MMF1928 Pricing Theory Project 2

Delta-Gamma Hedging

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Abstract

In this report, we applied *Delta* and *Delta-Gamma* hedging strategy to hedge a short position in an European put option. We shorted the underlying asset (stock) in our *Delta* hedging portfolio, and included an extra call option in our *Delta-Gamma* hedging portfolio. We implemented time-based and move-based hedging strategies to achieve *Delta* and *Delta-Gamma* neutrality respectively. We found out that with fixed per unit transaction costs, time-based and move-based strategy perform relatively the same with *Delta* hedging. The move-based strategy outperforms the time-based one with *Delta-Gamma* hedging. Overall, *Delta* hedging performs better in terms of profit and loss compared to *Delta-Gamma* hedging, even though *Delta-Gamma* hedging yields higher replication accuracy. When the real-world volatility deviates from risk-neutral volatility, it has a greater impact on the performance of *Delta* hedging than that of *Delta-Gamma* hedging. Also, as the rebalancing band gets wider, *Delta* hedging yields poorer performance, but *Delta-Gamma* hedging's profit and loss improves and its stability decreases.

Contents

1 Introduction

In this project, we aims to hedge the risk induced by selling a European put option in discrete time within the Black-Scholes model. We will use a call option written on the same asset as the put option does, the underlying asset (stock), and the bank account to construct self-financing hedging portfolios to achieve our goal. We will first look at *Delta* hedging, which is done through neutralizing *Delta* of the entire portfolio to hedge against price movement. We only use the underlying asset and the bank account to construct the hedging portfolio. It results in a hedging portfolio with only a linear approximation of the put option price. At the same time, we will use *Delta* - *Gamma* hedging to achieve a more precise approximation of the put option price by including a *Gamma* neutral condition of the put option using a call option. We divide the lifetime of the option until maturity into small time increments to approximate a dynamic hedging. After implementing the hedging strategy, we look at the impact of real-world volatility being different from the risk-neutral volatility. Also, we will compare move-based hedging with time-based hedging, and investigate how different rebalancing-bands set on *Delta* change our hedging outcome. The rest of the report is organized as follows. Chapter 2 will introduce the fundamental concepts needed for this project and the model setup. Chapter 3 will discuss the methodologies used to conduct dynamic hedging. Chapter 4 will present and analyze the results and findings. Chapter 5 will summarize and conclude this report.

2 Model Setup

2.1 Fundamental Settings

This project aims to discuss dynamic hedging within the Black-Scholes model under discrete time setting. We first assume that there exists an asset, $S = S_t$, where $t \geq 0$, and it follows the Black-Scholes model with stock price $S_0 = 100$, modelled volatility $\sigma = 0.2$, drift $\mu = 0.1$, and risk-free interest rate $r = 0.02$.

Since we have just sold an at-the-money put option with $\frac{1}{4}$ years of maturity written on this asset, a hedging strategy could be developed by trading a call option on the same asset with strike price $K = 100$ and $\frac{1}{2}$ years of maturity, a stock, as well as a bank account. For any European put option, the investor will not exercise the right to sell the underlying asset when $S_t > K$. In contrast, the investor will exercise the put option and the option payoff is $K - S_t$ when $S_t < K$.

Also, it's crucial to note that the transaction cost for every one unit of equity traded is \$ 0.005, whereas that for one unit of option traded is \$ 0.01. Since the rebalancing and hedging of *Delta* is done from time to time, transaction cost is an important source of loss.

Therefore, in this report, we will investigate the move-based as well as the time-based hedging strategy with delta-hedging and delta gamma-hedging, assuming a base band of 0.05. We will also examine the effect of changing real-world volatility to $\sigma = 0.15$ when the risk-neutral volatility is 0.2, as well as the role that the rebalancing band in delta (Δ) plays on the hedge.

2.2 Black-Scholes Model

The Black-Scholes model refers to the case where the drift and volatility parameters of the risky assets are modeled by constant parameter geometric Brownian motions (i.e. the drift and volatility are constant), and where the short rate of interest is also a constant. Therefore, the $\mathbb{P}$-dynamics of a risky asset S_t, the bank account B, and the risky asset S_t as the source of uncertainty are given by equations below. Also, f_t is a traded claim written on the source of uncertainty:

$$
\begin{aligned}
\frac{dB_t}{B_t} &= r(t, S_t)dt \\
\frac{dS_t}{S_t} &= \mu(t, S_t)dt + \sigma(t, S_t)dW_t^{\mathbb{P}} \quad \Rightarrow \quad \frac{dS_t}{S_t} = \mu^{S_t}(t, S_t)dt + \sigma^{S_t}(t, S_t)dW_t \\
\frac{df_t}{f_t} &= \mu^f(t, S_t)dt + \sigma^f(t, S_t)dW_t
\end{aligned}
$$

where $W^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion. The market price of risk is also a constant:

$$\lambda(t, S_t) = \frac{\mu(t, S_t) - r(t, S_t)}{\sigma(t, S_t)}.$$

A claim g written on the asset S with payoff function $G(S)$ must satisfy the Black-Scholes PDE:

$$
\begin{aligned}
\partial_t g(t, S) + rS \cdot \partial_S g(t, S) + \frac{1}{2}\sigma^2 S^2 \cdot \partial_{SS} g(t, S) &= r \cdot g(t, S) \\
g(t, S) &= G(S)
\end{aligned}
$$

And the general form of the contingent claim g can be expressed as the following SDE:

$$\frac{dg_t}{g_t} = \mu^g(t, S_t)dt + \sigma^g(t, S_t)dW_t$$

Since S_t is traded, the term $\mu_t^{S_t} - \lambda_t S_t \sigma_t^{S_t}$ was replaced by r where all parameters are constants, and a Feynman-Kac representation of the solution to this PDE is given by:

$$g(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}_{t,S_t}[G(S_T)]$$

where S satisfies the SDE:

$$\frac{dS_t}{S_t} = r(t, S_t)dt + \sigma(t, S_t)dW_t^{\mathbb{P}}.$$

The Black-Scholes model is widely used for option pricing. Given an asset price process $S = S_{t_k}$, where $k \in \{0, 1, 2, \cdots\}$, the option price C for an European call option written on this asset can be expressed as the following:

$$
\begin{aligned}
C(S_t, t) &= \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)} \\
d_1 &= \frac{(ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{(ln(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}} \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Using put-call parity, the price P of the European put option issued at time t and expiration date at time T can be expressed as the following:

$$
\begin{aligned}
C(S_t, t) - P(S_t, t) &= S_t - Ke^{-r(T-t)} \\
P(S_t, t) &= \Phi(-d_2)Ke^{-r(T-t)} - \Phi(-d_1)S_t
\end{aligned}
$$

2.3 Greeks

Greeks are quantities that can be used to measure the sensitivity of the price of a portfolio to a small change in underlying parameters. Using *Greeks*, component risks could be examined independently, and rebalancing of the portfolio could be achieved according to the required exposure. Two of the most commonly used *Greeks* are *Delta* and *Gamma*.

2.3.1 Delta

Delta measures the price sensitivity of the portfolio (option, stock or bank account) relative to the price change in the underlying asset. In other words, rate of change between the price of the portfolio and a $1 change in the underlying asset's price when holding other parameters fixed.

The *Delta* for a European call option is:

$$\Delta^{Call}(t, S_t) = \partial_{S_t} f^{Call}(t, S_t) = \Phi(d_+)$$

where $f^{Call}(t, S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$ with $d_\pm = \frac{log(\frac{S_t}{K})+(r\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ is the price of the European call option struck on an asset with value S_t at time t with strike price K by Black-Scholes Model.

By *Put-Call Parity*, we know that

$$f^{Put}(t, S_t) + S_t = f^{Call}(t, S_t) + Ke^{-r(T-t)}$$

where $Ke^{-r(T-t)}$ is the value of the strike price K, discounted from the value on the expiration date T to time t at the risk-free rate r, S_t is the spot price or the current market value of the asset at time t, $f^{Put}(t, S_t)$ and $f^{Call}(t, S_t)$ are the values of the European put and call options at time t respectively. Taking partial derivative of both sides with respect to S_t gives us the relation between *Delta* of a European put and call option as

$$\begin{aligned}\partial_{S_t} f^{Put}(t, S_t) + \partial_{S_t} S_t &= \partial_{S_t} f^{Call}(t, S_t) + \partial_{S_t}(Ke^{-r(T-t)}) \\ \Rightarrow \Delta^{Put}(t, S_t) &= \Delta^{Call}(t, S_t) - 1\end{aligned}$$

Therefore we have the *Delta* for a European put option as:

$$\Delta^{Put}(t, S_t) = \Phi(d_+) - 1$$

The *Delta* of the underlying asset (stock) is calculated as follows by definition:

$$\Delta^{S_t} = \frac{dS_t}{dS_t} = 1$$

Note that the asset (stock) is the underlying asset of itself, thus we have its *Delta* equals 1. Similarly, we calculate the *Delta* of the bank account by definition as follows:

$$\Delta^{B_t} = \frac{B_t}{dS_t} = 0$$

since by fundamental settings, we know $\frac{dB_t}{B_t} = r(t, S_t)dt$ where $r = (r_t)_{r\geq 0} = (r(t, S_t)_{t\geq 0})$ is the short rate process, which is independent of S_t.

2.3.2 Gamma

Gamma measures the amount the *Delta* changes given a \$1 change in the underlying asset. In other words, it represents the rate of change between an option's delta and the underlying asset's price when holding other parameters fixed.

The *Gamma* for a European call option is calculated as follows:

$$\begin{aligned}\Gamma^{Call}(t,S_t) &= \partial_{S_tS_t} f^{Call}(t,S_t)\\ &= \partial_{S_t}\Delta^{Call}(t,S_t)\\ &= \partial_{S_t}\Phi(d_+)\\ &= \phi(d_+)\partial_{S_t}d_+\\ &= \frac{\phi(d_+)}{S_t\sigma\sqrt{T-t}}\end{aligned}$$

where $\Phi'(d_+) = \frac{d(\Phi(d_+))}{d(d_+)} = \phi(d_+)$, and $d_+ = \frac{log(\frac{S_t}{K})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Similarly, we could obtain the *Gamma* of the European put option as follows:

$$\begin{aligned}\Gamma^{Put}(t,S_t) &= \partial_{S_tS_t} f^{Put}(t,S_t)\\ &= \partial_{S_t}\Delta^{Put}(t,S_t)\\ &= \partial_{S_t}(\Phi(d_+)-1)\\ &= \phi(d_+)\partial_{S_t}d_+\\ &= \frac{\phi(d_+)}{S_t\sigma\sqrt{T-t}}\end{aligned}$$

Moreover, the *Gamma* for underlying asset (stock) S_t and bank account B_t are 0 as the *Delta* for the underlying asset and bank account are both constants, which means they are independent of S_t. Therefore, taking derivative with respect to S_t results in 0.

