Exercises

Matthias Frerichs

March 12, 2024

Contents

1 Exercises Part 1

1.1 Irreducible topological spaces

Exercise 1.1.1 See [2, Exercise 1.3] Determine all irreducible Hausdorff spaces. Determine all noetherian Hausdorff spaces. Show that a topological space is noetherian if and only if every open subspace is quasi-compact.

Exercise 1.1.2 Prove Cayley Hamilton by using the irreducibility of $\mathbb{A}^{n^2}(\mathbb{k})$ and the representation of the determinant as a polynomial.

Hint: Use matrices with n distinct Eigenvalues and show that these fulfill the statement of Cayley Hamilton. Then show that this set is open and non-empty.

1.2 Localization of rings and modules

Remark 1.2.1 (Localization of rings and modules) See [3, p. 1.3.3.]. Another important example of a definition by universal property is the notion of localization of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A multiplicative subset S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}\mathsf{A}$. The elements of $S^{-1}A$ are of the form $\frac{a}{s}$ where $a \in \mathsf{A}$ and $s \in S$, and where $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ if (and only if) for some $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. We define $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1+s_1a_2}{s_1s_2}$, and $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}$. (If you wish, you may check that this equality of fractions really is an equivalence relation and the two binary operations on fractions are well-defined on equivalence classes and make $S^{-1}\mathsf{A}$ into a ring.) We have a canonical ring map $\mathsf{A} \to S^{-1}\mathsf{A}$ given by $a \mapsto \frac{a}{1}$. Note that if $0 \in S$, $S^{-1}\mathsf{A}$ is the 0-ring. There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, f^3, \cdots\}$ where $f \in \mathsf{A}$. This localization is denoted A_f. (Can you describe an isomorphism $\mathsf{A}_f \to \mathsf{A}[t]/(tf-1)$?) The second is $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}\mathsf{A}$ is denoted $\mathsf{A}_\mathfrak{p}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $\mathsf{A}_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in \mathsf{A}$, A_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $\mathsf{A}_f \neq \mathsf{A}_{(f)}$.)

Exercise 1.2.1 See [3, Exercise 1.3.C]. Show that $\mathsf{A} \to S^{-1}\mathsf{A}$ is injective if and only if S contains no zerodivisors. (A zerodivisor of a ring A is an element a such that there is a nonzero element b with $ab = 0$. The other elements of A are called non-zerodivisors. For example, an invertible element is never a zerodivisor. Counter-intuitively, 0 is a zerodivisor in every ring but the 0-ring. More generally, if M is an A-module, then $a \in \mathsf{A}$ is a zerodivisor for M if there is a nonzero $m \in M$ with $am = 0$. The other elements of A are called non-zerodivisors for M. Equivalently, and very usefully, $a \in \mathsf{A}$ is a non-zerodivisor for M if and only if $(-) \cdot a \colon M \to M$ is an injection, or equivalently if the sequence

$$0 \longrightarrow M \xrightarrow{(-)\cdot a} M \tag{1.2.1}$$

is exact.) If A is an integral domain and $S = \mathsf{A} \setminus \{0\}$, then $S^{-1}\mathsf{A}$ is called the fraction field of A, which we denote $K(\mathsf{A})$. The exercise shows that A is a subring of its fraction field.

PROOF: Assume that $A \mapsto S^{-1}A$, $a \mapsto \frac{a}{1}$ is not injective. Then there exist a_1, a_2 such that

$$\frac{a_1}{1} \sim \frac{a_2}{1} \Leftrightarrow s(a_1 - a_2) = 0$$

for some $s \in S$. Since $a_1 \neq a_2$ this means that S contains zero divisors.

Conversely, if S contains no zero divisors, the equation

$$\frac{a_1}{1} \sim \frac{a_2}{1} \Leftrightarrow s(a_1 - a_2) = 0$$

implies that $a_1 = a_2$, hence the inclusion is injective. $\square$

Exercise 1.2.2 See [3, Exercise 1.3.D.] Verify that $\mathsf{A} \to S^{-1}\mathsf{A}$ satisfies the following universal property:

$S^{-1}\mathsf{A}$ is initial among A-algebras $\mathscr{B}$ where every element of S is sent to an invertible element in $\mathscr{B}$.

(Recall: the data of "an A-algebra $\mathscr{B}$" and "a ring map $\mathsf{A} \to \mathscr{B}$" are the same.)

PROOF: Let $\mathscr{B}$ be an A-algebra such that S is invertible i.e. an algebra morphism $f\colon \mathsf{A} \to \mathscr{B}$ such that $f(S)$ consists of invertible elements of $\mathscr{B}$.

We construct a unique morphism $\hat{f}\colon S^{-1}\mathsf{A} \to \mathscr{B}$. We define

$$\hat{f}(\frac{a}{s}) \coloneqq f(a)f(s)^{-1}.$$

This is well defined since $f(s)$ is invertible. This is unique by construction, i.e. $S^{-1}\mathsf{A}$ is initial. $\square$

1.3 Algebraic varieties and prevarieties

Exercise 1.3.1 See [2, Exercise 1.4]. Show that the underlying topological space X of a prevariety is a T1-space (i.e., for all $x, y \in X$ there exist open neighborhoods U of x and V of y with $y \notin U$ and $x \notin V$).

Exercise 1.3.2 See [2, Exercise 1.5]. Consider the twisted cubic curve $C = \{(t, t^2, t^3) \mid t \in \Bbbk\} \subset \mathbb{A}^3(\Bbbk)$. Show that C is an irreducible closed subset of $\mathbb{A}^3(\Bbbk)$. Find generators for the ideal $I(C)$. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(\Bbbk)$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

PROOF: We have $I(C) = (Y - X^2, Z - X^3) \trianglelefteq \Bbbk[X, Y, Z]$ since $(x, y, z) \in C$ if and only if $y = x^2$ and $z = x^3$. We present two proofs to show that C is irreducible:

First proof: To show that $I(C)$ is prime, we compute $\Gamma(C) = \Bbbk[X, Y, Z]/I(C)$. We write $a := Y - X^2$ and $b = Z - X^3$, then we use $R/(a, b) = (R/(a))/(\bar{b})$ for $r = \Bbbk[X, Y, Z]$. The ring homomorphism $\varphi : \Bbbk[X, Y, Z] \to \Bbbk[X, Z]$ with $\varphi(X) = X$, $\varphi(Z) = Z$ and $\varphi(Y) = X^2$ has kernel $(Y - X^2)$, since for $f \in \ker\varphi$ we can do polynomial division by $Y - X^2$ and write $f = (Y - X^2)q + r$ with $\deg_Y r \geq 0$, i.e. $r \in \Bbbk[X, Z]$. So we get $\varphi(f) = \varphi(r) = r$ and therefore $r = 0$ and f is a multiple of $Y - X^2$. Now we apply the same trick for b and get $\Gamma(C) \cong \Bbbk[X]$. This ring is an integral domain and therefore C irreducible.

