微積分演習 その6

問題 1. $f\binom{x}{y}$, $g(t)$, $h(t)$ を次のように定めたとき，そのとき，$\dfrac{d}{dt}f\begin{pmatrix} g(t) \\ h(t) \end{pmatrix}$ を求めよ.

(1)$f\binom{x}{y} = ye^{\sqrt{x^2+y^2}}$, $g(t) = \cos t$, $h(t) = \sin t$.

(2) $f\binom{x}{y} = x^2y^2 - 2y^3$, $g(t) = t^3 - t^2 + 1$, $h(t) = t^2 - t$.

(3) $f\binom{x}{y} = \log(x^2 + xy + y^2 + 1)$, $g(t) = e^t + e^{-t}$, $h(t) = e^t - e^{-t}$.

(4) $f\binom{x}{y} = x^y$, $g(t) = \sin t$, $h(t) = \cos t$.

(5) $f\binom{x}{y} = x\cos\dfrac{x}{y}$, $g(t) = 2t + 1$, $h(t) = \sqrt{t^2 + 1}$.

(6) $f\binom{x}{y} = \dfrac{x - y}{1 - y}$, $g(t) = 3t + 2$, $h(t) = -4t + 3$.

(7) $f\binom{x}{y} = \dfrac{x(x^2 - y^2)}{x^2 + y^2}$, $g(t) = t\cos\theta$, $h(t) = t\sin\theta$.

問題 2. 以下の各問で与えられる関数 $f : R^2 \to R$ の各点における偏導関数 $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ を求めよ.

(1) $f\binom{x}{y} = \begin{cases} \dfrac{xy}{x^4 + y^2} & \binom{x}{y} \neq \binom{0}{0} \\ 0 & \binom{x}{y} = \binom{0}{0} \end{cases}$ (2) $f\binom{x}{y} = \begin{cases} \dfrac{2x^3 - y^3}{4x^2 + y^2} & \binom{x}{y} \neq \binom{0}{0} \\ 0 & \binom{x}{y} = \binom{0}{0} \end{cases}$

(3) $f\binom{x}{y} = \begin{cases} \dfrac{x^3 + y^4}{x^2 + 4y^2} & \binom{x}{y} \neq \binom{0}{0} \\ 0 & \binom{x}{y} = \binom{0}{0} \end{cases}$ (4) $f\binom{x}{y} = \begin{cases} \dfrac{e^{x^2+y^2} - 1}{x^2 + y^2} & \binom{x}{y} \neq \binom{0}{0} \\ 1 & \binom{x}{y} = \binom{0}{0} \end{cases}$

問題 3. 以下で定められる関数 f の 2 次偏導関数をすべて求め, それぞれの場合に, $\dfrac{\partial^2 f}{\partial y \partial x}$ と $\dfrac{\partial^2 f}{\partial x \partial y}$ が一致することを確かめよ. ただし, a, b, c は定数とする.

(1) $xy^3(1 + x^2 - y)$ (2) e^{x+y} (3) $\sin x^2 y$ (4) $e^{-(ax^2+2bxy+cy^2)}$ (5) $e^{3x}\cos(x + 2y)$

問題 4. 以下で定められる関数 f に対し, $\binom{0}{0}$ においてテイラーの定理を用いた場合に f を近似する x, y の () 内で指定される次数以下の多項式をそれぞれ求めよ.

(1) $f\binom{x}{y} = e^{-x}\log(1 + 2y)$ (3 次近似) (2) $f\binom{x}{y} = \log(1 + 3x + y^2)$ (3 次近似)

(3) $f\binom{x}{y} = x^3 - xy + y^2 + 2$ (2 次近似) (4) $\log(\cos(x + y))$ (3 次近似)

(5) $f\binom{x}{y} = e^{x-y}\sin x$ (4 次近似)

問題 5. 以下の関数の指定された点における接平面の式を求めよ．(2 変数関数の 1 次近似)

(1) $f\binom{x}{y} = 3x^2 - 4y$, $\binom{x}{y} = \binom{1}{2}$ (2) $f\binom{x}{y} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $\binom{x}{y} = \binom{a}{b}$

(3) $f\binom{x}{y} = e^x\sin y$, $\binom{x}{y} = \begin{pmatrix} -\log\pi \\ \frac{\pi}{2} \end{pmatrix}$ (4) $f\binom{x}{y} = \log(e^x + e^{2y})$, $\binom{x}{y} = \binom{0}{0}$

問題 6. 下記 (1)，(2) に答えよ.

(1)$z(x, y) = f\left(\dfrac{y}{x}\right)$ と記述できるとき，$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$ を示せ．ただし，$x \neq 0$ とする.

(2)$z(x, y) = \sqrt{x^2 + y^2}f\left(\dfrac{y}{x}\right)$ と記述できるとき，$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$ を示せ．ただし，$x > 0$ とする.

問題 7. 下記 (1)，(2) に答えよ.

(1) $z = \log\sqrt{x^2 + y^2}$，$z = \dfrac{x}{x^2 + y^2}$ および $z = \arctan\dfrac{y}{x}$ は，$\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ をみたすことを示せ.

(2) $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ および $u = \log(x^2 + y^2 + z^2 - xy - yz - zx)$ は，$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ をみたすことを示せ.

