AMERICAN OPTIONS

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Abstract. In this project, we discuss the discrete-time approximation of the Black-Scholes model for American put option pricing. After setting up a two-asset, discrete-time model, we construct the convergence of the Black-Scholes model. We evaluate the American put option pricing by using the numerical results of exercise boundary, exercise time and profit and loss.

1. Introduction

American option is the one that could be exercised at any time up to expiration, are considerably more complicated than European option. It is more difficulty to price or hedge American option since we have to account for a great number of different possible exercise policies. In this report, we shall consider the pricing and optimal exercise of American options in the simplest nontrivial setting by using building a discrete-time financial model with a stock and a bank account and see how it converge to the Black-Scholes model.

Suppose that an asset price process $S = (S_{t_k})_{k\in 0,1,\ldots,N}(with t_k = k\Delta t and \Delta t = \frac{T}{N}, for a fixed N)$ are given by the stochastic dynamics

$$S_{t_k} = S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k},$$

where ϵ_k are i.i.d random variable with $\epsilon_k \in \{+1,-1\}$ and

$$\mathbb{P}(\epsilon_k = \pm 1) = \tfrac{1}{2}(1 \pm \tfrac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}).$$

Here, r ≥ 0 and $\sigma > 0$ are constants.

Section 2 discusses the Fundamental Theorem of Asset Pricing and model set up. We will show the first step to find the convergence of this discrete-time model to Black-Scholes model. Then it will dig deeper into how to prove that there are no arbitrage opportunities in these measures defined in previous section. The measure of two assets helps estimate the distribution of the stock price S_T at maturity.

Section 3 constructs the exercise boundary for American put option. It is followed by the estimation of obtaining appropriate kernel density for the exercise time and profit of the simulated stock prices. MATLAB is used for implementing the order. We will show how the changing model variables affect the exercise boundary, the exercise time and the exercise profit and loss distributions for an American put option.

2. Methodology

Before starting the analysis, it is crucial to note that the absence of arbitrage should always hold, and it is typically reached through the First Fundamental Theorem of Asset Pricing, which we will cover in the beginning of methodology part.

2.1. **Fundamental Theorem of Asset Pricing (FTAP).**

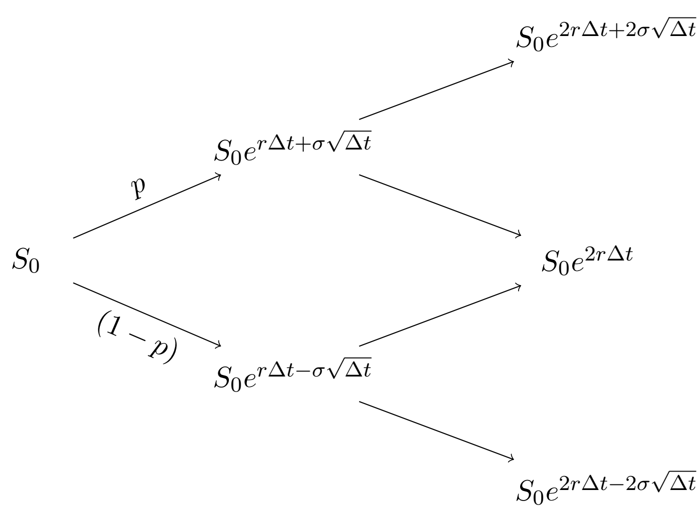
An economy does not admit any arbitrage opportunities if and only if there exists:
(i) a numeraire asset, i.e. a traded asset B satisfying $\mathbb{P}(B_t > 0) = 1$ for all t; an associated probability measure $\mathbb{Q}^B$ equivalent to the physical measure $\mathbb{P}$,

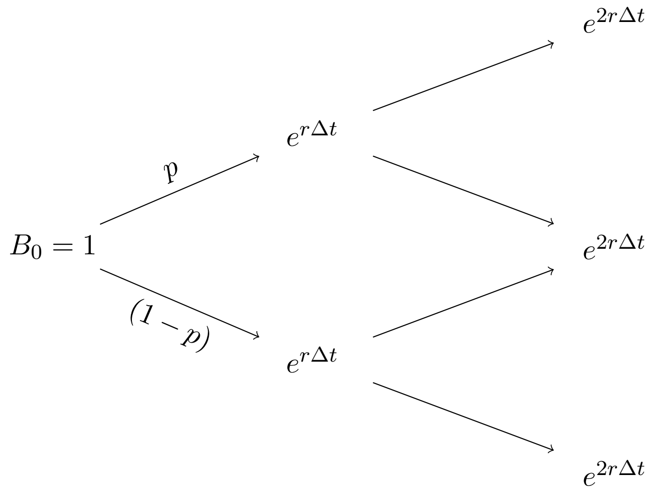
(ii) such that the prices of all traded assets relative to B are martingales under $\mathbb{Q}^B$, i.e. for any traded asset A we have: $\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}^B}[\frac{Au}{Bu}|\mathcal{F}_t]$ for u > t where $\mathcal{F}_t$ is the σ-algebra generated by all asset paths up to time t.

2.2. **Model Setup.**

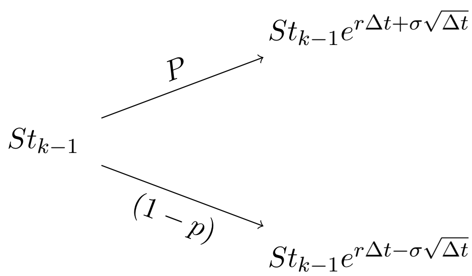
We assume that there are two assets in this economy:
(i) **A risk-free asset**, B_t, which is a money market account where the per-period interest rate is given by a constant r > 0. That is, the asset evolves according to:
$B_{k\Delta t} = B_{k-1\Delta t} \cdot (1 + r\Delta t) \approx B_{k-1\Delta t} \cdot e^{r\Delta t}$
$= B_0(1 + r\Delta t)^N \approx B_0 \cdot e^{r\Delta t}$
For simplicity, we assume that $B_0 = 1$.
(ii) **A risky asset**, which evolves according to the following model:
$S_{k\Delta t} = S_{k-1\Delta t}e^{r\Delta t+\sigma\sqrt{\Delta t}\epsilon_k}$
$= S_0 \cdot exp(Nr\Delta t + \sigma\sqrt{\Delta t}\sum_{k=1}^{N} \epsilon_k)$
where ϵ_k are i.i.d. random variables with $\mathbb{P}(\epsilon_k = 1)$ = p and $\mathbb{P}(\epsilon_k = 1)$ = 1 - p and $\sigma > 0$ is a known constant.