3 Methodology

3.1 Dynamic Hedging

Dynamic hedging of a short position in the claim g aims to construct a replicating portfolio for the claim g and invoke a no-arbitrage argument to find the PDE that the function of g satisfies. Therefore, the objective is to build an instantaneously risk-free portfolio with positions in f and B, as well as to short the claim g.

First, we set up a zero cost, self-financing portfolio with $(\alpha, \beta) = (\alpha_t, \beta_t)_{t \geq 0}$, which satisfies the equations below:

$$\begin{aligned} V_0 &= 0 \\ V_t &= \alpha_t f_t + \beta_t B_t - g_t \\ dV_t &= \alpha_t df_t + \beta_t dB_t - dg_t \quad \text{(by self-financing assumption)} \end{aligned}$$

We then substitute the dB_t, df_t as well as dg_t into the portfolio value SDE, dV_t:

$$dV_t = (\alpha_t f_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) d_t + (\alpha_t f_t \sigma_t^f - g_t \sigma_t^g) dW_t \quad (\star)$$

Setting the coefficient of dW_t to zero to achieve instantaneously risk-free portfolio:

$$\begin{aligned} \alpha_t^* f_t \sigma_t^f - g_t \sigma_t^g = 0 \\ \Rightarrow \alpha_t^* = \frac{g_t \sigma_t^g}{f_t \sigma_t^f} \end{aligned}$$

Therefore, the choice of α_t^* eliminates local stochasticity from the evolution of the portfolio value, resulting the portfolio value process with previsible drift. Plugging $\alpha_t^* = \frac{g_t \sigma_t^g}{f_t \sigma_t^f}$ into equation $(\star)$ gives us:

$$
\begin{aligned}
dV_t &= (\alpha_t^* f_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) d_t + (\alpha_t^* f_t \sigma_t^f - g_t \sigma_t^g) dW_t \\
&= (\frac{\sigma_t^g}{\sigma_t^f} \cdot g_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) d_t
\end{aligned}
$$

Note that the drift term for the value process must be identically zero to avoid arbitrage, since the portfolio value at the outset is zero. Setting the drift term equals zero gives us:

$$
\frac{\sigma_t^g}{\sigma_t^f} \cdot g_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g = 0 \quad (*)
$$

Therefore, as discussed previously, there is no randomness and zero drift along the process and $V_0 = 0$, we have $V_t = 0$ for all t. Thus, setting $V_t = 0$, and plugging in $\alpha_t^* = \frac{g_t \sigma_t^g}{f_t \sigma_t^f}$ we have:

$$
V_t = \alpha_t^* f_t + \beta_t B_t - g_t = 0 \Rightarrow \beta_t B_t \quad = g_t (1 - \frac{\sigma_t^g}{\sigma_t^f})
$$

Plugging this into equation $(*)$ gives us:

$$
\begin{aligned}
\frac{\sigma_t^g}{\sigma_t^f} \cdot g_t \mu_t^f + g_t (1 - \frac{\sigma_t^g}{\sigma_t^f}) r_t &= g_t \mu_t^g \\
\Rightarrow \frac{\mu_t^f - r_t}{\sigma_t^f} &= \frac{\mu_t^g - r_t}{\sigma_t^g}
\end{aligned}
$$

Let $\lambda_t = \frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}$, multiplying both sides of $\lambda_t = \frac{\mu_t^g - r_t}{\sigma_t^g}$ by σ_t^g and g_t gives us:

$$
g_t \mu_t^g - \lambda_t g_t \sigma_t^g = r_t g_t \quad (\dagger)
$$

This means that, through obtaining the market price of risk of f and g, any two contingent claims written on S in this economy must have the same market price of risk denoted by $\lambda_t = \lambda(t, S_t)$.

Secondly, all claims written on S satisfy the generalized Black-Scholes PDE. Since we are targeting to price g, we can substitute the expression of $(\dagger)$ into the given drift and volatility:

$$g_t\mu_t^g = \partial_t g + \mu_t^S S_t \cdot \partial_S g + \frac{1}{2}(\sigma_t^S)^2 S_t^2 \cdot \partial_{SS} g$$

$$g_t\mu_t^g = \sigma_t^S S_t \cdot \partial_S g$$

$$\Rightarrow \partial_t g + (\mu_t^S - \lambda_t \cdot \sigma_t^S) S_t \cdot \partial_S g + \frac{1}{2}(\sigma_t^S)^2 S_t^2 \cdot \partial_{SS} g = r_t g_t$$

$$\Rightarrow \partial_t g(t, S_t) + [\mu^S(t, S_t) - \lambda(t, S_t) \cdot \sigma^S(t, S_t)] S_t \cdot \partial_S g(t, S_t) + \frac{1}{2}[\sigma_S(t, S_t)]^2 S_t^2 \cdot \partial_S S g(t, S_t)$$

$$= r(t, S_t) \cdot g(t, S_t)$$

Since the above equation must hold for all possible paths of S_t, this implies that the function of g must satisfy the following PDE, which is the generalized Black-Scholes partial differential equation:

$$\partial_t g(t, S) + [\mu^S(t, S) - \lambda(t, S) \cdot \sigma^S(t, S)] S \cdot \partial_S g(t, S) + \frac{1}{2}[\sigma_S(t, S)]^2 S^2 \cdot \partial_S S g(t, S)$$
$$= r(t, S_t) \cdot g(t, S_t)$$

3.1.1 Delta Hedging Strategy

Recalling the instantaneously risk-free portfolio with:

$$dV_t = (\alpha_t^* f_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) d_t + (\alpha_t^* f_t \sigma_t^f - g_t \sigma_t^g) dW_t$$

$$\alpha_t^* f_t \sigma_t^f - g_t \sigma_t^g = 0 \Rightarrow \alpha_t^* = \frac{g_t \sigma_t^g}{f_t \sigma_t^f}$$

If S is traded now, this can be used as the hedging instrument:

$$\alpha_t^* = \frac{g_t \sigma_t^g}{S_t \sigma_t^S}$$

And according to Ito's lemma, the volatility of the claim price process g is given by $g_t \sigma_t^g = \partial_S g(t, S_t) \cdot \sigma_t^S S_t$, meaning that:

$$\alpha_t^* = \partial_S g(t, S_t)$$

This is known as the *Delta* of the option g, denoted as Δ^g, representing the change in the option value resulting from an increase in the value of underlying asset:

$$\begin{aligned} V_t &= \alpha_t^* S_t + \beta_t B_t - g_t \\ \Rightarrow \Delta^V &= \alpha_t \Delta^S + \beta_t \Delta^B - \Delta^g \\ &= \Delta^g \cdot 1 + \beta_t \cdot 0 - \Delta^g \\ \Rightarrow \Delta^V &= 0 \end{aligned}$$

Therefore, the hedging portfolio reaches *Delta* neutral position, whose value is unaffected by the changes in the value of underlying asset. By implementing the hedging strategy against short position of an option, it results in zero profit-and-loss (PnL) for $V_0 = 0$ and $V_t = 0$ for all t, which requires the frequent rebalancing of the hedging portfolio in the discrete time frame and leading to variance in PnL.

3.1.2 Delta-Gamma Hedging Strategy

Delta-Hedging immunizes the portfolio from 1^{st} order changes in the underlying asset's value. But the estimation of the slop may not be accurate simply using first-order changes. A better estimation could be implemented using a quadratic function, which builds up the *Delta-Gamma*-Hedging.

We want to construct a a portfolio with zero *Delta* and zero *Gamma*, which is also called *Delta-Gamma* neutral. As discussed in previous section, *Delta* for the underlying asset is 1 and *Gamma* for the underlying asset is 0. We will include a call option in our hedging portfolio to hedge both *Delta* and *Gamma* to hedge the sold put option.

Suppose we hold $(\alpha, \beta) = (\alpha_t, \beta_t)_{t \geq 0}$ units in (S_t, f_t) to hedge the sold put option g_t, where $f_t = f(t, S_t)$ is the value of the call option. The value of our whole portfolio is as follows:

$$V_t = B_t + \alpha_t S_t + \beta_t f_t - g_t$$

The *Delta* and *Gamma* of the whole portfolio are calculated by definition as follows:

$$\Delta_t^V = \alpha_t + \beta_t \Delta_t^f - \Delta_t^g$$
$$\Gamma_t^V = \beta_t \Gamma_t^f - \Gamma_t^g$$

Setting the whole portfolio to be *Delta-Gamma* neutral, we get the number of units of call options and underlying asset (stock) required to hold in our hedging portfolio:

$$\alpha_t^* = \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^f}\Delta_t^f$$
$$\eta_t^* = \frac{\Gamma_t^g}{\Gamma_t^f}$$

Note that by previous derivation, we know that the *Gamma* for an European put option is always positive, thus, a short position in a put option has negative *Gamma*, and we need a hedging portfolio with positive *Gamma* to obtain *Gamma* neutrality. At the same time, a short put option and a long call option both have positive *Delta*. We need a short position in the underlying asset (stock) in our hedging portfolio to obtain *Delta* neutrality. At this point, a hedging portfolio with a long position in call option and a short position in the underlying asset (stock) is implemented to hedge the short put option, obtaining *Delta-Gamma* neutrality.

3.2 Profit and Loss

The profit and loss of the hedging portfolio is net cash flow at maturity time T. It is made up of the intrinsic value of the call option longed, the value of the underlying asset shorted, and the transaction cost paid to rebalance at the maturity of the put option. Mathematically, we could denote the profit and loss of the portfolio as follows:

$$PnL = B_T + \alpha_T S_T + \beta_T f_T - G(S_T)$$

3.3 Implementation of Time-Based and Move-Based Hedging Strategy

We implemented two hedging strategies: *Delta* and *Delta-Gamma* hedging. We rebalance the position of the hedging portfolio in discrete time to maintain *Delta* and *Delta-Gamma* neutrality. The following table illustrates the evolution of the hedging portfolio value across the life time of the portfolio until the maturity of the put option with $Delta - Gamma$ hedging. For *Delta* hedging, we follow the same regime for the transactions of the underlying asset and the bank account, excluding transactions of the call option.