微積分演習 解答 その6

問題 1. (1) $\cos t \sin t e(-\sin t) + e(\sin^2 t + 1)\cos t$
(2) $2(t^3 - t^2 + 1)(t^2 - t)^2(3t^2 - 2t) + \{2(t^3 - t^2 + 1)^2(t^2 - t) - 6(t^2 - t)^2\}(2t - 1)$
(3) $\dfrac{2x + y}{x^2 + xy + y^2 + 1}(e^t - e^{-t}) + \dfrac{x + 2y}{x^2 + xy + y^2 + 1}(e^t + e^{-t})$
(4) $yx^{y-1}(\cos t) + x^y \log x(-\sin t)$
(5) $\{\cos\dfrac{x}{y} + \dfrac{1}{y}(-\sin\dfrac{x}{y})\}2 + x\dfrac{-x}{y^2}(-\sin\dfrac{x}{y})\dfrac{t}{\sqrt{t^2+1}}$
(6) $\dfrac{1}{1-y}3 + \dfrac{x-1}{(1-y)^2}4$
(7) $\cos\theta(\cos^2\theta - \sin^2\theta)$

問題 2. (1) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{-3x^4y + y^3}{(x^4+y^2)^2}$, $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x^5 - xy^2}{(x^4+y^2)^2}$, $\dfrac{\partial f}{\partial x}(0,0) = 0$, $\dfrac{\partial f}{\partial y}(0,0) = 0$
(2) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{8x^4 + 6x^2y^2 + 8xy^3}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{-4x^3y - 12x^2y^2 - y^4}{(x^2+y^2)^2}$ $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{1}{2}$, $\dfrac{\partial f}{\partial y}(0,0) = -1$
(3) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{x^4 + 12x^2y^2 - 2xy^4}{(x^2+4y^2)^2}$, $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{-8x^3y + 4x^2y^3 + 8y^5}{(x^2+4y^2)^2}$ $\dfrac{\partial f}{\partial x}(0,0) = 1$, $\dfrac{\partial f}{\partial y}(0,0) = 0$
(4) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{2x\left(1 + (x^2+y^2-1)e^{x^2+y^2}\right)}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{2y\left(1 + (x^2+y^2-1)e^{x^2+y^2}\right)}{(x^2+y^2)^2}$,
$\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$

問題 3. (1) $\dfrac{\partial^2 f}{\partial x^2} = 6xy^3$, $\dfrac{\partial^2 f}{\partial x \partial y} = 3y^2 + 9x^2y^2 - 4y^3$, $\dfrac{\partial^2 f}{\partial y^2} = 6xy + 6x^3y - 12xy^2$,
(2) $\dfrac{\partial^2 f}{\partial x^2} = e^{x+y}$, $\dfrac{\partial^2 f}{\partial x \partial y} = e^{x+y}$, $\dfrac{\partial^2 f}{\partial y^2} = e^{x+y}$
(3) $\dfrac{\partial^2 f}{\partial x^2} = 2y\cos x^2y - 4x^2y^2\sin x^2y$, $\dfrac{\partial^2 f}{\partial x \partial y} = 2x\cos x^2y - 2x^3y\sin x^2y$, $\dfrac{\partial^2 f}{\partial y^2} = -x^4\sin x^2y$
(4) $\dfrac{\partial^2 f}{\partial x^2} = (4(ax+by)^2 - 2a)e^{-(ax^2+2bxy+cy^2)}$, $\dfrac{\partial^2 f}{\partial x \partial y} = (4(ax+by)(bx+cy) - 2b)e^{-(ax^2+2bxy+cy^2)}$, $\dfrac{\partial^2 f}{\partial y^2} = (4(bx+cy)^2 - 2c)e^{-(ax^2+2bxy+cy^2)}$
(5) $\dfrac{\partial^2 f}{\partial x^2} = e^{3x}(9\cos(x+2y) - 6\sin(x+2y) - \cos(x+2y))$,
$\dfrac{\partial^2 f}{\partial x \partial y} = e^{3x}(-6\sin(x+2y) - 2\cos(x+2y))$, $\dfrac{\partial^2 f}{\partial y^2} = -6e^{3x}\cos(x+2y)$

問題 4. (1) $2y - 2xy - 2y^2 + x^2y + 2xy^2 + \dfrac{8}{3}y^3$ (2) $3x - \dfrac{9}{2}x^2 + y^2 + 9x^3 - 3xy^2$ (3) $-xy + y^2 + 2$
(4) $\dfrac{1}{2}(-x^2 - 2xy - y^2)$ (5) $x + x^2 - xy + \dfrac{1}{3}x^3 - x^2y + \dfrac{1}{2}xy^2 - \dfrac{1}{3}x^3y + \dfrac{1}{2}x^2y^2 - \dfrac{1}{6}xy^3$

問題 5. (1) $-5 + 6(x-1) - 4(y-2)$ (2)$2 + \dfrac{2}{a}(x-a) + \dfrac{2}{b}(y-b)$ (3) $\dfrac{1}{\pi} + \dfrac{1}{\pi}(x + \log\pi)$
(4) $\log 2 + \dfrac{1}{2}x + y$

問題 6. 省略

問題 7. 省略