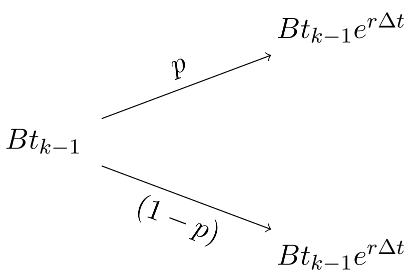
The trees produced by the CRR (Cox-Ross-Rubinstein) model begin as follows:





Thus, at any given step t the prices of the two assets evolve as follows:



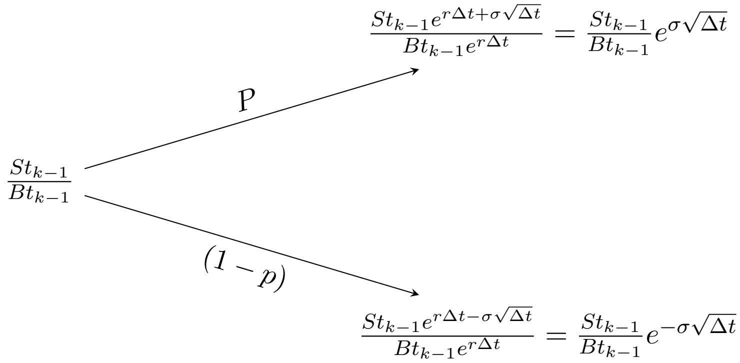


2.3. **Risk neutral probabilities and no-arbitrage conditions.**

First, we can derive the risk-neutral probabilities in this model. Using the money market as the numeraire asset produces the following relative price process.

Note that here we drop the superscript of B, since B is the money market account, and it is common to refer to the Equivalent Martingale Measure(EMM) associated with this asset as the risk-neutral measure, often denoted $\mathbb{Q}$.

After computing related measures, we can use the Black-Scholes model to compute the limiting distribution of stock price with $X^{(N)}$.



Recall that the risk-neutral probability of an up-move (when using B as the numeraire) in a binomial tree is given by:

$$\text{q} = \frac{S_0/B_0 - S_d/B_d}{S_u/B_u - S_d/B_d} \Rightarrow \text{q} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \in (0,1)$$

2.4. **Limiting Distribution under $\mathbb{P}$.**

We are now interested in the limiting distribution of asset prices under $\mathbb{P}$ in the limit as the number of subperiods approaches infinity. $St_k = St_{k-1}e^{r\Delta t+\sigma\sqrt{\Delta t}\epsilon_k}$.

Let $X^{(N)}$ denote the random variable $X^{(N)} := \log(\frac{S_T}{S_0})$. Prove that $X^{(N)} \xrightarrow[N\to\infty]{d} (\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z$, and Z $\overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$.

Proof. $S_t \overset{d}{\underset{\mathbb{P}}{=\!=}} S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$

Consider $X^{(N)} = \log(\frac{S_T}{S_0}) = \text{Nr}\Delta t + \sigma\sqrt{\Delta t}\sum_{k=1}^{N}\epsilon_k$

Denote the Moment Generating Function of the random variable $X^{(N)}$

$$\begin{aligned}
&\Phi_N(u) = \mathbb{E}^{\mathbb{P}}[e^{uX^{(N)}}]\\
&= \mathbb{E}^{\mathbb{P}}[e^{uNr\Delta t+\sigma\sqrt{\Delta t}\sum_{k=1}^{N}\epsilon_k}]\\
&= \mathbb{E}^{\mathbb{P}}[e^{uNr\Delta t}e^{\sigma\sqrt{\Delta t}\sum_{k=1}^{N}\epsilon_k}]\\
&= e^{uNr\Delta t}\mathbb{E}^{\mathbb{P}}[e^{u\sigma\sqrt{\Delta t}\sum_{k=1}^{N}\epsilon_k}] \text{ for each u} \in \mathbb{R}
\end{aligned}$$

Since $\{\epsilon_{k=1}^{N}\}$ are independent and identically distributed (i.i.d), we have that
$\Phi_N(u) = e^{uNr\Delta t}(\mathbb{E}^{\mathbb{P}}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}])^N$

By Talyor Expansion and the formula for the first and second moments of a Bernoulli random variable

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}] = \mathbb{E}^{\mathbb{P}}[1 + u\sigma\sqrt{\Delta t}\epsilon_1 + \tfrac{1}{2}u^2\sigma^2\Delta t(\epsilon_1)^2 + \tfrac{1}{3!}u^3\sigma^3\Delta t^{\frac{3}{2}}(\epsilon_1)^3 + ...]\\
&= 1 + u\sigma\sqrt{\Delta t}\epsilon_1\mathbb{E}^{\mathbb{P}}[\epsilon_1] + \tfrac{1}{2}u^2\sigma^2\Delta t\mathbb{E}^{\mathbb{P}}[(\epsilon_1)^2] + o(\Delta t)\\
&= 1 + u(\mu - \tfrac{1}{2}\sigma^2)\Delta t + \tfrac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)\\
&= 1 + [(\mu - \tfrac{1}{2}\sigma^2)u + \tfrac{1}{2}\sigma^2u^2]\Delta t + o(\Delta t)
\end{aligned}$$

$= 1 + \text{g(u)}\Delta t + o(\Delta t)$ where $\text{g(u)} = (\mu - \frac{1}{2}\sigma^2)u + \frac{1}{2}\sigma^2 u^2$
$\Rightarrow \mathbb{E}^{\mathbb{P}}[e^u X^{(N)}] = e^{urN\Delta t}[1 + g(u)\Delta t + o(\Delta t)]^N$
$= e^{urN\Delta t}(1 + \frac{g(u)T}{N} + o(\frac{1}{N}))^N \xrightarrow{N\to\infty} e^{(\mu - \frac{1}{2}\sigma^2)uT + \frac{1}{2}\sigma^2 u^2 T}$ which is the Moment Generating Function of a random variable

So we have that $X^{(N)} = log(\frac{S_T}{S_0}) \xrightarrow[N\to\infty]{d(under\mathbb{P})} \mathcal{N}((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T)$

$$\xrightarrow[N\to\infty]{d(under\mathbb{P})} (\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z, Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

□

2.5. Limiting Distribution under $\mathbb{Q}$.

We are now interested in the limiting distribution of asset prices under $\mathbb{Q}$ in the limit as the number of subperiods approaches infinity. $St_k = St_{k-1}e^{r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k}$.