Second proof: We show that C is isomorphic to $\mathbb{A}^1(\Bbbk)$, as affine algebraic sets. The maps $\varphi : C \to \mathbb{A}^1(\Bbbk)$, $(x, y, z) \mapsto x$ and $\psi : \mathbb{A}^1(\Bbbk) \to C$, $t \mapsto (t, t^2, t^3)$ are morphisms of algebraic sets and mutual inverses. Therefore they are isomorphisms and since $\mathbb{A}^1(\Bbbk)$ is irreducible C is also irreducible.

For the second part a case analysis shows that $V = V_1 \cup V_2 \cup V_3$, where

$$V_1 = V(X, Y), \quad V_2 = V(X, Z) \quad \text{and} \quad V_3 = (Y - X^2, Z - 1).$$

Similar to the arguments above, we can compute $\Gamma(V_1) \cong \Bbbk[Z]$, $\Gamma(V_2) \cong \Bbbk[Y]$ and $\Gamma(V_3) \cong \Bbbk[X]$ to see that each of these components is irreducible. $\square$

Exercise 1.3.3 See [2, Exercise 1.14]. Let X be a prevariety and let Y be an affine variety. Show that the map

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y),\Gamma(X)),\ f \mapsto (f^*\colon \phi \mapsto \phi \circ f),$$

is bijective. Deduce that $\mathrm{Hom}(X, \mathbb{A}^n(\Bbbk)) = \Gamma(X)^n$.

PROOF: Let us first consider the case that X and Y are affine varieties. Then a map $f\colon X \to Y$ is a component-wise polynomial map $f\colon \Bbbk^n \supset X \to Y \subset \Bbbk^m$ with

$$f(x_1,\cdots,x_n) = (f_1(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n)).$$

Then the result to show is that the contravariant functor

$$\Gamma(\cdot)\colon \mathrm{AffineVarieties} \to \mathrm{int-Alg}, \quad (f\colon X \to Y) \mapsto (f^*\colon \Gamma(Y) \to \Gamma(X))$$

from affine varieties into integral $\Bbbk$-algebras is full and faithful. We show this by constructing explicit isomorphisms

$$\Phi\colon\ \mathrm{Hom}(X,Y) \leftrightarrow \mathrm{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y),\Gamma(X))\colon\ \Psi.$$

We define $\Phi(f) = f^*$. To define Ψ we want to find for any $\alpha\colon \Gamma(Y) \to \Gamma(X)$ an element $f\colon X \to Y$ such that $f^* = \alpha$, since then $\Psi(\Phi(f)) = \Psi(\alpha) = f$ would be a left inverse map.

For this we define

$$\Psi(\alpha) = (\alpha(Y_1),\cdots,\alpha(Y_m))$$

where we denote with Y_n the map $\mathrm{pr}_n\big|_Y\colon \Bbbk^m \supset Y \to \Bbbk$ which is a morphism of affine varieties between Y and $\mathbb{A}^1$. Note, that we need to show that this map is well defined. I.e. that for the prime-ideal $\mathfrak{p}$ such that $Y = V(\mathfrak{p})$ we have $p(\Psi(\alpha)(x)) = 0$ for any $x \in X$ and $p \in \mathfrak{p}$. This however is clear since

$$(Y_1,\cdots,Y_m)\colon Y \to Y$$

is just the identity map and by the $\Bbbk$-algebra morphism property of α we get

$$\begin{aligned} p(\Psi(\alpha)(x)) &= p(\alpha(Y_1)(x),\cdots,\alpha(Y_m)(x)) \\ &= \alpha(p \circ (Y_1,\cdots,Y_m))(x) \\ &= \alpha(p)(x) \\ &= \alpha(0)(x) \end{aligned}$$

since any $p \in \mathfrak{p}$ is the 0-map restricted to Y.

Now to show that Φ and Ψ are bijections we show that they are mutually inverse.

For $f = (f_1,\cdots,f_m)\colon X \to Y$ with $f_i \in \Bbbk[T_1,\cdots,T_n]$ we calculate

$$\begin{aligned} \Psi(\Phi(f)) =& \Psi(f^*) \\ =& (f^*Y_1,\cdots,f^*Y_m) \\ =& (f_1,\cdots,f_m) \\ =& f. \end{aligned}$$

Conversely for $\alpha\colon \Gamma(Y) \to \Gamma(X)$ it is enough to show $\Phi \circ \Psi = \mathrm{id}$ on generators of $\Gamma(Y)$. These are given by the component functions Y_n. We calculate

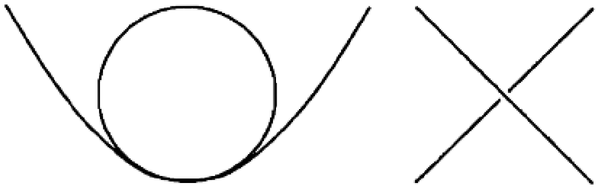
$$\Phi(\Psi(\alpha))(Y_n) = \Phi(\alpha(Y_1),\cdots,\alpha(Y_m))(Y_n)$$

$$
\begin{aligned}
&= (\alpha(Y_1), \cdots, \alpha(Y_m))^*(Y_n) \\
&= \mathrm{pr}_n(\alpha(Y_1), \cdots, \alpha(Y_m)) \\
&= \alpha(Y_n).
\end{aligned}
$$

Since we never needed that X is actually an affine variety, we can easily replace it with $\cup X_i$, where every X_i is an affine variety and the construction follows analogously.

Now for $\mathbb{A}^n$ this means that $\mathrm{Hom}(X, \mathbb{A}^n) \cong \mathrm{Hom}(\Gamma(\mathbb{A}^n), \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[T_1, \cdots, T_n], \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[x], \Gamma(X))^n \cong \mathrm{Hom}(\Gamma(\mathbb{A}^1), \Gamma(X))^n$. $\square$

Exercise 1.3.4 See [1, Exercise 1.10]. Find rings to represent the following figures.



The first represents the union of a circle and a parabola in the plane and the second shows the union if two skew lines in 3-space. (You may use the Nullstellensatz to prove your answer is right.)

Exercise 1.3.5 When is $\mathbb{S}^1 \subset \mathbb{R}^2$ irreducible?

2 Exercises Part 2

2.1 Projective varieties

Exercise 2.1.1 See [2, Exercise 1.22]. Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3(\Bbbk)$.

(i) Show that there exists a change of coordinates such that $L_1 = V_+(X_0, X_1)$ and $L_2 = V_+(X_2, X_3)$.

(ii) Let $Z = L_1 \cup L_2$. Determine the homogeneous radical ideal $\mathfrak{a}$ such that $V_+(\mathfrak{a}) = Z$ (Exercise 1.21).

