Exercises

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Contents

1 Exercises Part 1

1.1 Irreducible topological spaces

Exercise 1.1.1 See [2, Exercise 1.3] Determine all irreducible Hausdorff spaces. Determine all noetherian Hausdorff spaces. Show that a topological space is noetherian if and only if every open subspace is quasi-compact.

Exercise 1.1.2 Prove Cayley Hamilton by using the irreducibility of $\mathbb{A}^{n^2}(\Bbbk)$ and the representation of the determinant as a polynomial.

Hint: Use matrices with n distinct Eigenvalues and show that these fulfill the statement of Cayley Hamilton. Then show that this set is open and non-empty.

1.2 Localization of rings and modules

Remark 1.2.1 (Localization of rings and modules) See [3, p. 1.3.3.]. Another important example of a definition by universal property is the notion of localization of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A multiplicative subset S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}\mathsf{A}$. The elements of $S^{-1}A$ are of the form $\frac{a}{s}$ where $a \in \mathsf{A}$ and $s \in S$, and where $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ if (and only if) for some $s \in S$, $s(s_2 a_1 - s_1 a_2) = 0$. We define $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2}$, and $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$. (If you wish, you may check that this equality of fractions really is an equivalence relation and the two binary operations on fractions are well-defined on equivalence classes and make $S^{-1}\mathsf{A}$ into a ring.) We have a canonical ring map $\mathsf{A} \to S^{-1}\mathsf{A}$ given by $a \mapsto \frac{a}{1}$. Note that if $0 \in S$, $S^{-1}\mathsf{A}$ is the 0-ring. There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, f^3, \cdots\}$ where $f \in \mathsf{A}$. This localization is denoted A_f. (Can you describe an isomorphism $\mathsf{A}_f \to \mathsf{A}[t]/(tf-1)$?) The second is $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}\mathsf{A}$ is denoted $\mathsf{A}_\mathfrak{p}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $\mathsf{A}_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in \mathsf{A}$, A_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $\mathsf{A}_f \neq \mathsf{A}_{(f)}$.)

Exercise 1.2.1 See [3, Exercise 1.3.C]. Show that $\mathsf{A} \to S^{-1}\mathsf{A}$ is injective if and only if S contains no zerodivisors. (A zerodivisor of a ring A is an element a such that there is a nonzero element b with $ab = 0$. The other elements of A are called non-zerodivisors. For example, an invertible element is never a zerodivisor. Counter-intuitively, 0 is a zerodivisor in every ring but the 0-ring. More generally, if M is an A-module, then $a \in \mathsf{A}$ is a zerodivisor for M if there is a nonzero $m \in M$ with $am = 0$. The other elements of A are called non-zerodivisors for M. Equivalently, and very usefully, $a \in \mathsf{A}$ is a non-zerodivisor for M if and only if $(-) \cdot a \colon M \to M$ is an injection, or equivalently if the sequence

$$0 \longrightarrow M \xrightarrow{(-)\cdot a} M \tag{1.2.1}$$

is exact.) If A is an integral domain and $S = \mathsf{A} \setminus \{0\}$, then $S^{-1}\mathsf{A}$ is called the fraction field of A, which we denote $K(\mathsf{A})$. The exercise shows that A is a subring of its fraction field.

Proof: Assume that $A \mapsto S^{-1}A$, $a \mapsto \frac{a}{1}$ is not injective. Then there exist a_1, a_2 such that

$$\frac{a_1}{1} \sim \frac{a_2}{1} \Leftrightarrow s(a_1 - a_2) = 0$$

for some $s \in S$. Since $a_1 \neq a_2$ this means that S contains zero divisors.

Conversely, if S contains no zero divisors, the equation

$$\frac{a_1}{1} \sim \frac{a_2}{1} \Leftrightarrow s(a_1 - a_2) = 0$$

implies that $a_1 = a_2$, hence the inclusion is injective. □

Exercise 1.2.2 See [3, Exercise 1.3.D.] Verify that $\mathsf{A} \to S^{-1}\mathsf{A}$ satisfies the following universal property:

$S^{-1}\mathsf{A}$ is initial among A-algebras $\mathscr{B}$ where every element of S is sent to an invertible element in $\mathscr{B}$.

