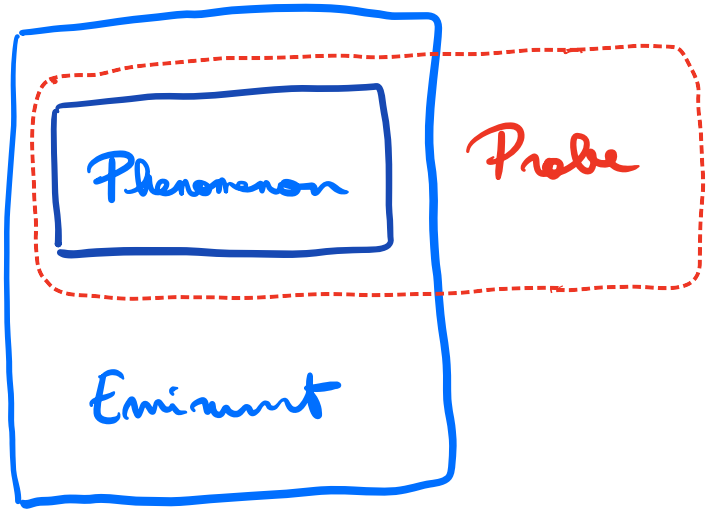
LECO4 Probabilistic modeling

# I. Data generating process

Eg.
- landing position of a ball
- images of galaxies taken by a telescope
- user preferences collected on a movie platform




Observational process ↳ ≠ model

⇓ ≳

Observations $x \in X$ := Random outcomes of the probe

$X$ ↓ Observational space ↳ Often unknown or uncertain!

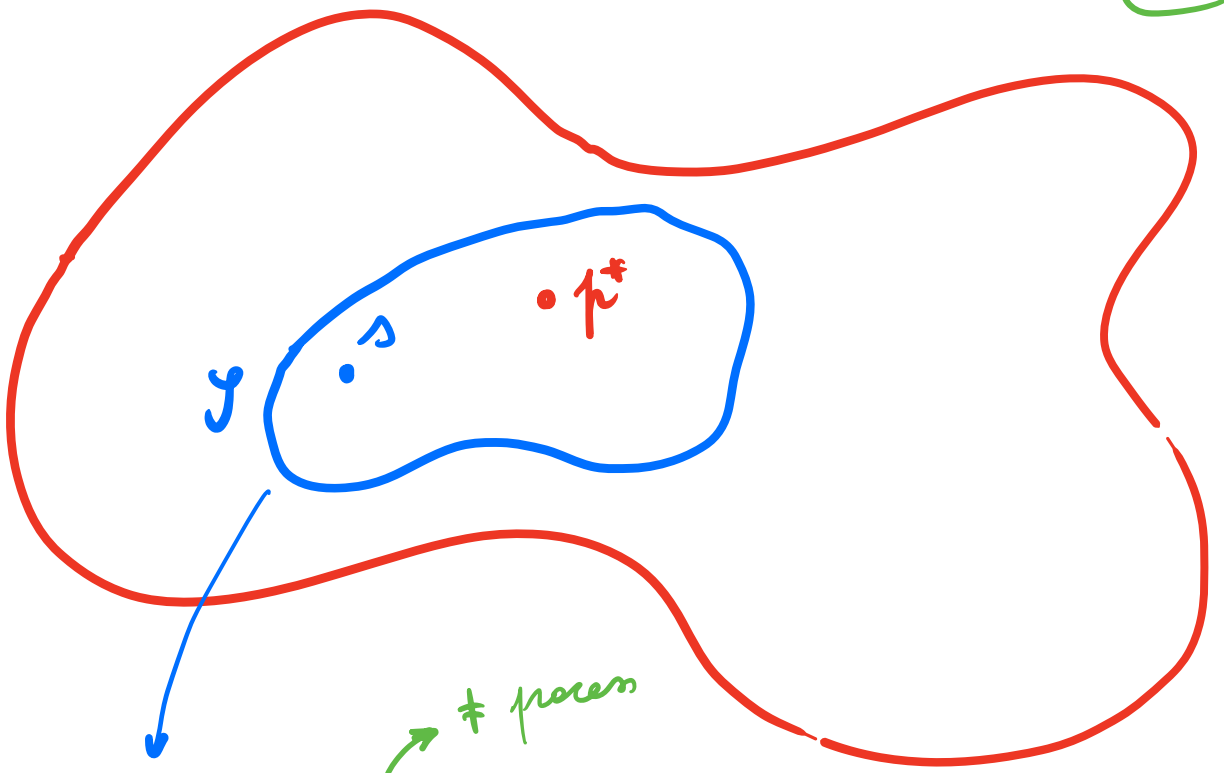
Data generating process = $p(x)$

True data generating process = $p^*(x)$

↳ Abus de notation, should be $\pi$

# II. The observational model

What if $p^* \notin S$?




