counting associated with a (small) expansion parameter
   - Use the separation of scales to form small ratios, which in the case just considered is $\frac{k}{\Lambda}$ with $\Lambda \sim 1/R \Longrightarrow ka_0 \ll 1$, etc.
   - This recovers the effective range expansion order-by-order with diagrams

$$f_0(k) \propto \frac{1}{k\cot\delta_0(k) - ik} \longrightarrow a_0\big[1 - ia_0k - (a_0^2 - a_0r_0/2)k^2 + \mathcal{O}(k^3a_0^3)\big] \tag{58}$$

$$\longrightarrow R[1 - ikR - 2k^2R^2/3 + \mathcal{O}(k^3R^3)] \quad \text{[hard sphere]} \tag{59}$$

     – With DR/MS, there is one power of k per diagram, *natural* coefficients.
     – We can estimate the *truncation error* from (naive) dimensional analysis if we assume that the coefficient is close to one.
     – We know when it breaks down (when k gets close to the underlying physics scale).
     – This is valid for *any* natural short-range interaction! (That is, not just for our example of hard-sphere scattering.)

In a later lecture, we will extend the discussion to the interesting case of large scattering lengths and elaborate on the discussion here.

d. Adding spin (optional)

What if we have spin dependence? If we restrict ourselves to central forces, from the discussion of general potentials we can have a term in the effective potential proportional to $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$. So it would seem that we need to introduce another independent coupling besides the C_0. But it turns out not to be independent (if all we have are neutrons)!

Because the spatial wave function must be symmetric to have a non-zero amplitude at the same $\mathbf{x}$, one must be spin up and one spin down in an anti-symmetric combination, i.e., in a $^1\mathrm{S}_0$ state. This means that of the two possible S-wave partial waves $^1\mathrm{S}_0$ and $^3\mathrm{S}_1$ that could have couplings (or scattering lengths) associated with them, only one of them is non-zero. One can show using something called a "Fierz rearrangement" (if you've taken QFT, you might have seen this)

$$(\psi^\dagger\boldsymbol{\sigma}\psi)\cdot(\psi^\dagger\boldsymbol{\sigma}\psi) = -3(\psi^\dagger\psi)^2\,. \tag{60}$$

Therefore

$$C_S(\psi^\dagger\psi)^2 + C_T(\psi^\dagger\boldsymbol{\sigma}\psi) = (C_S - 3C_T)(\psi^\dagger\psi)^2\,, \tag{61}$$

and we can eliminate the C_T term entirely. (Or we can eliminate the C_S term!)

e. Introduction to 3-body forces from "integrating out" dofs

e.1 Classical analogy: tidal forces

Why do we have three-body forces in a low-energy effective theory? Let's start by considering a classical analog of describing the interaction of the sun, moon, and earth by gravity. Our low-energy

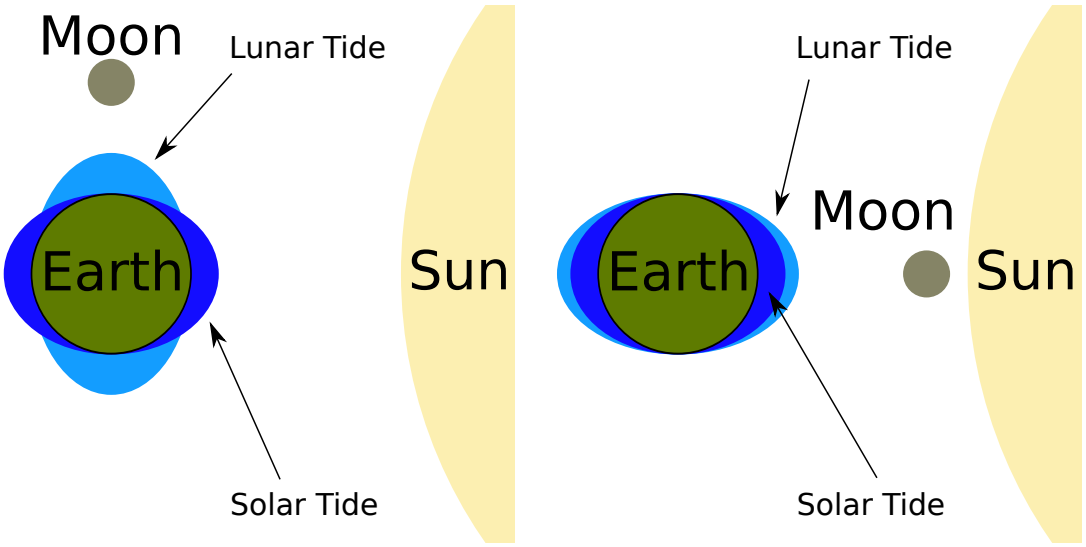


Figure 6: Analog of quantum three-body forces from tidal forces. [credit: K. Wendt]

theory replaces these finite composite systems by point masses at the center-of-mass of each body. This gives us a first approximation to energy of the system. But if we want to be more accurate, we have to recognize that because of tidal forces, the gravitational force on the Earth is not just the pairwise sum of the point-like Earth-Moon and Earth-Sun forces. Because the Earth is actually a composite object, it gets "polarized" and the force between the Earth and the Sun in Fig. 6 depends on the location of the Moon. Thus to take this into account we need a force that depends on the coordinates of all three bodies; in our low-energy theory of point masses, this is a three-body force between them.

