MMF1928 Pricing Theory Project 3

Stochastic Interest Rates and Swaptions

by:

Manyi Luo (1003799419)

Xinyue Zhang (1003237712)

Yitian Zhang (1003838519)

December 2021

Abstract

In this report, we investigated various aspects of stochastic interest rates and interest rate products. Based on a two-factor Vasicek model, we obtained the term structure of interest rates as well as a analytical formula for T-maturity bond price. Then, the T-maturity bond yield curve is generated through Monte Carlo simulation of risk-neutral interest rate paths, and is compared with solutions calculated analytically. Afterwards, we study the impact of various parameters on the term structure using the analytical formula. Also, we compare the bond price under both risk-neutral, forward-neutral and using analytical formula for a collection of trikes. Lastly, we investigate the Black implied volatility of an interest rate swaption based on LSM model and Monte Carlo simulations.

Contents

1 Introduction

In this project, we aim to investigate the stochastic interest rates modelled by a two-factor Vasicek model. The bond price is assumed to take an exponential affine form and the term structure of interest rates can be obtained by computing this affine form of equation using the Multi-Dimensional Feynman-Kac theorem. Meanwhile, a Euler discretization method is used to simulation risk-neutral interest rate paths and numerically approximate the bond yield. The change in term structure induced by change in model parameters will also be studied. In addition, we will investigate the pricing of a bond option. A bond option is a contract that gives the holder the right but not the obligation to buy a bond on a date with a pre-specified price. We will derive the analytical expression for the bond option price by using a bond of maturity T1 as a numeraire asset, as well as use Monte Carlo simulation to numerically compute the price. Finally, we will investigate the Black implied volatility of an interest rate swap (IRS) as a function of strike using numerical simulation.

The rest of the report is organized as follows. chapter 2 will define the theoretical frame needed for this project. Chapter 3 will describe the implementation of the numerical simulations, discuss the results and findings. Chapter 4 will summarize the conclude this report.

2 Theoretical Framework

2.1 Fundamental Settings for Stochastic Interest Rate

Given the fact that the dynamic hedging argument claims were written on the source of uncertainty, X_t, the interest rate itself is considered as deterministic. However, in the real world settings, the source of uncertainty can also include a stochastic interest rate process, r_t, which refers to the interest rate derivative.

Therefore, within the stochastic interest rate model that satisfies the continuous time setting, the bank account is described as $B = (B_t)_{t\geq 0}$ with $B_0 = 1$. After going forward with a short time step later, the amount in the bank account, B_t, grows at some interest rate, and its increment can be described as the following:

$$dB_t = r_t B_t dt$$
$$B_{t+\Delta t} = B_t(1 + r_t \Delta t)$$
$$B_{t+\Delta t} - B_t = r_t \Delta t B_t$$

whereas $B_t = e^{\int_0^t r_u du}$ and $r = (r_t)_{t\geq 0}$.

This describes a continuous time analog of the accumulation of the bank account, and the short rate of interest process satisfies the following SDE:

$$dr_t = \mu^r(t, r_t)dt + \sigma^r(t, r_t)dW_t^{\mathbb{P}}$$

Here, $W_t^{\mathbb{P}}$ is a standard Brownian motion under the probability measure $\mathbb{P}$. μ_r and σ_r are the drift and volatility functions of the short rate process of interest.

Furthermore, the model assumes that the price process of a claim written on r with payoff function $F(r)$ with $f = (f_t)_{t\geq 0}$, which is Markovian in r, as there exists a function f such that $f_t = f(t, r_t)$. As a result, by the dynamic hedging argument, the function f satisfies the following PDE:

$$\begin{cases} \partial_t f(t,r) + (\mu^r(t,r) - \lambda(t,r)\sigma^r(t,r))\partial_r(t,r) + \dfrac{1}{2}(\sigma(t,r))^2 \partial_{rr} f(t,r) = rf(t,r) \\ f(T,r) = F(r) \end{cases}$$

This admits the following stochastic representation with $W^{\mathbb{Q}}$ as a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$:

$$f(t,r) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} F(r)]$$
$$dr_t = (\mu^r(t,r_t) - \lambda(t,r_t)\sigma^r(t,r_t))dt + \sigma^r(t,r_t)dW_t^{\mathbb{Q}}$$

As a result, given a specific interest rate model with specific μ_r and σ_r, the price of the claim can be obtained by solving the PDE or computing the expectation, and the drift adjustment, λ is selected to maintain the functional form of the interest rate model.

2.2 Interest Rate Model

2.2.1 One-Factor Vasicek model

In the One-Factor Vasicek model, the short rate and its reversion level, r_t and θ_t, are modeled as the Ornstein-Uhlenbeck process:

$$dr_t = \alpha(\theta - r_t)dt + \sigma dW_t$$

This implies that the short rate process of interest is mean-reverting with mean reversion rate and level given by α and θ respectively. The solution to the Vasicek model and the distribution of r_T are given by the following:

$$r_T = r_t e^{-\alpha(T-t)} + \theta(1 - e^{-\alpha(T-t)}) + \sigma \int_t^T e^{-\alpha(T-t)} dW_s$$
$$r_T \sim N(r_t e^{-\alpha(T-t)} + \theta(1 - e^{-\alpha(T-t)}), \frac{\sigma^2}{2\alpha}(1 - e^{-\alpha(T-t)}))$$

Therefore, by integrating the SDE can directly obtain an expression for the integral $\int_t^T r_s ds$ and find its distribution:

$$\int_t^T r_s ds = \theta(T-t) - \frac{1 - e^{-\alpha(T-t)}}{\alpha}(\theta - r_t) + \frac{\theta}{\alpha}\int_t^T (1 - e^{-\alpha(T-t)})dW_s$$
$$\int_t^T r_s ds \sim N(\theta(T-t) - \frac{1 - e^{-\alpha(T-t)}}{\alpha}(\theta - r_t), \frac{\theta^2}{\alpha^2}\int_t^T (1 - e^{-\alpha(T-t)})^2 ds)$$

Note that r_t and $\int_t^T r_s ds$ are jointly normally distributed. Therefore, in the case of the

Vasicek model, the $\mathbb{Q}$-dynamics of the interest rate process are given by:

$$dr_t = (\alpha^{\mathbb{P}}(\theta^{\mathbb{P}} - r_t) - \lambda_t\sigma)dt + \sigma dW_t^{\mathbb{Q}}$$

Whereas we can choose the factor λ such that the functional form of the process is maintained, which is the Q-dynamics of the interest rate process still follow an Ornstein-Uhlenbeck process. For example, when $\lambda_t = a$:

$$\begin{aligned} dr_t &= \alpha^{\mathbb{Q}}(\theta_t^{\mathbb{Q}} - r_t)dt + \sigma dW_t^{\mathbb{Q}} \\ \alpha^{\mathbb{Q}} &= \alpha^{\mathbb{P}} \\ \theta^{\mathbb{Q}} &= \theta^{\mathbb{P}} - \frac{a\sigma}{\alpha^{\mathbb{P}}} \end{aligned}$$

When $\lambda_t = a + br_t$:

$$\begin{aligned} dr_t &= \alpha^{\mathbb{Q}}(\theta_t^{\mathbb{Q}} - r_t)dt + \sigma dW_t^{\mathbb{Q}} \\ \alpha^{\mathbb{Q}} &= \alpha^{\mathbb{P}} + b\sigma \\ \theta^{\mathbb{Q}} &= \theta^{\mathbb{P}} - \frac{a\sigma}{\alpha^{\mathbb{P}} + b\sigma} \end{aligned}$$

As a result, the parameters, $\alpha^{\mathbb{P}}$, $\theta^{\mathbb{P}}$ and σ are the real-world dynamics of the asset price and variance processes, which can be determined through simulation or historical data. Whereas the parameters, a and b are degrees of freedom that are derived through calibration between the model and the observed market price, which also helps to minimize the difference between each two. Finally, $\alpha^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ represents the market-implied mean reversion level and rate, and the derivative valuation is done under the $\mathbb{Q}$ - dynamics of r_t.

2.2.2 Two-Factor Vasicek model

In the Two-Factor Vasicek model, the short rate and its reversion level, r_t and θ_t, are modeled as the Ornstein-Uhlenbeck process, which is similar to the One Factor Vasicek model by adding one more Brownian motion θ_t:

$$dr_t = \alpha(\theta_t - r_t)dt + \sigma dW_t^1$$
$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^2$$

The Two=Factor Vasicek model implies that the short rate process of interest and it's reversion level are mean-reverting with mean reversion rate and level given by β and ϕ respectively. These two Brownian motions are potentially correlated and interact with one another, which are captured by the coefficient α: the greater the coefficient is, the closer of these two stochastic processes.

In the later section of the bond pricing, we will further elaborate on its integration result using multi-dimensional Feynman-Kac Theorem, which provides a more convenient approach comparing to directly solving the integral.

2.3 Multi-Dimensional Feynman-Kac Theorem

The Feynman-Kac theorem establishes the link between partial differential equations and stochastic processes. First, it provides a solution to a certain class of PDEs. Second, it gives a way of computing certain expectations by solving an associated PDE.

Since the interest rate model contains two stochastic processes, the multi-dimensional Feynman-Kac Theorem is implemented, and it suggests the solution to the PDE as the following.

If $f(t, x, y)$ satisfies the relationship, whereas X_t and Y_t are two sources of uncertainty, we have:

$$\begin{cases} (\partial_t + L)f(t, x, y) = c(t, x, y)f(t, x, y) \\ f(T, x, y) = \phi(x, y) \end{cases}$$

Here:

$L = \mu^x(t, x, y)\partial_x + \mu^y(t, x, y)\partial_y + \frac{1}{2}(\sigma^x(t, x, y))^2\partial_{xx} + \frac{1}{2}(\sigma^y(t, x, y))^2\partial_{yy} + \rho\sigma^x(t, x, y)\sigma^y(t, x, y)\partial_{xy}$

Then, the following results can be obtained:

$$\begin{aligned}
f(t,x,y) &= \mathbb{E}^{\mathbb{P}^*}[e^{-\int_t^T c(u,X_u,Y_u)du}\phi(X_T,Y_T)|X_t=x,Y_t=y] \\
dX_t &= \mu_t^X dt + \sigma_t^X dW_t^{1,\mathbb{P}^*} \\
dY_t &= \mu_t^Y dt + \sigma_t^Y dW_t^{2,\mathbb{P}^*} \\
d[W^{1,\cdot},W^{2,\cdot}]_t &= \rho
\end{aligned}$$

Therefore, for $P_t(T) = f(t, r_t, \theta_t)$, we have:

$$\begin{cases} (\partial_t + L)f(t,r,\theta) = rP(t,r,\theta) \\ P(T,r,\theta) = 1 \end{cases}$$

Here:
$L = \alpha(\theta - r)\partial_r + \beta(\phi - \theta)\partial_\theta + \frac{1}{2}\sigma^2\partial_{rr} + \frac{1}{2}\eta^2\partial_{\theta\theta} + \rho\sigma\eta d_{\theta r}$

The advantage of formulating the equation in this manner is that it leads to an affine PDE in r and θ, which is the PDE having coefficients that are at most linear in the state variables. Therefore, it is typically easier to work with and to arrive at closed-form solutions.

We thus assume the price of the bond takes an exponential affine form $P_t(T) = e^{A_t^T + B_t^T + C_t^T}$. The expression for A, B and C will be obtained in the next section. Note that $\rho = 0$ in this case, since we assume the stochastic terms in r_t and θ_t are independent under the risk-neutral measure.