Observational model / Model configuration space $\mathcal{P}$ ← ≠ process

= set of observed distributions $s$, each obtained via a possible mathematical mechanism of how the data could be generated.

set of all distributions over $X$.

Parameterization

⚠ Not unique!

$$\mathcal{S} = \{ p_\theta(x; \theta) \}$$

Family of PDFs ($s$) indexed by $\theta \in \Theta$.

Example:

$X = \mathbb{R}$

$\mathcal{P}$ = set of distributions Gaussian density.

a) $\Theta = \mathbb{R} \times \mathbb{R}_0^+$

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

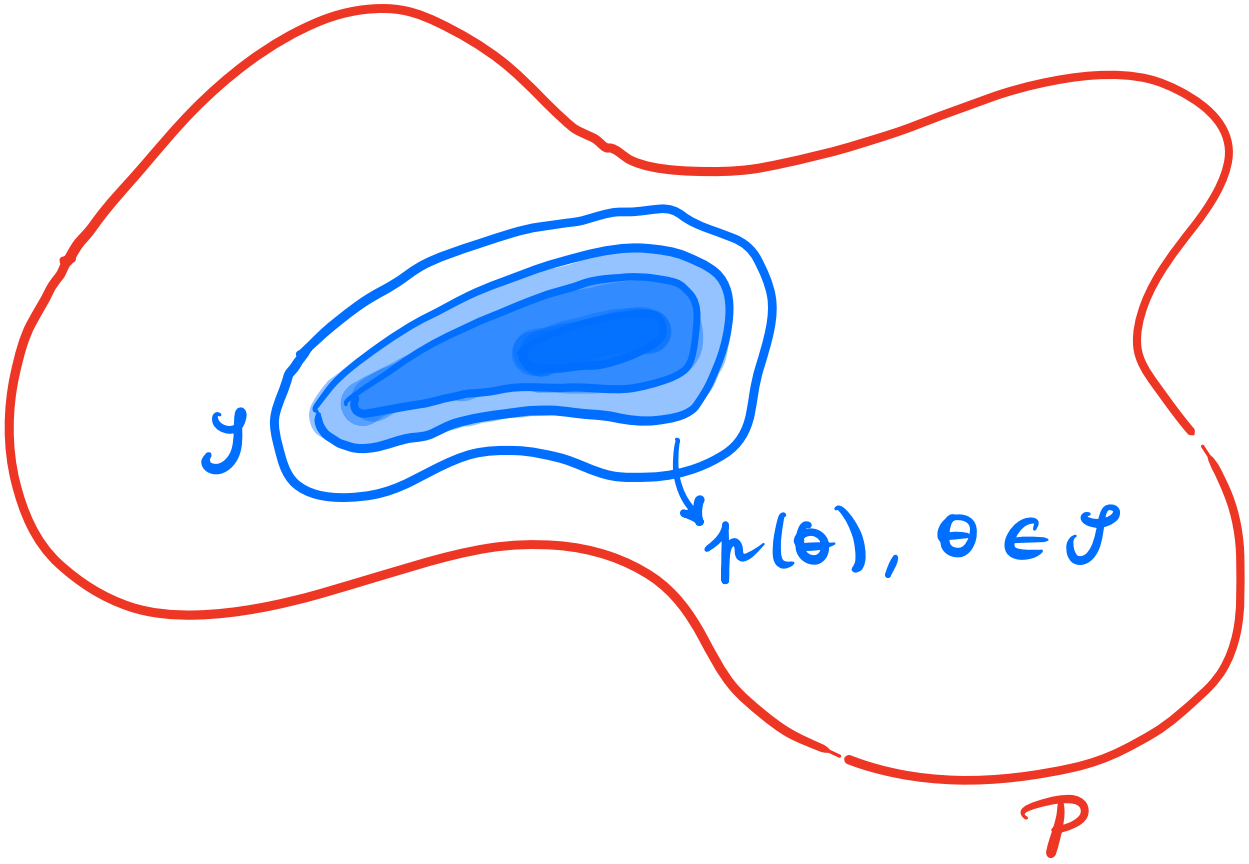
b) $\Theta = \mathbb{R} \times \mathbb{R}$

$$p(x; \mu, \lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda} \exp\left(-\frac{1}{2}(x-\mu)^2 e^{-2\lambda}\right)$$

$\lambda = \log \sigma$

## III. Bayesian models

All uncertainties are captured in a single unified framework, including epistemic uncertainty summarizing what we know (and don't) about the model configurations. ($\Rightarrow$ $\Theta$ becomes a R.V.