2.2 Veronese embedding

Exercise 2.2.1 [2, Exercise 1.30]. Let $n, d > 0$ be integers. Let $M_0, \cdots, M_N \in \Bbbk[X_0, \cdots, X_n]$ be all monomials in $X_0, \cdots, X_n$ of degree d.

(i) Define a $\Bbbk$-algebra homomorphism

$$\theta\colon \Bbbk[Y_0, \cdots, Y_N] \to \Bbbk[X_0, \cdots, X_n], \quad Y_i \mapsto M_i$$

and let $\mathfrak{a} = \ker\theta$. Show that $\mathfrak{a}$ is a homogeneous prime ideal (Exercise 1.20). Let $V_+(\mathfrak{a}) \subset \mathbb{P}^N(\Bbbk)$ the projective variety defined by $\mathfrak{a}$ (Exercise 1.21).

(ii) Consider the morphism

$$v_d\colon \mathbb{P}^n(\Bbbk) \to \mathbb{P}^N(\Bbbk), \quad (x_0 : \cdots : x_n) \mapsto (M_0(x_0, \cdots, x_n) : \cdots : M_N(x_0, \cdots, x_n)),$$

and show that v_d induces an isomorphism $\mathbb{P}^n(\Bbbk) \cong V_+(\mathfrak{a})$ of prevarieties. Is $V_+(\mathfrak{a})$ a linear subspace of $\mathbb{P}^N(\Bbbk)$?

(iii) Let $f \in \Bbbk[X_0, \cdots, X_n]$ be homogeneous of degree d. Show that $v_d(V_+(f))$ is the intersection of $V_+(\mathfrak{a})$ and a linear subspace of $\mathbb{P}^N(\Bbbk)$. The morphism v_d is called the d-Uple embedding or d-fold Veronese embedding.

2.3 Spectra of rings

Exercise 2.3.1 Find two classes of rings or your favourite two specific rings and classify their spectrum.

Exercise 2.3.2 See [2, Exercise 2.9]. For valuation rings see [2, (B.13) (p562)]. Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is called isolated if $0 \leq \gamma \leq \delta$ and $\delta \in \Delta$ implies $\gamma \in \Delta$.

(i) Let A be a valuation ring, $K = \mathrm{Frac} A$. Show that $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ is an inclusion reversing bijection from $\operatorname{Spec} A$ onto the set of rings B with $A \subset B \subset K$ and that such a ring B is a valuation ring of K. Its inverse is given by sending B to its maximal ideal (which is contained in A).

(ii) Let A be a valuation ring with value group Γ. For every isolated subgroup Δ of Γ set $\mathfrak{p}_\Delta := \{a \in A \mid v(a) \notin \Delta\}$. Show that $\Delta \mapsto \mathfrak{p}_\Delta$ defines a order reversing bijection between the set of isolated subgroups of Γ and $\operatorname{Spec} A$ (both totally ordered by inclusion).

(iii) Show that the value groups of the valuation ring $A_{\mathfrak{p}_\Delta}$ is isomorphic to Γ/Δ and the value groups of the valuation ring $A/\mathfrak{p}_\Delta$ is isomorphic to Δ.

(iv) For an arbitrary totally ordered abelian group Γ let $R = \Bbbk[\Gamma]$ be the group algebra of Γ over some field $\Bbbk$. Write elements of $u \in A$ as finite sums $u = \sum_\gamma a_\gamma e_\gamma$. Define a map $v\colon R\backslash\{0\} \to \Gamma$ by sending u to the minimal $\gamma \in \Gamma$ such that $\alpha_\gamma \neq 0$. Show that R is an integral domain and that v can be extended to a valuation v on $K = \mathrm{Frac}R$ with value group Γ. In particular $A \coloneqq \{x \in K \mid v(x) \geq 0\}$ is a valuation ring with value group Γ.

(v) Let I be a well-ordered set. Endow $\mathbb{Z}^I$ with the lexicographic order (i.e., we set $(n_i)_{i\in I} < (m_i)_{i\in I}$ if and only if $J \coloneqq \{i \in I \mid m_i \neq n_i\}$ is non-empty and $n_{i_J} < m_{i_J}$ where i_J is the smallest element of J). Show that $\mathbb{Z}^I$ is a totally ordered abelian group and that for all $k \in I$ the subsets $\Gamma_{\geq k}$ of $(n_i)_i \in \mathbb{Z}^I$ such that $n_i = 0$ for all $i < k$ are isolated subgroups of $\mathbb{Z}^I$.

(vi) Deduce that for every cardinal number k there are valuation rings whose spectrum has cardinality $\geq k$.

Proof: For (i) consider the map $\mathfrak{p} \mapsto A_\mathfrak{p}$. This reverses orientation, i.e. if $q \subset p$ for two prime ideals, then $(A \setminus \mathfrak{p})^{-1} = A_\mathfrak{p} \subset A_\mathfrak{q} = (A \setminus \mathfrak{q})^{-1}A$. This is also a valuation ring since for any $a \in \mathrm{Frac}A = K$ either $a \in A$ or $a^{-1} \in A$ is naturally satisfied for $A \subset A_\mathfrak{p}$.

To see that this is a bijection, we consider a local ring $A \subset B \subset K$, which we map to the preimage under the inclusion $A \subset B$ of its unique maximal ideal $\mathfrak{m}_B$. This is inverse to the localization above since

$$\mathfrak{m}_{A_\mathfrak{p}} = \mathfrak{p}A_\mathfrak{p}$$

for a ring $A_\mathfrak{p}$ this is given by $\mathfrak{p}$.

On the other hand for a local ring $A \subset B \subset K$ the localization at the preimage of the maximal ideal $\mathfrak{m}_B$ has an inclusion into B by the universal property of the localization. Since both of these rings are maximal with respect to domination and local they have to coincide.

For (ii) we can easily see that the set of isolated subgroups is totally ordered. Consider the map $\Delta \mapsto \mathfrak{p}_\Delta$. The image is a prime ideal since for $ab \in \mathfrak{p}_\Delta$ we have $v(ab) = v(a) + v(b) \notin \Delta$. This means that either a or b already have to be in $\mathfrak{p}_\Delta$.

We define the following inverse:

$$\mathfrak{p} \mapsto v(A \setminus \mathfrak{p}) \cup -v(A \setminus \mathfrak{p}).$$

This is a group since for $p, q \notin \mathfrak{p}$ we have $v(p) + v(q) = v(pq) \in v(A \setminus \mathfrak{p})$. We also need to check that it is isolated. Assume it is not isolated. Since $\mathfrak{p}$ is prime, the set $A \setminus \mathfrak{p}$ is multiplicatively closed. This means that $v(A\setminus\mathfrak{p}) \cup -v(A\setminus\mathfrak{p})$ is a group. On the other hand it contains any element between $-\alpha$ and α for $\alpha \in v(A \setminus \mathfrak{p})$ since otherwise for $0 \leq \beta \leq \alpha$ with $\beta \in v(\mathfrak{p})$ we would also have $v(A) + \beta \in v(\mathfrak{p})$. But since for $p \in \mathfrak{p}$ and $q \in A \setminus \mathfrak{p}$ with $v(p) = \beta$ and $v(q) = \alpha$ the element $v(p^{-1}q) = \alpha - \beta$ is in $v(A)$ or $-v(A)$ we would get that $\alpha \in v(\mathfrak{p})$ which is a contradiction.