2.4 Pricing of the Bond

Let the T-maturity bond price process be denoted as $P(T) = (P_t(T))_t$, whereas $t \in [0,T]$. Starting with the simplest case for a T - maturity bond, which pays 1 when it matures, $P_T(T) = 1$. We can compute the price of such bonds with the fundamental theorem of finance under the probability measure $\mathbb{Q}$ using the money market account B_t as a numeraire:

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]$$

Assuming the two-factor Vasicek model for interest rates and given the fact that the model is affine, i.e. the coefficients of the generator are at most linear, there exist deterministic functions, $A_t(T), B_t(T), C_t(T)$ such that the following expression exists:

$$P_t(T) = e^{A_t(T) - B_t(T) r_t - C_t(T)\theta_t}$$

We will now determine the function forms of A, B and C by using the PDE approach, using the solution form that the PDE is affine with a terminal condition.

By Feynman Kac Theorem, we have:

$$\begin{cases} \partial_t f(t,r,\theta) + [\alpha(\theta - r)\partial_r + \frac{1}{2}\sigma^2 \partial_{rr} + \beta(\phi - \theta)\partial_\theta + \frac{1}{2}\eta^2 \partial_{\theta\theta}] f(t,r,\theta) = r f(t,r,\theta) \quad (\star) \\ f(T,r,\theta) = 1 \end{cases}$$

where $f(t,r,\theta) = e^{A_t - B_t r_t - C_t \theta_t}$.

Then we have:

$$\begin{cases} \partial_t f(t,r,\theta) = (\frac{\partial A_t}{\partial t} - r_t \frac{\partial B_t}{\partial t} - \theta_t \frac{\partial C_t}{\partial t}) e^{A_t - B_t r_t - C_t \theta_t} = (\dot{A}_t - r_t \dot{B}_t - \theta_t \dot{C}_t) f(t,r,\theta) \\ \partial_r f(t,r,\theta) = \frac{\partial (A_t - B_t r_t - C_t \theta_t)}{\partial r_t} e^{A_t - B_t r_t - C_t \theta_t} = -B_t f(t,r,\theta) \\ \partial_\theta f(t,r,\theta) = \frac{\partial (A_t - B_t r_t - C_t \theta_t)}{\partial \theta_t} e^{A_t - B_t r_t - C_t \theta_t} = -C_t f(t,r,\theta) \\ \partial_{rr} f(t,r,\theta) = \partial_r(\partial_r f(t,r,\theta)) = -B_t \partial_r f(t,r,\theta) = -B_t(-B_t f(t,r,\theta)) = B_t^2 f(t,r,\theta) \\ \partial_{\theta\theta} f(t,r,\theta) = \partial_\theta(\partial_\theta f(t,r,\theta)) = -C_t \partial_\theta f(t,r,\theta) = -C_t(-C_t f(t,r,\theta)) = C_t^2 f(t,r,\theta) \end{cases}$$

where $\dot{A}_t = \frac{\partial A_t}{\partial t}$, $\dot{B}_t = \frac{\partial B_t}{\partial t}$, and $\dot{C}_t = \frac{\partial C_t}{\partial t}$.

Plugging the above equations into $(\star)$ gives us:

$$\begin{aligned}
&(\dot{A}_t - r_t\dot{B}_t - \theta_t\dot{C}_t)f - \alpha(\theta - r_t)B_t f + \frac{1}{2}\sigma^2 B_t^2 f - \beta(\phi - \theta_t)C_t f + \frac{1}{2}\eta^2 C_t^2 f = r_t f \\
&\Rightarrow (\dot{A}_t - r_t\dot{B}_t - \theta_t\dot{C}_t) - \alpha(\theta - r_t)B_t + \frac{1}{2}\sigma^2 B_t^2 - \beta(\phi - \theta_t)C_t + \frac{1}{2}\eta^2 C_t^2 = r_t \\
&\Rightarrow \dot{A}_t - \theta_t\dot{C}_t - \alpha\theta_t B_t + \frac{1}{2}\sigma^2 B_t^2 - \beta\phi C_t + \beta\theta_t C_t + \frac{1}{2}\eta^2 C_t^2 = (\dot{B}_t - \alpha B_t + 1)r_t \\
&\Rightarrow \dot{A}_t + \frac{1}{2}\sigma^2 B_t^2 - \beta\phi C_t + \frac{1}{2}\eta^2 C_t^2 + (-\dot{C}_t - \alpha B_t + \beta c_t)\theta_t = (\dot{B}_t - \alpha B_t + 1)r_t
\end{aligned}$$

Note that the above equation holds $\forall t$, $\forall \theta$ and $\forall r$, thus, we have:

$$\begin{cases}
\dot{A}_t + \frac{1}{2}\sigma^2 B_t^2 - \beta\phi C_t + \frac{1}{2}\eta^2 C_t^2 + (-\dot{C}_t - \alpha B_t + \beta C_t)\theta_t = 0 & (1) \\
\dot{C}_t + \alpha B_t - \beta C_t = 0 & (2) \\
\dot{B}_t - \alpha B_t + 1 = 0 & (3)
\end{cases}$$

Note that we have $f(T, r, \theta) = e^{A_T - B_T r - C_T\theta} = 1$ for $\forall r, \theta$, i.e. $A_T - B_T r - C_T\theta = 0$ for $\forall r, \theta$. Therefore, we have $A_T = B_T = C_T = 0$.

By equation (3), we have:

$$\begin{aligned}
&\dot{B}_t = \alpha B_t - 1 \\
&\Rightarrow \frac{\dot{B}_t}{1 - \alpha B_t} = -1 \quad (\text{where } \dot{B}_t = \frac{\partial B_t}{\partial t}) \\
&\Rightarrow \frac{\frac{\partial B_t}{\partial t}}{1 - \alpha B_t} = -1 \\
&\Rightarrow -\frac{1}{\alpha}(\frac{\partial(-\alpha B_t)}{\partial t})\frac{1}{1 - \alpha B_t} = -1 \\
&\Rightarrow -\frac{1}{\alpha}(\frac{\partial(1 - \alpha B_t)}{\partial t})\frac{\partial}{\partial(1 - \alpha B_t)}log(1 - \alpha B_t) = -1 \\
&\Rightarrow -\frac{1}{\alpha}\frac{\partial}{\partial t}log(1 - \alpha B_t) = -1 \\
&\Rightarrow \frac{\partial}{\partial t}log(1 - \alpha B_t) = \alpha
\end{aligned}$$

$$\Rightarrow \int_t^T \frac{\partial}{\partial u} log(1-\alpha B_u)du = \int_t^T \alpha du$$
$$\Rightarrow log(1-\alpha B_u)\Big|_t^T = \alpha u\Big|_t^T$$
$$\Rightarrow log(\frac{1-\alpha B_T}{1-\alpha B_t}) = \alpha(T-t)$$
$$\Rightarrow 1-\alpha B_t = e^{-\alpha(T-t)}$$
$$\Rightarrow B_t = \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

As a result, using the derived formula of B_t, we can plug it into (2) to obtain the expression of C_t:

$$\begin{cases} \dot{C}_t + \alpha B_t - \beta C_t = 0 & (2) \\ B_t = \dfrac{1-e^{-\alpha(T-t)}}{\alpha} & (4) \end{cases}$$

Inspired by recursion, we take the first order derivative of (2) regarding the variable t, after obtaining the expression that only contains the single variable C_t:

$$\dot{C}_t + 1 - e^{-\alpha(T-t)} - \beta C_t = 0 \Rightarrow \ddot{C}_t - \alpha e^{-\alpha(T-t)} - \beta \dot{C}_t = 0$$

$$\begin{cases} \dot{C}_t + 1 - \beta C_t = e^{-\alpha(T-t)} & (5) \\ \ddot{C}_t - \beta \dot{C}_t = \alpha e^{-\alpha(T-t)} & (6) \end{cases}$$

Through plugging (5) into (6) by replacing the term $e^{-\alpha(T-t)}$, we can get the relationship as following:

$$\Rightarrow \ddot{C}_t - \beta \dot{C}_t = \alpha(\dot{C}_t + 1 - \beta C_t)$$
$$\Rightarrow \ddot{C}_t - (\alpha+\beta)\dot{C}_t - \alpha - \alpha\beta C_t = 0 \qquad (7)$$

The form coincides with the 2^{nd} order ODE, which can be re-written into the form below, and C_1, C_2 are non-zero terms:

$$C_t = C_1 e^{\beta t} + C_2 e^{\alpha t} + \frac{1}{\beta} \Rightarrow \dot{C}_t = C_1 \beta e^{\beta t} + C_2 \alpha e^{\alpha t} \tag{8}$$

Therefore, by plugging (8) derived just now into (1), we can obtain the expression of C_2:

$$\begin{aligned}
&\Rightarrow C_1 \beta e^{\beta t} + C_2 \alpha e^{\alpha t} + 1 - \beta(C_1 e^{\beta t} + C_2 e^{\alpha t} + \frac{1}{\beta}) = e^{-\alpha(T-t)} \\
&\Rightarrow C_2 \alpha e^{\alpha t} - C_2 \beta e^{\alpha t} = e^{-\alpha(T-t)} \\
&\Rightarrow C_2(\alpha - \beta) e^{\alpha t} = e^{-\alpha(T-t)} \\
&\Rightarrow C_2(\alpha - \beta) = e^{-\alpha T} \\
&\Rightarrow C_2 = \frac{e^{-\alpha T}}{\alpha - \beta}
\end{aligned}$$

As mentioned earlier, we have $f(T, r, \theta) = e^{A_T - B_T r - C_T \theta} = 1$ for $\forall r, \theta$ from the initial condition, i.e. at $t = T$, $A_T - B_T r - C_T \theta = 0$ for $\forall r, \theta$. Therefore, we have $A_T = B_T = C_T = 0$.

Hence, we can tell that $C_1 e^{\beta t} + C_2 e^{\alpha t} + \frac{1}{\beta} = 0$. Therefore, we can utilize the known C_2 to solve for the expression of C_1:

$$\begin{aligned}
&\Rightarrow C_1 = \frac{-\frac{1}{\beta} - C_2 e^{\alpha T}}{e^{\beta T}} \\
&\Rightarrow C_1 = \frac{-\frac{1}{\beta} - (\frac{e^{-\alpha T}}{\alpha - \beta}) e^{\alpha T}}{e^{\beta T}} \\
&\Rightarrow = \frac{-\frac{1}{\beta} - \frac{1}{\alpha - \beta}}{e^{\beta T}} \\
&\Rightarrow = -\frac{\alpha}{\beta(\alpha - \beta) e^{\beta T}}
\end{aligned}$$

As a result, $C_t = C_1 e^{\beta t} + C_2 e^{\alpha t} + \frac{1}{\beta}$, whereas $C_1 = -\frac{\alpha}{\beta(\alpha-\beta)e^{\beta T}}$ and $C_2 = \frac{e^{-\alpha T}}{\alpha - \beta}$. This can be further transformed as the following:

$$
\begin{aligned}
C_t &= -\frac{\alpha}{\beta(\alpha-\beta)e^{\beta T}}e^{\beta t} + \frac{e^{-\alpha T}}{\alpha-\beta}e^{\alpha t} + \frac{1}{\beta} \\
&= \frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{\alpha e^{-\beta(T-t)}}{\beta(\alpha-\beta)} + \frac{1}{\beta} \\
&= \frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta}
\end{aligned}
$$

Therefore, we are able to leverage the solutions for B_t and C_t to obtain the expression of A_t, through bringing them into (1). Since we know that $\dot{C}_t + \alpha B_t - \beta C_t = 0$ according to (3), we can simplify (1) as the following:

$$
\dot{A}_t + \frac{1}{2}\sigma^2 B_t^2 - \beta\phi C_t + \frac{1}{2}\eta^2 C_t^2 + (-\dot{C}_t - \alpha B_t + \beta C_t)\theta_t = 0 \Rightarrow \dot{A}_t + \frac{1}{2}\sigma^2 B_t^2 - \beta\phi C_t + \frac{1}{2}\eta^2 C_t^2 = 0
$$

$$
\begin{aligned}
\Rightarrow \dot{A}_t + & \frac{1}{2}\sigma^2 \left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right)^2 \\
& -\beta\phi\left(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta}\right) \\
& +\frac{1}{2}\eta^2\left(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta}\right)^2 \\
& = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \dot{A}_t = & -\frac{1}{2}\sigma^2\left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right)^2 && (9) \\
& +\beta\phi\left(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta}\right) && (10) \\
& -\frac{1}{2}\eta^2\left(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta}\right)^2 && (11)
\end{aligned}
$$