Let $X^{(N)}$ denote the random variable $X^{(N)} := \log(\frac{S_t}{S_0})$. Prove that $X^{(N)} \xrightarrow[\mathbb{Q}]{d} (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z^{\mathbb{Q}}$, and $Z^{\mathbb{Q}} \overset{\mathbb{Q}}{\sim} \mathcal{N}$(0,1).

Proof. $S_t \overset{d}{\underset{\mathbb{Q}}{=\!=}} S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z^{\mathbb{Q}}}$

Denote the Moment Generating Function of the random variable $X^{(N)}$

$\Psi_N(u) = \mathbb{E}^{\mathbb{Q}}[e^{uX^{(N)}}]$
$= \mathbb{E}^{\mathbb{Q}}[e^{uNr\Delta t + \sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$
$= \mathbb{E}^{\mathbb{Q}}[e^{uNr\Delta t}e^{\sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$
$= e^{uNr\Delta t}\mathbb{E}^{\mathbb{Q}}[e^{u\sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$ for each $u \in \mathbb{R}$

Since $\{\epsilon_{k=1}^N\}$ are independent and identically distributed (i.i.d), we have that
$\Psi_N(u) = e^{uNr\Delta t}(\mathbb{E}^{\mathbb{Q}}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}])^N$

Returning to the risk-neutral probability,q, and using the fact that $e^k = \sum_{k=0}^n \frac{x^k}{k!} + o(x^n)$, we have that:

$\text{q} = \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$
$= \frac{1 - (1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + o((\Delta t)^{3/2})}{(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + o((\Delta t)^{3/2}) - (1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + o((\Delta t)^{3/2})}$
$= \frac{\sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t + o((\Delta t)^{3/2})}{2\sigma\sqrt{\Delta t} + o((\Delta t)^{3/2})}$
$= \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t} + o(\Delta t)$
$\Rightarrow$ 1-q $= \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t} + o(\Delta t)$

By Taylor Expansion:

$\mathbb{E}^{\mathbb{Q}}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}]$
$= [\frac{1}{2} - \frac{1}{4}\sigma\sqrt{\Delta t} + o(\Delta t)][1 + u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)]$
$+ [\frac{1}{2} + \frac{1}{4}\sigma\sqrt{\Delta t} + o(\Delta t)][1 - u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)]$
$= (\frac{1}{2} + \frac{1}{2}u\sigma\sqrt{\Delta t} + \frac{1}{4}u^2\sigma^2\Delta t - \frac{1}{4}\sigma\sqrt{\Delta t} - \frac{1}{4}u\sigma^2\Delta t)$
$+ (\frac{1}{2} - \frac{1}{2}u\sigma\sqrt{\Delta t} + \frac{1}{4}u^2\sigma^2\Delta t + \frac{1}{4}\sigma\sqrt{\Delta t} - \frac{1}{4}u\sigma^2\Delta t)$
$= 1 + \Delta t(\frac{1}{2}u^2\sigma^2 - \frac{1}{2}\sigma^2 u) + o(\Delta t)$

$\Rightarrow \Psi_N(u) = e^{uNr\Delta t}(1 + \frac{(\frac{1}{2}u^2\sigma^2 - \frac{1}{2}\sigma^2 u)T}{N} + o(\frac{1}{N}))^N \xrightarrow{N\to\infty} e^{u(r-\frac{1}{2}\sigma^2)T+\frac{1}{2}(\sqrt{T}\sigma)^2u^2}$ which is the Moment Generating Function of a random variable

So we have that $X^{(N)} = log(\frac{S_t}{S_0}) \xrightarrow[N\to\infty]{d(under\mathbb{Q})} \mathcal{N}((r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$

□

2.6. Limiting Distribution under $\mathbb{Q}^S$.

The analysis in the previous section can be repeated with the risky asset as the numeraire to derive an analogous result involving the limiting distribution of S_t under $\mathbb{Q}^S$.

Let $X^{(N)}$ denote the random variable $X^{(N)} := \log(\frac{S_t}{S_0})$. Prove that $X^{(N)} \xrightarrow[\mathbb{Q}^S]{d} (r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z^{\mathbb{Q}^S}$, and $Z^{\mathbb{Q}^S} \overset{\mathbb{Q}^S}{\sim} \mathcal{N}(0,1)$.

Proof. $S_t \overset{d}{\underset{\mathbb{Q}^S}{=}} S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z^{\mathbb{Q}^S}}$

Denote the Moment Generating Function of the random variable $X^{(N)}$

$\Lambda_N(u) = \mathbb{E}^{\mathbb{Q}^S}[e^{uX^{(N)}}]$
$= \mathbb{E}^{\mathbb{Q}^S}[e^{uNr\Delta t+\sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$
$= \mathbb{E}^{\mathbb{Q}^S}[e^{uNr\Delta t}e^{\sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$
$= e^{uNr\Delta t}\mathbb{E}^{\mathbb{Q}^S}[e^{u\sigma\sqrt{\Delta t}\sum_{k=1}^N \epsilon_k}]$ for each u $\in \mathbb{R}$

Since $\{\epsilon_{k=1}^N\}$ are independent and identically distributed (i.i.d), we have that
$\Lambda_N(u) = e^{uNr\Delta t}(\mathbb{E}^{\mathbb{Q}^S}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}])^N$

When using S as the numeraire, the risk-neutral probability is equal to:
$q^S = \frac{B_0/S_0 - B_d/S_d}{B_u/S_u - B_d/S_d} \Rightarrow q^S = \frac{e^{\sigma\sqrt{\Delta t}}-1}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}} \in(0,1)$
$= \frac{(1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+o((\Delta t)^{3/2})-1}{(1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+o((\Delta t)^{3/2})-(1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+o((\Delta t)^{3/2})}$
$= \frac{\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+o((\Delta t)^{3/2})}{2\sigma\sqrt{\Delta t}+o((\Delta t)^{3/2})}$
$= \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t} + o(\Delta t)$
$\Rightarrow$ 1-q^S $= \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t} + o(\Delta t)$

By Taylor Expansion:

$\mathbb{E}^{\mathbb{Q}^S}[e^{u\sigma\sqrt{\Delta t}\epsilon_1}]$
$= [\frac{1}{2} + \frac{1}{4}\sigma\sqrt{\Delta t} + o(\Delta t)][1 + u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)]$
$+ [\frac{1}{2} - \frac{1}{4}\sigma\sqrt{\Delta t} + o(\Delta t)][1 - u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)]$
$= (\frac{1}{2} + \frac{1}{2}u\sigma\sqrt{\Delta t} + \frac{1}{4}u^2\sigma^2\Delta t + \frac{1}{4}\sigma\sqrt{\Delta t} - \frac{1}{4}u\sigma^2\Delta t)$
$+ (\frac{1}{2} - \frac{1}{2}u\sigma\sqrt{\Delta t} + \frac{1}{4}u^2\sigma^2\Delta t - \frac{1}{4}\sigma\sqrt{\Delta t} - \frac{1}{4}u\sigma^2\Delta t)$
$= 1+\Delta t(\frac{1}{2}u^2\sigma^2 + \frac{1}{2}\sigma^2 u) + o(\Delta t)$
$\Rightarrow \Lambda_N(u) = e^{uNr\Delta t}(1 + \frac{(\frac{1}{2}u^2\sigma^2 + \frac{1}{2}\sigma^2 u)T}{N} + o(\frac{1}{N}))^N \xrightarrow{N\to\infty} e^{u(r+\frac{1}{2}\sigma^2)T+\frac{1}{2}(\sqrt{T}\sigma)^2u^2}$ which is the Moment Generating Function of a random variable

So we have that $X^{(N)} = log(\frac{S_t}{S_0}) \xrightarrow[N\to\infty]{d(under\mathbb{Q}^S)} \mathcal{N}((r + \frac{1}{2}\sigma^2)T, \sigma^2 T)$

□

3. Numerical Simulation Results

Through the process of numerical simulation, the exercise boundary of American put option can be determined, while special attention can be also placed toward exercise time and exercise profit, depending on the various parameters used in the discrete modelling of binomial tree. Simulations are done using $T = 1$, $S_0 = 10$ and $N = 5000$.

3.1. **Exercise Boundary.**

In this section, we analyse the relationship of changing in volatility and risk-free rate with respect to the exercise boundary of American put option, since the martingale measure is independent from dividend rate, μ.

As shown in Figure 1 below, the trend of exercise boundary follows positive exponential growth before the maturity, while it starts at a non-zero price at $T = 0$. This means that at the initial time when the purchaser owns the American put option, the stock price doesn't have enough time to decrease. As the exponential growth reaches to it's maturity, the holder might have more chance to lose the put option premium if he fails to exercise the contract, leading to a wider range of exercise boundary, as he might be more flexible about the profit.

Since put option benefits the option holder while price drops, a higher volatility indicates more fluctuations in the price, which is therefore possible for the holder to exercise at the an extremely low price. As a result, higher volatility is reflected in Figure 2, as holder exercises the option when observing a lower stock price as compared.

In contrast, through observing the relationship between risk-free rate and exercise boundary, increasing interest rate makes the price of money more expensive, so the holder of American put option will more likely to exercise earlier, which is shown in Figure 3.