(Recall: the data of "an A-algebra $\mathscr{B}$" and "a ring map $\mathsf{A} \to \mathscr{B}$" are the same.)

Proof: Let $\mathscr{B}$ be an A-algebra such that S is invertible i.e. an algebra morphism $f\colon \mathsf{A} \to \mathscr{B}$ such that $f(S)$ consists of invertible elements of $\mathscr{B}$.

We construct a unique morphism $\hat{f}\colon S^{-1}\mathsf{A} \to \mathscr{B}$. We define

$$\hat{f}(\frac{a}{s}) \coloneqq f(a)f(s)^{-1}.$$

This is well defined since $f(s)$ is invertible. This is unique by construction, i.e. $S^{-1}\mathsf{A}$ is initial. □

1.3 Algebraic varieties and prevarieties

Exercise 1.3.1 See [2, Exercise 1.4]. Show that the underlying topological space X of a prevariety is a T1-space (i.e., for all $x, y \in X$ there exist open neighborhoods U of x and V of y with $y \notin U$ and $x \notin V$).

Exercise 1.3.2 See [2, Exercise 1.5]. Consider the twisted cubic curve $C = \{(t, t^2, t^3) \mid t \in \Bbbk\} \subset \mathbb{A}^3(\Bbbk)$. Show that C is an irreducible closed subset of $\mathbb{A}^3(\Bbbk)$. Find generators for the ideal $I(C)$. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(\Bbbk)$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Proof: We have $I(C) = (Y - X^2, Z - X^3) \trianglelefteq \Bbbk[X, Y, Z]$ since $(x, y, z) \in C$ if and only if $y = x^2$ and $z = x^3$. We present two proofs to show that C is irreducible:

First proof: To show that $I(C)$ is prime, we compute $\Gamma(C) = \Bbbk[X, Y, Z]/I(C)$. We write $a := Y - X^2$ and $b = Z - X^3$, then we use $R/(a, b) = (R/(a))/(\bar{b})$ for $r = \Bbbk[X, Y, Z]$. The ring homomorphism $\varphi : \Bbbk[X, Y, Z] \to \Bbbk[X, Z]$ with $\varphi(X) = X$, $\varphi(Z) = Z$ and $\varphi(Y) = X^2$ has kernel $(Y - X^2)$, since for $f \in \ker \varphi$ we can do polynomial division by $Y - X^2$ and write $f = (Y - X^2)q + r$ with $\deg_Y r \geq 0$, i.e. $r \in \Bbbk[X, Z]$. So we get $\varphi(f) = \varphi(r) = r$ and therefore $r = 0$ and f is a multiple of $Y - X^2$. Now we apply the same trick for b and get $\Gamma(C) \cong \Bbbk[X]$. This ring is an integral domain and therefore C irreducible.

Second proof: We show that C is isomorphic to $\mathbb{A}^1(\Bbbk)$, as affine algebraic sets. The maps $\varphi : C \to \mathbb{A}^1(\Bbbk)$, $(x, y, z) \mapsto x$ and $\psi : \mathbb{A}^1(\Bbbk) \to C$, $t \mapsto (t, t^2, t^3)$ are morphisms of algebraic sets and mutual inverses. Therefore they are isomorphisms and since $\mathbb{A}^1(\Bbbk)$ is irreducible C is also irreducible.

For the second part a case analysis shows that $V = V_1 \cup V_2 \cup V_3$, where

$$V_1 = V(X, Y), \quad V_2 = V(X, Z) \quad \text{and} \quad V_3 = (Y - X^2, Z - 1).$$

Similar to the arguments above, we can compute $\Gamma(V_1) \cong \Bbbk[Z]$, $\Gamma(V_2) \cong \Bbbk[Y]$ and $\Gamma(V_3) \cong \Bbbk[X]$ to see that each of these components is irreducible. □

Exercise 1.3.3 See [2, Exercise 1.14]. Let X be a prevariety and let Y be an affine variety. Show that the map

$$\operatorname{Hom}(X,Y) \to \operatorname{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y),\Gamma(X)),\ f \mapsto (f^*\colon \phi \mapsto \phi \circ f),$$

is bijective. Deduce that $\operatorname{Hom}(X,\mathbb{A}^n(\Bbbk)) = \Gamma(X)^n$.