As a result, we take the integrals of expressions (9), (10) and (11):

$$
\begin{aligned}
\int_t^T \dot{A}_s ds = A_s\Big|_t^T = A_T - A_t = &-\int_t^T \frac{1}{2}\sigma^2(\frac{1-e^{-\alpha(T-s)}}{\alpha})^2 ds \\
&+\int_t^T \beta\phi(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta})ds \\
&-\int_t^T \frac{1}{2}\eta^2(\frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta})^2 ds \\
\Rightarrow A_t = &\int_t^T \frac{\sigma^2\left(1-\mathrm{e}^{-\alpha(T-s)}\right)^2}{2\alpha^2} ds \quad (A_T = 0 \text{ as mentioned previously}) \\
&-\int_t^T \beta\phi\left(\frac{\mathrm{e}^{-\alpha(T-s)}-\mathrm{e}^{-\beta(T-s)}}{\alpha-\beta} + \frac{1-\mathrm{e}^{-\beta(T-s)}}{\beta}\right) ds \\
&+\int_t^T \frac{\eta^2\left(\frac{\mathrm{e}^{-\alpha(T-s)}-\mathrm{e}^{-\beta(T-s)}}{\alpha-\beta} + \frac{1-\mathrm{e}^{-\beta(T-s)}}{\beta}\right)^2}{2} ds
\end{aligned}
$$

According to the proof in lecture using Ito's isometry, we can rewrite the integrals for (9) and (11) as following:

$$
\begin{aligned}
\int_t^T \frac{\sigma^2\left(1-\mathrm{e}^{-\alpha(T-s)}\right)^2}{2\alpha^2} ds \Rightarrow &\frac{1}{2}\sigma^2\int_t^T B_s^2 ds \\
\Rightarrow &\frac{1}{2}\frac{\sigma^2}{\alpha^2}\int_t^T (1-e^{-\alpha(T-s)})^2 ds \\
\Rightarrow &\Omega_{t,T}^1
\end{aligned}
$$

$$
\begin{aligned}
\int_t^T \frac{\eta^2\left(\frac{\mathrm{e}^{-\alpha(T-s)}-\mathrm{e}^{-\beta(T-s)}}{\alpha-\beta} + \frac{1-\mathrm{e}^{-\beta(T-s)}}{\beta}\right)^2}{2} ds \Rightarrow &\frac{1}{2}\eta^2\int_t^T C_s^2 ds \\
\Rightarrow &\frac{1}{2}\frac{\eta^2}{\beta^2}\int_t^T (\frac{\beta e^{-\alpha(T-s)}}{\alpha-\beta} - \frac{\beta e^{-\beta(T-s)}}{\alpha-\beta} - e^{-\beta(T-s)} + 1)^2 ds \\
\Rightarrow &\Omega_{t,T}^2
\end{aligned}
$$

Note that the expressions for $\Omega_{t,T}^1$ and $\Omega_{t,T}^2$ can be shown explicitly through the utilization of integral calculator:

$$
\begin{aligned}
\Omega^1_{t,T} &= \frac{1}{2}\frac{\sigma^2}{\alpha^2}\int_t^T (1-e^{-\alpha(T-s)})^2 ds \\
&= \frac{\sigma^2\left(\frac{2\alpha T-3}{2\alpha}-\frac{\mathrm{e}^{-2\alpha T}\left(-4\mathrm{e}^{\alpha T+\alpha t}+2\alpha t\mathrm{e}^{2\alpha T}+\mathrm{e}^{2\alpha t}\right)}{2\alpha}\right)}{2\alpha^2} \\
&= \frac{\sigma^2\mathrm{e}^{-2\alpha T}\left(\mathrm{e}^{\alpha T}\left(\left(2\alpha\left(T-t\right)-3\right)\mathrm{e}^{\alpha T}+4\mathrm{e}^{\alpha t}\right)-\mathrm{e}^{2\alpha t}\right)}{4\alpha^3} \\
\Omega^2_{t,T} &= \frac{1}{2}\frac{\eta^2}{\beta^2}\int_t^T (\frac{\beta e^{-\alpha(T-s)}}{\alpha-\beta}-\frac{\beta e^{-\beta(T-s)}}{\alpha-\beta}-e^{-\beta(T-s)}+1)^2 ds \\
&= \frac{\eta^2\left(\frac{\alpha\beta T-\beta-\alpha}{\alpha\beta}-\frac{\left(\alpha^2\mathrm{e}^{\alpha T+\beta t}+\left(\left(\alpha\beta^2-\alpha^2\beta\right)t\mathrm{e}^{\alpha T}-\beta^2\mathrm{e}^{\alpha t}\right)\mathrm{e}^{\beta T}\right)\mathrm{e}^{-\beta T-\alpha T}}{\alpha\beta^2-\alpha^2\beta}\right)}{2\beta^2} \\
&= \frac{\eta^2\mathrm{e}^{-(\beta+\alpha)T}\left(\left(\left(\beta-\alpha\right)\left(\beta\left(\alpha\left(T-t\right)-1\right)-\alpha\right)\mathrm{e}^{\alpha T}+\beta^2\mathrm{e}^{\alpha t}\right)\mathrm{e}^{\beta T}-\alpha^2\mathrm{e}^{\alpha T+\beta t}\right)}{2\alpha\beta^3\left(\beta-\alpha\right)}
\end{aligned}
$$

Therefore, the expression for A_t can be written as following:

$$
A_t = \phi(\frac{\alpha}{(\alpha-\beta)\beta}(1-e^{-\beta(T-t)})-\frac{\beta}{(\alpha-\beta)\alpha}(1-e^{-\alpha(T-t)})-(T-t))+\frac{1}{2}(\Omega^1_{t,T}+\Omega^2_{t,T})
$$

In conclusion, assuming the Vasicek model for interest rates, the price of the T-maturity bond is given by:

$$
P_t(T) = e^{A_t(T)-B_t(T)r_t-C_t(T)\theta_t}
$$

where:

$$
\begin{aligned}
A_t &= \phi(\frac{\alpha}{(\alpha-\beta)\beta}(1-e^{-\beta(T-t)}) - \frac{\beta}{(\alpha-\beta)\alpha}(1-e^{-\alpha(T-t)}) - (T-t)) + \frac{1}{2}(\Omega^1_{t,T}+\Omega^2_{t,T}) \\
B_t &= \frac{1-e^{-\alpha(T-t)}}{\alpha} \\
C_t &= \frac{e^{-\alpha(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\alpha-\beta} - \frac{e^{-\beta(T-t)}}{\beta} + \frac{1}{\beta} \\
\Omega^1_{t,T} &= \frac{1}{2}\sigma^2\int_t^T B_s^2 ds = \frac{\sigma^2 \mathrm{e}^{-2\alpha T}\left(\mathrm{e}^{\alpha T}\left((2\alpha\,(T-t)-3)\,\mathrm{e}^{\alpha T}+4\mathrm{e}^{\alpha t}\right)-\mathrm{e}^{2\alpha t}\right)}{4\alpha^3} \\
\Omega^2_{t,T} &= \frac{1}{2}\eta^2\int_t^T C_s^2 ds = \frac{\eta^2 \mathrm{e}^{-(\beta+\alpha)T}\left(\left((\beta-\alpha)\,(\beta\,(\alpha\,(T-t)-1)-\alpha)\,\mathrm{e}^{\alpha T}+\beta^2\mathrm{e}^{\alpha t}\right)\mathrm{e}^{\beta T}-\alpha^2\mathrm{e}^{\alpha T+\beta t}\right)}{2\alpha\beta^3\,(\beta-\alpha)}
\end{aligned}
$$

Since bonds are traded assets, they can be used as numeraires, in some cases this leads to considerable simplifications. Using the same set of parameters as the Vasicek model, the equivalent martingale measure induced by a T-maturity bond is referred to as the T-forward neutral measure, denoted by Q_T. This links to the relationship between Q to Q_T, and we define the Radon-Nikodym derivative in the usual manner by using the T-maturity bond as a numeraire:

$$
(\frac{d\mathbb{Q}^T}{d\mathbb{Q}})_t = \frac{P_t(T)/P_0(T)}{M_t/M_0}
$$

2.5 Term Structure of Interest Rate

To obtain the term structure of interest rates with the base parameters as suggested in the previous part, the process begins by computing the approximation for each simulating path and calculate the expectation of paths under the probability measure $\mathbb{P}$:

$$
\frac{1}{N}\sum_{n=1}^{N} e^{-\int_t^T r_s ds} = \mathbb{E}_t^{\mathbb{P}}[e^{-\int_t^T r_s ds}]
$$

Therefore, the evolution of the interest rate model under the physical measure $\mathbb{P}$ can be transformed into the evaluation of bond price under probability measure $\mathbb{Q}$:

$$\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}] = \mathbb{E}_t^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}} e^{-\int_t^T r_s ds}]$$

Whereas the process $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is considered as $e^{-\frac{1}{2}\int_t^T \lambda_s^2 ds + \int_t^T \lambda_s dW_s}$, and the process of bond price evaluation can be obtained through getting an estimate and taking the average of different scenarios.

With respect to simulations, the expression of bond price can also be rewritten into the form of yield, which is a discount factor at the current time until the bond maturity date. As a result, the bond yield represents the equivalent constant rate of discount that matches the prices in the market given a whole range of bond prices of different maturities:

$$P_t(T) = e^{-y_t(T)(T-t)}$$

Moving forward with the yield at a fixed time, the distribution of the bond price and the yield curve, which is the real world evolution of the term structure of interest rates, can be potentially realized in later sections, while simulating against the rate of interest.

Consequently, we can use the result from previous section, $P_t(T) = e^{A_t(T) - B_t(T)r_t - C_t(T)\theta_t}$ to derive the explicit formula as below:

$$y_t(T) = \frac{B_t(T)r_t + C_t(T)\theta_t - A_t(T)}{T-t}$$

As a result, the **term structure of interest rate** can be shown when $t = 0$ with varying T:

$$y_0(T) = \frac{B_0(T)r_0 + C_0(T)\theta_0 - A_0(T)}{T}$$

2.6 Spot Rate and Forward Rate

As mentioned earlier, a bond yield and a spot rate are associated with every bond price for a given term, which are defined as follows:

$$P_t(T) = e^{-y_t(T)(T-t)} = [1 + l_t^T(T-t)]^{-1}$$

Whereas $y_t(T)$ is the bond yield at time t with maturity T and l_t^T is the spot rate associated with this bond. Since these are functions of time and maturity, we can write spot rate in terms of bond price as follows through rearranging the terms:

$$l_t^T = \frac{1}{T-t}[\frac{1}{P_t(T)} - 1]$$

Also, forward rate relates spot rate of the bond price with two different maturities, T_1 and T_2:

$$(1 + l_t^{T_2}(T_2 - t)) = (1 + l_t^{T_1}(T_1 - t))(1 + l_t^{T_1,T_2}(T_2 - T_1))$$

Since $l_t^{T_1,T_2}$ is the forward rate for the period T_1 to T_2. We can express the forward rate in terms of bond prices as well:

$$\begin{aligned}(1 + l_t^{T_2}(T_2 - t))^{-1} =& (1 + l_t^{T_1}(T_1 - t))^{-1}(1 + l_t^{T_1,T_2}(T_2 - T_1))^{-1} \\ l_t^{T_1,T_2} =& \frac{1}{T_1 - t}[\frac{P_t(T_1)}{P_t(T_2)} - 1]\end{aligned}$$

Since the forward rate involves a relative price process with a T_2 - maturity bond in the denominator, the forward rate, $l_t^{T_1,T_2}$, is a martingale under the T_2 - forward neutral measure, $\mathbb{Q}^{T_2}$. This acts as the basis of the log-normal forward rate model, in which forward rate is modelled as the following:

$$\frac{dl_t^{T_1,T_2}}{l_t^{T_1,T_2}} = \sigma_t^{T_1,T_2} dW_t^{\mathbb{Q}^{T_2}}$$

When we have a partition for time, we let T_k denote the various time points and let $\tau_k =$

$T_K - T_{K-1}$ and denote $l_t^{T_{K-1},T_K} = l_t^{(k)}$.