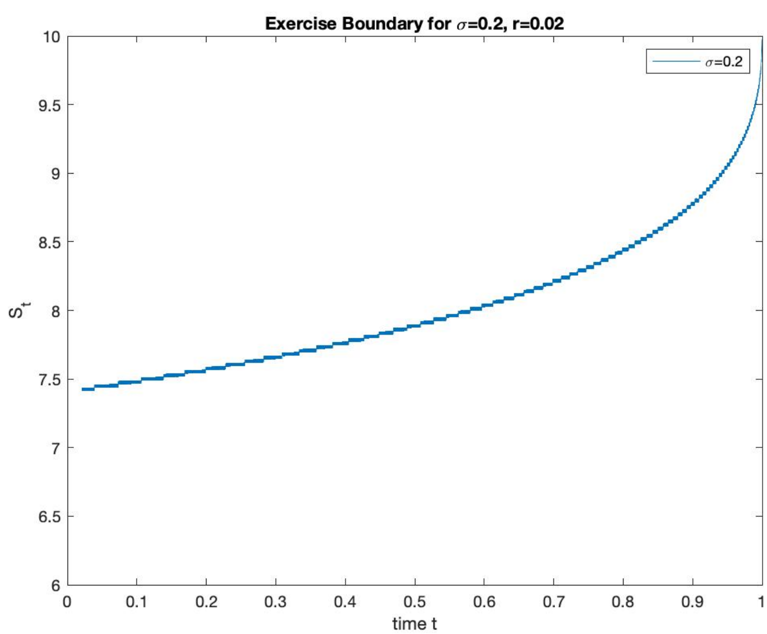


FIGURE 1. This diagram shows the exercise boundary

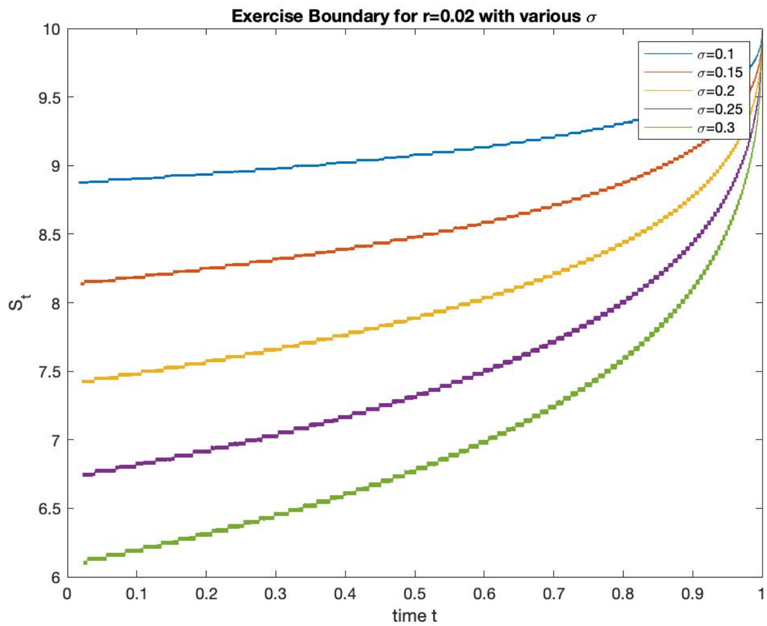


FIGURE 2. This diagram shows the exercise boundary with various volatility

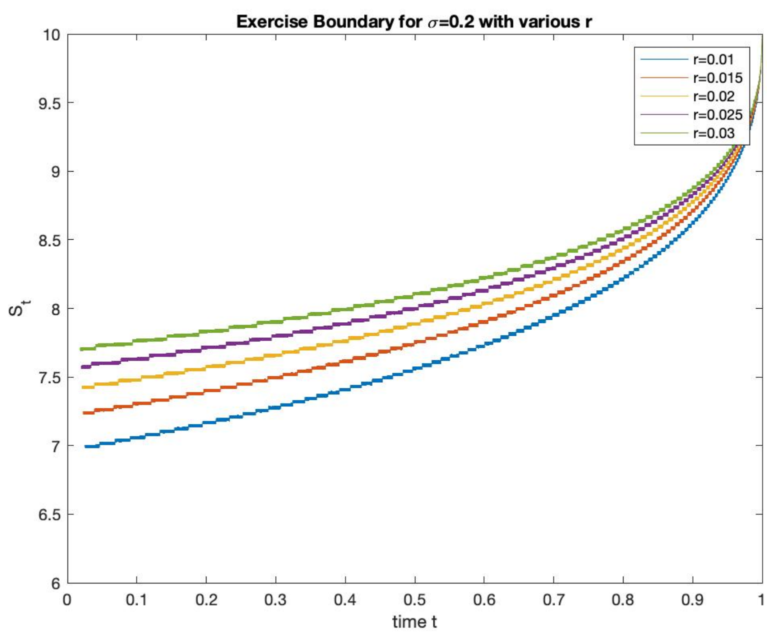


FIGURE 3. This diagram shows the exercise boundary with various interest rate

3.2. **Exercise Time and Exercise Profit Depend on Parameters.**

We apply the same intuition used to show influence of the volatility and risk-free rate on exercise boundary to study how do volatility and risk-free affect exercise time and profit for an American put option. Below are two graphs of Kernel Density Estimate(KDE) of exercise time and profit received for an American put option with various volatility while r = 2%, $\mu = 5\%$ and $K = 10$.

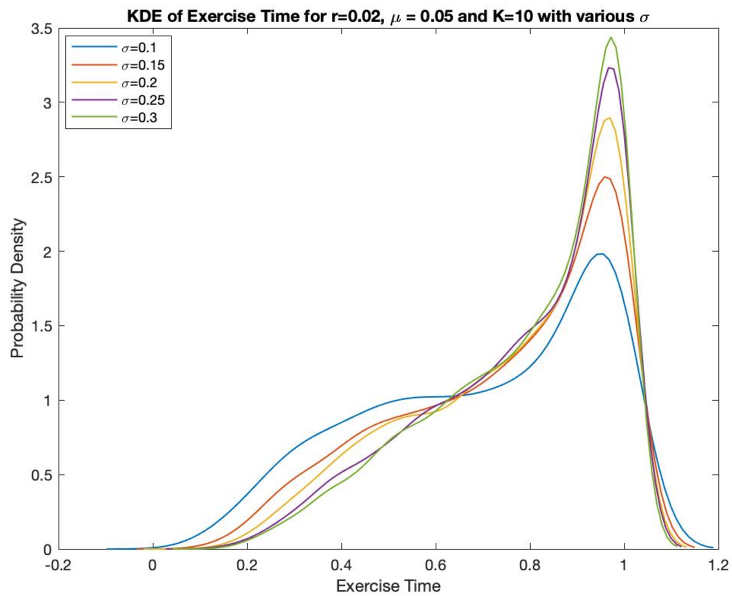


FIGURE 4. This diagram shows the exercise time with various volatility

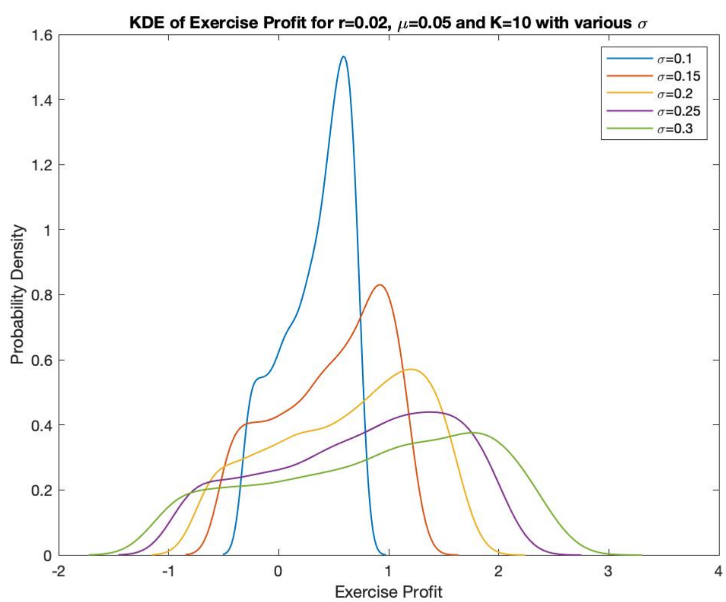


FIGURE 5. This diagram shows the exercise profit with various volatility

Figure 4, KDE of exercise time starts with a horizontal line from zero at $T = 0$, followed by a slightly upward curve. A sharp increase occurs afterwards, then it drops heavily to the initial value zero. The general trend of Figure 4 can be interpreted that there is a higher probability for exercising American put option when we increase the volatility of the asset, which also gives the potential for the holder to obtain above-average profits and losses, indicating a more risky investing environment. The mean of exercise time increases when the volatility increases, as exercise boundary decreases.

Figure 5 follows transformed normal distribution with mean value between zero and one. We could see that the higher the volatility, the higher the probability of extreme gain and loss as well as the value of them: when the volatility is low, the exercise boundary is higher due to early exercise; whereas when the volatility is high, exercise boundary is lower, since average exercise time increases. This is because higher volatility represents riskier market.

We then move on to study the effect of changing parameter risk-free interest rate on the exercise boundary of American put option. The Figure 6 below tells that the probability of exercise increases when risk-free interest rate increases, whereas the average exercise time declines under this circumstance. Since raising interest rate represents higher price of money, this also denotes the positive correlation between higher exercise boundary and higher value of money.