Table 1: Evolution of Hedging portfolio

Time	Call Option Value	Underlying Asset Value	Bank Account Value
$t = 0$	Buy β_0 units of call option: pay $\beta_0 f_0 = f(0, S_0)$	Buy α_0 units of S: receive $\alpha_0 S_0$	$B_0 = g_0 - \alpha_0 S_0 - \beta_0 f_0 - \epsilon$
$t = \Delta t^-$	Before rebalancing: $\beta_0 f_{\Delta t} = \beta_0 f(\Delta t, S_{\Delta t})$	Before rebalancing: $\alpha_0 S_{\Delta t}$	$B_{\Delta t} = B_0 e^{r\Delta t} - \alpha_0 S_{\Delta t} - \beta_0 f_{\Delta t}$
$t = \Delta t$	Rebalance to $\beta_{\Delta t}$ units: $\beta_{\Delta t} f_{\Delta t} = \beta_{\Delta t} f(\Delta t, S_{\Delta t})$	Rebalance to $\alpha_{\Delta t}$ units: sell $\alpha_0 S$, buy $\alpha_{\Delta t} S$ $\Rightarrow$ cash flow $= (\alpha_{\Delta t} - \alpha_0) S_{\Delta t}$	$B_{\Delta t} = B_0 e^{r\Delta t} - (\alpha_{\Delta t} - \alpha_0) S_{\Delta t}$ $-(\beta_{\Delta t} - \beta_0) f_{\Delta t} - \epsilon$
$\vdots$			
$t = t_k^-$	Before rebalancing: $\beta_{t_{k-1}} f_{t_k} = \beta_{t_{k-1}} f(t_k, S_{t_k})$	Before rebalancing: $\alpha_{t_{k-1}} S_{t_k}$	$B_{t_k} = B_{t_{k-1}} e^{r\Delta t} - \alpha_{t_{k-1}} S_{t_k} - \beta_{t_{k-1}} f_{t_k}$
$t = t_k$	Rebalance to β_{t_k} units: $f_{t_k} = f(t_k, S_{t_k})$	Rebalance to $\alpha_{t_{k-1}}$ units: sell $\alpha_{t_{k-1}S}$, buy $\alpha_{t_k} S$: $\Rightarrow$ cash flow $= (\alpha_{t_k} - \alpha_{k-1}) S_{t_k}$	$B_{t_k} = B_{t_{k-1}} e^{r\Delta t} - (\alpha_{t_k} - \alpha_{t_{k-1}}) S_{t_k}$ $-(\beta_{t_k} - \beta_{t_{k-1}}) f_{t_k}$ - ϵ
$\vdots$			
$t = T$	Option holding value: $\beta_{T-\Delta t} f_T = \beta_{T-\Delta t} f(T, S_T)$	Asset value owed: $\alpha_{T-\Delta t} S_T$	$B_T = B_{T-\Delta t} e^{r\Delta t} - (\alpha_T - \alpha_{T-\Delta t}) S_T$ $-(\beta_T - \beta_{T-\Delta t}) f_T - \epsilon$

At $t = 0$, the hedging portfolio includes α_0 [1] units of the underlying asset and β_0 [2] units of

[1] Note that if $\alpha_0 > 0$, it means that we are buying α_0 units of the asset (stock). Similarly, if $\alpha_0 < 0$, then we are selling α_0 units of the asset (stock)

[2] Note that $\beta_0 = \frac{\Gamma_0^f}{\Gamma_0^g} > 0$, as *Gamma* of both call and put option are positive as calculated in previous section. This means we hold long position in the call option in our hedging portfolio

call option f in bank account, $B_0 = g_0 - \alpha_0 S_0 - \beta_0 f_0 - \epsilon$, where ϵ is the transaction cost that equals $\epsilon_s|\alpha_0| + \epsilon_f|\beta_0|$ and ϵ_s is the per unit transaction cost of the asset and ϵ_f is the per unit transaction cost of the call option.

At $t = \Delta t$, the bank account grows from B_0 to $B_0 e^{r\Delta t}$, and the portfolio is rebalanced: $\alpha_0 \to \alpha_{\Delta t}$ and $\beta_0 \to \beta_{\Delta t}$. Thus, the bank account at time Δt is $B_{\Delta t} = B_0 e^{r\Delta t} - (\alpha_{\Delta t} - \alpha_0)S_{\Delta t} - (\beta_{\Delta t} - \beta_0)f_{\Delta t} - \epsilon_s|\alpha_{\Delta t} - \alpha_0| - \epsilon_f|\beta_{\Delta t} - \beta_0|$.

Therefore, in the general case, at time t_k, the bank account grows from $B_{t_{k-1}}$ to $B_{t_{k-1}} e^{r\Delta t}$, and the portfolio is rebalanced: $\alpha_{t_{k-1}} \to \alpha_{t_k}$ and $\beta_{t_{k-1}} \to \beta_{t_k}$. Thus, the bank account at time t_k is $B_{t_k} = B_{t_{k-1}} e^{r\Delta t} - (\alpha_{t_k} - \alpha_{t_{k-1}})S_{t_k} - (\beta_{t_k} - \beta_{t_{k-1}})f_{t_k} - \epsilon_s|\alpha_{t_k} - \alpha_{t_{k-1}}| - \epsilon_f|\beta_{t_k} - \beta_{t_{k-1}}|$.

At maturity $t = T$, there are B_T in the bank account, where $B_T = B_{T-\Delta t} e^{r\Delta t} - (\alpha_T - \alpha_{T-\Delta t})S_T - (\beta_T - \beta_{T-\Delta t})f_T - \epsilon_s|\alpha_T - \alpha_{T-\Delta t}| - \epsilon_f|\beta_T - \beta_{T-\Delta t}|$. Note that at the end of the life time of the portfolio, we have to liquidate the positions, which means we have to make the position of the underlying asset as well as the call option to be zero. Therefore, we have to add $\alpha_T S_T$ and $\beta_T f_T$ back. As a result, the total profit and loss would be $PnL = B_T + \alpha_T S_T + \beta_T f_T - G(S_T)$.

As discussed previously, the time-based hedging strategy partitions the time from 0 to maturity T into equally-spaced discrete time steps. We check for the change in prices and rebalance the portfolio to hedge *Delta* and *Gamma* at every time step, as shown in Table 1. In contrast, the move-based hedging strategy provides an alternative way of hedging the *Delta* (and *Gamma*) of the option. Starting from *Delta* at time 0, we put a band around the current *Delta*, and keep the current portfolio position until the new *Delta* exceeds the certain band upper or lower boundary. When touching the boundary, we rebalance the portfolio to neutralize *Delta* and *Gamma* and set a new band around the new *Delta*. We Repeat the same strategy along the lifetime of the portfolio until the put option matures.

For the move-based hedging strategy, we use the same time steps as we set for the time-based strategy to check whether the portfolio *Delta* touches the pre-determined band boundary. It is only at these discrete points in time when *Delta* has changed large enough that we rebalance the portfolio. We follow the same pattern in Table 1 to hedge and rebalance.

4 Implementation and Findings

In this section, we compare move-based hedging with time-based hedging as well as the difference between *Delta* and *Delta-Gamma* hedging strategy performance. We implement a rebalancing-band on *Delta* for move-based hedging. The rebalancing frequency for time-based hedging is set to be 100. We will look into how different real-world volatilities affect our heding outcome. Also, we will investigate how different rebalancing-band affect hedging performances.

4.1 Results

4.1.1 Delta Hedging

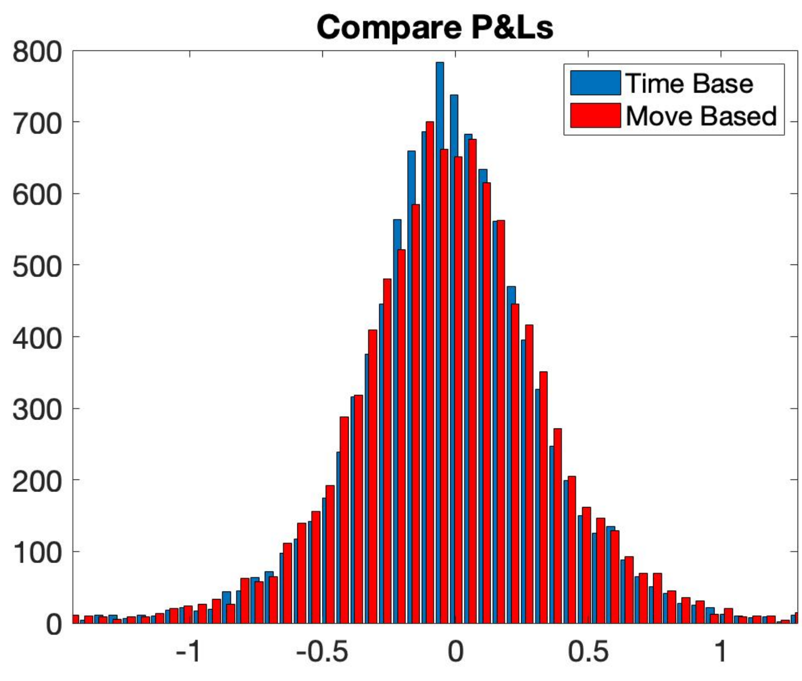

Figure 1: Profit & Loss Distribution with Delta Hedging

Figure 1 shows the profit and loss distributions of the time-based hedging strategy and move-based hedging strategy with *Delta* hedging. Both the *PnL*s for the time-based and move-based hedging strategies are relatively symmetric with a single peak at a value close to zero. The distribution of the move-based hedging strategy has a fatter right tail compared to that of the time-based, while the left tails of both distributions are of relative same size.

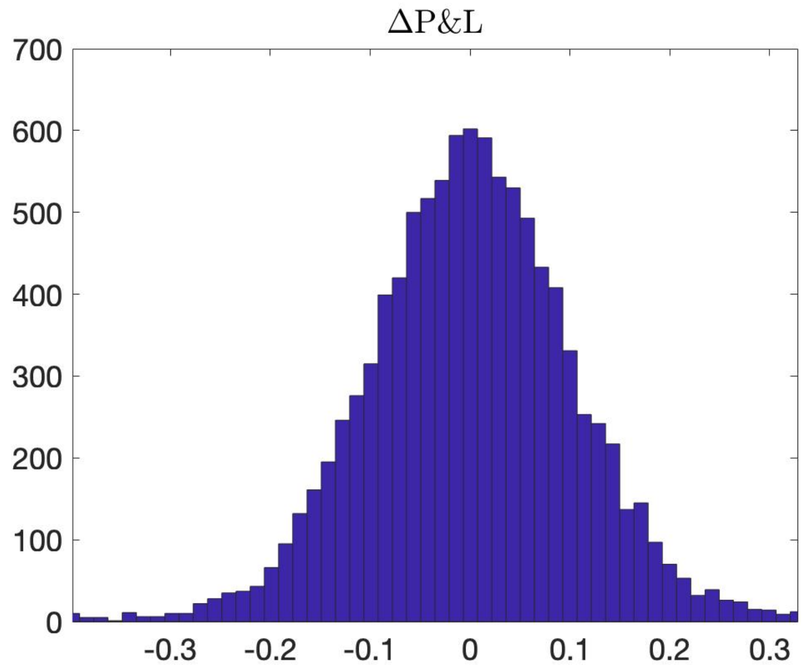

Figure 2: Distribution of the Difference in Profit & Loss with Delta Hedging

Figure 2 shows the distribution of the PnL of the move-based hedging strategy subtracts the PnL of the move-based hedging strategy with *Delta* hedging. The distribution is slightly left-skewed with a peak at a value slightly higher than zero.

Table 2: Mean and Standard Deviation of Profit & Loss with Delta Hedging

	Mean of P&L	**Stdev of P&L**
Time-Based	-0.023022	0.342934
Move-Based	-0.021654	0.358849
Difference in P&L	0.000313	0.001013

Table 2 provides a more detailed description of the PnL with different statistics. The move-based hedging strategy has a slightly higher mean PnL compared to the time-based one, and a slightly larger standard deviation of PnL compared to the time-based one. The average of the difference between the move-based and time-based p&ls is positive but very close to 0.