2.7 Pricing of the Bond Option

As a continuation of the previous theoretical framework, this part is going to investigate the pricing of the bond option under the context of the two-factor interest rate model.

In this part, we would like to further use a bond of maturity T_1 as a numeraire asset and it is possible to determine an analytic expression for the price of a bond option paying $(P_{T_1}(T_2) - K)_+$ at T_1 where $K = P_0(T_2)/P_0(T_1)$.

The construction of the call (or put) bond option is similar with respect to usual call (or put) option. For example, at the option maturity time, T_1, the call bond option pays $(P_{T_1}(T_2) - K)_+$, with the option maturity time smaller than the bond maturity time, T_2. Similarly, at the option maturity time, T_1, the put bond option pays $(K - P_{T_1}(T_2))_+$. Therefore, this derivative gives the holder of the option to purchase a T_2-maturity bond at time T_1 for the strike price K.

Since we are comparing the price of the bond at time T_1 but the bond matures at time T_2, we can compute the price of this option as the expected discounted payoff by the following value function:

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[e^{\int_t^T r_s ds}(P_{T_1}(T_2) - K)_+]$$

Using a bond of maturity T_1 as a numeraire asset, by the fundamental theory of asset pricing, we have:

$$\frac{V(t)}{P_t(T_1)} = \mathbb{E}_t^{\mathbb{Q}_{T_1}}[\frac{(P_{T_1}(T_2) - K)^+}{P_{T_1}(T_1)}] = \mathbb{E}_t^{\mathbb{Q}_{T_1}}[(P_{T_1}(T_2) - K)^+] \quad \text{(Note that } P_{T_1}(T_1) = 1)$$

Now we introduce $X = (X_t)_{t\in[0,T]}$ such that $X_t = \frac{P_t(T_2)}{P_t(T_1)}$. Then, we have $X_{T_1} = \frac{P_{T_1}(T_2)}{P_{T_1}(T_1)} =$

$P_{T_1}(T_2)$. Also, note that X is a $\mathbb{Q}_{T_1}$-martingale, since X is a relative price process with the numeraire in the denominator. It's clear that the X_{T_1} is equal to one part of the term in the max function at time T_1 in this process, and the other term is unaffected since $P_{T_1}(T_1) = 1$ and $\frac{K}{P_{T_1}(T_1)} = K$. Therefore, we have the equation as the following and the next step is to find the dynamics of X under $\mathbb{Q}_{T_1}$:

$$V_t = P_t(T_1) \cdot \mathbb{E}_t^{\mathbb{Q}_{T_1}}[(X_{T_1} - K)^+]$$

Based on our previous calculation and analysis and the Vasicek model for interest rate, we have the T_1 and T_2 maturity bond price as:

$$P_t(T_2) = e^{A_t(T_2) - B_t(T_2)r_t - C_t(T_2)\theta_t}$$
$$P_t(T_1) = e^{A_t(T_1) - B_t(T_1)r_t - C_t(T_1)\theta_t}$$

Then, we have:

$$\begin{aligned} X_t &= \frac{P_t(T_2)}{P_t(T_1)} \\ &= e^{A_t(T_2) - A_t(T_1) - (B_t(T_2) - B_t(T_1))r_t - (C_t(T_2) - C_t(T_1))\theta_t} \\ &= e^{(A_t(T_2) - A_t(T_1)) + (B_t(T_1) - B_t(T_2)r_t + (C_t(T_1) - C_t(T_2))\theta_t} \\ &= f(t, r_t, \theta_t) \end{aligned}$$

we could write

$$f(t, r, \theta) = e^{\Delta A_t + \Delta B_t r + \Delta C_t \theta}$$

where $\Delta A_t = A_t(T_2) - A_t(T_1), \Delta B_t = B_t(T_1) - B_t(T_2)$, and $\Delta C_t = C_t(T_1) - C_t(T_2)$.

Computed by an application of Ito's Lemma, we have:

$$
\begin{aligned}
dX_t &= df(t, r_t, \theta_t) \\
&\quad \text{(Note that } X \text{ is a } \mathbb{Q}^{T_1} \text{ martingale, thus, coefficients before } dt \text{ term would be zero)} \\
&= \sigma \partial_r f(t, r, \theta)\Big|_{r=r_t, \theta=\theta_t} dW_t^{1\mathbb{Q}_{T_1}} + \eta \partial_\theta f(t, r, \theta)\Big|_{r=r_t, \theta=\theta_t} dW_t^{2\mathbb{Q}_{T_1}} \\
&\quad \text{(where } W_t^{1\mathbb{Q}_{T_1}} \text{ and } W_t^{2\mathbb{Q}_{T_2}} \text{ are two independent Brownian motions under } \mathbb{Q}^{T_1}\text{)} \\
&= (B_t(T_1) - B_t(T_2))\sigma f(t, r_t, \theta_t) dW_t^{1\mathbb{Q}_{T_1}} + (C_t(T_1) - C_t(T_2))\eta f(t, r_t, \theta_t) dW_t^{2\mathbb{Q}_{T_1}} \\
&= \sigma \Delta B_t f(t, r_t, \theta_t) dW_t^{1\mathbb{Q}_{T_1}} + \eta \Delta C_t f(t, r_t, \theta_t) dW_t^{2\mathbb{Q}_{T_1}} \\
&= \sigma \Delta B_t X_t dW_t^{1\mathbb{Q}_{T_1}} + \eta \Delta C_t X_t dW_t^{2\mathbb{Q}_{T_1}}
\end{aligned}
$$

Note that since under measure $\mathbb{Q}$, we have

$$dr_t = \alpha(\theta_t - r_t)dt + \sigma dW_t^1$$

$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^2$$

By change of numeriare, there exists λ_t^1 and λ_t^2 such that:

$$dr_t = (\alpha(\theta_t - r_t) - \lambda_t^1 \sigma)dt + \sigma dW_t^{\mathbb{Q}_{T_1}}$$

$$d\theta_t = (\beta(\phi - \theta_t) - \lambda_t^2 \eta)dt + \eta dW_t^{\mathbb{Q}_{T_1}}$$

By Girsanov's Theorem, we have:

$$\lambda_t^1 = \sigma B_t(T_1)$$

$$\lambda_t^2 = \eta C_t(T_1)$$

Therefore, under measure $\mathbb{Q}_{T_1}$, we have:

$$dr_t = (\alpha(\theta_t - r_t) - \sigma^2 B_t(T_1))dt + \sigma dW^{1\mathbb{Q}_{T_1}}$$

$$d\theta_t = (\beta(\phi - \theta_t) - \eta^2 C_t(T_1))dt + \eta dW^{2^{\mathbb{Q}_{T_1}}}$$

Suppose $Y_t = log(X_t)$, then we could write:

$$dY_t = \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}d[X,X]_t$$

where $d[X,X]_t = \sigma^2(\Delta B_t)^2 X_t^2 dt + \eta^2(\Delta C_t)^2 X_t^2 dt$

Then, we have:

$$\begin{aligned}
dY_t &= \frac{1}{X_t}(\sigma\Delta B_t X_t dW_t^{1^{\mathbb{Q}_{T_1}}} + \eta\Delta C_t X_t dW_t^{2^{\mathbb{Q}_{T_1}}}) - \frac{1}{2}\frac{1}{X_t^2}(\sigma^2(\Delta B_t)^2 X_t^2 dt + \eta^2(\Delta C_t)^2 X_t^2 dt) \\
&= \sigma\Delta B_t dW_t^{1^{\mathbb{Q}_{T_1}}} + \eta\Delta C_t dW_t^{2^{\mathbb{Q}_{T_1}}} - \frac{1}{2}(\sigma^2(\Delta B_t)^2 dt + \eta^2(\Delta C_t)^2 dt) \\
\Rightarrow \int_t^{T_1} 1 dY_s &= \int_t^{T_1} \sigma\Delta B_s dW_s^{1^{\mathbb{Q}_{T_1}}} ds + \int_t^{T_1} \eta\Delta C_s dW_s^{2^{\mathbb{Q}_{T_1}}} ds - \frac{1}{2}\int_t^{T_1} \sigma^2(\Delta B_s)^2 ds - \frac{1}{2}\int_t^{T_1} \eta^2(\Delta C_s)^2 ds \\
\Rightarrow Y_{T_1} - Y_t &= \int_t^{T_1} \sigma\Delta B_s dW_s^{1^{\mathbb{Q}_{T_1}}} ds + \int_t^{T_1} \eta\Delta C_s dW_s^{2^{\mathbb{Q}_{T_1}}} ds - \frac{1}{2}\int_t^{T_1} \sigma^2(\Delta B_s)^2 ds - \frac{1}{2}\int_t^{T_1} \eta^2(\Delta C_s)^2 ds \\
\Rightarrow log(\frac{X_{T_1}}{X_t}) &= \int_t^{T_1} \sigma\Delta B_s dW_s^{1^{\mathbb{Q}_{T_1}}} ds + \int_t^{T_1} \eta\Delta C_s dW_s^{2^{\mathbb{Q}_{T_1}}} ds - \frac{1}{2}\int_t^{T_1} \sigma^2(\Delta B_s)^2 ds - \frac{1}{2}\int_t^{T_1} \eta^2(\Delta C_s)^2 ds \\
\Rightarrow X_{T_1} &= X_t e^{\int_t^{T_1} \sigma\Delta B_s dW_s^{1^{\mathbb{Q}_{T_1}}} ds + \int_t^{T_1} \eta\Delta C_s dW_s^{2^{\mathbb{Q}_{T_1}}} ds - \frac{1}{2}\int_t^{T_1} \sigma^2(\Delta B_s)^2 ds - \frac{1}{2}\int_t^{T_1} \eta^2(\Delta C_s)^2 ds}
\end{aligned}$$

According to Ito isometry, we have:

$$\begin{aligned}
\mathbb{V}[\int_t^{T_1} \sigma\Delta B_u dW_u^{1^{\mathbb{Q}_{T_1}}}] &= \mathbb{E}[\sigma^2 \int_t^{T_1} \Delta B_u^2 du] \\
&= \sigma^2 \int_t^{T_1} \Delta B_u^2 du
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}[\int_t^{T_1} \eta\Delta C_u dW_u^{2^{\mathbb{Q}_{T_1}}}] &= \mathbb{E}[\eta^2 \int_t^{T_1} \Delta C_u^2 du] \\
&= \eta^2 \int_t^{T_1} \Delta C_u^2 du
\end{aligned}$$

Denote

$$\Omega^2 := \mathbb{V}[\int_t^{T_1} \sigma \Delta B_u dW_u^{1\mathbb{Q}_{T_1}} + \int_t^{T_1} \eta \Delta C_u dW_u^{2\mathbb{Q}_{T_1}}]$$

$$\text{(Note that } W_t^1 \text{ and } W_t^2 \text{ are stochastically independent)}$$

$$= \mathbb{V}[\int_t^{T_1} \sigma \Delta B_u dW_u^{1\mathbb{Q}_{T_1}}] + \mathbb{V}[\int_t^{T_1} \eta \Delta C_u dW_u^{2\mathbb{Q}_{T_1}}]$$

$$= \sigma^2 \int_t^{T_1} (\Delta B_u)^2 du + \eta^2 \int_t^{T_1} (\Delta C_u)^2 du$$

Therefore, we have:

$$X_{T_1} \stackrel{d}{=} X_t e^{-\frac{1}{2}\Omega^2 + \Omega Z} \quad \text{where } Z \stackrel{\mathbb{Q}_{T_1}}{\sim} N(0,1)$$

Since X_t is lognormal, as we discussed in class, we can simply apply the Black-Scholes option pricing formula with $\sigma \to \frac{\Omega}{\sqrt{T_1 - t}}$ and $r \to 0$. Therefore, we have

$$\mathbb{E}_t^{\mathbb{Q}_{T_1}}[(X_{T_1} - K)^+] = X_t \Phi(d_+) - K\Phi(d_-)$$

where $d_\pm = \frac{log(\frac{X_t}{K}) \pm \frac{1}{2}\Omega^2}{\Omega}$.