Compared to the previous figure when we have varying volatility, it is less likely to have extreme gain and loss at an increased risk-free interest rate, which is shown in Figure 7, and average profit decreases. It is obvious that change of interest rate and volatility has an opposite effect on our model and investor with risk-averse preference might favour varying risk-free interest rate option than the other one.

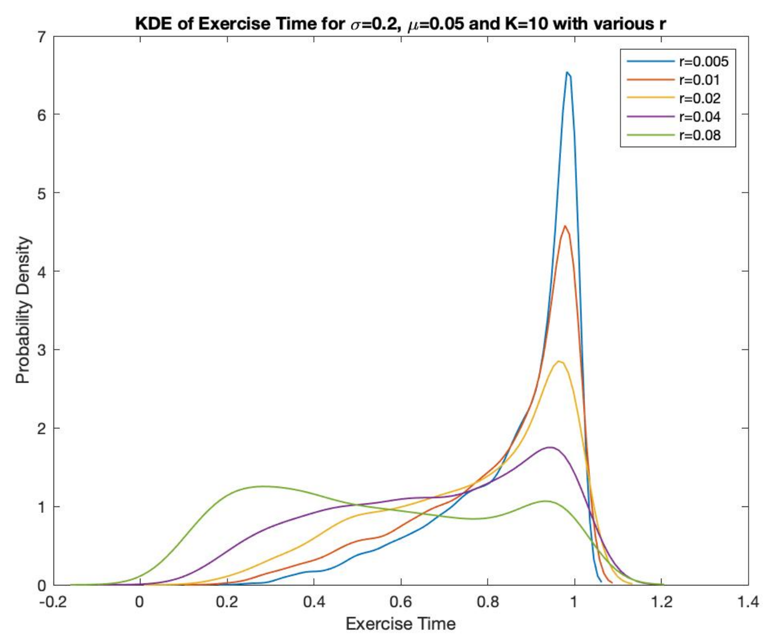


Figure 6. This diagram shows the exercise time with various risk-free rate

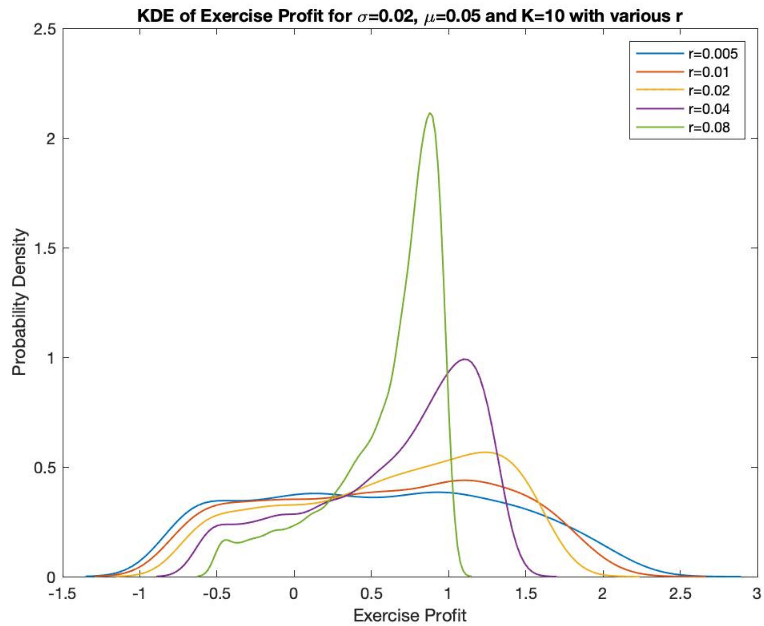


FIGURE 7. This diagram shows the profit with various risk-free rate

Next, we change the variable drift, μ, to investigate its effect on the exercise boundary of American put option. The Figure 8 and 9 illustrate that there is a lower probability of option exercise and a higher average exercise time when we increase the value of the μ.

Since the option price increases as we raise the value of drift and price decreases as drift lowers, the investor will only benefit by a lower drift, which also leads to lower exercise time and higher profit.