Proof: Let us first consider the case that X and Y are affine varieties. Then a map $f\colon X \to Y$ is a component-wise polynomial map $f\colon \Bbbk^n \supset X \to Y \subset \Bbbk^m$ with

$$f(x_1,\cdots,x_n) = (f_1(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n)).$$

Then the result to show is that the contravariant functor

$$\Gamma(\cdot)\colon \mathrm{AffineVarieties} \to \mathrm{int}-\mathrm{Alg}, \quad (f\colon X \to Y) \mapsto (f^*\colon \Gamma(Y) \to \Gamma(X))$$

from affine varieties into integral $\Bbbk$-algebras is full and faithful. We show this by constructing explicit isomorphisms

$$\Phi\colon\ \operatorname{Hom}(X,Y) \leftrightarrow \operatorname{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y),\Gamma(X))\colon\ \Psi.$$

We define $\Phi(f) = f^*$. To define Ψ we want to find for any $\alpha\colon \Gamma(Y) \to \Gamma(X)$ an element $f\colon X \to Y$ such that $f^* = \alpha$, since then $\Psi(\Phi(f)) = \Psi(\alpha) = f$ would be a left inverse map.

For this we define

$$\Psi(\alpha) = (\alpha(Y_1),\cdots,\alpha(Y_m))$$

where we denote with Y_n the map $\mathrm{pr}_n\big|_Y\colon \Bbbk^m \supset Y \to \Bbbk$ which is a morphism of affine varieties between Y and $\mathbb{A}^1$. Note, that we need to show that this map is well defined. I.e. that for the prime-ideal $\mathfrak{p}$ such that $Y = V(\mathfrak{p})$ we have $p(\Psi(\alpha)(x)) = 0$ for any $x \in X$ and $p \in \mathfrak{p}$. This however is clear since

$$(Y_1,\cdots,Y_m)\colon Y \to Y$$

is just the identity map and by the $\Bbbk$-algebra morphism property of α we get

$$\begin{aligned} p(\Psi(\alpha)(x)) &= p(\alpha(Y_1)(x),\cdots,\alpha(Y_m)(x)) \\ &= \alpha(p \circ (Y_1,\cdots,Y_m))(x) \\ &= \alpha(p)(x) \\ &= \alpha(0)(x) \end{aligned}$$

since any $p \in \mathfrak{p}$ is the 0-map restricted to Y.

Now to show that Φ and Ψ are bijections we show that they are mutually inverse.

For $f = (f_1,\cdots,f_m)\colon X \to Y$ with $f_i \in \Bbbk[T_1,\cdots,T_n]$ we calculate

$$\begin{aligned} \Psi(\Phi(f)) =& \Psi(f^*) \\ =& (f^*Y_1,\cdots,f^*Y_m) \\ =& (f_1,\cdots,f_m) \\ =& f. \end{aligned}$$

Conversely for $\alpha\colon \Gamma(Y) \to \Gamma(X)$ it is enough to show $\Phi \circ \Psi = \mathrm{id}$ on generators of $\Gamma(Y)$. These are given by the component functions Y_n. We calculate

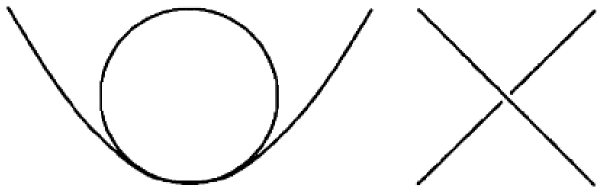
$$\Phi(\Psi(\alpha))(Y_n) = \Phi(\alpha(Y_1),\cdots,\alpha(Y_m))(Y_n)$$

$$= (\alpha(Y_1), \cdots, \alpha(Y_m))^*(Y_n)$$
$$= \mathrm{pr}_n(\alpha(Y_1), \cdots, \alpha(Y_m))$$
$$= \alpha(Y_n).$$

Since we never needed that X is actually an affine variety, we can easily replace it with $\cup X_i$, where every X_i is an affine variety and the construction follows analogously.