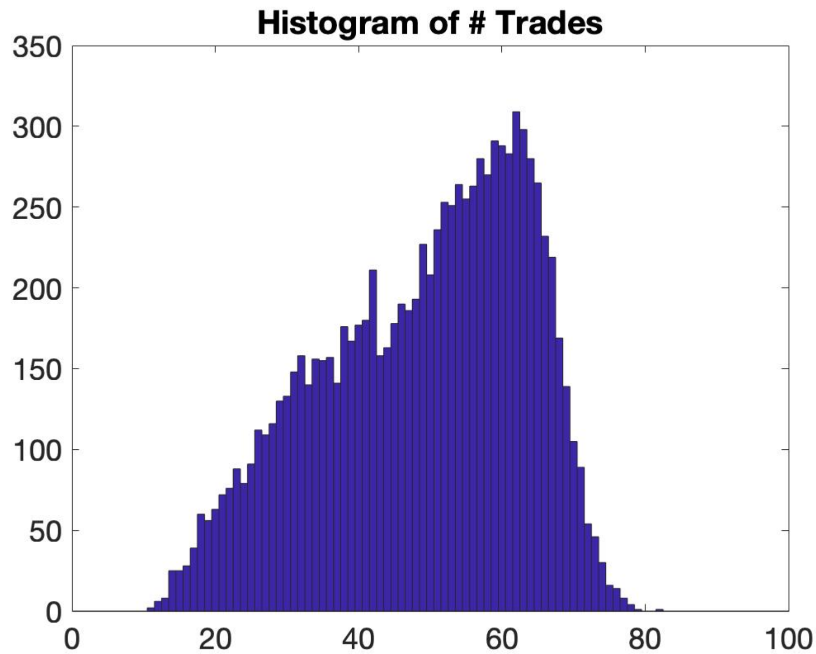

Figure 3: Histogram of Number of Trades

Figure 3 is a histogram showing the number of trades conducted in the move-based hedging strategy simulations. The distribution is left skewed with a peak at approximately 65 trades. Since we used a time step of 100 in the time-based hedging, all simulations with move-based hedging have made fewer transactions than the time-based ones, thus had less transaction costs in total.

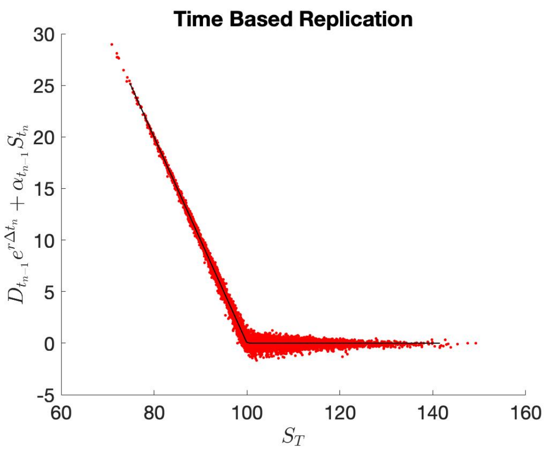

Figure 4: Payoff of Time-Based Replicating Portfolio

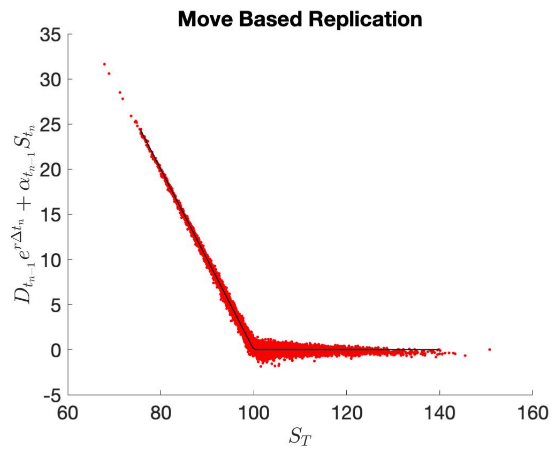

Figure 5: Payoff of Move-Based Replicating Portfolio

Figure 4 and 5 show the payoffs of the hedging portfolio with time-based and move-based *Delta* hedging strategy. The black lines show the exact replication position, and the red dots show the resulting positions of the simulations. The more closely aligned the red dots are with the black line, the more accurate the hedging is. The replications are more accurate when the put option is either deep out-of-money or deep in-the-money. When the option is at the money, the move-based hedging strategy has a better performance.

4.1.2 Delta-Gamma Hedging

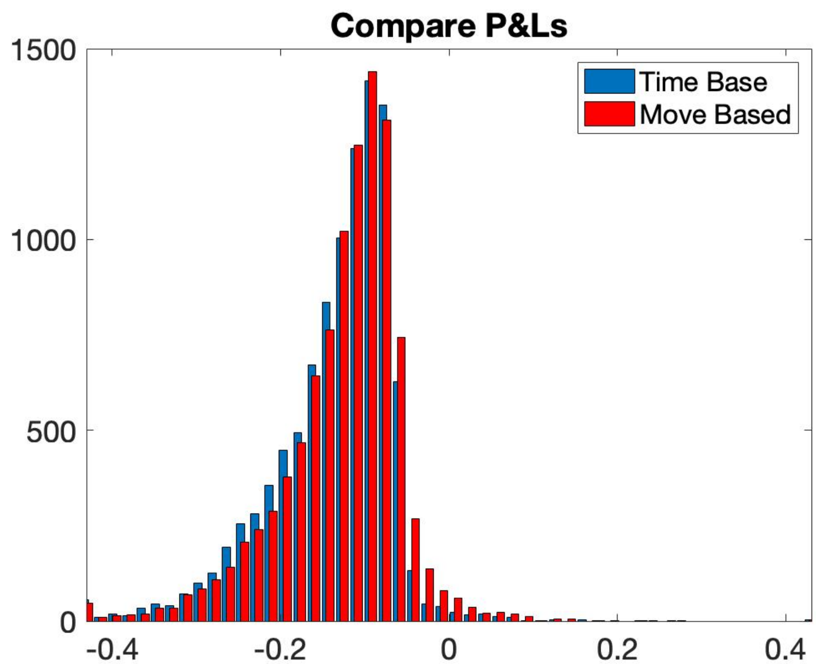

Figure 6: Profit & Loss Distribution with Delta-Gamma Hedging

Figure 6 plots the profit and loss distributions of the time-based hedging strategy and move-based hedging strategy with $Delta-Gamma$ hedging. Both the PnLs for the time-based and move-based hedging strategies peak at approximately -0.1. The left tails of the distributions are fatter than the right tails. The distribution of the time-based hedging has a fatter left tail and a thinner right tail compared to that of the move-based. This indicates that the move-based hedging strategy results in higher PnL in distribution.

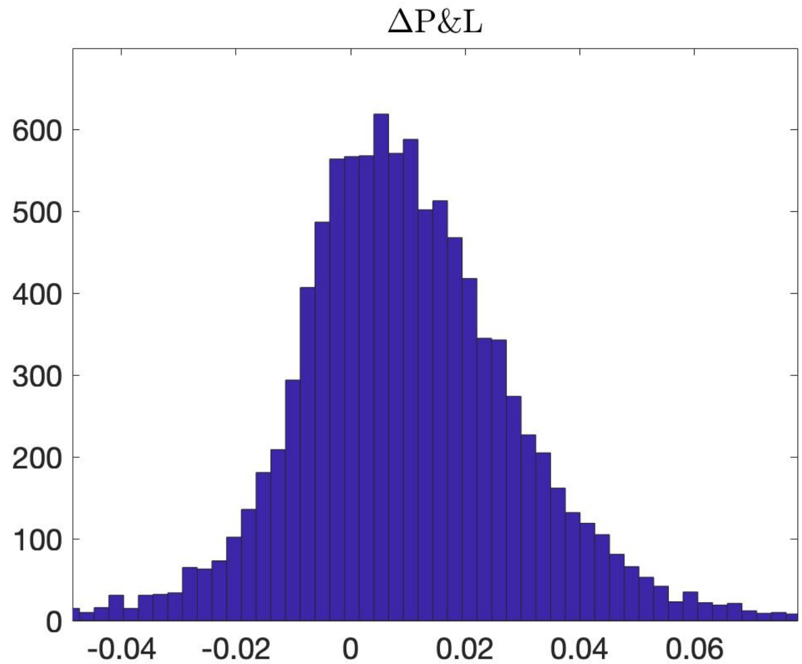

Figure 7: Distribution of the Difference in Profit & Loss with Delta-Gamma Hedging

Figure 7 plots the distribution of the difference in PnL between the move-based hedging strategy and the time-based hedging strategy with *Delta-Gamma* hedging. The distribution is slightly right-skewed with a peak at a value slightly higher than zero, indicating the move-based one is more profitable than the time-based one.

Table 3: Mean and Standard Deviation of Profit & Loss with Delta-Gamma Hedging

	Mean of P&L	**Stdev of P&L**
Time-Based	-0.136342	0.077027
Move-Based	-0.126418	0.076628
Difference in P&L	0.010073	0.021181

Table 3 provides a more detailed description of the PnLs. The move-based hedging strategy has a slightly higher mean PnL compared to the time-based one, and a slightly smaller standard deviation of PnL compared to the time-based one. The mean difference in p&l is also positive, meaning the p&l of the move-based strategy is higher than the time-based one on average.

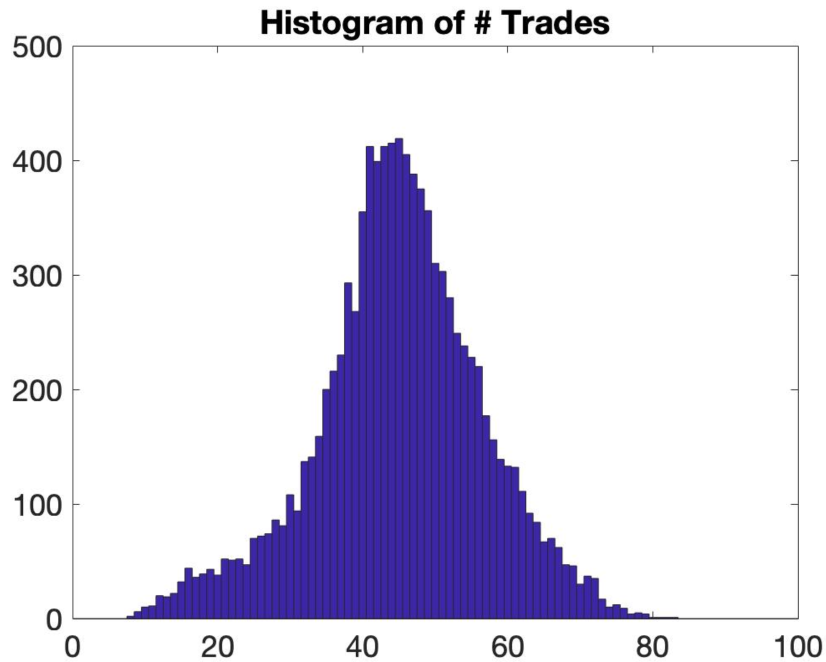

Figure 8: Histogram of Number of Trades

Figure 8 is a histogram showing the number of trades conducted in the move-based hedging strategy simulations. The distribution is approximately symmetric with a peak at around 45 trades. Since we used a time step of 100 in the time-based hedging, all simulations with move-based hedging have made fewer transactions, thus had less transaction costs in total.