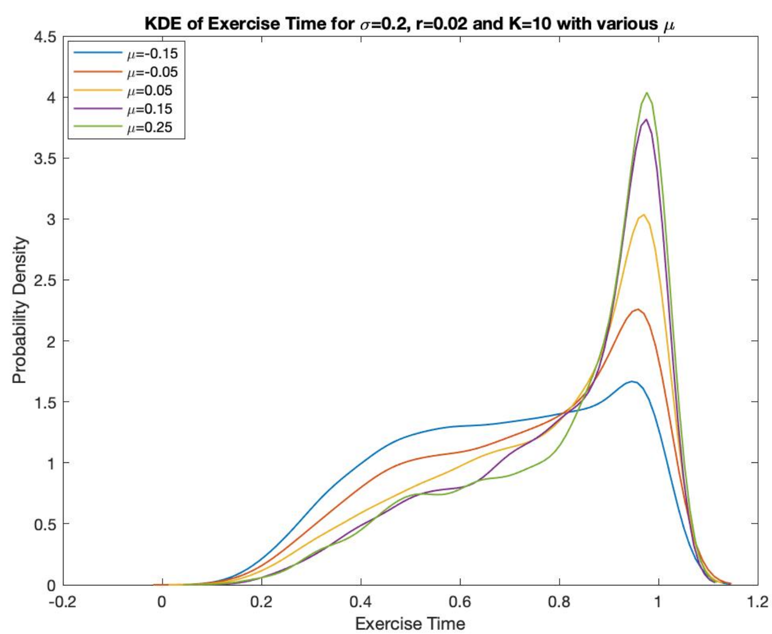


FIGURE 8. This diagram shows the exercise time with various μ

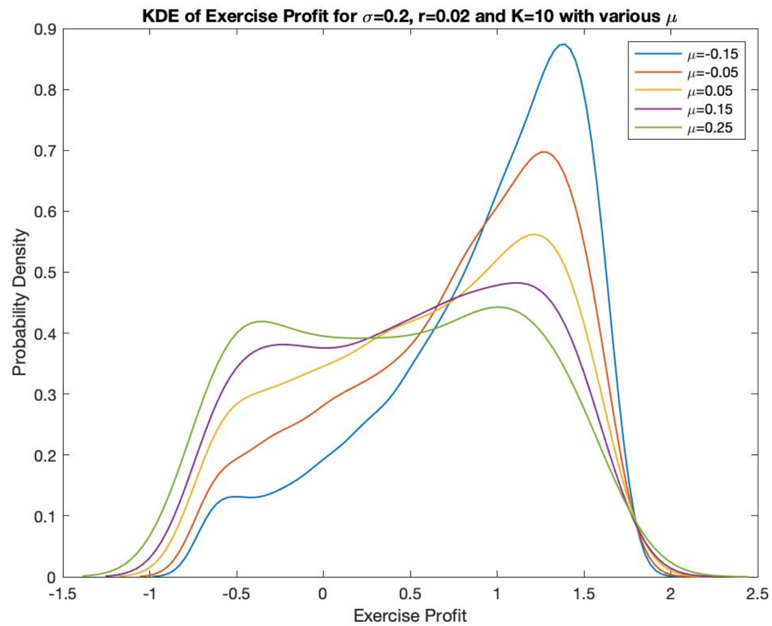


FIGURE 9. This diagram shows the exercise profit with various μ

The strike price K affecting the exercise time and profit is the last changing variable in this section. It is characterized from Figure 10 that when strike price increases, the exercise probability increases while the exercise time decreases. Increasing strike price will lead to a higher payoff, so the option is more likely to be exercised. Figure 11 shows that the changing strike price reflects that when strike price increases, average exercise profit decreases. This is because that a higher strike price will increase the trading premium, which lower the average profits.

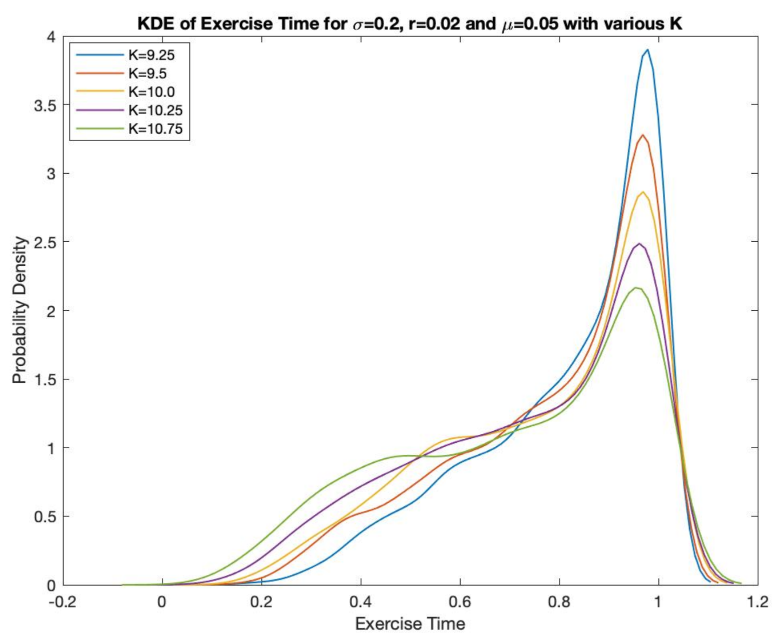


FIGURE 10. This diagram shows the exercise time with various Strike Price

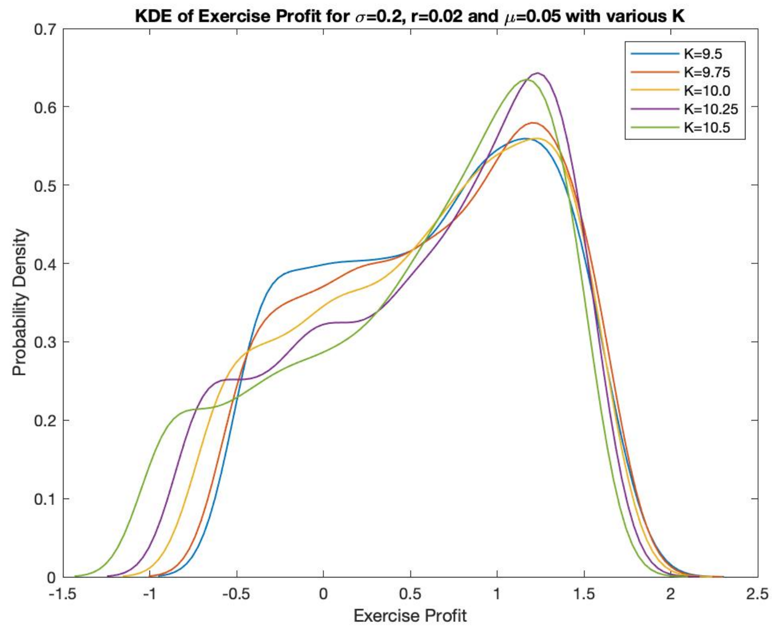


FIGURE 11. This diagram shows the profit with various Strike Price

3.3. **Exercise Time and Profit Depend on Realized Volatility.**

In this last section, we aim to find the influence of the changing variable, realized volatility, which is the volatility in real life σ. It is clear from Figure 12 that when realized volatility is higher than the model volatility, investors are more likely to exercise the option with shorter exercise time and expected return is higher comparing to expected volatility. Also, Figure 13 illustrates a higher profit in this case.

