Homework 3 of CS520 Theory of Programming Languages
Deadline: 6:00pm on 25 November (Wednesday)

Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

The numbers in the questions refer to exercise questions in the textbook of the course, i.e. "Theories of Programming Languages" by John C. Reynolds.

Question 1

This question is concerned with the categorical fixed-point theorem that we studied in the class. The theorem appears in note3/recursively-defined-domains, which can be found in the course webpage. Prove that the commutativity diagram about $\eta : F(x_{\mathrm{fix}}) \to x_{\mathrm{fix}}$, $\eta' : F(y) \to y$, and $\rho : x_{\mathrm{fix}} \to y$ in the theorem holds. That is, show that $\rho \circ \eta$ and $\eta' \circ F(\rho)$ are the same morphism. In your answer, you do not need to repeat the construction of η, η' and ρ, which we covered in lectures. Also, you may use any properties about η, η' and ρ that I proved in lectures.

(30 marks)

Question 2

You need to do two things for this question. First, solve the parts (a), (b) and (c) of 10.1. Among the three expressions in 10.1, ignore the third and solve these parts with the first two expressions. Second, for these two expressions, write down the canonical forms that you would get if those expressions are evaluated according to the eager evaluation.

[Hint 1] Using the following abbreviations made it much easier for me to do the required calculations.

$$N \stackrel{\mathrm{def}}{=} \lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x, \qquad T \stackrel{\mathrm{def}}{=} \lambda z.\, \lambda w.\, z,$$

$$D \stackrel{\mathrm{def}}{=} \lambda d.\, d\ d, \qquad F \stackrel{\mathrm{def}}{=} \lambda f.\, \lambda x.\, f\ (f\ x).$$

[Hint 2] N is the standard encoding of the negation of booleans in the lambda calculus, and T that of true. The expression F takes a function f and composes it with itself. Thus, the first expression composes the negation operation with itself, and then applies it to true. The second expression applies F to F. Intuitively, this should lead to the composition of the second F with itself because of the first F.

(40 marks)

Question 3

Solve 10.12. You may assume the substitution lemma stated in 10.11. Using this assumption, you have to prove that

$$[\![(\lambda v.\, e)\ z]\!]\,\eta = [\![e/(v \to z)]\!]\,\eta$$

for all variables v, expressions e, canonical expressions or variables z, and environments η. Recall that an environment η is an element in $V^{\langle \mathrm{var} \rangle}$ (i.e. function from the set of variables $\langle \mathrm{var} \rangle$ to

V), and that every canonical form is a lambda expression $\lambda w.\, e'$. As a result, if z is a canonical expression or a variable,

$$[\![z]\!]\,\eta \neq \perp.$$

(30 marks)

Model answers

Answer to question 1

In this answer, we use the notation of the course lecture notes (note3/recursively-defined-domains). It is possible that students adopt a different notation because I did not follow the notation of the lecture notes strictly in my lectures.

Here is a quick reminder of what I explained in my lectures. I considered the following two ω-chains in the category $\mathcal{C}$:

$$\Delta \stackrel{\text{def}}{=} x_0 \xrightarrow{f_0} F(x_0) \xrightarrow{F(f_0)} F^2(x_0) \xrightarrow{F^2(f_0)} F^3(x_0) \xrightarrow{F^3(f_0)} \ldots$$
$$\Delta' \stackrel{\text{def}}{=} F(x_0) \xrightarrow{F(f_0)} F^2(x_0) \xrightarrow{F^2(f_0)} F^3(x_0) \xrightarrow{F^3(f_0)} F^4(x_0) \xrightarrow{F^4(f_0)} \ldots$$

where f_0 is the unique morphism from the initial object x_0 to the object $F(x_0)$. The primary object x_{fix} of the theorem was constructed by taking the co-limit of the first chain. That is,

$$(x_{\text{fix}}, \{g_n : F^n(x_0) \to x_{\text{fix}}\}_{n\geq 0})$$

is the co-limit of Δ, where F^n means the n-repeated application of the functor F. The morphism $\eta : F(x_{\text{fix}}) \to x_{\text{fix}}$ was then built and the required property about it was shown by noting that both

$$(x_{\text{fix}}, \{g_{n+1} : F^{n+1}(x_0) \to x_{\text{fix}}\}_{n\geq 0}) \qquad \text{and} \qquad (F(x_{\text{fix}}), \{F(g_n) : F^{n+1}(x_0) \to F(x_{\text{fix}})\}_{n\geq 0})$$