We check the inverses:

$$v(\mathfrak{p}_\Delta) = v(A \setminus \mathfrak{p}_\Delta) \cup -v(A \setminus \mathfrak{p}_\Delta) = \Delta$$
$$\mathfrak{p}_{v(A\setminus\mathfrak{p})\cup -v(A\setminus\mathfrak{p})} = \{a \in A \mid v(a) \notin v(A \setminus \mathfrak{p}) \cup -v(A \setminus \mathfrak{p}_\Delta)\} = \mathfrak{p}$$

For (iii) the value group of A the kernel of the group morphism $K^\times \to v(K^\times)$ is given by $A^\times$. Thus for the value group of $A_{\mathfrak{p}_\Delta}$ this kernel is given by $(A \setminus \mathfrak{p}_\Delta)$. Thus the map $v\colon K^\times/(A \setminus \mathfrak{p}_\Delta) \to v(K^\times)$ is an isomorphism with $K^\times/(A \setminus \mathfrak{p}_\Delta) \cong \Gamma/\Delta$ since $v(A \setminus \mathfrak{p}_\Delta) = \Delta$.

For (iv) consider the group algebra $R = \Bbbk[\Gamma]$. Assume there are zero-divisors, i.e. $u = \sum_\gamma a_\gamma e_\gamma$ and $w = \sum_\eta b_\eta e_\eta$ with $uw = 0$. Then $v(u) = \gamma_0$ and $v(w) = \eta_0$. Then $uw = \sum_{\eta,\gamma} a_\gamma b_\eta e_{\gamma+\eta}$ where the summand $a_{\gamma_0} b_{\eta_0} e_{\gamma_0+\eta_0}$ is non-zero since $\Bbbk$ is integral.

Extending v to $K = \mathrm{Frac}R$ in the obvious way, i.e. $v(\frac{u}{w}) = v(u) - v(w)$ we have that $v(K^\times) \cong \Gamma$ by definition. And v fulfills

- $v(ab) = v(a) + v(b)$,
- $v(a + b) \geq \min\{v(a), v(b)\}$ with $>$ if and only if $v(a) = v(b)$ and the coefficients cancel,
- $v(0) = \infty$ by definition, $\square$

and is thus a valuation on K.

For (v) consider $(n_i)_{i\in I}$ and $(m_i)_{i\in I}$. If $J = \emptyset$ we have $(n_i) = (m_i)$ and $(n_i) < (m_i)$ as well as $(n_i) > (m_i)$. Thus assume that $J \neq \emptyset$. Then there exists i_J as a minimal element in J. Then, since $\mathbb{Z}$ are totally ordered, we get that (m_i) and (n_i) are comparable. Reflexivity, antisymmetry and transitivity are clear by total order in $\mathbb{Z}$.

The structure of an abelian group in $\mathbb{Z}^I$ is just given by the pointwise group structure.

The subsets $\Gamma_{\geq k}$ of $(n_i)_{i\in I}$ such that $n_i = 0$ for all $i < k$ are isolated. Let $0 \leq (n_i) \leq (m_i)$ for $(m_i) \in \Gamma_{\geq k}$. Then we have $n_j = 0$ for all $j < k$ since it is “sandwiched”. Thus $(n_i) \in \Gamma_{\geq k}$.

For (vi) we just combine the previous parts. Thus from $(n_i) \in \Gamma_{\geq k}\Bbbk[\mathbb{Z}^I]$ we get a $(n_i) \in \Gamma_{\geq k}$ valuation ring with spectrum cardinality $(n_i) \in \Gamma_{\geq k}$given by $|I|$.

3 Part 3

3.1 Collection of exercises on the basics of sheaves

Exercise 3.1.1 (2.1.A) See [3, 2.1.A]. Show that for the sheaf of $\mathscr{C}^\infty$ functions on a topological space denoted with $\mathscr{O}$ the stalk $\mathscr{O}_p$ is a local ring with maximal ideal given by

$$\mathfrak{m}_p = \{(f, U) \mid p \in U,\ f \in \mathscr{O}(U),\ f(p) = 0\}.$$

PROOF: Let $I' \in \mathscr{O}_p$ be another maximal ideal not equal to $\mathfrak{m}_p$. Then there exists a germ (f, U) such that $f(p) \neq 0$ in I'. This however is invertible, since for $f(p) \neq 0$ there exists $V \subset U$ such that $f(x) \neq 0$ for all $x \in V$. Then (f^{-1}, V) is a well defined germ in $\mathscr{O}_p$ with $(f, U) \cdot (f^{-1}, V) = (1, V)$ which is a representative of $1 \in \mathscr{O}_p$. This means that $I' = \mathscr{O}_p$ and we are done. $\square$

Exercise 3.1.2 (2.2.A) See [3, 2.2.A]. Show that the data of a $\mathfrak{C}$ valued presheaf on a topological space X is equivalent to a contravariant functor

$$\mathscr{F} : \mathrm{Opens}(X)^{\mathrm{op}} \to \mathfrak{C}$$

from the category $\mathrm{Opens}(X)$ of the open sets of X with morphisms given by the inclusion of subsets. This is especially a directed set and thus a nice category to work with.

PROOF: The conditions that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ is and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ are just equivalent to the functoriality of above functor. $\square$

Exercise 3.1.3 (2.2.B) See [3, 2.2.B]. Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves:

- bounded functions,
- holomorphic functions admitting a holomorphic square root.

PROOF: For the bounded functions it is clear that the bounded open subsets cover $\mathbb{C}$, just consider the discs with radius $r \in \mathbb{N}$. On each of those the function $f(z) = z$ is bounded, but it does not glue to a globally bounded function.

Similarly the function $f(z) = z$ admits a holomorphic square root for the two open subsets $\mathbb{C}\setminus(0, \infty)$ and $\mathbb{C} \setminus (-\infty, 0)$. These cover $\mathbb{C}$. But it does not admit a global holomorphic square root. $\square$

Exercise 3.1.4 (2.2.C) See [3, p. 2.2.C]. The identity and gluability axioms may be interpreted as saying that $(\cup_{i\in I} U_i)$ is a certain limit. What is that limit?