Now for $\mathbb{A}^n$ this means that $\mathrm{Hom}(X, \mathbb{A}^n) \cong \mathrm{Hom}(\Gamma(\mathbb{A}^n), \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[T_1, \cdots, T_n], \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[x], \Gamma(X))^n \cong \mathrm{Hom}(\Gamma(\mathbb{A}^1), \Gamma(X))^n$. $\square$

Exercise 1.3.4 See [1, Exercise 1.10]. Find rings to represent the following figures.



The first represents the union of a circle and a parabola in the plane and the second shows the union if two skew lines in 3-space. (You may use the Nullstellensatz to prove your answer is right.)

Exercise 1.3.5 When is $\mathbb{S}^1 \subset \mathbb{R}^2$ irreducible?

2 Exercises Part 2

2.1 Projective varieties

Exercise 2.1.1 See [2, Exercise 1.22]. Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3(\Bbbk)$.

(i) Show that there exists a change of coordinates such that $L_1 = V_+(X_0, X_1)$ and $L_2 = V_+(X_2, X_3)$.

(ii) Let $Z = L_1 \cup L_2$. Determine the homogeneous radical ideal $\mathfrak{a}$ such that $V_+(\mathfrak{a}) = Z$ (Exercise 1.21).

2.2 Veronese embedding

Exercise 2.2.1 [2, Exercise 1.30]. Let $n, d > 0$ be integers. Let $M_0, \cdots, M_N \in \Bbbk[X_0, \cdots, X_n]$ be all monomials in $X_0, \cdots, X_n$ of degree d.

(i) Define a $\Bbbk$-algebra homomorphism

$$\theta \colon \Bbbk[Y_0, \cdots, Y_N] \to \Bbbk[X_0, \cdots, X_n], \quad Y_i \mapsto M_i$$

and let $\mathfrak{a} = \ker \theta$. Show that $\mathfrak{a}$ is a homogeneous prime ideal (Exercise 1.20). Let $V_+(\mathfrak{a}) \subset \mathbb{P}^N(\Bbbk)$ the projective variety defined by $\mathfrak{a}$ (Exercise 1.21).

(ii) Consider the morphism

$$v_d \colon \mathbb{P}^n(\Bbbk) \to \mathbb{P}^N(\Bbbk), \quad (x_0 : \cdots : x_n) \mapsto (M_0(x_0, \cdots, x_n) : \cdots : M_N(x_0, \cdots, x_n)),$$

and show that v_d induces an isomorphism $\mathbb{P}^n(\Bbbk) \cong V_+(\mathfrak{a})$ of prevarieties. Is $V_+(\mathfrak{a})$ a linear subspace of $\mathbb{P}^N(\Bbbk)$?

(iii) Let $f \in \Bbbk[X_0, \cdots, X_n]$ be homogeneous of degree d. Show that $v_d(V_+(f))$ is the intersection of $V_+(\mathfrak{a})$ and a linear subspace of $\mathbb{P}^N(\Bbbk)$. The morphism v_d is called the d-Uple embedding or d-fold Veronese embedding.

2.3 Spectra of rings

Exercise 2.3.1 Find two classes of rings or your favourite two specific rings and classify their spectrum.

Exercise 2.3.2 See [2, Exercise 2.9]. For valuation rings see [2, (B.13) (p562)]. Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is called isolated if $0 \leq \gamma \leq \delta$ and $\delta \in \Delta$ implies $\gamma \in \Delta$.