## Ingredients

* Prior model $p(\theta)$

* Observational model $p(x|\theta)$

The complete Bayesian model is the joint

$$\boxed{p(\theta, x) = p(\theta)\, p(x|\theta)}$$

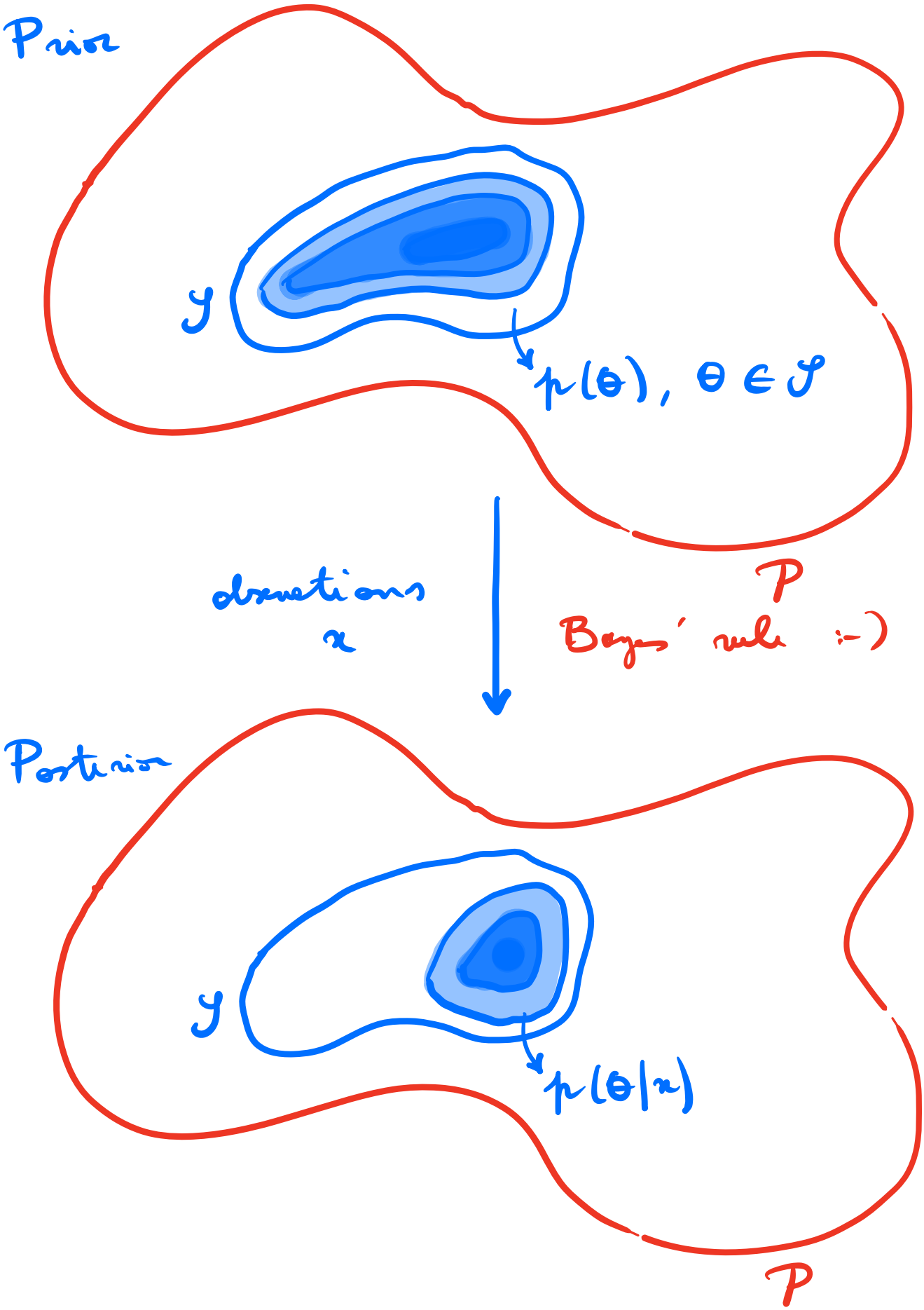
The joint probability perspective makes it easy to develop generative models that follow the structure of the true data generating process.