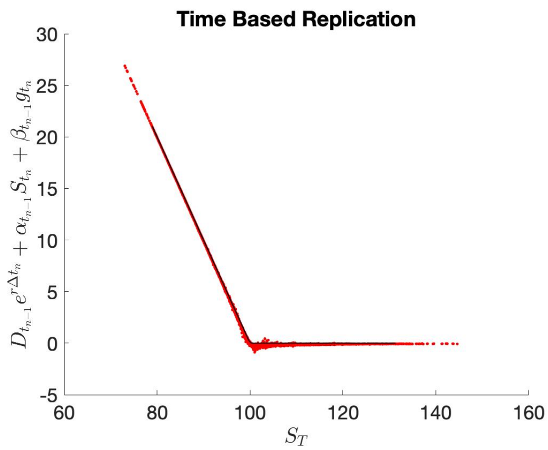

Figure 9: Payoff of Time-Based Replicating Portfolio

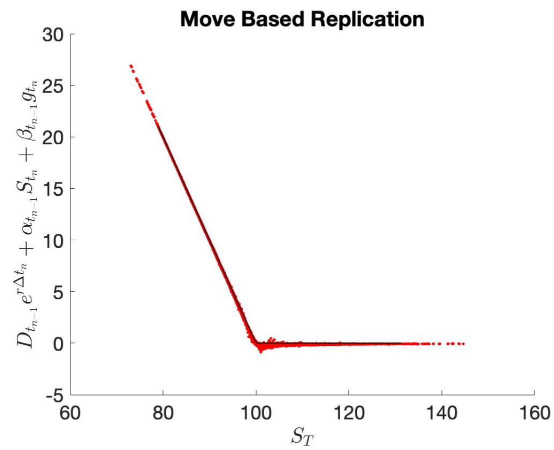

Figure 10: Payoff of Move-Based Replicating Portfolio

Figure 9 and 10 show the payoffs of the hedging portfolio with the time-based and move-based hedging strategy with *Delta-Gamma* hedging. The black lines show the exact replication position, and the red dots show the positions of the simulations. The more closely aligned the red dots are with the black line, the more accurate the replication is. The replications are more accurate when the put option is either deep out-of-money or deep in-the-money. Both strategies yield very similar and fairly accurate replications.

4.1.3 Deviating Real-world Volatility

Table 4: Mean and Standard Deviation of Profit & Loss with Delta Hedging

Real-World Volatility	**Hedge Type**	**Mean of P&L**	**Stdev of P&L**
15%	Time-Based	0.968122	0.410877
15%	Move-Based	0.968990	0.424728
20%	Time-Based	-0.018274	0.343614
20%	Move-Based	-0.018550	0.357854

Table 4 shows the mean and standard deviation of the profit and loss with *Delta* hedging when the real-world volatility is 15% and 20% respectively. The risk-neutral volatility is fixed to be 20%. When the real-world volatility is 15%, both time-based and move-based hedging strategy result in positive mean PnL that deviates further from 0, comparing to when the real-world volatility and risk-neutral volatility match. In addition, the standard deviation also increases when the real-world volatility decreases, indicating the hedging is less stable.

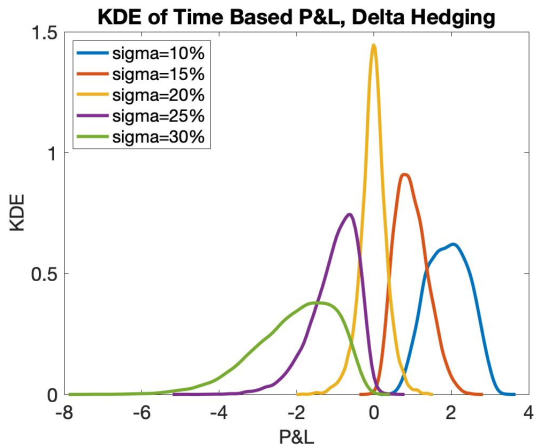

Figure 11: KDE of P&Ls with Different Real-World Volatilities of Time-Based Delta Hedging

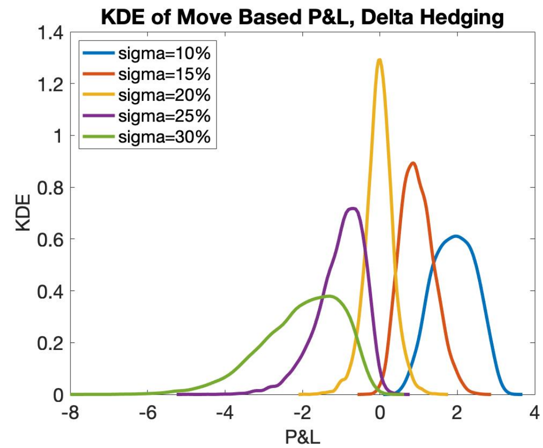

Figure 12: KDE of P&Ls with Different Real-World Volatilities of Move-Based Delta Hedging

Figure 11 and 12 plot the kernel density estimates of the profit and loss distributions of time-based and move-based hedging strategy with *Delta* hedging when the real-world volatility is different from the risk-neutral volatility. The risk-neutral volatility is fixed to be 20%, while the legend in the figures indicates various real-world volatilities. When the risk-neutral volatility equals the real-world volatility, the KDE is symmetric with a peak at approximately zero, which matches our findings in section 4.1.1. When the real-world volatility is smaller than the risk-neutral one, the KDE shifts to the right as volatility decreases. The peak also becomes less sharp. When the real-world volatility is larger than the risk-neutral one, the KDE shifts to the left as volatility increases. The peak becomes less sharp and the distribution becomes increasingly left-skewed.

Table 5: Mean and Standard Deviation of Profit & Loss with Delta-Gamma Hedging

Real-World Volatility	Hedge Type	Mean of P&L	Stdev of P&L
15%	Time-Based	-0.120589	0.060211
15%	Move-Based	-0.109877	0.060247
20%	Time-Based	-0.136222	0.074542
20%	Move-Based	-0.126497	0.074039

Table 5 shows the mean and standard deviation of the profit and loss with delta-gamma hedging when the real-world volatility is 15% and 20% respectively. The risk-neutral volatility is fixed to be 20%. When the real-world volatility is 15%, both time-based and move-based hedging strategy result in a mean PnL that is slightly closer to 0 compared to when the real-world volatility and risk-neutral volatility match. In addition, the standard deviation also decreases when the real-world volatility decreases, indicating the hedging is more stable.

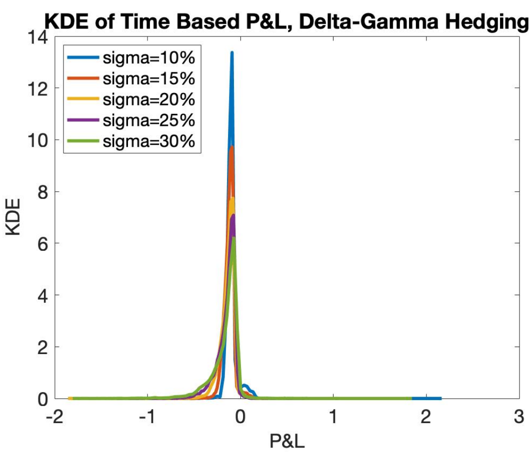

Figure 13: KDE of P&Ls with Different Real-World Volatilities of Time-Based Delta-Gamma Hedging

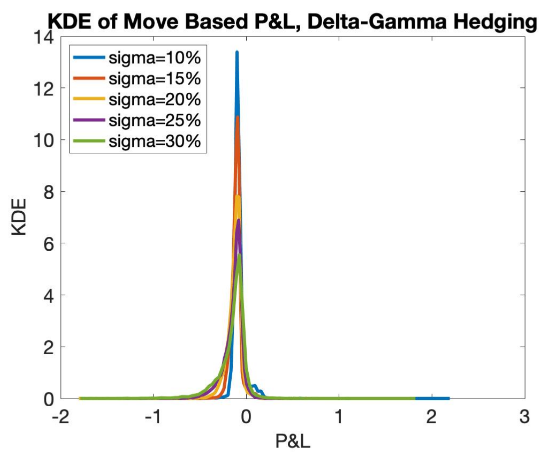

Figure 14: KDE of P&Ls with Different Real-World Volatilities of Move-Based Delta-Gamma Hedging

Figure 13 and 14 plot the kernel density estimates of the profit and loss distributions of time-based and move-based hedging strategy with $Delta - Gamma$ hedging when the real-world volatility is different from the risk-neutral volatility. The risk-neutral volatility is fixed to be 20%, while the legend in the figures indicates real-world volatilities. When the risk-neutral volatility equals the real-world volatility, the KDE is symmetric with a peak at approximately

-0.1 which matches our findings in section 4.1.2. When the real-world volatility deviates the risk-neutral one, the kurtosis of the KDE increases as volatility decreases, the peak becomes sharper and the left tail becomes thinner. However, the mode remains very close to the mode when the two volailities match.

4.1.4 Different Rebalancing Bands

Now we are interested in the exploring how the change in rebalancing band of delta affects move-based *Delta* hedging and $Delta - Gamma$ hedging.

For the move-based *Delta* hedging strategy with different rebalancing bands, table 6 shows the mean and standard deviation of the PnL, the mean and standard deviation of the transaction cost, and the mean number of transactions for each bandwidth. The mean PnL decreases and standard deviation increases as the band gets larger, which indicates the hedging becomes less effective. In addition, the mean transaction costs and mean number of transactions both decrease as the band gets larger, which makes sense because it is harder for the *Delta* of the hedging portfolio to reach the limit of the band when the width is larger. In the meantime, the standard deviation of the transaction cost does not any exhibit clear trend.