e.2 Atomic three-body forces

This would seem to be a general phenomenon that we would expect when we describe any composite particles as point particles in a low-energy theory. In particular, at the quantum mechanics level we can think about the forces between atoms in such a theory. You'll recall that there is an effective potential between two atoms from the mutual polarization, which leads to the intermediate-range van der Waals force, which is described (for example) by the familiar Lennard-Jones potential where the atoms are represented as point particles (with the potential being a function of the distance between the center-of-masses of the atoms). Note that the qualitative form of the potential is the same as the local nuclear potential: attraction at intermediate distances and strong repulsion at short distances (in the case of atoms from the overlap of electron clouds).

So now one would expect from the analogy to the Earth-Moon-Sun system that the force between two atoms would be modified by the polarization due to a third atom. This is very natural, but we don't seem to be taught about such three-body forces! However, they are real and in fact were described by Axilrod and Teller way back in 1943. To be specific, the three-body potential for

atoms or molecules is from triple-dipole mutual polarization and arises as a 3rd-order perturbation theory correction (cf. dipole-dipole mutual polarization at 2nd order).

Referring to Fig. 7, the three-body potential is

$$V(i,j,k) = \frac{\nu(1+3\cos\theta_i\cos\theta_j\cos\theta_k)}{(r_{ij}r_{ik}r_{jk})^3} \ , \tag{62}$$

although the specific details are not important for us. Some comments on this three-body force:

- It is usually negligible in metals and semiconductors because the fine-structure constant is so small. So we rarely hear about it.
- However, it can be important for the ground-state energy of solids bound by van der Waals potentials.
- It is used to improve the accuracy of calculations performed on Van der Waals clusters such as those formed by the noble gases.
- Bell and Zuker (1976) find it to be 10% of energy in solid xenon. (Note: the typical size of the three-body contribution to the triton binding energy is also about 10%!)
- See "Local Density Theory of Polarizability" by Mahan and Subbaswamy for more details.

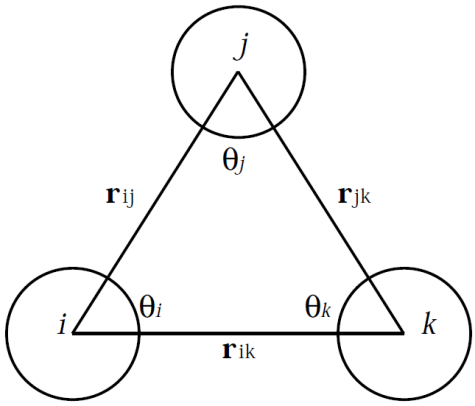


Figure 7: Diagram defining the angles and distance for the Axilrod-Teller three-body force.

We can say more generally that these three-body forces arise from the elimination of degrees of freedom in our low-energy theory. If we included the positions of all the masses in the Earth-Moon-Sun or the electrons (and nuclei) in the atoms, we would get the result from just summing two-body forces. But by eliminating those variables (degrees of freedom) in favor of collective coordinates (center-of-mass position), we introduce three-body forces.

e.3 Preview of nuclear three-body forces

Now consider a system of three (or more nucleons). In our low-energy effective theory we have also eliminated degrees of freedom. Some examples are shown in Fig. 8):

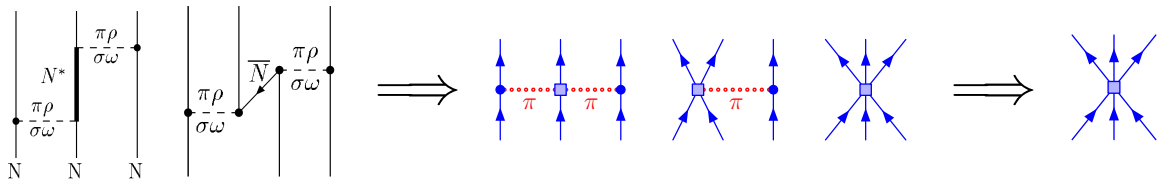


Figure 8: Sources of three-body forces and the diagrams for their contributions in chiral effective field theory and pionless EFT.

- excited states of nucleon, generically denoted N^* in the figure;
- relativistic effects (eliminating anti-particles);
- integrating out high-momentum intermediate states (next week!).

When we omit these contributions, the effects of the heavy meson exchanges are shrunk to points and included in a derivative expansion, as in our example at the beginning. If the pion is treated as a heavy degree of freedom, it will also shrink and we will only have diagrams like the contact interaction on the right (but with vertices with more and more powers of momentum or more and more derivatives, as in the two-body case). If we resolve the pion, we will have mid-range (middle), and long-range (left) three-body forces, which we have characteristic spin- and isospin-dependence because of the pion-nucleon interaction. We will discuss the impact of such forces on nuclear structure in later lectures.