$$\theta = (\theta_{phenom}, \theta_{env}, \theta_{probe})$$

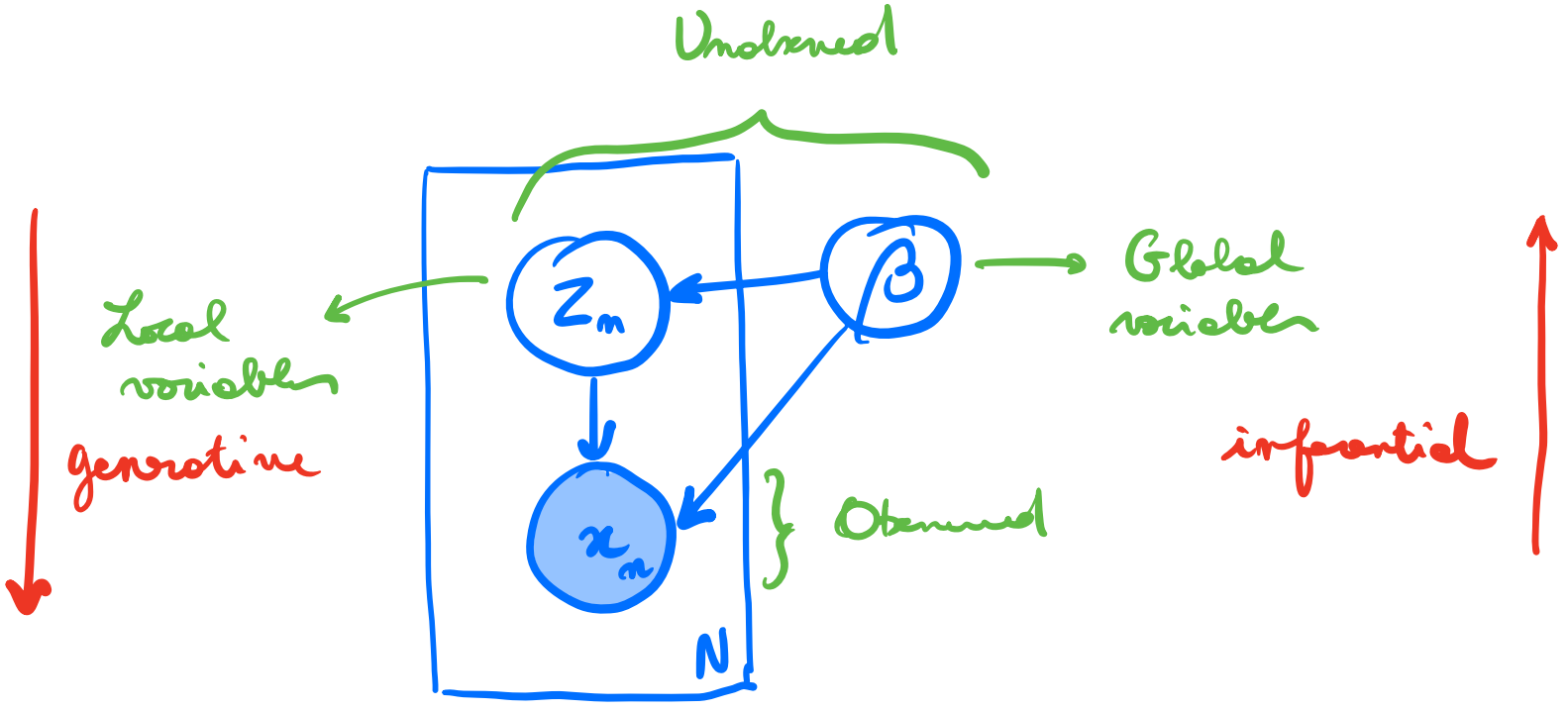
$$p(x, \theta) = p(x, \theta_{phenom}, \theta_{env}, \theta_{probe})$$
$$= p(x | \theta_{phenom}, \theta_{env}, \theta_{probe})$$
$$\times p(\theta_{probe} | \theta_{phenom}, \theta_{env})$$
$$\times p(\theta_{env} | \theta_{phenom})$$
$$\times p(\theta_{phenom})$$

$\Leftrightarrow$ Formulate the observational model as a series of generative steps, from the phenomena of interest down to the measurements.

