Advanced Linear Algebra
2022 Fall
Final Exam
2022.6.11
Exam Duration: 3 Hours

Name: ______________________

Student ID: ______________________

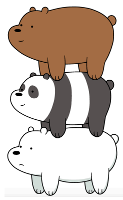
This exam includes 6 pages (including this page) and 5 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, but do not use anything with an internet connection. Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the provided **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers or write down a process, unless the problem states otherwise.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 50. If your total score exceeds 50 (there are 52 points in total), it will be recorded as 50.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	12	
Total:	52	

1.  Three bears stack together. The polar bear is holding the panda bear, and the panda bear is holding the brown bear. Now, the polar bear jumped a meters forward, then throw the two bears on top of it forward by b meters. When the panda bear hit the ground, it would then throw the brown bear on top of it forward by c meters. In the end, the polar bear, panda bear and brown bear end up at x, y, z meters away from the initial position of the polar bear. Then we have a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ sending $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

   (a) (4 points) Find the matrix A, and then find the matrix X such that $A = XJX^{-1}$ and J is in Jordan canonical form.

   (b) (3 points) Calculate $\sin(\frac{\pi}{2}A)$, and find its minimal polynomial.

   (c) (3 points) Find all solutions X to the matrix equation $(\sin(A))^2X + X(\cos(A))^2 = I$.

2. For an unknown real number x, consider $A(x) = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$. This matrix may change value as x changes value.

   (a) (4 points) Find the Jordan canonical form $J(x)$ of $A(x)$. Does $J(x)$ depends on x continuously? If not, find all locations of discontinuity.

   (b) (3 points) Find the minimal polynomial $p_x(t)$ of $A(x)$. Does the coefficients of $p_x(t)$ depend on x continuously? If not, find all locations of discontinuity.

   (c) (3 points) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{(A(x)-2I)^2})$.

3. Let V be the space of real polynomials on two variables x, y with degree at most two. In particular, a typical element of V looks like $p(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ for some $a, b, c, d, e, f \in \mathbb{R}$. Let W be the space of polynomials on the variable y with degree at most 2. We define a map $S : V \to W$ that sends $p(x,y)$ to $\int_0^1 p(x,y)\,\mathrm{d}x$. (For this integration, we hold y constant.) Then we also have a dual map $S^* : W^* \to V^*$.

   (a) (2 points) Consider $\mathrm{ev}_5 \in W^*$, which sends each $q(y) \in W$ to $q(5)$. What would $S^*(\mathrm{ev}_5)$ send $p(x,y)$ to? Describe it as a formula involving the input $p(x,y)$, then pick any basis and find the coordinates for $S^*(\mathrm{ev}_5)$.

   (b) (2 points) Consider $\int_0^1 \in W^*$, which sends each $q(y) \in W$ to $\int_0^1 q(y)\,\mathrm{d}y$. What would $S^*(\int_0^1)$ send $p(x,y)$ to? Describe it as a formula involving the input $p(x,y)$, then pick any basis and find the coordinates for $S^*(\int_0^1)$.

   (c) (3 points) We give W the inner product defined as $\langle q_1(y), q_2(y)\rangle = \int_0^1 q_1(y)q_2(y)\,\mathrm{d}y$. (You don't have to verify that this is indeed an inner product.) Let $B : W \to W^*$ be the bra map. Pick any basis and find the coordinates for $S^*BS(p(x,y))$ when $p(x,y) = 2xy$. Also find the rank of the linear map S^*BS.

   (d) (3 points) Let $D : W \to W$ be the derivative map. Then we have four bilinear maps $(p,q) \mapsto \langle Dp, q\rangle, (p,q) \mapsto \langle p, Dq\rangle, (p,q) \mapsto p(1)q(1), (p,q) \mapsto p(0)q(0)$. So they are four $\binom{0}{2}$-tensors on W. Find the dimension of the subspace in $W^* \otimes W^*$ spanned by these four tensors.

4. Let $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \alpha = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. In particular, $v^i = 1, w^i = i, \alpha_i = 1, \beta_i = i$ for all $i = 1, 2, 3$.

   Let $\tau = \boldsymbol{w} \otimes \alpha \otimes \beta + \boldsymbol{v} \otimes \beta \otimes \beta$. Note that Einstein's notation is used throughout this problem.

   (a) (2 points) Find the expression of the entry τ^i_{jk} for τ in terms of i, j, k.

   (b) (2 points) Calculate $\alpha(\boldsymbol{w})\beta + \beta(\boldsymbol{v})\beta$ and $\beta(\boldsymbol{w})\alpha + \beta(\boldsymbol{v})\beta$.

   (c) (3 points) Write out the row vector with coordinates τ^i_{ik}, and the row vector with coordinates τ^i_{ji}.

   (d) (3 points) Let ω be a tensor whose entries are $\alpha_i \tau^i_{jk}$. Find the (j, k) entry of the alternization of ω in terms of j, k.

5. For any $n \times 3$ matrix $A = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$, set $\omega_A = \boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$. For any 3×3 matrix M, we define $(\omega_A)_M$ as the wedges of the three columns of AM, in the obvious order.
   (a) (2 points) Show that $((\omega_A)_M)_N = (\omega_A)_{MN}$ for all $n \times 3$ matrix A and 3×3 matrices M, N.

   (b) (3 points) If M is an elementary matrix (i.e., column permutation, column scaling, column shearing), then find the relation between ω_A and $(\omega_A)_M$.

   (c) (2 points) Show the relation between ω_A and $(\omega_A)_M$ in terms of M.

   (d) (2 points) Show the relation between $(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}) \wedge (\boldsymbol{u} - \boldsymbol{v}) \wedge (\boldsymbol{u} + \boldsymbol{v} - 2\boldsymbol{w})$ and $\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$.

   (e) (3 points) What is the dimension of $\bigwedge_3(\mathbb{R}^3)$? For any row vectors $\alpha, \beta, \gamma \in (\mathbb{R}^3)^*$, find the relation between $\alpha \wedge \beta \wedge \gamma$ and the determinant tensor.