For now, we can ask for phenomenological evidence that we will need three-body forces. To do so we can turn to the set of high-precision NN interactions we discussed in an earlier lecture. These describe the two-body physics (nucleon-nucleon scattering and the deuteron) essentially perfectly up to a rather high energy. But now that us take all of the ones available and calculate accurately the binding energies of the triton (^{3}H) and the alpha particle (^{4}He). If we plot the results and also the experimental point, we find that none of the potentials agree with experiment, but interestingly they all lie close to a line, which is called the Tjon line (see Fig. 9). To agree with experiment, for each potential we need to add a three-body force and the energy we need is generally different in each case. What about our expectations for 4-body and higher body?

e.4 Three-body forces in pionless EFT

Finally, let's return to the pionless EFT and some more formal considerations. Feynman rules and power counting predict a three-body force is possible and because it is allowed by the symmetries, general EFT principles say it *will* appear. If we have only neutrons, the value is zero from the Pauli principle because we can't have a non-zero wave function for all three particle at the same point $\mathbf{x}$ since at least two spins must be the same.

But when we have protons and neutrons, our simple perturbative case tells us we *need* a three-body contact force to fix the contribution from our two-body interactions at high energy when we consider the scattering of three nucleons. At low resolution, we don't resolve a series of two-body

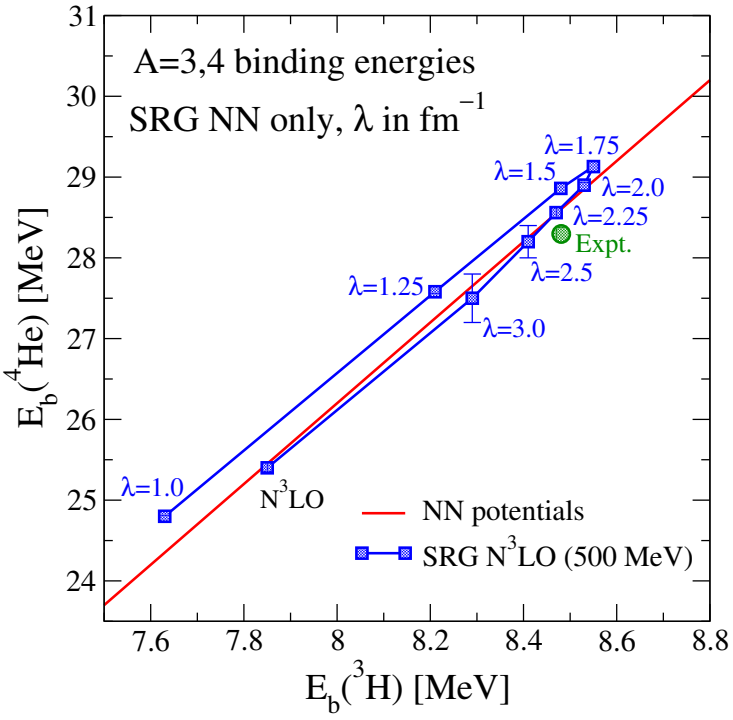
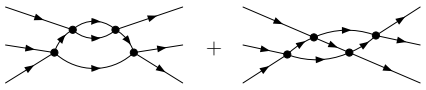


Figure 9: Tjon line: correlation of alpha particle and triton binding energies with different potentials. Note where experiment lies.

scatterings at high energy, and this implies that we need three-body, even if the underlying potential we are trying to reproduce (e.g., hard-sphere scattering) is two-body only! This shows up as new *logarithmic* divergences in 3–3 scattering in these diagrams:

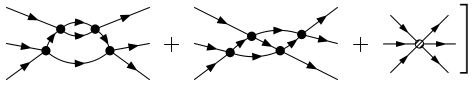


$$\propto (C_0)^4 \ln(k/\Lambda_c) \tag{63}$$

That is, if you evaluate the contribution of these Feynman diagrams, you will find that it contains a dependence on the momentum cutoff as indicated. The changes in Λ_c *must* be absorbed by a *3-body* coupling $D_0(\Lambda_c)$ where

$$D_0(\Lambda_c) \propto (C_0)^4 \ln(a_0\Lambda_c) + \text{const.} \quad \text{[from Braaten \& Nieto]} \tag{64}$$

. The requirement that the total (which is part of a measurable quantity) be independent of Λ_c (which is an auxiliary parameter that must disappear in the final result) tells us that any change with Λ_c in the diagrams must be compensated by $D_0(\Lambda_c)$:

$$\frac{d}{d\Lambda_c}$$  $$= 0 \,,$$ (65)

and this fixes the coefficient! This is an example of using the *renormalization group* running of a coupling to derive new information, seemingly at no cost!