Therefore, we have:

$$V_t = P_t(T_1)\mathbb{E}_t^{\mathbb{Q}_{T_1}}[(X_{T_1} - K)^+]$$

$$= P_t(T_1)(X_t\Phi(d_+) - K\Phi(d_-)) \quad \text{(plugging in } X_t = \frac{P_t(T_2)}{P_t(T_1)}\text{)}$$

$$= P_t(T_1) \cdot \frac{P_t(T_2)}{P_t(T_1)} \Phi(d_+) - P_t(T_1)K\Phi(d_-)$$

$$= P_t(T_2)\Phi(d_+) - P_t(T_1)\frac{P_0(T_2)}{P_0(T_1)}\Phi(d_-) \quad \text{(plugging in } K = \frac{P_0(T_2)}{P_0(T_1)}\text{)}$$

where $d_\pm = \frac{log(\frac{P_t(T_2)}{KP_t(T_1)} \pm \frac{1}{2}\Omega^2)}{\Omega} = \frac{log(\frac{P_t(T_2)P_0(T_1)}{P_t(T_1)P_0(T_2)}) \pm \frac{1}{2}\Omega^2}{\Omega}$.

2.8 Interest Rate Swap

An interest rate swap (IRS) is a financial derivative which allows one party to exchange a set of fixed cash flows, called fixed leg, with another set of cash flows that are based on a floating interest rate, called floating leg. The IRS payer is the one who pays the fixed amounts, while the IRS receiver receives the fixed amounts and pays the floating amount.

Regarding IRS, the payments are all proportional to a notional amount N and the elapsed time τ_k. Specifically, the fixed leg payments are proportional to a fixed rate F, while the floating leg payments are proportional to the prevailing forward rate. This means that, for a payment at time T_k, the payment is proportional to the forward rate at the time of the previous payment $l^{(k)}_{T_{k-1}}$, which is addressed in earlier section, spot rate and forward rate.

As a result, the swap rate represents the value of F such that the floating leg and fixed leg of the IRS have equal value. For a fixed leg, it's value can be found by discounting all the cash flows to the present time using bond price; whereas for a floating leg, the value is obtained by computing the value of each payment individually using the $\mathbb{Q}^{T_k}$ measure, since we known that $P_{T_k}(T_k) = 1$ and $l^{(k)}$ is a $\mathbb{Q}^{T_k}$ - martingale:

$$
\begin{aligned}
V_t^{fixed} =& N \cdot F \cdot \sum_{k=1}^{n} \tau_k \cdot P_t(T_k) \\
\frac{V_t^{floating,k}}{P_t(T_k)} =& \mathbb{E}_t^{\mathbb{Q}^{T_k}}[\frac{N \cdot \tau_k \cdot l^{(k)}_{T_{k-1}}}{P_{T_k}(T_k)}] = N \cdot \tau_k \cdot \mathbb{E}_t^{\mathbb{Q}^{T_k}}[l^{(k)}_{T_{k-1}}] = N \cdot \tau_k \cdot l_t^{(k)}
\end{aligned}
$$

Using the relationship between forward rates and price, we can write the value of the k - th payment of the floating leg as the following, and it's value can be then written as the sum of the individual cash flows, which follows a telescoping sum:

$$
\begin{aligned}
V_t^{floating,k} =& N \cdot (P_t(T_{k-1}) - P_t(T_k)) \\
V_t^{floating} =& N \cdot \sum_{k=1}^{n} (P_t(T_{k-1}) - P_t(T_k)) = N \cdot (P_t(T_0) - P_t(T_n))
\end{aligned}
$$

Therefore, by setting the value of the two legs equal, we can derive the value of the swap rate at time t as:

$$
S_t = \frac{P_t(T_0) - P_t(T_n)}{\sum_{k=1}^{n} P_t(T_k)\tau_k}
$$

2.9 Lognormal Swap-Rate Model

Before moving onto the lognormal swap-rate, it's important to introduce the swaption model. Swaption is described as a type of financial instrument that gives the holder the right but not the obligation to enter into an interest rate swap between fixed and floating rate at given time in the future (swaption maturity date at T), and make or receive payments based on the rate K, depending on the status of a payer/receiver. As a result, the swaption payoff for the swaption payer who pays the fixed rate can be shown as:

$$\phi = \mathbb{1}(S_T > K) \cdot (V_T^{floating} - V_T^{fixed})$$

The indicator function represents that the swaption will only exercise as the fixed interest rate K is smaller comparing to the prevailing swap rate S_T when the option matures. Therefore, by exercising the swaption, the payer has to pay smaller amounts regarding rate difference, which is also shown in the present value difference between the floating leg and the fixed leg. By using the results derived in previous section, interest rate swap, we can re-write the value function by using equal payments ranging with individual time increments ΔT:

$$\begin{aligned}
\phi =& \mathbb{1}(S_T > K) \cdot [P_T(T_0) - P_T(T_k)) - K \cdot \Delta T \sum_{k=1}^{n} P_T(T_k)] \\
=& (\Delta T \cdot \sum_{k=1}^{n} P_T(T_k)) \cdot \mathbb{1}(S_T > K) \cdot (\frac{P_T(T_0) - P_T(T_n)}{\Delta T \cdot \sum_{k=1}^{n} P_T(T_k)} - K) \\
=& A_T \cdot (S_T - K)_+
\end{aligned}$$

Here, $A_T = \Delta T \sum_{k=1}^{n} P_T(T_k)$, which is the value of an annuity at time T. With the change in numeraire, the value of the swaption can thus be calculated as the following:

$$\begin{aligned}
V_t =& \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}(S_T - K)_+ A_T] \\
\Rightarrow \frac{V_t}{A_t} =& \mathbb{E}_t^{\mathbb{Q}^A}[\frac{V_T}{A_T}] = \mathbb{E}_t^{\mathbb{Q}^A}[\frac{(S_T - K)_+ A_T}{A_T}] = \mathbb{E}_t^{\mathbb{Q}^A}[(S_T - K)_+] \\
\Rightarrow V_t =& A_t \cdot \mathbb{E}_t^{\mathbb{Q}^A}[(S_T - K)_+]
\end{aligned}$$

Given the fact that, $S_T = \frac{P_T(T_0)-P_T(T_n)}{A_T}$ is a relative price of a self-financing trading strategy under the annuity numeraire asset, it is a martingale under the probability measure $\mathbb{Q}^A$. As a result, the lognormal swap-rate is created as a lognormal process under the measure $\mathbb{Q}^A$ for interest rate swap:

$$\frac{dS_t}{S_t} = \sigma_t^S dW_t^{\mathbb{Q}^A}$$

Furthermore, as its name suggests, the lognormal swap-rate model assumes that the swap rate is a continuous-time stochastic process in which the logarithm of the stochastic term follows a normal distribution. The mathematical expression is as follows.

$$\begin{aligned} \mathcal{S}_T &= \mathcal{S}_t \exp\left(-\frac{1}{2}\int_t^T \beta_u^2 du + \int_t^T \beta_u dW_u^{\mathbb{Q}^A}\right) \\ &\overset{d}{=} \mathcal{S}_t e^{-\frac{1}{2}\Omega^2 + \Omega Z} \end{aligned}$$

where $\beta = (\beta_t)_{t\geq 0}$ is deterministic under the risk-neutral measure, $\Omega = (\int_t^T \beta_u^2 du)^{1/2}$ and $Z \overset{\mathbb{Q}}{\sim}$ N(0,1). As a result, the swap rate is lognormally distributed and the swaption value can be computed using the Black-Scholes option price formula: $S_T|_{F_T}$ is as in Black-Scholes with $r = 0$ and $\sigma = \frac{\Omega}{\sqrt{T-t}}$.

$$V_t = A_t[S_t\Phi(d_+) - K\Phi(d_-)]$$

Whereas $d_\pm = \frac{log(\frac{S_t}{K}) \pm \frac{1}{2}\Omega^2}{\Omega}$.

Note that, the **Black Implied Volatility** is the $\frac{\Omega}{\sqrt{T-t}}$ such that the Lognormal Swap-Rate equals to the market price.

3 Implementation, Results and Analysis

We have setup the theoretical framework need for this project and worked out the analytical expressions for the bond price, term structure, bond option price, interest rate swaption, and Black implied volatility within the two-factor Vasicek model framework. Now we are going to conduct Monte Carlo simulations to numerically compute the bond price, term structure, bond option price, swaption price, and Black implied volatility and compare them with the analytical results. We will also study how parameter changes affect the term structure of interest rates. The following base parameters are used for the study:

$$r_o = 2\%, \alpha = 3, \sigma = 1\%, \theta_0 = 3\%, \beta = 1, \phi = 5\%, \eta = 0.5\%.$$

For all simulations, we use a time step of 0.01, and a number of simulations of 100.

3.1 Term Structure of Interest Rates

To numerically approximate the bond price with different maturities, we first use an Euler discretization to simulate the first-order discrete approximation to short rate process of the two-factor Vasicek model described in Equation 2.2.2 above. The idea is to approximate the SDE for the short rate r_t by its first order Taylor approximation. The time interval [0, T] is divided into 100 equal intervals of length Δt such that $T = 100\Delta t$.

Then, the SDEs for θ and r become the following:

$$\begin{aligned}\theta_{t+\Delta t} - \theta_t &= \beta(\phi - \theta_t)\Delta t + \eta(W^2_{t+\Delta t} - W^2_t) \\ &= \beta(\phi - \theta_t)\Delta t + \eta\sqrt{\Delta t}Z^2_t\end{aligned}$$

$$
\begin{aligned}
r_{t+\Delta t} - r &= \alpha(\theta_t - r_t)\Delta t + \sigma(W^1_{t+\Delta t} - W^1_t) \\
&= \alpha(\theta_t - r_t)\Delta t + \sigma\sqrt{\Delta t} Z^2_t
\end{aligned}
$$

where the Brownian motion W^1_t and W^2_t have mean zero and variance t, thus is replaced by the processes where Z^1_t and Z^1_t are independent standard normal variables under the risk-neutral measure $\mathbb{Q}$.

The following equation shows the how the price of the bond with maturity T at time 0 is calculated.

$$
\begin{aligned}
P_0(T) &= \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_u\, du} | \mathcal{F}_0) \\
&= \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_u\, du})
\end{aligned}
\tag{1}
$$

The integral term $I = \int_0^T r_u\, du$ in the above equation can be approximated using a left Riemann sum as follows.

$$
I = \Delta t(r(\theta) + r(\Delta t) + \cdots + r(T - \Delta t))
$$

Finally, the yield of bond with maturity T at time 0 can be calculated as follows.

$$
y_0(T) = -\frac{ln(P_0(T))}{T} = -\frac{ln(\mathbb{E}^{\mathbb{Q}}(e^{-I}))}{T}
$$

To numerically simulate the price of the bond with a given maturity, the above process is repeated 1000 times and the average of the bond yields will be used as the expected bond yield with maturity T.