Table 6: P&L, Transaction Costs and Number of Transactions with Move-Based Delta Hedging

Band Width	**0.01**	**0.05**	**0.10**	**0.50**
Mean P&L	-0.017070	-0.019371	-0.022305	-0.045219
Stdev P&L	0.349386	0.359696	0.417865	1.319596
Mean Transaction Cost	0.017865	0.016005	0.013233	0.005731
Stdev Transaction Cost	0.005935	0.005872	0.005330	0.002333
Mean # Transactions	80.1219	48.8952	28.1840	4.3895

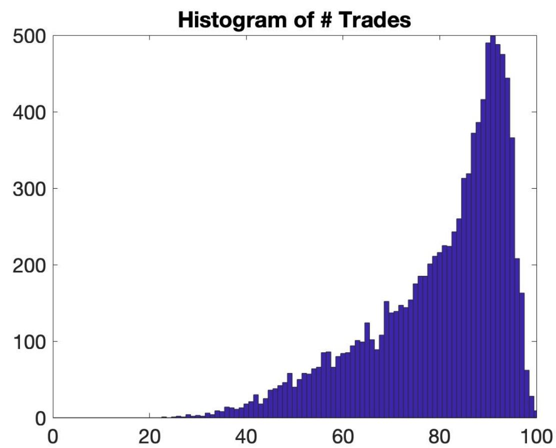

Figure 15: Rebalancing Bandwidth = 0.01

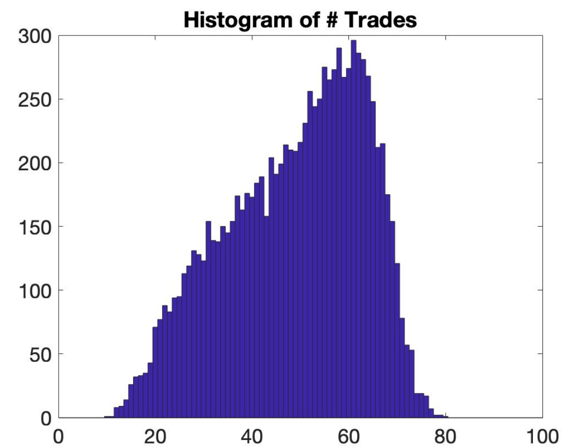

Figure 16: Rebalancing Bandwidth = 0.05

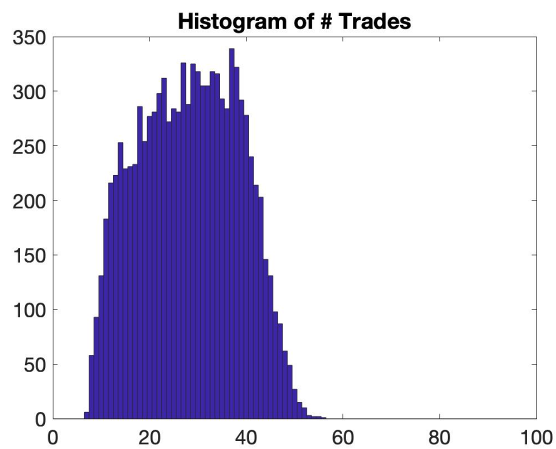

Figure 17: Rebalancing Bandwidth = 0.1

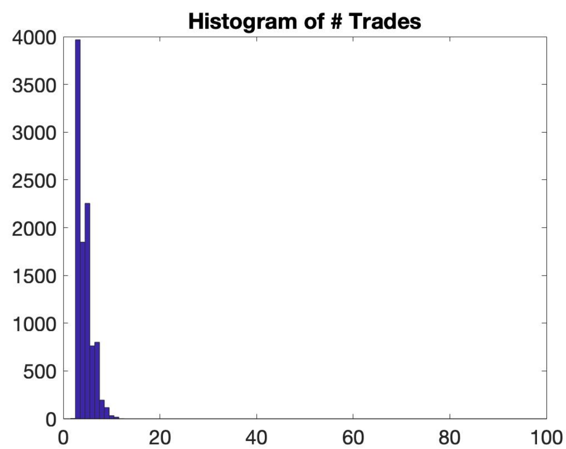

Figure 18: Rebalancing Bandwidth = 0.5

Figure 15 to 18 show the histograms of number of trades with different bandwidths. The peak shifts to the left as the bandwidth increases and the distribution changes from left-skewed to right-skewed. This indicates a decrease in the number of rebalancing made as the bandwidth increases.

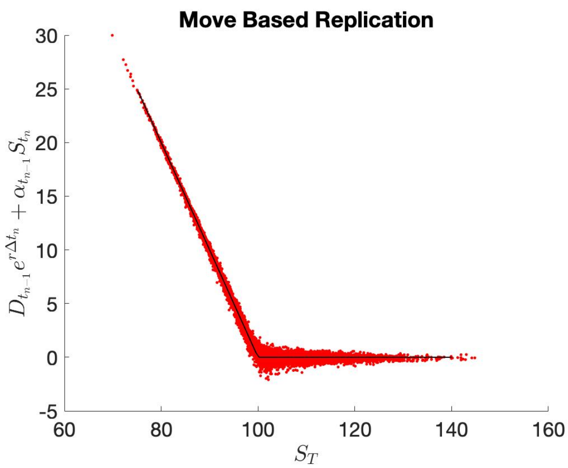

Figure 19: Rebalancing Bandwidth = 0.01

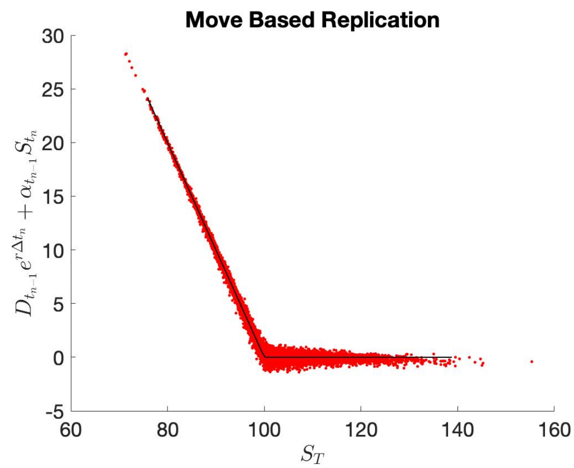

Figure 20: Rebalancing Bandwidth = 0.05

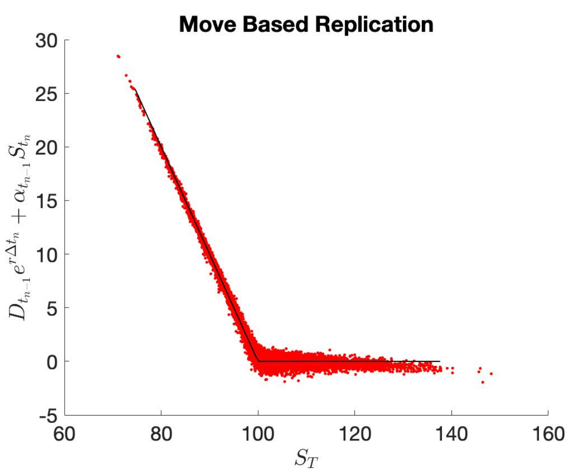

Figure 21: Rebalancing Bandwidth = 0.1

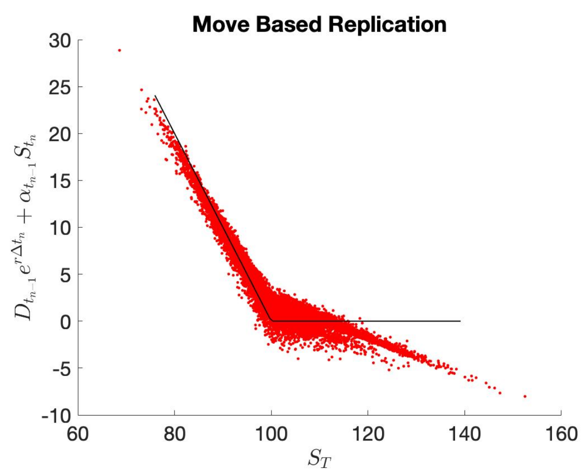

Figure 22: Rebalancing Bandwidth = 0.5

Figure 19 to 22 show how the hedging portfolio can replicate the ideal hedging payoff that yields zero PnL. The black line in each graph represents the ideal hedging payoff at different ending asset price and the red dots represent the payoff of the hedging portfolio in the simulations. As the bandwidth increases, the replication becomes less accurate, especially when the ending asset price is at-the-money and out-of-money.

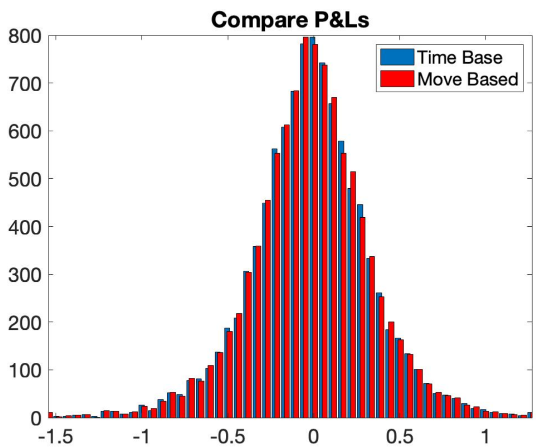

Figure 23: Rebalancing Bandwidth = 0.01

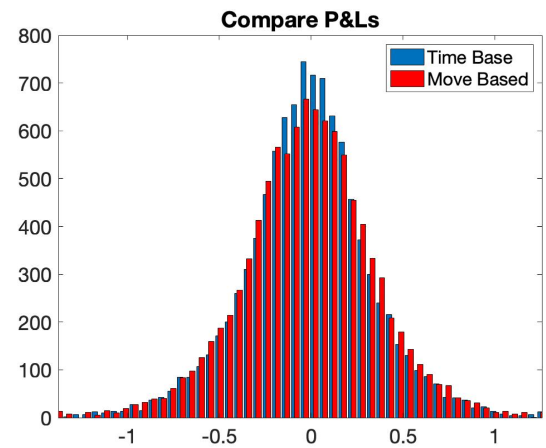

Figure 24: Rebalancing Bandwidth = 0.05

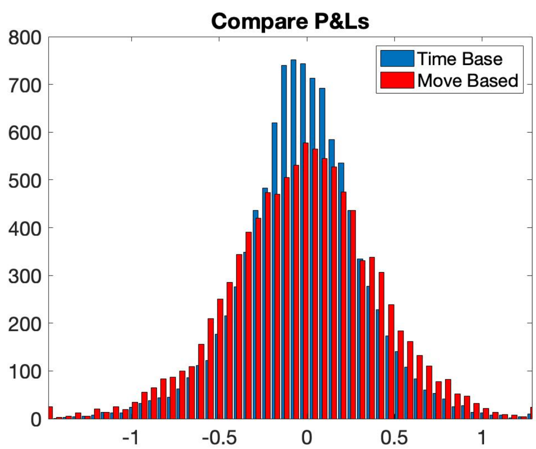

Figure 25: Rebalancing Bandwidth = 0.1

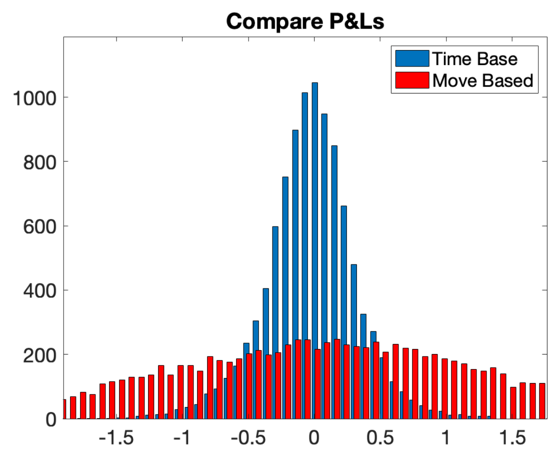

Figure 26: Rebalancing Bandwidth = 0.5

Figure 23 to 26 show the distributions of the profit and loss with move-based *Delta* hedging strategy versus time-based *Delta* hedging strategy with different rebalancing bandwidths. When the bandwidth equals to 0.01, the distributions of the time-based and move-based are very similar because the move-based one rebalanced almost as frequently as the time-based one. However, as the bandwidth increases, the move-based one rebalances less frequently and deviates from the time-based one. The peak becomes less sharp and the left and right tails become fatter. Since an exact replication will result in a distribution with 100% probability that the PnL is 0, the hedging becomes less effective as the bandwidth gets larger. Especially when the bandwidth equals to 0.5, the PnL approaches a uniform distribution and the replication is invalid. On the other hand, the transaction cost decreases as the rebalancing becomes less frequent. Overall, according to the figures, too large a rebalancing band leads to poorer PnL in comparison to the time-based one.