are co-limits of Δ'. In particular, we have

$$\eta \circ F(g_n) = g_{n+1} \quad \text{for all } n \geq 0. \tag{1}$$

For the construction of ρ, we noted that if we let g'_0 be the unique morphism from the initial object x_0 to y, and set, for each $n \geq 1$,

$$g'_n \stackrel{\text{def}}{=} F^0(\eta') \circ \ldots \circ F^{n-1}(\eta') \circ F^n(g'_0)$$

then

$$(y, \{g'_n : F^n(x_0) \to y\}_{n\geq 0})$$

is a co-cone of Δ, and derived the existence of a unique morphism from the co-limit x_{fix} of Δ to the co-cone y, which we used as ρ. We recall the following commutativity property of ρ:

$$\rho \circ g_n = g'_n \quad \text{for all } n \geq 0. \tag{2}$$

We now answer the question. We observe that

$$(y, \{g'_{n+1} : F^{n+1}(x_0) \to y\}_{n\geq 0})$$

is a co-cone of Δ'. Since $(F(x_{\text{fix}}), \{F(g_n) : F^{n+1}(x_0) \to F(x_{\text{fix}})\}_{n\geq 0})$ is a co-limit of Δ', there exists a unique morphism satisfying the commutativity property in the definition of co-limit. We show that both

$$\rho \circ \eta : F(x_{\text{fix}}) \to y \qquad \text{and} \qquad \eta' \circ F(\rho) : F(x_{\text{fix}}) \to y$$

satisfy the mentioned commutativity property. Then, the uniqueness implies that these two morphisms should be the same. For each $n \geq 0$,

$$(\rho \circ \eta) \circ F(g_n) = \rho \circ (\eta \circ F(g_n)) = \rho \circ g_{n+1} = g'_{n+1}$$

where the second equality uses (1) and the last uses (2), and

$$(\eta' \circ F(\rho)) \circ F(g_n) = \eta' \circ (F(\rho) \circ F(g_n)) = \eta' \circ F(\rho \circ g_n) = \eta' \circ F(g'_n) = g'_{n+1}$$

where the third equality uses (2) and the last uses the definition of g'_{n+1} and the $\circ$-preservation of F. We have just shown that both $\rho \circ \eta$ and $\eta' \circ F(\rho)$ satisfy the desired commutativity property.

Answer to question 2

[Part (a)] Here are the normal-order reduction sequences for the first two expressions in the question 10.1 in the textbook.

$$
\begin{aligned}
&\Big(\lambda f.\, \lambda x.\, f\ (f\ x)\Big)\ (\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ (\lambda z.\, \lambda w.\, z) \\
&\longrightarrow \Big(\lambda x.\, (\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ \Big((\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ x\Big)\Big)\ (\lambda z.\, \lambda w.\, z) \\
&\longrightarrow (\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ \Big((\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ (\lambda z.\, \lambda w.\, z)\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, \Big((\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ (\lambda z.\, \lambda w.\, z)\Big)\ y\ x \\
&\longrightarrow \lambda x.\, \lambda y.\, \Big(\lambda x.\, \lambda y.\, (\lambda z.\, \lambda w.\, z)\ y\ x\Big)\ y\ x \\
&\longrightarrow \lambda x.\, \lambda y.\, \Big(\lambda y'.\, (\lambda z.\, \lambda w.\, z)\ y'\ y\Big)\ x \\
&\longrightarrow \lambda x.\, \lambda y.\, (\lambda z.\, \lambda w.\, z)\ x\ y \\
&\longrightarrow \lambda x.\, \lambda y.\, (\lambda w.\, x)\ y \\
&\longrightarrow \lambda x.\, \lambda y.\, x
\end{aligned}
$$

$$
\begin{aligned}
&(\lambda d.\, d\ d)\ (\lambda f.\, \lambda x.\, f\ (f\ x)) \\
&\longrightarrow (\lambda f.\, \lambda x.\, f\ (f\ x))\ (\lambda f.\, \lambda x.\, f\ (f\ x)) \\
&\longrightarrow \lambda x.\, (\lambda f.\, \lambda y.\, f\ (f\ y))\ \Big((\lambda f.\, \lambda z.\, f\ (f\ z))\ x\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, \Big((\lambda f.\, \lambda z.\, f\ (f\ z))\ x\Big)\ \Big(((\lambda f.\, \lambda z.\, f\ (f\ z))\ x)\ y\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, (\lambda z.\, x\ (x\ z))\ \Big(((\lambda f.\, \lambda z.\, f\ (f\ z))\ x)\ y\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, x\ \Big(x\ \Big(((\lambda f.\, \lambda z.\, f\ (f\ z))\ x)\ y\Big)\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, x\ \Big(x\ \Big((\lambda z.\, x\ (x\ z))\ y\Big)\Big) \\
&\longrightarrow \lambda x.\, \lambda y.\, x\ (x\ (x\ (x\ y)))
\end{aligned}
$$