The standard error of the simulations at maturity T is used as the confidence band of the corresponding simulation. The standard error at maturity T is calculated as follows.

$$\text{SE(T)} = \frac{\sigma(y_0(T)_i)}{\sqrt{\text{number of simulations}}}$$

where $\sigma(\cdot)$ is the standard deviation, $y_0(T)_i$ is the simulated bond yield at simulation i and i = 1,2,...,1000, and the number of simulations is 1000. The confidence band will be $[y_0(T) - SE(T),\, y_0(T) + SE(T)]$.

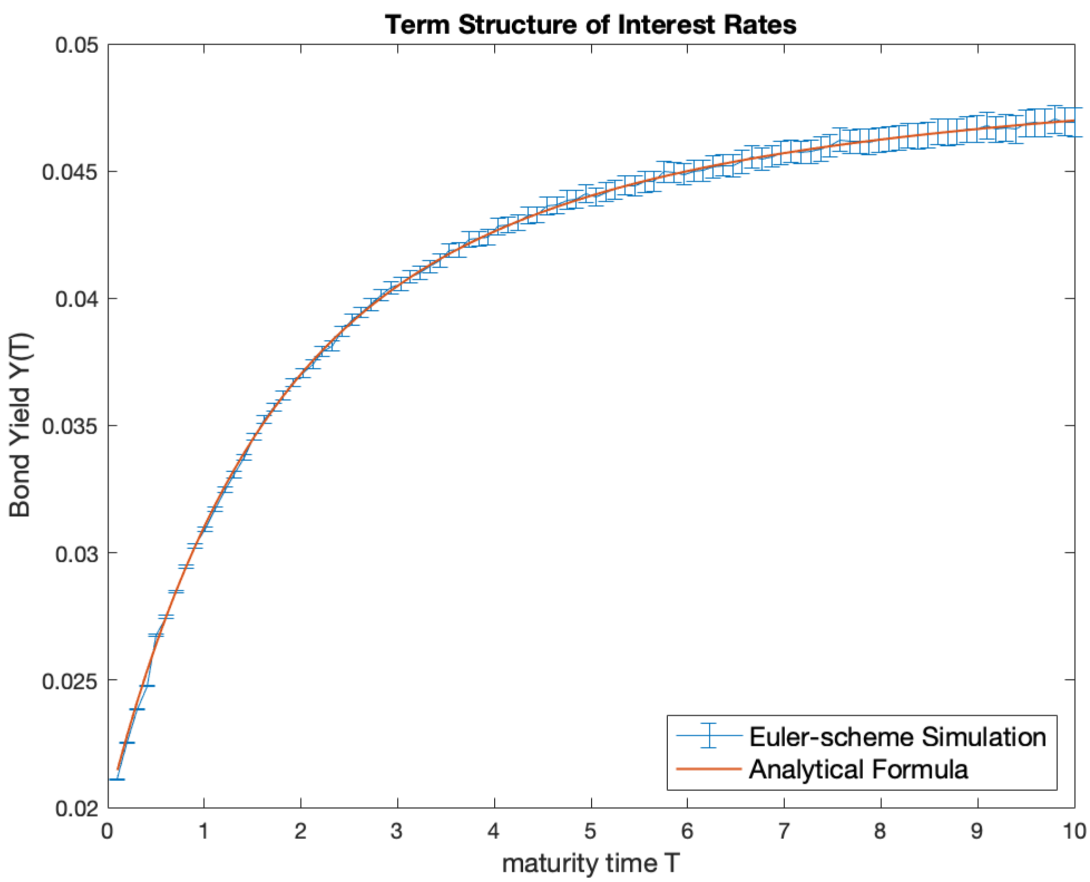

Figure 1: Term Structure of Interest Rates

Figure 1 plots the term structure of interest rates using the Euler-scheme simulation and the associated confidence band as well as the analytical formula derived in section 2.4 and 2.5. According to the plot, the results generated by simulation and the analytical formula closely align with each other. There are visible differences between the two results only when the maturity is close to 0, where the simulated bond yields are slightly lower than the analytical ones. This is likely because the left Riemann sum underestimated the area under the integral term in equation 1, which consequently underestimated the bond yield. This underestimation is more obvious with a small maturity because the change in bond yield is larger, thus the Riemann sum approach cannot estimate the integral as accurately as it can when the maturity is longer.

In addition, the width of the confidence band increases as maturity time increases. This makes sense because the longer the maturity, the more uncertainty we have about the yield at the maturity.

3.2 Varying Parameters on the Term Structure

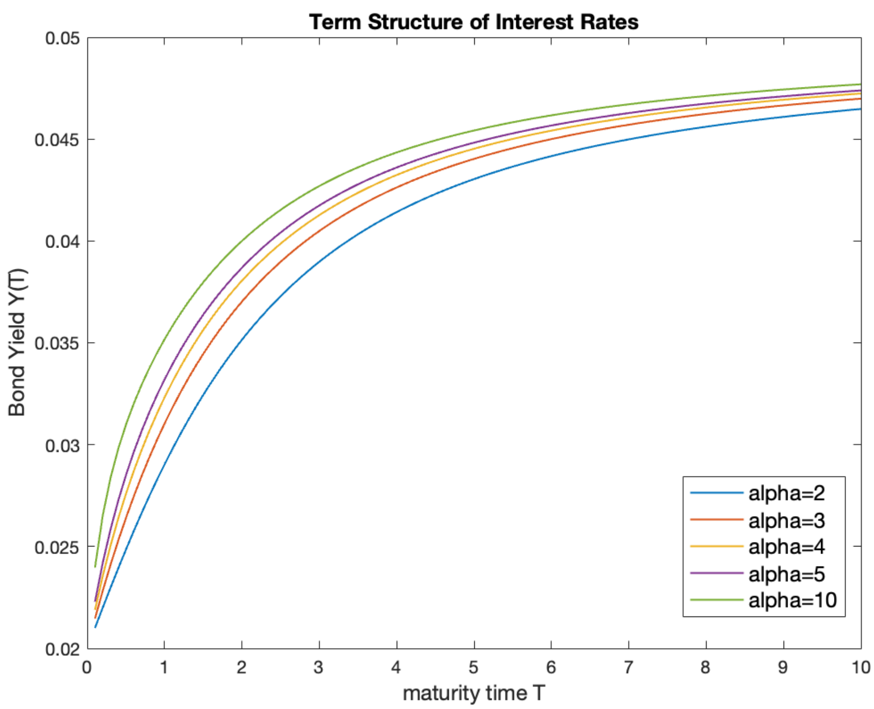

Figure 2: Term Structure with Different α's

Figure 2 plots the term structure of interest rates with different α's. According to the graph, the yield curve shifts up as α increases. According to equation 2.2.2, α is the mean-reversion rate at which the short rate r reverts to θ_t, and θ_t has a mean-reverting level of 5%. Since the short rate r at time 0 is 2%, the bond yield will revert from 2% to 5% as time increases. Therefore, as the mean-reversion rate increases, the short rate increases at a faster rate toward 5%. Since the bond yield is proportional to the integral of short rate, the bond yield will also increase at a faster rate as α increases.

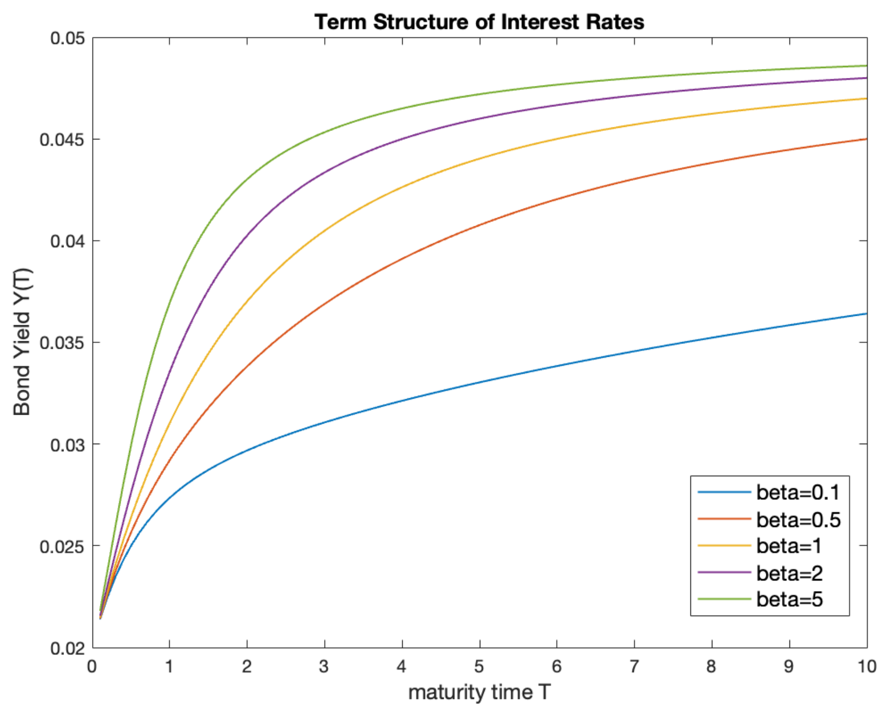

Figure 3: Term Structure with Different β's

Figure 3 plots the term structure of interest rates with different β's. According to the graph, the yield curve converges to 5% faster with a larger β. β is the mean-reversion rate of the SDE for θ_t. Similar to α, the larger the β, the faster that θ_t reverts from 3% to 5%, and thus the faster that r reverts from 2% to 5%. Since the bond yield is proportional to the integral of short rate, the bond yield will also increase at a faster rate as α increases.

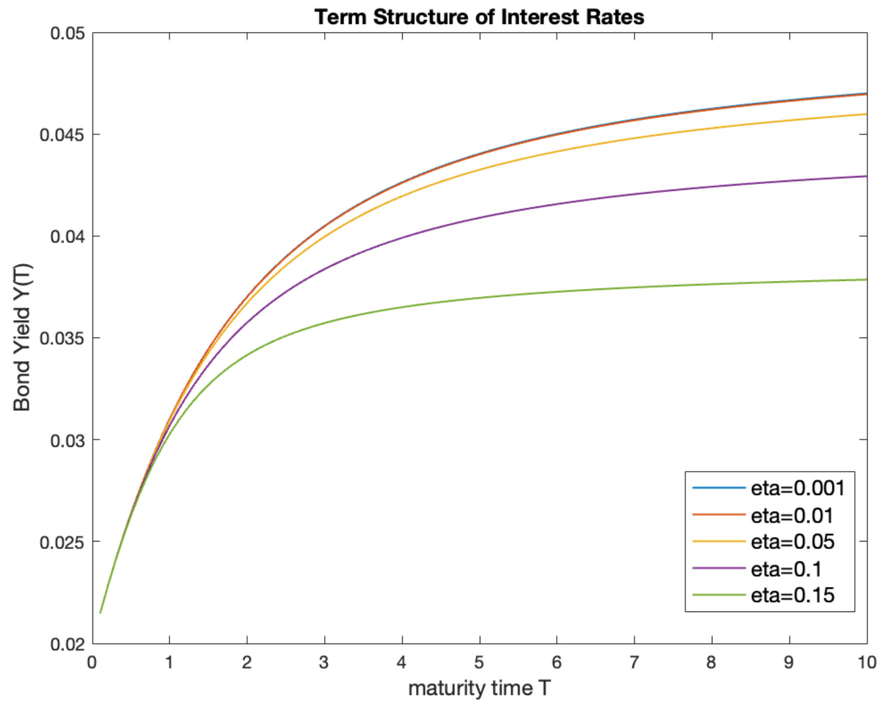

Figure 4: Term Structure with Different η's

Figure 4 plots the term structure of interest rates with different η's. Similar to previous plots, the structure of the yield curve behaves in an increasing trend that grows at a decaying growth rate when maturity gets larger. Also, the yield curve begins at the starting point of $Y_0(T) = 2\%$, which is the short-run interest rate given in the prompt for simulations.