PROOF: Gluability corresponds to the limit

$$\mathscr{F}(\cup_{i\in I} U_i) = \lim(\cup_{i,j}\mathscr{F}(U_i \cap U_j) \leftleftarrows \cup_{k\in I}\mathscr{F}(U_k))$$

where the two arrows are given by $\mathrm{res}_{U_j, U_j \cap U_i}$ and $\mathrm{res}_{U_i, U_j \cap U_i}$, see the notes for the equalizer definition of sheaves for more detail. $\square$

Exercise 3.1.5 (2.2.D) See [3, p. 2.2.D].

- Verify that smooth functions, continuous functions, real-analytic funcitons, plain real-valued functions, on a manifold or Rn are indeed sheaves.
- Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

PROOF: Trivial. $\square$

Exercise 3.1.6 (2.2.E) See [3, 2.2.E]. Let $\mathscr{F}(U)$ be the maps to S that are locally constant, i.e., for any point p in U, there is an open neighborhood of p where the function is constant. Show that this is a sheaf. (A better description is this: endow S with the discrete topology, and let $\mathscr{F}(U)$ be the continuous maps $U \to S$.) This is called the constant sheaf (with values in S); do not confuse it with the constant presheaf. We denote this sheaf $\underline{S}$.

First we show that the constant presheaf is not a sheaf.

PROOF: Let

$$\mathscr{F} : \mathrm{Opens}_X^{\mathrm{op}} \to \{S\}$$

be a functor with $|S| \geq 2$ defined by $\mathscr{F}(U) = S$ and $\mathrm{res}_{V,U} = \mathrm{id}_S$ for all V, U. Then this is obviously a presheaf. This is not a sheaf since for $x, y \in S$ and $U, V \subset X$ with $U \cap V = \emptyset$ the sections $x\big|_U$ and $y\big|_V$ have no glueing on $V \cup U$.

For the second part see the next exercise. $\square$

Exercise 3.1.7 (2.2.F) See [3, 2.2.F]. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of U to Y. Show that this forms a sheaf.

PROOF: We define the sheaf in more detail:

$$\mathscr{C}^0(\cdot, Y)\colon \mathrm{Opens}_X^{\mathrm{op}} \to \mathsf{Set}$$

is defined by mapping an object $U \subset X$ open to $\mathscr{C}^0(U, Y)$ and a morphism $\iota_{V,U}\colon U \to V$ in $\mathrm{Opens}_X^{\mathrm{op}}$ to $\iota_{V,U}^*\colon \mathscr{C}^0(V, Y) \to \mathscr{C}^0(U, Y)$ via pullback. This is a sheaf since:

- $\mathrm{res}_{V,U} = \iota_U^*$ fulfill the functoriality, i.e. for $V \subset W \subset U$ we have $\mathrm{res}_{U,V} = \iota_{V,U}^* = (\iota_{W,U} \circ \iota_{V,W})^* = \iota_{V,W}^* \circ \iota_{W,U}^* = \mathrm{res}_{W,V} \circ \mathrm{res}_{W,U}$ and $\mathrm{res}_{U,U} = \mathrm{id}_U^* = \mathrm{id}$.
- Identity axiom holds, since it can be checked pointwise for functions.
- For some open cover $\{U_i\}_{i \in I}$ of X and $f_i \in \mathscr{C}^0(U_i, Y)$ with $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$ for all i, j, we define $f \in \mathscr{C}^0(X, Y)$ by $f(p) = f_i(p)$ for $p \in U_i$. This is well defined since for $p \in U_i \cap U_j$ we have $f_i(p) = f_j(p)$. This is continuous, since $f^{-1}(W) = \cup_{i \in I} f_i^{-1}(W)$ by definition for some $W \subset Y$ open.

We have thus checked that this indeed gives a sheaf. For $Y = S$ with the discrete topology this reproduces the last exercise. $\square$

Exercise 3.1.8 (2.2.G) See [3, 2.2.G]. This is a fancier version of the previous exercise.

- (sheaf of sections of a map) Suppose we are given a continuous map $\mu\colon Y \to X$. Show that "sections of μ" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps $s\colon U \to Y$ such that $\mu \circ s = \mathrm{id}\big|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.) This is motivation for the phrase "section of a sheaf".

- (This exercise is for those who know what a topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that Y is a topological group. Show that continuous maps to Y form a sheaf of groups.

PROOF: We define the following sheaf:

$$\Gamma_s(\cdot)\colon \mathrm{Opens}_X^{\mathrm{op}} \to \mathsf{Set}$$

by setting

$$\Gamma_\mu(U) = \{f \in \mathscr{C}^0(U,Y) \mid \mu \circ f = \mathrm{id}\big|_U\}$$

and mapping an inclusion $\iota_{V,U}\colon U \to V$ in $\mathrm{Opens}_X^{\mathrm{op}}$ to $\iota_{V,U}^*\colon \mathscr{C}^0(V,Y) \to \mathscr{C}^0(U,Y)$ via pullback. We need to show that this is well defined. We see this by calculating $\mu \circ \mathrm{res}_{U,V} f = \mu \circ \iota_{V,U}^* f = \mu \circ f\big|_V = \mathrm{id}\big|_V$ for $f \in \Gamma_\mu(U)$. The functoriality follows analogously to the previous exercise.

The identity axiom also holds by equal argument.

For the gluability it remains to see that f defined as before is contained in $\Gamma_\mu(X)$. This can be checked pointwise:

For all $p \in X$ there is a U_i in the open cover such that $p \in U_i$. But then $\mu \circ f(p) = \mu \circ f\big|_{U_i}(p) = \mu \circ f_i(p) = \mathrm{id}\big|_{U_i}(p) = p$. Since this holds for all $p \in X$ we are done.

For the second part we can consider the same sheaf as in the previous exercise.

We define the group operation by

$$m\colon \mathscr{C}^0(U,Y) \times \mathscr{C}^0(U,Y) \to \mathscr{C}^0(U,Y), \quad (f(p), g(p)) \mapsto f(p) \cdot g(p)$$

which is the pointwise operation. This is obviously well defined. It remains to show that inverse exist. This is the case since for $f \in \mathscr{C}^0(U,Y)$ we can define $f^{-1} \in \mathscr{C}^0(U,Y)$ by $f^{-1}(p) = (f(p))^{-1}$. This is continuous since forming inverses is continuous. The neutral element is obviously given by the constant map to the group unit. $\square$

Exercise 3.1.9 (2.2.H) See [3, 2.2.H]. Suppose $\pi\colon X \to Y$ is a continuous map, and $\mathscr{F}$ is a presheaf on X. Then define $\pi_*\mathscr{F}$ by $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$, where V is an open subset of Y. Show that $\pi_*\mathscr{F}$ is a presheaf on Y, and is a sheaf if $\mathscr{F}$ is. This is called the pushforward or direct image of $\mathscr{F}$. More precisely, $\pi_* F$ is called the pushforward of $\mathscr{F}$ by π.