[Part (b)] The first sequence would end after the third contraction, and give:

$$\lambda x.\, \lambda y.\, \Big((\lambda b.\, \lambda x.\, \lambda y.\, b\ y\ x)\ (\lambda z.\, \lambda w.\, z)\Big)\ y\ x$$

The second sequence would stop after the second contraction, and give:

$$\lambda x.\,(\lambda f.\,\lambda y.\,f\ (f\ y))\ \Big((\lambda f.\,\lambda z.\,f\ (f\ z))\ x\Big)$$

[Part (c)] Let

$$N \stackrel{\text{def}}{=} \lambda b.\,\lambda x.\,\lambda y.\,b\ y\ x, \qquad T \stackrel{\text{def}}{=} \lambda z.\,\lambda w.\,z,$$
$$D \stackrel{\text{def}}{=} \lambda d.\,d\ d, \qquad F \stackrel{\text{def}}{=} \lambda f.\,\lambda x.\,f\ (f\ x).$$

Using the inference rules for the normal-order evaluation, we need to derive

$$(F\ N)\ T \Rightarrow \lambda x.\,\lambda y.\,(N\ T)\ y\ x \qquad \text{and} \qquad D\ F \Rightarrow \lambda x.\,F\ (F\ x).$$

Here are the derivations of these evaluation relationships:

$$\dfrac{\dfrac{\overline{F \Rightarrow F} \quad \overline{\lambda x.\,N\ (N\ x) \Rightarrow \lambda x.\,N\ (N\ x)}}{(F\ N) \Rightarrow \lambda x.\,N\ (N\ x)} \qquad \dfrac{\overline{N \Rightarrow N} \quad \overline{\lambda x.\,\lambda y.\,(N\ T)\ y\ x}}{N\ (N\ T) \Rightarrow \lambda x.\,\lambda y.\,(N\ T)\ y\ x}}{(F\ N)\ T \Rightarrow \lambda x.\,\lambda y.\,(N\ T)\ y\ x}$$

Let

$$\dfrac{\overline{D \Rightarrow D} \qquad \dfrac{\overline{F \Rightarrow F} \quad \overline{\lambda x.\,F\ (F\ x) \Rightarrow \lambda x.\,F\ (F\ x)}}{F\ F \Rightarrow \lambda x.\,F\ (F\ x)}}{D\ F \Rightarrow \lambda x.\,F\ (F\ x)}$$

[Second Question] If we use eager evaluation, we would get the following results:

$$\begin{aligned}&\Big(\lambda f.\,\lambda x.\,f\ (f\ x)\Big)\ (\lambda b.\,\lambda x.\,\lambda y.\,b\ y\ x)\ (\lambda z.\,\lambda w.\,z)\\ &\quad \Rightarrow_E \Big(\lambda x.\,\lambda y.\,\Big(\lambda x.\,\lambda y.\,(\lambda z.\,\lambda w.\,z)\ y\ x\Big)\ y\ x\Big)\end{aligned}$$

$$\begin{aligned}&(\lambda d.\,d\ d)\ (\lambda f.\,\lambda x.\,f\ (f\ x))\\ &\quad \Rightarrow_E \Big(\lambda x.\,(\lambda f.\,\lambda y.\,f\ (f\ y))\ \Big((\lambda f.\,\lambda z.\,f\ (f\ z))\ x\Big)\Big)\end{aligned}$$

Answer to question 3

Note that

$$[\![(\lambda v.\,e)]\!]\,\eta = \psi(\lambda a \in V.\ [\![e]\!]\,[\eta|v:a]) \neq \bot.$$

Also, since z is a variable or a canonical expression, $[\![z]\!]\,\eta \neq \bot$. Thus,

$$\begin{aligned}[\![(\lambda v.\,e)\,z]\!]\,\eta = \phi([\![(\lambda v.\,e)]\!]\,\eta)\,([\![z]\!]\,\eta) &= \phi(\psi(\lambda a \in V.\ [\![e]\!]\,[\eta|v:a]))([\![z]\!]\,\eta)\\ &= (\lambda a \in V.\ [\![e]\!]\,[\eta|v:a])([\![z]\!]\,\eta)\\ &= [\![e]\!]\,[\eta|v:[\![z]\!]\,\eta]\\ &= [\![e/v \to z]\!]\,\eta.\end{aligned}$$

The third equality uses the fact that $\phi \circ \psi = \text{id}$, and the last equality uses the substitution lemma described in the question 10.11 of the textbook.