Now, we will repeat the same analysis on the effect of rebalancing bands on delta-gamma hedging.

Table 7: P&L, Transaction Costs and Number of Transactions with Move-Based Delta-Gamma Hedging

Band Width	**0.01**	**0.05**	**0.10**	**0.50**
Mean P&L	-0.135219	-0.126931	-0.116852	-0.080285
Stdev P&L	0.074794	0.078030	0.081077	0.247227
Mean Transaction Cost	0.134753	0.127392	0.117673	0.081365
Stdev Transaction Cost	0.053810	0.054429	0.054579	0.047320
Mean # Transactions	76.2733	30.3539	28.1840	8.3227

For the move-based $Delta - Gamma$ hedging strategy with different rebalancing bands, table 7 shows the mean and standard deviation of the PnL, the mean and standard deviation of the transaction costs, and the mean number of transactions for each bandwidth. As the bandwidth increases, the mean PnL increases and gets closer to zero. However, the standard deviation increases as the band gets larger, meaning there is more uncertainty involved in the performance of the hedging.

In addition, the mean transaction cost and mean number of transaction both decrease as the band gets larger, which makes sense because it is harder for the $Delta$ of the option to reach the limit of the band when the width is larger. In the meantime, the standard deviation of the transaction cost remains relatively constant and does not any exhibit clear trend.

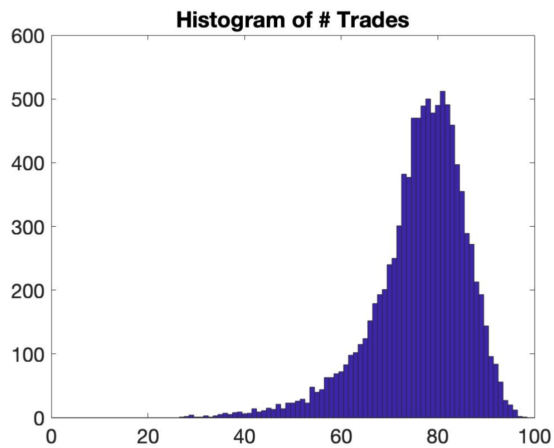

Figure 27: Rebalancing Bandwidth = 0.01

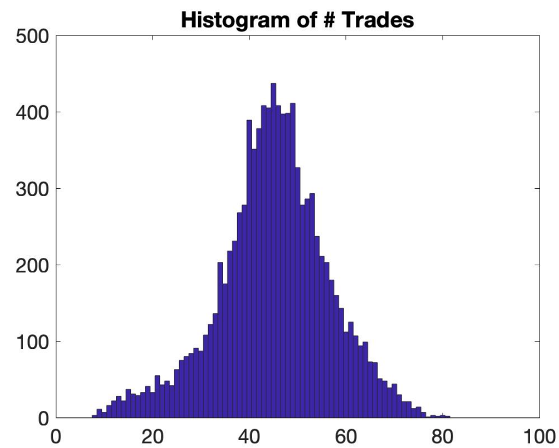

Figure 28: Rebalancing Bandwidth = 0.05

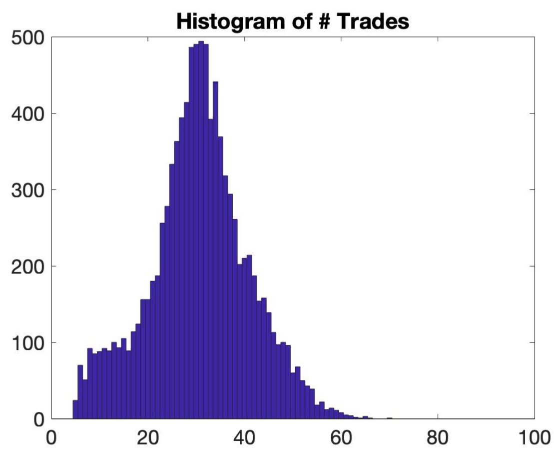

Figure 29: Rebalancing Bandwidth = 0.1

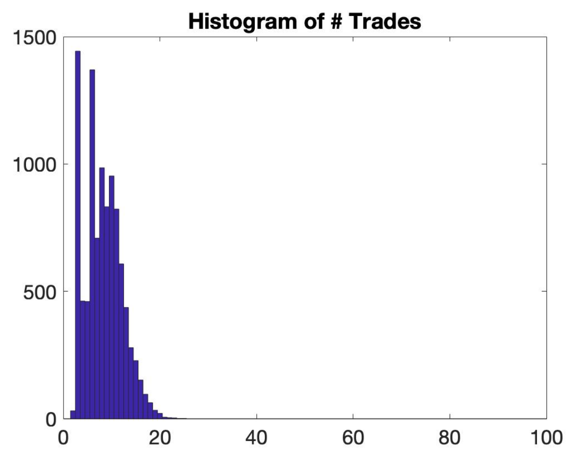

Figure 30: Rebalancing Bandwidth = 0.5

Figure 27 to 30 show the histograms of number of trades with different bandwidths. The peak shifts to the left as the bandwidth increases and the distribution changes from left-skewed to right-skewed. This indicates a decrease in the number of rebalancing made.

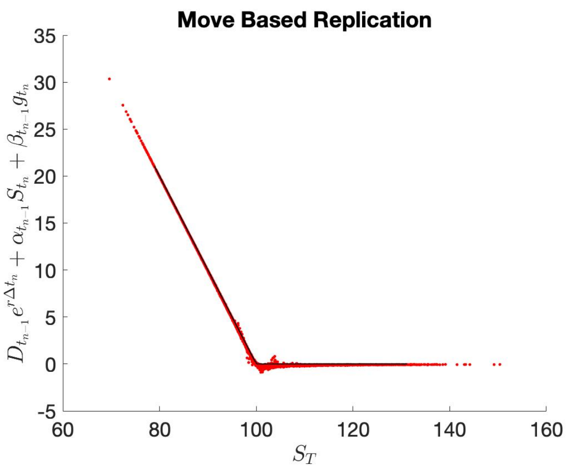

Figure 31: Rebalancing Bandwidth = 0.01

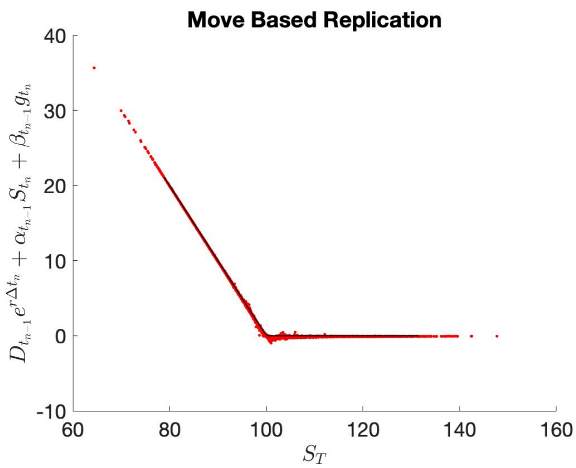

Figure 32: Rebalancing Bandwidth = 0.05

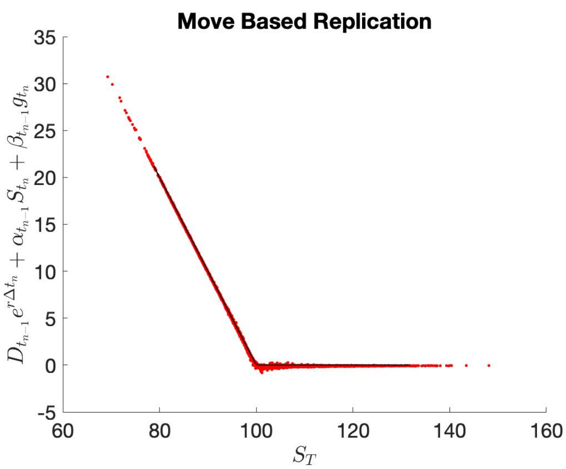

Figure 33: Rebalancing Bandwidth = 0.1

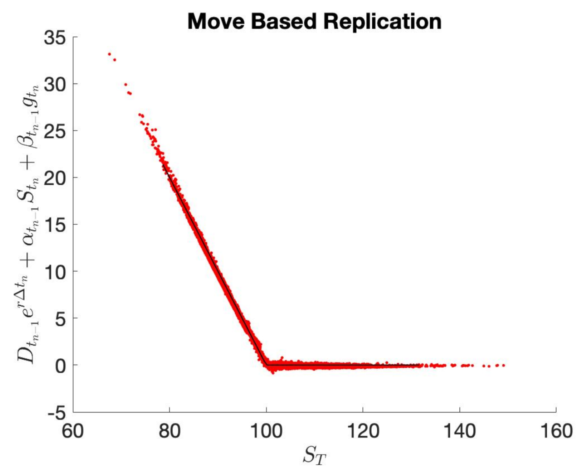

Figure 34: Rebalancing Bandwidth = 0.5

Figure 31 to 34 show how the hedging portfolio can replicate the ideal hedging payoff that yields zero PnL. The black line in each graph represents the ideal hedging payoff at different ending asset price and the red dots represent the payoff of the hedging portfolio in the simulations. As the bandwidth increases, the replication becomes slightly less accurate, especially when the ending asset price is at-the-money. However, the decreases in replication accuracy is not significant and it still provides a reasonably good replication when the bandwidth is large (i.e. when the bandwidth equals to 0.5).