According to the graph, the yield curve is lower with larger η. η is the volatility of the mean-reverting process $d\theta_t$. As the volatility increases, the more difficult it is to revert to the mean ϕ. Since the starting point of θ_t, θ_0 is smaller than the mean-reversion level ϕ, the integral $\int_0^T r_u du$ will be smaller with a larger volatility, the bond price will thus be higher. Consequently, the bond yield will be smaller.

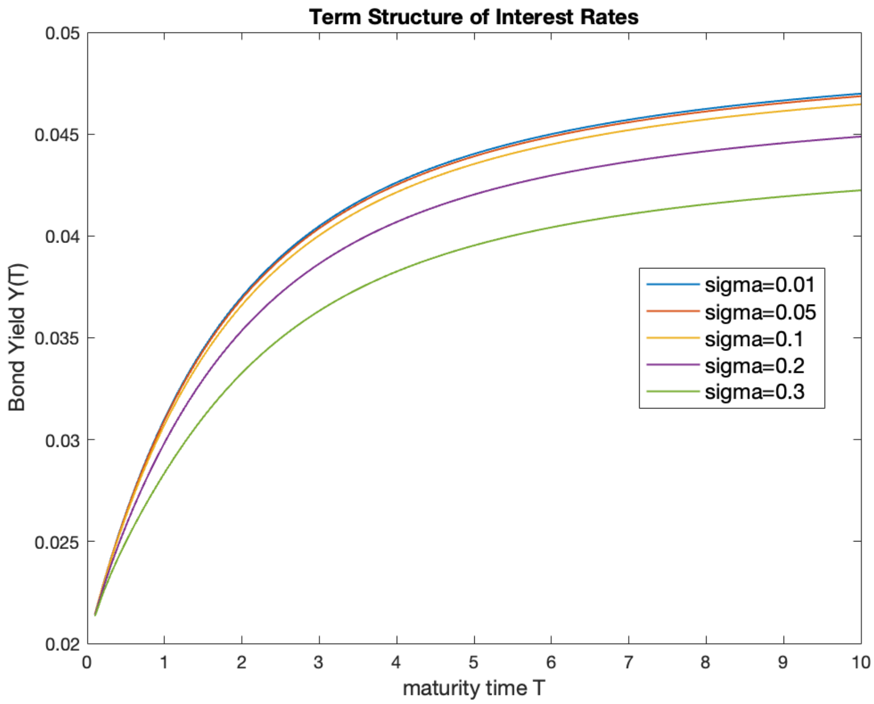

Figure 5: Term Structure with Different σ's

Figure 5 plots the term structure of interest rates with different σ's. The bond yield is smaller with a larger σ. σ is the volatility of the mean-reverting process of dr_t. As the volatility increases, the more difficult it is to revert to the mean θ_t. Since θ_t will revert to ϕ as maturity increases, dr_t will also revert to ϕ. In addition, because the starting point of r_t is r_0, which is lower than ϕ, the integral $\int_0^T r_u du$ will be smaller with a larger volatility, the bond price will thus be higher. Consequently, the bond yield will be smaller.

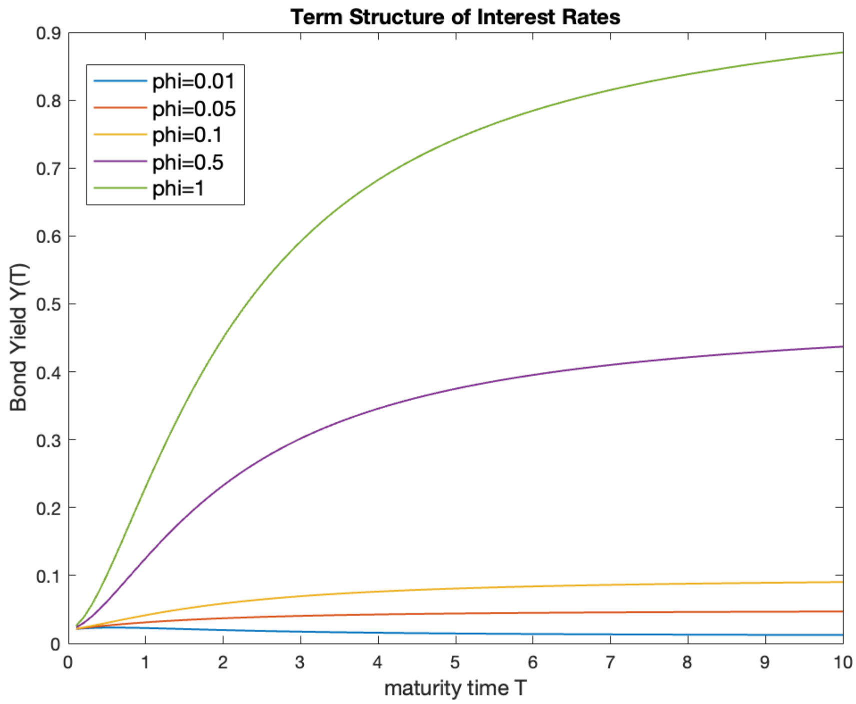

Figure 6: Term Structure with Different ϕ's

Figure 6 plots the term structure of interest rates with different ϕ's. The bond yield increases as ϕ increases. ϕ is the mean-reversion level for θ_t and θ_t is the mean-reversion level for r_t, thus the short rate r_t will revert towards ϕ. Since bond yield is proportional to the integral of the short rate over time, it makes sense for bond yield to be larger when the mean-reversion level of r_t is larger as r_t gets larger as maturity time increases.

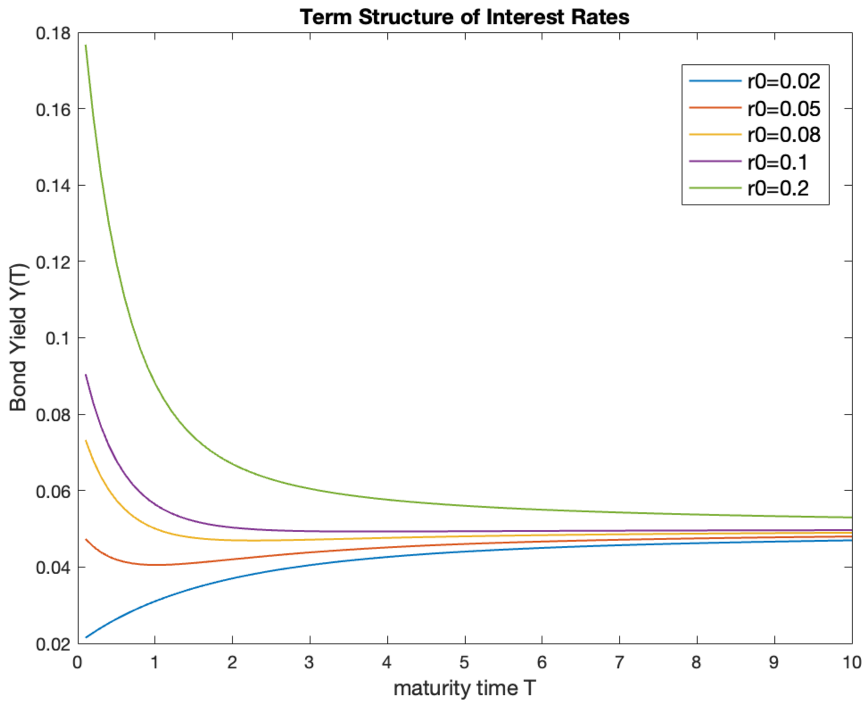

Figure 7: Term Structure with Different r_0's

Figure 7 plots the term structure of interest rates with different r_0's. The bond yield all converge to ϕ, which is approximately 5%, as maturity time increases, regardless of the initial bond yield. If r_0 is smaller than ϕ, the yield curve will increase and have a concave shape. If r_0 is larger than ϕ, the yield curve will decrease and have a convex shape. This make senses because dr_t is a mean-reverting process, it reverts to θ_t while θ_t reverts to ϕ.

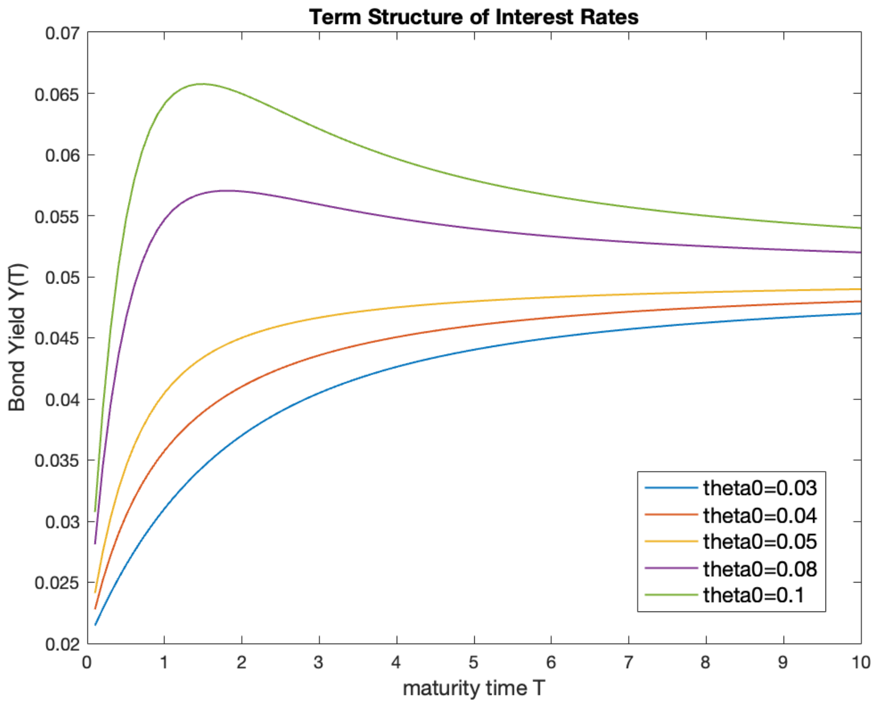

Figure 8: Term Structure with Different θ_0's

Figure 8 plots the term structure of interest rates with different θ_0's. θ_0 is the initial value of $d\theta_t$. θ_t will move from its initial value to ϕ as maturity increases, and thus r_t will revert to θ_t at the same time. If θ_0 is smaller than ϕ, then θ_t will increase and get closer to ϕ, thus r_t will also increase from 2% and get closer to θ_t as maturity increases. As bond yield is proportional to short rate, bond yield will consequently increase as maturity increases. On the other hand, if θ_0 is larger than ϕ, then θ_t will decrease in order to get closer to ϕ when maturity is relatively small. Consequently, r_t will initially revert to a level that is higher than ϕ when θ_t has not reached ϕ yet, and then decrease to ϕ as θ_t decreases. This is why the term structure shows a peak at a maturity around 1-2 years.

3.3 Bond Option Price For Various Strikes

We will analytical calculate and numerically simulate the price of a bond option paying $(P_{T_1}(T_2) - \alpha K)_+$ where K $= P_0(T_2)/P_0(T_1)$ for a collection of strikes αK , for $\alpha = 0.95$, 0.96,

. . . , 1.05, $T_1 = 3$, and $T_2 = 5$. The Monte Carlo simulation will be conducted under both the risk-neutral and forward-neutral measures.