(i) Let A be a valuation ring, $K = \mathrm{Frac} A$. Show that $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ is an inclusion reversing bijection from $\operatorname{Spec} A$ onto the set of rings B with $A \subset B \subset K$ and that such a ring B is a valuation ring of K. Its inverse is given by sending B to its maximal ideal (which is contained in A).

(ii) Let A be a valuation ring with value group Γ. For every isolated subgroup Δ of Γ set $\mathfrak{p}_\Delta := \{a \in A \mid v(a) \notin \Delta\}$. Show that $\Delta \mapsto \mathfrak{p}_\Delta$ defines a order reversing bijection between the set of isolated subgroups of Γ and $\operatorname{Spec} A$ (both totally ordered by inclusion).

(iii) Show that the value groups of the valuation ring $A_{\mathfrak{p}_\Delta}$ is isomorphic to Γ/Δ and the value groups of the valuation ring $A/\mathfrak{p}_\Delta$ is isomorphic to Δ.

(iv) For an arbitrary totally ordered abelian group Γ let $R = \Bbbk[\Gamma]$ be the group algebra of Γ over some field $\Bbbk$. Write elements of $u \in A$ as finite sums $u = \sum_\gamma a_\gamma e_\gamma$. Define a map $v\colon R\backslash\{0\} \to \Gamma$ by sending u to the minimal $\gamma \in \Gamma$ such that $\alpha_\gamma \neq 0$. Show that R is an integral domain and that v can be extended to a valuation v on $K = \mathrm{Frac}R$ with value group Γ. In particular $A \coloneqq \{x \in K \mid v(x) \geq 0\}$ is a valuation ring with value group Γ.

(v) Let I be a well-ordered set. Endow $\mathbb{Z}^I$ with the lexicographic order (i.e., we set $(n_i)_{i\in I} < (m_i)_{i\in I}$ if and only if $J \coloneqq \{i \in I \mid m_i \neq n_i\}$ is non-empty and $n_{i_J} < m_{i_J}$ where i_J is the smallest element of J). Show that $\mathbb{Z}^I$ is a totally ordered abelian group and that for all $k \in I$ the subsets $\Gamma_{\geq k}$ of $(n_i)_i \in \mathbb{Z}^I$ such that $n_i = 0$ for all $i < k$ are isolated subgroups of $\mathbb{Z}^I$.

(vi) Deduce that for every cardinal number k there are valuation rings whose spectrum has cardinality $\geq k$.

PROOF: For (i) consider the map $\mathfrak{p} \mapsto A_\mathfrak{p}$. This reverses orientation, i.e. if $q \subset p$ for two prime ideals, then $(A \setminus \mathfrak{p})^{-1} = A_\mathfrak{p} \subset A_\mathfrak{q} = (A \setminus \mathfrak{q})^{-1}A$. This is also a valuation ring since for any $a \in \mathrm{Frac}A = K$ either $a \in A$ or $a^{-1} \in A$ is naturally satisfied for $A \subset A_\mathfrak{p}$.

To see that this is a bijection, we consider a local ring $A \subset B \subset K$, which we map to the preimage under the inclusion $A \subset B$ of its unique maximal ideal $\mathfrak{m}_B$. This is inverse to the localization above since

$$\mathfrak{m}_{A_\mathfrak{p}} = \mathfrak{p}A_\mathfrak{p}$$

for a ring $A_\mathfrak{p}$ this is given by $\mathfrak{p}$.

On the other hand for a local ring $A \subset B \subset K$ the localization at the preimage of the maximal ideal $\mathfrak{m}_B$ has an inclusion into B by the universal property of the localization. Since both of these rings are maximal with respect to domination and local they have to coincide.

For (ii) we can easily see that the set of isolated subgroups is totally ordered. Consider the map $\Delta \mapsto \mathfrak{p}_\Delta$. The image is a prime ideal since for $ab \in \mathfrak{p}_\Delta$ we have $v(ab) = v(a) + v(b) \notin \Delta$. This means that either a or b already have to be in $\mathfrak{p}_\Delta$.