Posterior inference → for the next lecture



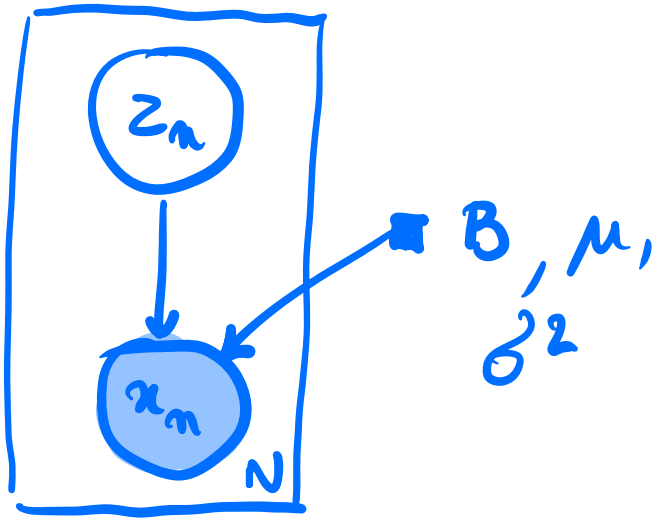
# IV. Latent variable models




$$p(\beta, z, x) = p(\beta) \prod_{n=1}^{N} p(z_n|\beta)\, p(x_n|z_n, \beta)$$

a) PCA → Check lec03 notes

↳ Dim reduction = inference on PPCA




Latent representation

Data point

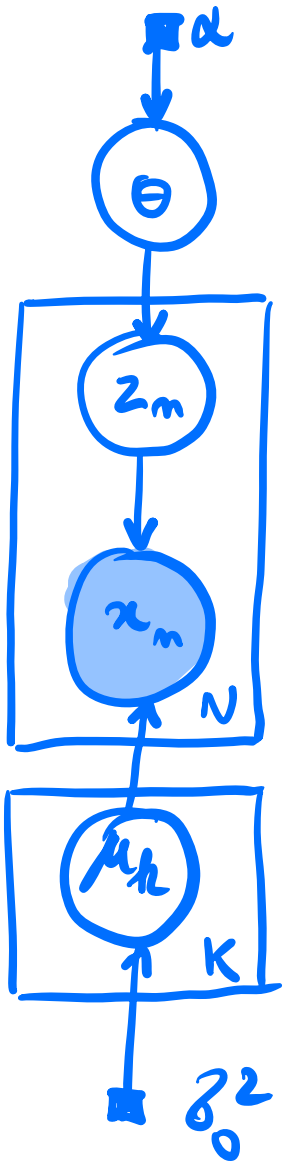
$$p(z, x \mid B, \mu, \sigma^2)$$

$$= p(z)\, p(x \mid z, B, \mu, \sigma^2)$$

$$= \mathcal{N}(z \mid 0, I)\, \mathcal{N}(x \mid Bz + \mu, \sigma^2 I)$$

→ Clustering = inference on mixture model

## 2) Mixture models




$\theta \in \Delta^K$

Mixture proportion — $\text{Dirichlet}_K(\alpha)$

Mixture component assigned to $n$ — $\text{Discrete}(\theta)$ / Multinomial

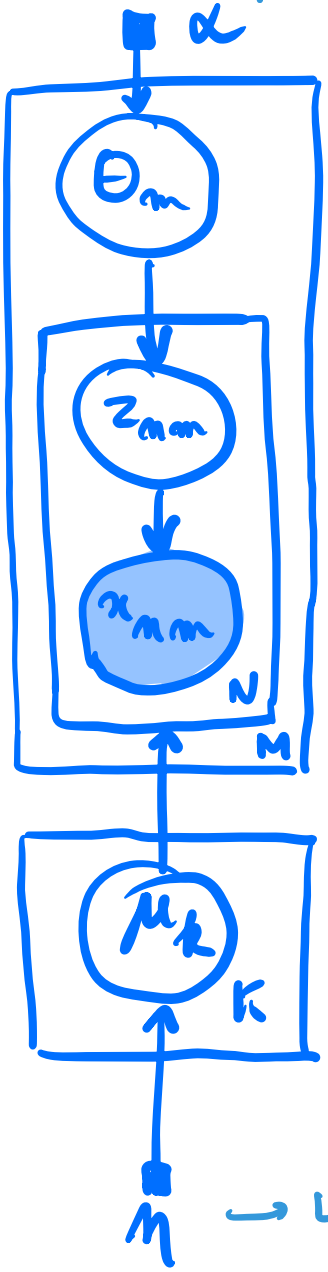
Data point — $\mathcal{N}(\mu_{z_n}, \sigma^2)$