3.3.1 Monte Carlo Simulation of Bond Option Price

We will simulate the bond option price under risk-neutral using the following SDE:

$$dr_t = \alpha(\theta_t - r_t)dt + \sigma dW_t^1$$

$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^2$$

and simulate the bond option price under forward-neutral using the following SDE:

$$dr_t = (\alpha(\theta_t - r_t) - \sigma^2 B_t(T_1))dt + \sigma dW_t^{\mathbb{Q}_{T_1}}$$

$$d\theta_t = (\beta(\phi - \theta_t) - \eta^2 C_t(T_1))dt + \eta dW_t^{\mathbb{Q}_{T_1}}$$

The strike of the option $K = P_0(T_2)/P_0(T_1)$ can be computed using the analytical expression for bond price derived in section 2.4. The only term in the payoff that involves stochasticity is $P_{T_1}(T_2)$ which is the term that will be simulated.

The Euler discretization method described in section3.1 is first used to simulate the interest rate path from t=0 to t=T2. A Riemann sum approximation is used to compute the integral of the interest rate curve from t=T1 to t=T2, where

$$I_2 = \Delta t(r(T1) + r(T1 + \Delta t) + \cdots + r(T2 - \Delta t)).$$

The price of the bond with maturity T2 at time T1 can then be calculated as follows.

$$P_{T_1}(T_2) = e^{-I_2}$$

Consequently, the price of the bond option at time 0 is:

$$V_0 = e^{-I_1} max(P_{T_1}(T_2) - K, 0) = \mathbb{E}^{\mathbb{Q}}(e^{-I_1} max(e^{-I_2} - K, 0))$$

where e^{-I_1} is the discount rate with $I_1 = \Delta t(r(0) + r(\Delta t) + \cdots + r(T1 - \Delta t))$.

3.3.2 Bond Option Price Analytical Expression

To get the bond option price at time 0, the starting time t in the analytical formula derived in section 2.7 with the strike scaled by a factor of α can be written as follows.

$$\begin{aligned} V_0 &= P_0(T_1) \cdot \frac{P_0(T_2)}{P_0(T_1)} \Phi(d_+) - P_0(T_1)\alpha K \Phi(d_-) \\ &= P_0(T_2)\Phi(d_+) - P_0(T_2)\alpha\Phi(d_-) \\ &= P_0(T_2)(\Phi(d_+) - \alpha\Phi(d_-)) \end{aligned}$$

where $d_\pm = \frac{log(\frac{P_0(T_2)}{\alpha K P_0(T_1)} \pm \frac{1}{2}\Omega^2)}{\Omega} = \frac{log(\frac{P_0(T_2)P_0(T_1)}{\alpha P_0(T_1)P_0(T_2)}) \pm \frac{1}{2}\Omega^2}{\Omega} = \frac{log(\frac{1}{\alpha}) \pm \frac{1}{2}\Omega^2}{\Omega}$,
and $\Omega^2 = \sigma^2 \int_t^{T_1} (\Delta B_u)^2 du + \eta^2 \int_t^{T_1} (\Delta C_u)^2 du$.

3.3.3 Bond Option Price vs. Strike Plots

Figure 9 plots the bond option price with different strikes under the risk-neutral measure, forward-neutral measure, and calculated using the analytical expression.

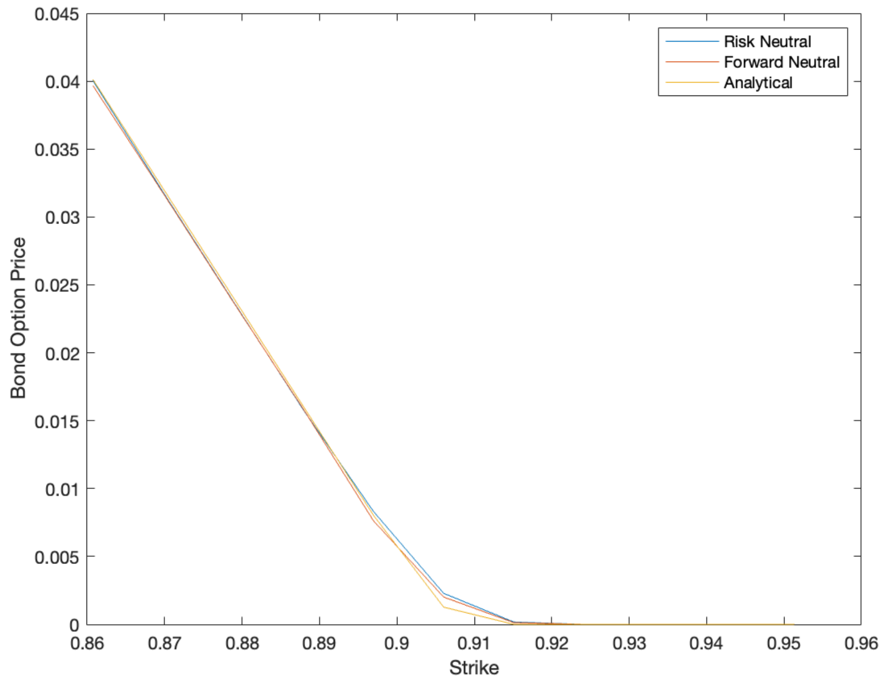

Figure 9: Risk Neutral and Forward Neutral Monte Carlo Simulation

According to the plot, the bond option decreases in value as the strike increases, and becomes zero when strike is greater than approximately 0.915. The bond option price decreases in an almost linear pattern as strike price rises. This makes sense because it is a call option, the higher the strike, the less likely the holder will be able to exercise the option. In addition, the simulated bond option prices are very close to each other under risk neutral, forward neutral measures and using analytical formula. There are two obvious turning points, one around 0.905, and another around 0.915. The bond option value decreases to zero when strike price reaches around 0.915, after which the bond option prices are relatively stable around zero. It's worth mentioning that the bond price calculated (simulated) under risk neutral, forward neutral and using analytical formula deviates relatively obviously when strike price is around 0.905. This means that the Riemann sum cannot accurate approximate the integral of r_t when the bond option price is close to 0.

3.4 Black Implied Volatility as a Function of Strike

Now, we are going compute the Black implied volatility as a function of strike for an interest rate swaption (IRS) with tenor structure $\tau = \{3, 3.25, \cdots, 6\}$. Suppose we enter into the IRS at T = 3, which is the same time as the first reset date. The strike of the swaption will be the swap rate at t=0 and it will be scaled by a factor α, where α = 0.95, 0.96, . . . , 1.05.

We first simulate the interest rate process under the risk-neutral measure using the following SDE:

$$dr_t = \alpha(\theta_t - r_t) + \sigma dW_t^1$$

$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^2$$

Then the strike of the swaption, which is equal to today's swap rate (i.e. $S_0 = F$), can be calculated by varying equation 2.8 as follows.

$$F = S_0 = \frac{P_0(\tau_0) - P_0(\tau_n)}{\sum_{k=1}^{n} P_0(\tau_k)\Delta\tau_k}$$

where $\Delta\tau_k = 0.25$, and $P_t(\tau) = e^{A_t(\tau) - B_t(\tau)r_t - C_t(\tau)\theta_t}$ as defined before.

Afterwards, we simulated the value of the IRS V_0 using the following equation:

$$V_0 = \mathbb{E}_0^{\mathbb{Q}}[e^{-\int_o^T r_u du}(S_T - \alpha F)^+ A_T]$$

where $\mathbb{Q}$ is the risk-neutral measure and $A_T = \sum_{k=1}^{n} P_T(\tau_k)\Delta\tau_k$.

By the LSM model, we also know that using the Black-Scholes equation, the the value of

IRS can be expressed as follows.

$$\begin{aligned} V_0 &= A_0(S_0\Phi(d+) - \alpha F\Phi(d-)) \\ &= A_0F(\Phi(d+) - \alpha\Phi(d-)) \end{aligned}$$

where $d\pm = \frac{-log(\alpha)\pm\frac{1}{2}\Omega^2}{\Omega}$, $F = S_0$.

Since V_0 is know, we can compute Ω and the Black implied volatility is then:

$$\sigma_{imp} = \frac{\Omega}{\sqrt{T}}$$

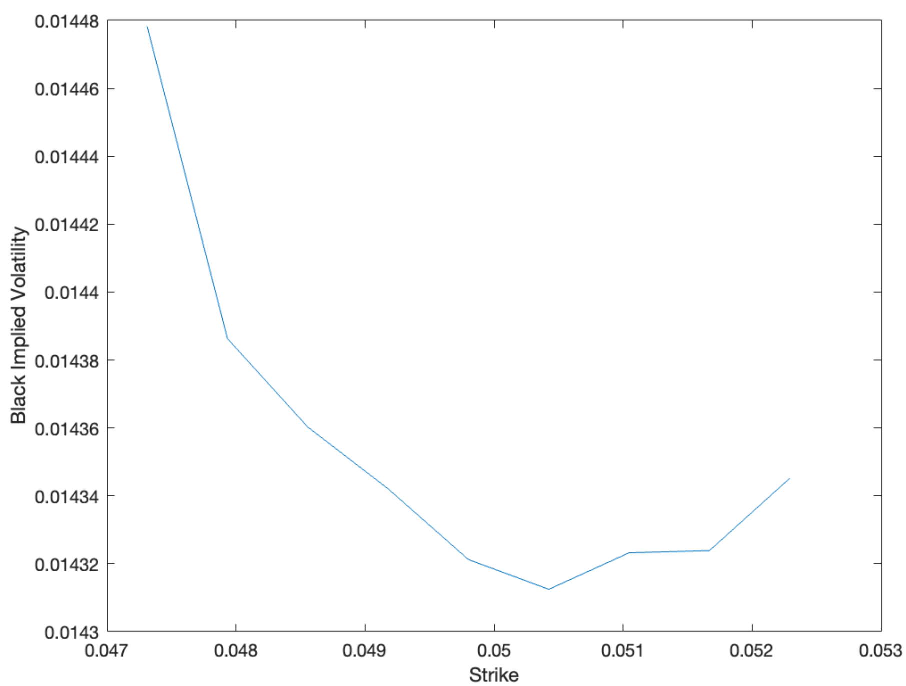

Figure 10: Black Implied Volatility vs. Strike

Figure 10 plots the Black Implied volatility with different strikes. For larger strike, the implied volatility first decreases drastically and then increases slightly, thus shows an asymmetric implied volatility smile. The minimal volatility occurs at approximately K = 0.0505.

4 Conclusion

In this project, we studied the analytical and numerical derivations of bond price, term structure of interest rates, bond option price, interest rate swaption, and Black implied volatility within the two-factor Vasicek model framework. We also varied the parameters of the Vasicek model to examine their impacts on the term structure.

We observed that the Monte Carlo simulated term structure of interest rates almost overlap with the term structure constructed using the analytical expression, which proved that numerical simulation is a very good approximation for the bond yield. The bond yield curve increases as maturity time increases. The speed of the increase is fastest when the maturity is close to zero, and is much slower as maturity is larger. We then explored how changes in the parameters of the Vasicek model can affect the yield curve. We found that increase in the mean-reverting rates will make the yield curve higher and also more concave. The increase in the mean-reverting level of θ_t will increase the long-run bond yield. The increase in the volatilities will make the yield curve lower. The change in the initial values of r_t and θ_t will have different impacts depending on the mean-reverting levels. If the initial value is higher than the mean-reverting level, the bond yield will initially reach a higher value and gradually decay to the mean-reverting level. If the initial value is lower than the mean-reverting level, the bond yield will be increasing towards the mean-reverting level.

We further validated the numerical simulations for the bond option price both under the risk-neutral and forward-neutral measures are good approximations as they closely align with each other. The bond option price decreases as strike increases, and become zero after it is higher than 0.92. Finally, we investigated the black implied volatility as a function of strike. As strike increases, the implied volatility curve first decreases significantly and then increases slightly after reaching its local minimum, thus shows a “smile” shape.