We define the following inverse:

$$\mathfrak{p} \mapsto v(A \setminus \mathfrak{p}) \cup -v(A \setminus \mathfrak{p}).$$

This is a group since for $p, q \notin \mathfrak{p}$ we have $v(p) + v(q) = v(pq) \in v(A \setminus \mathfrak{p})$. We also need to check that it is isolated. Assume it is not isolated. Since $\mathfrak{p}$ is prime, the set $A\setminus\mathfrak{p}$ is multiplicatively closed. This means that $v(A\setminus\mathfrak{p})\cup -v(A\setminus\mathfrak{p})$ is a group. On the other hand it contains any element between $-\alpha$ and α for $\alpha \in v(A \setminus \mathfrak{p})$ since otherwise for $0 \leq \beta \leq \alpha$ with $\beta \in v(\mathfrak{p})$ we would also have $v(A) + \beta \in v(\mathfrak{p})$. But since for $p \in \mathfrak{p}$ and $q \in A \setminus \mathfrak{p}$ with $v(p) = \beta$ and $v(q) = \alpha$ the element $v(p^{-1}q) = \alpha - \beta$ is in $v(A)$ or $-v(A)$ we would get that $\alpha \in v(\mathfrak{p})$ which is a contradiction.

We check the inverses:

$$v(\mathfrak{p}_\Delta) = v(A \setminus \mathfrak{p}_\Delta) \cup -v(A \setminus \mathfrak{p}_\Delta) = \Delta$$
$$\mathfrak{p}_{v(A\setminus\mathfrak{p})\cup -v(A\setminus\mathfrak{p})} = \{a \in A \mid v(a) \notin v(A \setminus \mathfrak{p}) \cup -v(A \setminus \mathfrak{p}_\Delta)\} = \mathfrak{p}$$

For (iii) the value group of A the kernel of the group morphism $K^\times \to v(K^\times)$ is given by $A^\times$. Thus for the value group of $A_{\mathfrak{p}_\Delta}$ this kernel is given by $(A \setminus \mathfrak{p}_\Delta)$. Thus the map $v\colon K^\times/(A \setminus \mathfrak{p}_\Delta) \to v(K^\times)$ is an isomorphism with $K^\times/(A \setminus \mathfrak{p}_\Delta) \cong \Gamma/\Delta$ since $v(A \setminus \mathfrak{p}_\Delta) = \Delta$.

For (iv) consider the group algebra $R = \Bbbk[\Gamma]$. Assume there are zero-divisors, i.e. $u = \sum_\gamma a_\gamma e_\gamma$ and $w = \sum_\eta b_\eta e_\eta$ with $uw = 0$. Then $v(u) = \gamma_0$ and $v(w) = \eta_0$. Then $uw = \sum_{\eta,\gamma} a_\gamma b_\eta e_{\gamma+\eta}$ where the summand $a_{\gamma_0} b_{\eta_0} e_{\gamma_0+\eta_0}$ is non-zero since $\Bbbk$ is integral.

Extending v to $K = \mathrm{Frac} R$ in the obvious way, i.e. $v(\frac{u}{w}) = v(u) - v(w)$ we have that $v(K^\times) \cong \Gamma$ by definition. And v fulfills

- $v(ab) = v(a) + v(b)$,
- $v(a+b) \geq \min\{v(a), v(b)\}$ with $>$ if and only if $v(a) = v(b)$ and the coefficients cancel,
- $v(0) = \infty$ by definition, □

and is thus a valuation on K.

For (v) consider $(n_i)_{i\in I}$ and $(m_i)_{i\in I}$. If $J = \emptyset$ we have $(n_i) = (m_i)$ and $(n_i) < (m_i)$ as well as $(n_i) > (m_i)$. Thus assume that $J \neq \emptyset$. Then there exists i_J as a minimal element in J. Then, since $\mathbb{Z}$ are totally ordered, we get that (m_i) and (n_i) are comparable. Reflexivity, antisymmetry and transitivity are clear by total order in $\mathbb{Z}$.

The structure of an abelian group in $\mathbb{Z}^I$ is just given by the pointwise group structure.