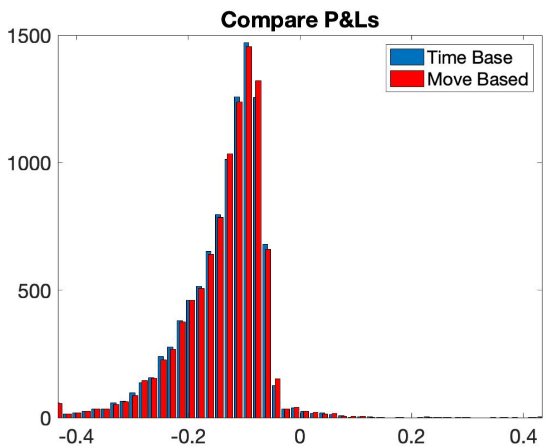

Figure 35: Rebalancing Bandwidth = 0.01

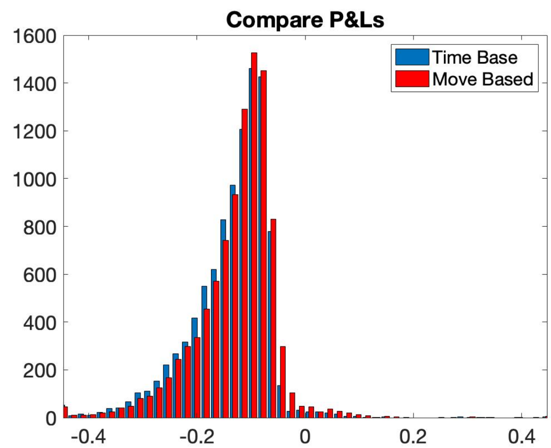

Figure 36: Rebalancing Bandwidth = 0.05

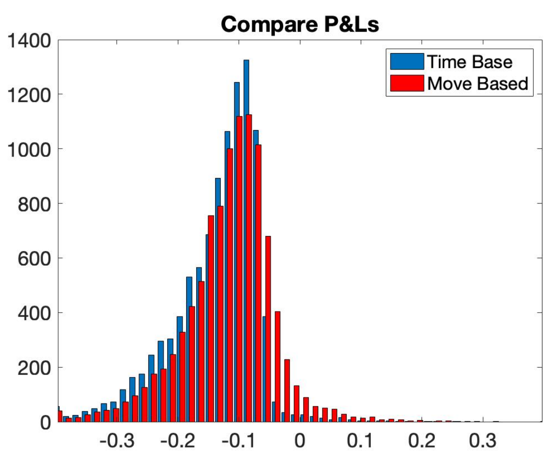

Figure 37: Rebalancing Bandwidth = 0.1

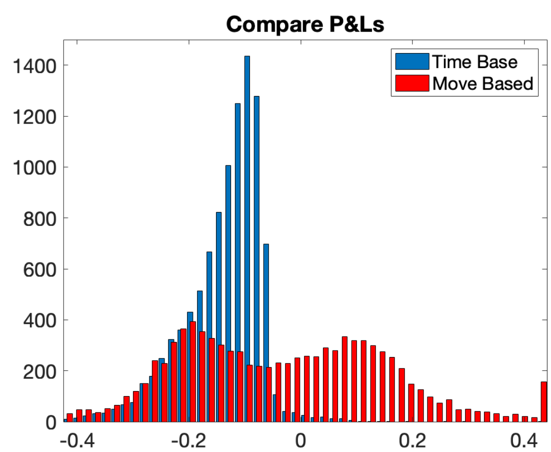

Figure 38: Rebalancing Bandwidth = 0.5

Figure 35 to 38 show the distributions of the profit and loss with move-based $Delta - Gamma$ hedging strategy versus time-based $Delta - Gamma$ hedging strategy with different rebalancing bandwidths. When the bandwidth equals to 0.01, the distributions of the time-based and move-based are very similar because the move-based one rebalanced almost as frequently as the time-based one. However, as the bandwidth increases, the move-based one rebalances less frequently and deviates from the time-based one. The peak of the move-based distribution becomes less sharp, its left tail becomes thinner and its right tail becomes fatter. In figure 35, both distributions peak at a PnL of -0.1. As the bandwidth increases to 0.1, the distribution shifts towards zero, meaning the hedging becomes more effective. However, as the rebalancing band goes to 0.5, the PnL approaches a uniform distribution and the replication is invalid.

4.2 Analysis

With a base band of 0.05, the move-based *Delta* hedging strategy yields a very similar hedging performance compared to the time-based *Delta* hedging strategy. According to Figure 1, The distributions of the PnLs for both hedging strategies are closely aligned with each other. Based on Table 2, the mean of the difference of PnL between move-based and time-based is relatively small (0.000313), which indicates the move-based and time-based hedging performance is similar for *Delta* hedging. This is consistent with the results shown in the replication diagrams in Figure 4 and 5, where the replicating performance is similar between the two.

As for the $Delta - Gamma$ hedging strategy, with a base band of 0.05, the move-based hedging strategy outperforms the time-based hedging strategy. According to Figure 6, both time-based and move-based strategy have their PnL distributions peaked around -0.1. However, the distribution of the move-based has thinner left tail and fatter right tail, meaning its distribution is centered closer to zero, the point of perfect hedging. The mean and standard deviation further justifies this observation. The move-based hedging has a mean PnL that's closer to 0, while both strategies have relatively small standard deviations. Looking at the replication plots in Figure 9 and 10, both strategies provide fairly accurate replications of the exact hedging payoff. Therefore, the reason for them not being able to get a PnL closer to zero is because of the transaction costs associated with trading the asset and the call option. Since the time-based strategy makes more frequent rebalancing than the move-based does, it leads to higher total transaction cost which lowers its hedging effectiveness.

Theoretically speaking, $Delta - Gamma$ hedging should yield a better result than *Delta* hedging as it results in a more accurate approximation of the target put option price. This can be validated by comparing the replication accuracy in Figure 4 and 5 for *Delta* hedging with Figure 9 and 10 for $Delta - Gamma$ hedge. The red dots in the latter figures are more closely aligned with the black lines, indicating a better hedging accuracy.

However, in our base case, both time-based and move-based *Delta* hedging yield PnLs closer to 0 than the time-based and move-based $Delta - Gamma$ hedging do. The relatively high transaction costs associated with $Delta - Gamma$ hedging is the main reason for its sub-optimal performance. Because the hedging portfolio with $Delta - Gamma$ hedging requires the trading of a call option, which results in a higher per-unit transaction cost. Since the purpose of the *Delta-Gamma* hedging is to allow the hedger to maintain the *Delta* hedging position for a longer period, which lowers the number of rebalancing and total transaction cost, using a relatively small band of 0.05 is not taking the full advantage of the purpose.

Next, we looked at what happens if the real-world volatility is different from the risk-neutral volatility. We first examined the case of *Delta* hedging. When the real-world volatility is 15% and the risk-neutral volatility is 20%, the mean PnL for both the time-based and move-based hedging increases from a negative value to a positive value. Yet the absolute value of the mean increases and the standard deviation of the PnL increases, which indicates a less accurate and less stable hedging. According to Figure 11 and 12, the PnL distribution shifts leftward and decreases in kurtosis when the real-world volatility increases from the risk-neutral volatility; and shifts rightward and decreases in kurtosis when the real-world volatility decreases from the risk-neutral volatility. This holds for both move-based and time-based hedging. As real-world volatility decreases relative to the risk neutral probability of 20 %, the rightward movement in PnL is mainly because of the put option is overpriced when we short it at time 0, thus we received an extra risk premium that is not properly hedged by the underlying asset until maturity.

We then examined the case of $Delta - Gamma$ hedging. The mean of PnL slightly increases and the standard deviation of PnL slightly decreases when the real-world volatility is 15%.

This is because a smaller real-world volatility results in smaller price movement in the asset and option, thus leads to smaller change in the positions every time when the rebalancing is required. Since the main reason for a $Delta-Gamma$ hedge to have negative PnL is the high transaction costs involved. A smaller change in the asset prices will also lead to more stable PnL. In contrast to $Delta$ hedging, the mean of PnL for $Delta-Gamma$ hedging does not change significantly when real-world volatility changes because we short an overpriced put option and purchased an overpriced call option. The premiums of the two options approximately cancelled out and we received little risk premium.

Finally, we looked at how the change in rebalancing-band in $Delta$ affects the hedging. Since the rebalancing-band has nothing to do with time-based hedging strategy, it will not affect the results of time-based hedging. We thus only focused on examining the move-based hedging strategy. As the bandwidth increases, it is more difficult for the change in asset price leads to a change in $Delta$ that reaches the upper or the lower boundaries of the band, therefore the number of rebalancing decreases with move-based hedging strategy. Firstly, for $Delta$ hedging, since the hedging position is only accurate when the change in $Delta$ lies within a relative small range, an increase in bandwidth leads to poorer hedging performance (as shown in Figure 19 to 22). The PnL distributions in Figure 23 to 26 is consistent with the findings as the distribution deviates from 0 when bandwidth increases. Therefore, for $Delta$ hedging, a smaller bandwidth is preferred because it allows the hedging portfolio to hedge the position more accurately while the transaction cost plays a minor role in affecting the PnLs. Secondly, for $Delta-Gamma$ hedging, $Gamma$ neutrality allows the hedger to keep the $Delta$ neutral position for a longer period without rebalancing. Therefore, the hedging performance is still good even when the bandwidth is large. In addition, a larger bandwidth leads to fewer change in hedging portfolio positions, and thus leads to less total transaction cost. Since the transaction cost associated with trading the call option that is used to hedge the position is large, the decrease in number of transactions significantly lowers the total transaction cost and increases the PnL. Yet the increase in bandwidth also significantly increases the standard deviation of the PnL, thus the stability of the hedging decreases and hedger might easily run into extreme PnL and receives poor hedging performance.

In summary, for $Delta - Gamma$ hedging, there is a trade-off between accurate hedging position and lower transaction cost. It is important to find a balance point where the hedging position is relatively accurate and stable while the transaction cost is not too high to result in a low PnL.

5 Conclusion

In this project, we investigated *Delta* and *Delta-Gamma* hedging in discrete time within the Black-Scholes model. For *Delta* hedging, we short sold the underlying asset of the put option. For $Delta - Gamma$ hedging, we longed a call option written on the same asset and short sold the asset. We implemented dynamic hedging under the time-based strategy by rebalancing the portfolio in each time step. For the move-based strategy, we rebalanced the hedging portfolio when the change in *Delta* touches the boundaries of the rebalancing-band.

For *Delta* hedging with a base band of 0.05, the move-based and time-based strategy exhibit similar hedging performance, while the time-based strategy has a slightly better accuracy in tracking the exact hedging position, which also yields higher transaction cost. Therefore, their final $PnLs$ are similar. For $Delta - Gamma$ hedging with a base band of 0.05, the move-based strategy outperforms the time-based one, while both of them result in fairly accurate replication of the hedging position. However, due to high transaction cost per unit associated with trading the call option, the move-based strategy with fewer rebalancing yields lower transaction cost, and thus a higher final PnL.

When the real-world volatility deviates from the risk-neutral volatility, the *Delta* hedging strategy becomes less effective. If the real-world volatility is smaller than the risk-neutral one, the PnL gets larger, and vice versa. The $Delta - Gamma$ hedging strategy remains effective, but smaller real-world volatility leads to more stable PnL.

Increasing the width of rebalancing band can significantly worsen the hedging accuracy for *Delta* hedging, since it's only effective when the change in *Delta* is within a relatively small range. It has less impact on the hedging accuracy for $Delta-Gamma$ hedging, since *Gamma* neutrality leads to a higher tolerance for change in *Delta*. The increase in bandwidth also reduces transaction cost as transaction becomes less frequent, improving the *PnLs*.