Random position of the mixture components — $\mathcal{N}(0, \sigma_0^2)$

$$p(\theta, \mu, z, x \mid \alpha, \sigma^2)$$

$$= p(\theta \mid \alpha) \prod_{k=1}^{K} p(\mu_k \mid \sigma_0^2) \prod_{n=1}^{N} p(z_n \mid \theta)\, p(x_n \mid z_n, \mu)$$

c) Mixed membership models




→ LDA: proportion parameter

→ LDA: per-document topic proportions

Mixture proportion

$\text{Dirichlet}_K(\alpha)$

→ LDA: per-word topic assignment

Mixture component for $x_{mn}$

$\text{Discrete}_K(\theta_m)$

→ LDA: Observed word

Data point

$p(\cdot \mid \mu_{z_{mn}})$

$(\text{Discrete}_W(\mu_{z_{mn}}))$

→ LDA: Topics

Mixture component parameter

$p(\cdot \mid \eta)$

$(\text{Dirichlet}_W(\beta))$

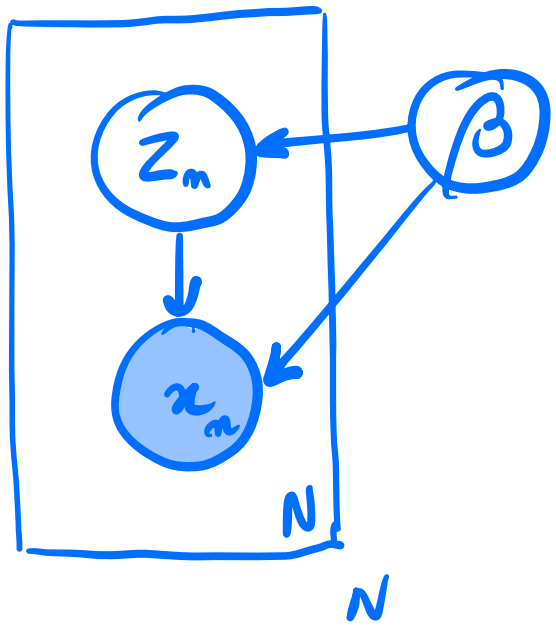
↓ number of words

→ LDA: topic parameter

$$p(\theta, \mu, z, x \mid \alpha, \eta)$$

$$= \prod_{k=1}^{K} p(\mu_k \mid \eta) \prod_{m=1}^{M} p(\theta_m \mid \alpha) \prod_{n=1}^{N} p(z_{mn} \mid \theta_m)\, p(x_{mn} \mid \mu_{z_{mn}})$$

# d) Probabilistic programs

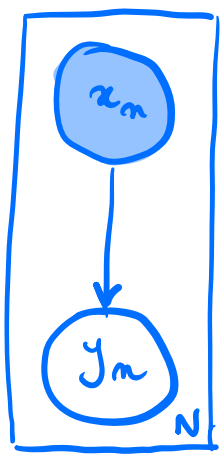


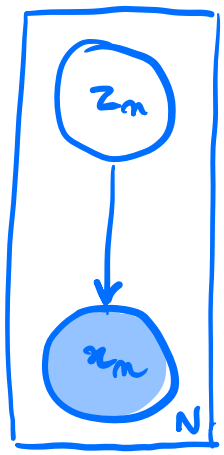

Show Naman's Metaprob's

$$p(\beta, z, x) = p(\beta) \prod_{n=1}^{N} \underbrace{p(z_n|\beta)\, p(x_n|z_n, \beta)}$$

forward generative model / compute simulations

# e) Discriminative / generative ML models




Also...

Linear factor models

Time series models

Matrix factorization

Multilevel regression

...

# V. Prior predictive checks

Often, the observational model is understood much better than the prior model.

⇒ We need to investigate the consequences of the prior model in the context of the observational model.

## Summary functions

$t : X \to T$

⟺ "statistics"

$t : \Theta \to T$

↓

some intermediate quantity

## Prior predictive checks

$x \sim p(x) \to \theta \sim p(\theta),\ x \sim p(x \mid \theta)$

Then plot $t(x)$ to evaluate the appropriate

# Summary

* Bayesian data analysis provides a principled framework for modeling and inference.

* Generative models capture the generative structure of the true data generating process (essence).

They make all assumptions explicit.