PROOF: We define $\pi_* \mathrm{res}_{U,V}\colon \pi_*\mathscr{F}(U) \to \pi_*\mathscr{F}(V)$ by $\pi_* \mathrm{res}_{U,V} = \mathrm{res}_{\pi^{-1}(U),\pi^{-1}(V)}$. This is well defined, since for $V \subset U$ also $\pi^{-1}(V) \subset \pi^{-1}(U)$. Then we fulfill the conditions for a presheaf since res fulfills these conditions.

To show that it is even a sheaf, we check identity and glueability: For identity it is clear that for a cover $\{W_i\}_{i\in I}$ of Y the set $\{U_i \coloneqq \pi^{-1}(W_i)\}_{i\in I}$ is an open cover of X. Let $f, g \in \pi_*\mathscr{F}(Y)$ with $\pi_* \mathrm{res}_{Y,U_i} f = \pi_* \mathrm{res}_{Y,W_i} g$ for all W_i. Then $f \in \mathscr{F}(X)$ with $\mathrm{res}_{X,U_i} f = \mathrm{res}_{X,U_i} g$. Since the U_i are an open cover $f = g \in \mathscr{F}(X)$ follows, which implies that $f = g \in \pi_*^{-1}\mathscr{F}(Y)$.

For the gluability pick $f_i \in \pi_*\mathscr{F}(W_i)$ with $\pi_* \mathrm{res}_{W_i,W_i\cap W_j} f_i = \pi_* \mathrm{res}_{W_j,W_i\cap W_j} f_j$. This means that for $f_i \in \mathscr{F}(U_i)$ we have $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_i\cap U_j} f_j$. Since $\mathscr{F}$ is a sheaf, we can glue the f_i to get $f \in \mathscr{F}(X)$ which corresponds to $f \in \pi_*\mathscr{F}(Y)$ with $\pi_* \mathrm{res}_{Y,W_i} f = \mathrm{res}_{X,U_i} f = f_i$ by construction. Thus $\pi_*\mathscr{F}$ is a sheaf if $\mathscr{F}$ is. $\square$

Exercise 3.1.10 (2.2.I) See [3, p. 2.2.I]. Suppose $\pi\colon X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf of sets (or rings or A-modules) on X. If $\pi(p) = q$, describe the natural morphism of stalks $(\pi_*\mathscr{F})_q \to F_p$.

Proof: Using explicit forms of stalks we get

$$(\pi_*\mathscr{F})_q = \{(g, V) \mid q \in V,\ g \in \pi_*\mathscr{F}(V)\}/\sim$$

and we map this to

$$\{(g \circ \pi, \pi^{-1}(V)) \mid q \in V, \mid g \in \pi_* g \in \pi_*\mathscr{F}(V)\}/\sim \subset \mathscr{F}_p.$$

This is well defined since $f \sim g$ in $(\pi_*\mathscr{F})_q$ means by continuity that there is $\pi^{-1}(W) \subset X$ with $q \in W$ such that $f \circ \pi = g \circ \pi$ on $\pi^{-1}(W)$.

For the universal approach we consider $\mathscr{F}_x = \operatorname{colim}_x \mathscr{F}(U)$ where the colimit is taken over the directed set of open subsets containing x. Then we can consider

$$\begin{tikzcd} \operatorname{colim}_q \pi_*\mathscr{F}(V) \arrow[r, "="] \arrow[rr, bend left, "\exists!"] & \operatorname{colim}_q \mathscr{F}(\pi^{-1}(V)) \arrow[r, hook] & \operatorname{colim}_p \mathscr{F}(U) \end{tikzcd} \tag{3.1.1}$$

since the condition that π is continuous is equivalent to the condition that the subsets $\pi^{-1}(V)$ for neighbourhoods of q are a cofinal subset of the neighbourhood basis of p, i.e. for all $q \in W$ there exists $p \in V$ such that $\pi(V) \subset W$.

Then we could also derive this result since cofinal functors preserve colimits, see notes. $\square$

Exercise 3.1.11 (2.2.J) See [3, 2.2.J]. If $(X, \mathscr{O}_X)$ is a ringed space, and $\mathscr{F}$ is an $\mathscr{F}_X$-module, describe how for each $p \in X$, $\mathscr{F}_p$ is an $\mathscr{O}_{X,p}$-module.

Proof: We can take the limit over the following diagram

$$\begin{tikzcd} \mathscr{O}_X(V) \times \mathscr{F}(V) \arrow[r, "\triangleright"] \arrow[d, "\mathrm{res}_{V,U} \times \mathrm{res}_{V,U}"] & \mathscr{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathscr{O}_X(U) \times \mathscr{F}(U) \arrow[r, "\triangleright"] \arrow[d, "\mathrm{res}_{U,W} \times \mathrm{res}_{U,W}"] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,W}"] \\ \mathscr{O}_X(W) \times \mathscr{F}(W) \arrow[r, "\triangleright"] \arrow[d, dashed] & \mathscr{F}(W) \arrow[d, dashed] \\ \operatorname{colim}_p \mathscr{O}_X \times \mathscr{F} \arrow[r, "\exists!"] & \operatorname{colim}_p \mathscr{F} \end{tikzcd} \tag{3.1.2}$$

Then the action of $\mathscr{O}_{X,p}$ is induced by the lowest map. Explicitly we define $\sigma \triangleright f$ for $\sigma \in \mathscr{O}_p$ and $f \in \mathscr{F}_p$ by defining for two representatives (σ, V) and (f, U) on the restriction $(\sigma, V) \triangleright (f, U) := (\sigma\big|_{U \cap V} \triangleright f\big|_{U\cap V}, U \cap V)$. This is well defined, since the action commutes with the restriction maps. $\square$

Exercise 3.1.12 (2.3.A) See [3, 2.3.A]. If $\phi\colon \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, and $p \in X$, describe an induced morphism of stalks $\phi_p\colon \mathscr{F}_p \to \mathscr{G}_p$.

Proof: Again by the property that ϕ is a natural transformation, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\phi"] \arrow[d, "\mathrm{res}_{V,U}"] & \mathscr{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
\mathscr{F}(U) \arrow[r, "\phi"] \arrow[d, "\mathrm{res}_{U,W}"] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,W}"] \\
\mathscr{F}(W) \arrow[r, "\phi"] \arrow[d, dashed] & \mathscr{G}(W) \arrow[d, dashed] \\
\operatorname{colim}_p \mathscr{F} \arrow[r, "\exists !\, \phi_p"] & \operatorname{colim}_p \mathscr{G}
\end{tikzcd} \tag{3.1.3}$$

and thus for $\tau_U : \mathscr{G}(U) \to \operatorname{colim}_p \mathscr{G} = \mathscr{G}_p$ the maps $\tau_U \circ \phi$ form a cone $\{\tau_U \circ \phi \colon \mathscr{F}(U) \to \mathscr{G}_p\}_{p \in U}$ which induces the morphism $\phi_p \colon \mathscr{F}_p \to \mathscr{G}_p$ by universal property. □

Exercise 3.1.13 (2.3.B) See [3, 2.3.B]. Suppose $\pi \colon X \to Y$ is a continuous map of topological spaces (i.e., a morphism in the category of topological spaces). Show that pushforward gives a functor $\pi_* \colon \mathsf{Set}_X \to \mathsf{Set}_Y$. Here Sets can be replaced by other categories.