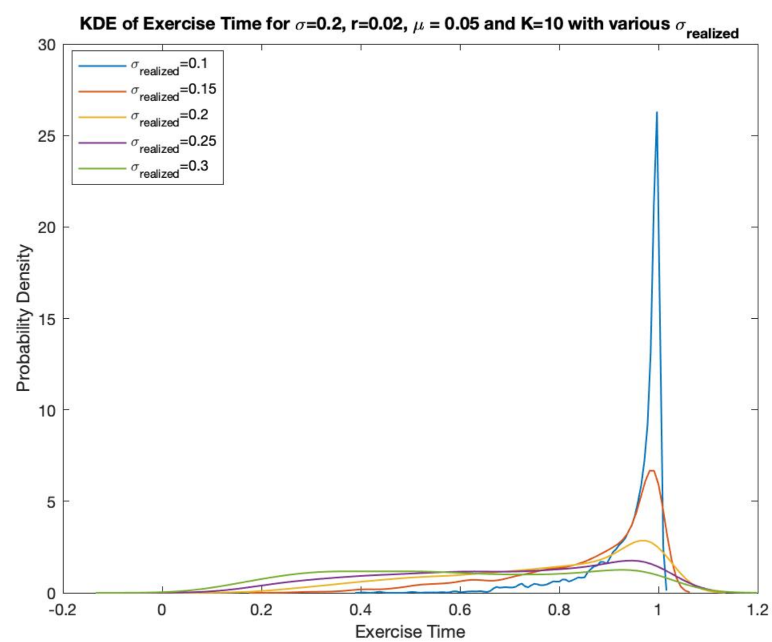


FIGURE 12. This diagram shows the exercise time with realized volatility

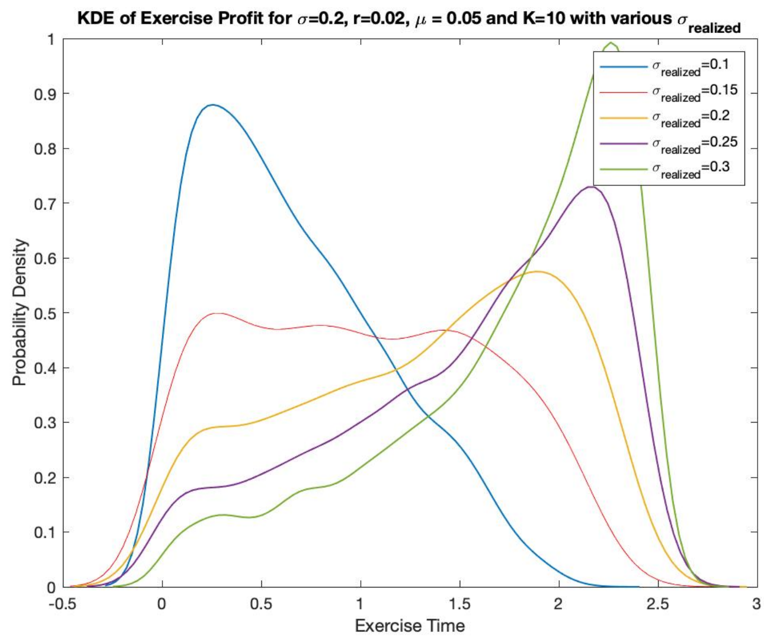


FIGURE 13. This diagram shows the profit with various Strike Price

To expand it with intuition, this observation makes sense in real world. This is because when realized volatility is higher than volatility used to compute exercise boundary, put options will be traded when there is a slight drop in price. Given the greater real-life volatility, a higher trend in price drop, there occurs a higher exercise probability and profits in a short time or lower exercise time. Vice versa, there is low probability for exercise boundary when realized volatility is lower than model volatility, indicating a riskier investing strategy. Investors might wait and see what is going on until maturity, however, with lower volatility, the price is unlikely to decline a lot and there might be a great loss upon maturity.

4. Conclusion

In this project, we use Black-Scholes pricing model and binomial tree to consider the pricing of American put option within discrete time frame. We derive the valuation process of American put option and further build on analysis regrading the exercise boundary, exercise time and exercise profit. All numerical simulations and graphings are done in Matlab.

- From previous analysis, exercise probability, time and profit are highly correlated with model parameters in discrete time frame: these parameters include volatility, risk-free rate, drift and strike price. Taking higher volatility as an example, this will yield lower exercise boundary, longer exercise time as well as higher chance to obtain extreme profit and loss; Also, higher risk-free rate will lead to lower exercise time as well as slightly decreasing profit. Therefore, using the discrete model, we reach the conclusions for both risky trading strategy represented by higher volatility and risk-averse strategy represented by higher risk-free rate.

Drift remains in the same direction with exercise time but moves in the opposite direction with exercise probabilities and profits. Also, according to previous analysis on graphs, drift doesn't place influence on drift. Thus, the expected payoff of this option increases as the decline in drift provides less exercise time and higher exercise boundary, which benefits investors. Strike price has a positive correlation with exercise probabilities, and a negative correlation with exercise times and profits from our simulated graphs. However, a higher strike price does not guarantee a profit as the premium might increases as well. Lastly, the real-world volatility has a great influence on the variables we study as well. A higher realized volatility leads to lower exercise probabilities, longer exercise times and less profits, and vice versa.

It is clear that the exercise probability, the exercise time and the profit in the discrete-time model all react sensitively to any changes of the model's parameters for American put option. While in real life, investors might care more about whether to exercise the American put option and when is the optimal time for it. We seem to sub-estimate the complicity of the strategy as we don't perfectly take the parameter into consideration and assume they remain the same as theoretical model. Although we conduct the fitting and simulating process accurately regarding the Black-Scholes model, we still need to keep in mind about the complexity and efficiency of implementing it to real-life scenarios.

5. Appendix

The following five tables clearly show that how average exercise time, average exercise profit and exercise probability changing with various σ, r, μ, K and $\sigma_{realized}$.

Table 1. Average Exercise Time & Exercise Profit with various σ

σ	$Mean_{time}$	Exercise Probability	$Mean_{profit}$
0.10	0.6572	0.3477	0.3520
0.15	0.6967	0.3999	0.3925
0.20	0.7081	0.4400	0.4169
0.25	0.7397	0.4770	0.4457
0.30	0.7505	0.4904	0.4936

Table 2. Average Exercise Time & Exercise Profit with various r

r	$Mean_{time}$	Exercise Probability	$Mean_{profit}$
0.005	0.8726	0.4265	0.5204
0.01	0.7994	0.4353	0.4686
0.02	0.7250	0.4409	0.4176
0.04	0.6576	0.4631	0.3915
0.08	0.6233	0.5088	0.4075

Table 3. Average Exercise Time & Exercise Profit with various μ

μ	$Mean_{time}$	Exercise Probability	$Mean_{profit}$
-0.15	0.6934	0.8028	0.4353
-0.05	0.7042	0.6369	0.4257
0.05	0.7309	0.4510	0.4179
0.15	0.7261	0.2596	0.4037
0.25	0.7428	0.1197	0.3876

Table 4. Average Exercise Time & Exercise Profit with various K

K	$Mean_{time}$	Exercise Probability	$Mean_{profit}$
9.5	0.7645	0.3417	0.4978
9.75	0.7495	0.3897	0.4638
10	0.7330	0.4408	0.4102
10.25	0.7023	0.4912	0.3680
10.5	0.6678	0.5153	0.3139

TABLE 5. Average Exercise Time & Exercise Profit with Realized Volatility

$\sigma_{realized}$	$Mean_{time}$	Exercise Probability	$Mean_{profit}$
0.1	1.1403	0.3732	0.7257
0.15	0.8669	0.4043	0.7533
0.2	0.7203	0.4397	0.7745
0.25	0.6627	0.4744	0.7808
0.3	0.6378	0.5098	0.8124