The subsets $\Gamma_{\geq k}$ of $(n_i)_{i\in I}$ such that $n_i = 0$ for all $i < k$ are isolated. Let $0 \leq (n_i) \leq (m_i)$ for $(m_i) \in \Gamma_{\geq k}$. Then we have $n_j = 0$ for all $j < k$ since it is "sandwiched". Thus $(n_i) \in \Gamma_{\geq k}$.

For (vi) we just combine the previous parts. Thus from $(n_i) \in \Gamma_{\geq k}\Bbbk[\mathbb{Z}^I]$ we get a $(n_i) \in \Gamma_{\geq k}$ valuation ring with spectrum cardinality $(n_i) \in \Gamma_{\geq k}$given by $|I|$.

3 Part 3

3.1 Collection of exercises on the basics of sheaves

Exercise 3.1.1 (2.1.A) See [3, 2.1.A]. Show that for the sheaf of $\mathscr{C}^\infty$ functions on a topological space denoted with $\mathscr{O}$ the stalk $\mathscr{O}_p$ is a local ring with maximal ideal given by

$$\mathfrak{m}_p = \{(f, U) \mid p \in U,\ f \in \mathscr{O}(U),\ f(p) = 0\}.$$

Proof: Let $I' \in \mathscr{O}_p$ be another maximal ideal not equal to $\mathfrak{m}_p$. Then there exists a germ (f, U) such that $f(p) \neq 0$ in I'. This however is invertible, since for $f(p) \neq 0$ there exists $V \subset U$ such that $f(x) \neq 0$ for all $x \in V$. Then (f^{-1}, V) is a well defined germ in $\mathscr{O}_p$ with $(f, U) \cdot (f^{-1}, V) = (1, V)$ which is a representative of $1 \in \mathscr{O}_p$. This means that $I' = \mathscr{O}_p$ and we are done. □

Exercise 3.1.2 (2.2.A) See [3, 2.2.A]. Show that the data of a $\mathfrak{C}$ valued presheaf on a topological space X is equivalent to a contravariant functor

$$\mathscr{F} : \mathrm{Opens}(X)^{\mathrm{op}} \to \mathfrak{C}$$

from the category $\mathrm{Opens}(X)$ of the open sets of X with morphisms given by the inclusion of subsets. This is especially a directed set and thus a nice category to work with.

Proof: The conditions that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ is and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ are just equivalent to the functoriality of above functor. □

Exercise 3.1.3 (2.2.B) See [3, 2.2.B]. Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves:

- bounded functions,
- holomorphic functions admitting a holomorphic square root.

Proof: For the bounded functions it is clear that the bounded open subsets cover $\mathbb{C}$, just consider the discs with radius $r \in \mathbb{N}$. On each of those the function $f(z) = z$ is bounded, but it does not glue to a globally bounded function.

Similarly the function $f(z) = z$ admits a holomorphic square root for the two open subsets $\mathbb{C} \setminus (0, \infty)$ and $\mathbb{C} \setminus (-\infty, 0)$. These cover $\mathbb{C}$. But it does not admit a global holomorphic square root. □

Exercise 3.1.4 (2.2.C) See [3, p. 2.2.C]. The identity and gluability axioms may be interpreted as saying that $(\cup_{i \in I} U_i)$ is a certain limit. What is that limit?

Proof: Gluability corresponds to the limit

$$\mathscr{F}(\cup_{i \in I} U_i) = \lim(\cup_{i,j} \mathscr{F}(U_i \cap U_j) \leftleftarrows \cup_{k \in I} \mathscr{F}(U_k))$$

where the two arrows are given by $\mathrm{res}_{U_j, U_j \cap U_i}$ and $\mathrm{res}_{U_i, U_j \cap U_i}$, see the notes for the equalizer definition of sheaves for more detail. □

Exercise 3.1.5 (2.2.D) See [3, p. 2.2.D].

- Verify that smooth functions, continuous functions, real-analytic funcitons, plain real-valued functions, on a manifold or Rn are indeed sheaves.
- Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

Proof: Trivial. □

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