Proof: It is clear that $\pi_* \mathscr{F}(U)$ is a sheaf. It remains to show, that for morphisms of Set_X given by $\phi \colon \mathscr{F} \to \mathscr{G}$ and $\psi \colon \mathscr{G} \to \mathscr{H}$ we have for the composition

$$\pi_*(\psi \circ \phi) = \pi_*(\psi) \circ \pi_*(\phi). \tag{3.1.4}$$

We define this functor on a morphism $\{\phi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)\}_{U \in \mathrm{Opens}_X}$ by

$$\pi_* \phi \coloneqq \{\pi_* \phi_V \coloneqq \phi_{\pi^{-1}(V)} \colon \pi_* \mathscr{F}(V) \to \pi_* \mathscr{G}(V)\}_{V \in \mathrm{Opens}_Y}. \tag{3.1.5}$$

Then the functoriality is given by the following diagram:

$$\begin{tikzcd}
\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V)) \arrow[r, "\phi_{\pi^{-1}(V)}"] \arrow[d, "\pi_* \mathrm{res}_{V,W} = \mathrm{res}_{\pi^{-1}(V),\pi^{-1}(W)}"] & \pi_*\mathscr{G}(V) = \mathscr{G}(\pi^{-1}(V)) \arrow[r, "\psi_{\pi^{-1}(V)}"] \arrow[d, "\pi_* \mathrm{res}_{V,W} = \mathrm{res}_{\pi^{-1}(V),\pi^{-1}(W)}"] & \pi_*\mathscr{H}(V) = \mathscr{H}(\pi^{-1}(V)) \arrow[d, "\pi_* \mathrm{res}_{V,W} = \mathrm{res}_{\pi^{-1}(V),\pi^{-1}(W)}"] \\
\pi_*\mathscr{F}(W) = \mathscr{F}(\pi^{-1}(W)) \arrow[r, "\phi_{\pi^{-1}(V)}"] & \pi_*\mathscr{G}(W) = \mathscr{G}(\pi^{-1}(W)) \arrow[r, "\psi_{\pi^{-1}(V)}"] & \pi_*\mathscr{H}(W) = \mathscr{H}(\pi^{-1}(W))
\end{tikzcd} \tag{3.1.6}$$

and the fact that $\mathrm{id}_* \, \mathrm{id}_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$ by construction. □

Exercise 3.1.14 (2.3.C) See [3, p. 2.3.C]. Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X. (In fact, it will suffice that $\mathscr{F}$ is a presheaf.) Let $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ be the collection of data

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G})(U) \coloneqq \mathrm{Hom}(\mathscr{F}\big|_U, \mathscr{G}\big|_U).$$

(Recall the notation $\mathscr{G}\big|_U$, the restriction of the sheaf to the open set U) Show that this is a sheaf of sets on X. This sheaf is called "sheaf Hom".

Proof: Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ a sheaf. We define the restriction maps for $V \subset U$ by

$$\mathrm{res}_{U,V} \colon \mathrm{Hom}(\mathscr{F}, \mathscr{G})(U) \to \mathrm{Hom}(\mathscr{F}, \mathscr{G})(V), \quad \{\phi_W\}_{W \in \mathrm{Opens}_U} \mapsto \{\phi_W\}_{W \in \mathrm{Opens}_V}$$

This is well defined, since $\{\phi_W\}_{W \in \mathrm{Opens}_V}$ is still a natural transformation $\mathscr{F}\big|_V \to \mathscr{G}\big|_V$.

Now for the functoriality it is clear that this holds since we just make our set of morphisms in the natural transformation smaller, i.e. forgetting stuff is functorial.

For the identity we consider a covering $\{U_i\}_{i\in I}$ of X. Then for two natural transformations $f, g\colon \mathscr{F} \to \mathscr{G}$ such that $\{f_W\}_{W\in \mathrm{Opens}_{U_i}} = \{f_W\}_{W\in \mathrm{Opens}_{U_i}}$ we need to show that for any open set $V \subset X$ we have $f_V = g_V$. Then the set $\{U_i \cap V\}_{\in I}$ is an open covering of U and $\mathrm{res}_{V,U_i\cap V} f_V = \mathrm{res}_{V,U_i\cap V} g_V$ since we assumed that $\{f_W\}_{W\in \mathrm{Opens}_{U_i}} = \{f_W\}_{W\in \mathrm{Opens}_{U_i}}$ and $U_i \cap V$ is open in U_i. Thus, since $\mathscr{G}\big|_V$ is a sheaf, we have $f_V = g_V$, which means that $\{f_V\}_{V\in\mathrm{Opens}_X} = \{g_V\}_{V\in\mathrm{Opens}_X}$.

For the gluability we consider the set $\{\{f^i_W\}_{W\in\mathrm{Opens}_{U_i}}\}_{i\in I}$ such that

$$\{f^i_W\}_{W\in\mathrm{Opens}_{U_i\cap U_j}} = \{f^j_W\}_{W\in\mathrm{Opens}_{U_i\cap U_j}}.$$

We have to construct a global $\{f_V\}_{V\in\mathrm{Opens}_X}$, i.e. we need to define an f_V for all open subsets of X. It is clear again that $\{U_i \cap V\}_{i\in I}$ is an open cover of V. Then the maps $\{f^i_{U_i\cap V}\}_{i\in I}$ fulfill the condition that on $U_i \cap U_j \cap V$ we have $f^i_{U_i\cap V}\big|_{U_j\cap V} = f^i_{U_i\cap U_j\cap V} = f^j_{U_i\cap U_j\cap V} = f^j_{U_j\cap V}\big|_{U_i}$ and we can thus glue these to a map $f\big|_V$. Then the set $\{f_V\}_{V\in\mathrm{Opens}_X}$ fulfills the demands. ☐

Exercise 3.1.15 (2.3.D) See [3, p. 2.3.D].

- If $\mathscr{F}$ is a sheaf of sets on X, then show that $\mathrm{Hom}(\{p\}, \mathscr{F}) \cong \mathscr{F}$, where $\{p\}$ is the constant sheaf "with values in the one element set $\{p\}$".
- If $\mathscr{F}$ is a sheaf of abelian groups on X, then show that $\mathrm{Hom}_{\mathsf{Ab}}(\underline{Z}, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of sheaves of abelian groups).
- If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then show that $\mathrm{Hom}_{\mathsf{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of OX-modules).

Proof: For the first part notice that for any $U \in \mathrm{Opens}_X$ the set $\mathrm{Hom}(\{p\}\big|_U, \mathscr{F}\big|_U)(V)$ is given by the elements of $\mathscr{F}\big|_U(V)$. The isomorphism is thus given by $\mathrm{Hom}(\{p\}\big|_X, \mathscr{F}\big|_X)(V) \cong \mathscr{V}$.

The similar argument holds in the second case, since the morphisms of $\underline{Z}\big|_U(V) \to \mathscr{F}\big|_U(V)$ corresponds to the subgroups generated by a unique element.

The same argument holds for the last statement. ☐

3.2 Compatible germs

Exercise 3.2.1 (2.4.A) See [3, 2.4.A]. The natural map

$$\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$$

is injective.

Proof: Consider for each $U \in \mathrm{Opens}_X$ and $\mathscr{F} \in \mathfrak{C}_X$ the set $\prod_{p\in U} \mathscr{F}_p$. This mapping

$$\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$$

is injective. For $f, g \in \mathscr{F}$ such that $f_p = g_p$ for all $p \in U$. Then there are open subsets $U_p \subset U$ and representatives $(\tilde{f}, U_p)$ and $(\tilde{g}, U_p)$ (do this by picking representatives and restricting to the intersection of their domain of definition) such that $\tilde{f} = \tilde{g}$. The set $\{U_p\}_{p\in U}$ is an open cover of U and thus by the identity axiom of a sheaf $f = g$ holds. ☐

Definition 3.2.1 *We say that an element* $(s_p)_{p\in U}$ *of the right side* $\prod_{p\in U}\mathscr{F}_p$ *of the previous exercise consists of compatible germs if for all* $p\in U$*, there is some representative*

$$(\tilde{s}_p \in \mathscr{F}(U_p),\ U_p \text{ open in } U);$$

for s_p *(where* $p\in U_p$*) such that the germ of* $\tilde{s}_p$ *at all* $q\in U_p$ *is* s_q*. Equivalently, there is an open cover* $\{U_i\}$ *of* U*, and sections* $f_i\in\mathscr{F}(U_i)$*, such that if* $p\in U_i$*, then* s_p *is the germ of* f_i *at* p*. Clearly any section* s *of* F *over* U *gives a choice of compatible germs for* U*.*

We denote the set of $(s_p)_{p\in U}\in\prod_{p\in U}\mathscr{F}_p$ *such that it consists of compatible germs by* $\mathrm{CGerms}_{\mathscr{F}}(U)$*.*

Exercise 3.2.2 (2.4.B) See [3, 2.4.B]. Prove that any choice of compatible germs for a sheaf of sets $\mathscr{F}$ over U is the image of a section of F over U.

Proof: For an element $(s_p)_{p\in U}\in\prod_{p\in U}\mathscr{F}_p$ in the set of compatible germs, we can for all $p\in U$ find a representative $\tilde{s}_p\in\mathscr{F}(U_p)$ for some U_p. The sets U_p are an open cover of U. We need to check that for $U_q\cap U_p\neq\emptyset$ we have $\tilde{s}_p\big|_{U_q\cap U_p}=\tilde{s}_q\big|_{U_q\cap U_p}$. We see this, since for all $x\in U_q\cap U_p$ we have $(\tilde{s}_p\big|_{U_q\cap U_p})_x=(\tilde{s}_q\big|_{U_q\cap U_p})_x=s_x$. This means that there is a neigbourhood V_x such that $\tilde{s}_p\big|_{V_x}=\tilde{s}_q\big|_{V_x}$ and these V_x cover $U_p\cap U_q$. Thus applying the gluability, we get $\tilde{s}_p\big|_{U_q\cap U_p}=\tilde{s}_q\big|_{U_q\cap U_p}$. Thus we can again apply gluability to obtain a section $\tilde{s}\in\mathscr{F}(U)$. ☐

Proposition 3.2.2 *We can consider the functor defined by*

$$\mathrm{CGerms}_{\mathscr{F}}\colon \mathrm{Opens}_X^{\mathrm{op}}\to\mathsf{Set},\ U\mapsto\mathrm{CGerms}_{\mathscr{F}}(U),$$

An inclusion $\iota V\to U$ *is mapped to the map* $\prod_{p\in V}\mathrm{pr}_p$ *of sets. This functor is naturally isomorphic to the sheaf* $\mathscr{F}\in\mathsf{Set}_X$*.*

Proof: First notice that $\mathrm{CGerms}_{\mathscr{F}}$ is a functor at all. This holds since $\prod_{p\in U}\mathrm{pr}_p\prod_{p\in U}\{s_p\}=\prod_{p\in U}\{s_p\}$ and $\prod_{p\in W}\mathrm{pr}_p\prod_{p\in V}\mathrm{pr}_p=\prod_{p\in W}\mathrm{pr}_p$. We define the natural isomorphism

$$\eta_U\colon\mathrm{CGerms}_{\mathscr{F}}(U)\to\mathscr{F}(U)$$

by mapping an element of compatible germs to the unique section defined in the previous exercise. The inverse for this is given by the canonical map $\mathscr{F}(U)\to\prod_{p\in U}\mathscr{F}_p$ as we have seen in the previous exercises.

Now to check that η is a natural transformation:

$$\begin{tikzcd} \mathrm{CGerms}_{\mathscr{F}}(U) \arrow[r, "\prod_{p\in W}\mathrm{pr}_p"] \arrow[d, "\eta_U"] & \mathrm{CGerms}_{\mathscr{F}}(W) \arrow[d, "\eta_W"] \\ \mathscr{F}(U) \arrow[r, "\mathrm{res}_{U,W}"] & \mathscr{F}(W) \end{tikzcd} \tag{3.2.1}$$

For an element $(s_p)_{p\in U}$ there is a unique section $\eta_U((s_p)_{p\in U})\in\mathscr{F}(U)$ which we restrict to W. This restriction does not change the germs $\eta_U((s_p)_{p\in U})_p$ for all $p\in W$.

Thus by a similar argument to (2.4.B) we obtain $\eta_w(s_p)_{p\in W}=\mathrm{res}_{U,W}\,\eta_U((s_p)_{p\in U})$ which shows that η is a natural isomorphism. ☐

Bibliography

[1] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995. ISBN: 9780387942698. URL: https://books.google.de/books?id=Fm_yPgZBucMC.

[2] U. Görtz and T. Wedhorn. *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010. ISBN: 9783834897220. URL: https://books.google.de/books?id=XEiLudn6sq4C.

[3] Ravi Vakil. *THE RISING SEA: Foundations of Algebraic Geometry*. 2024. URL: https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.