Homework 10

Problem 1.

(a) Let X be a random variable taking values in $[0,1]$. Prove that if $\mathbb{E}(X) = \varepsilon$, then

$$\Pr\left(X \geq \frac{\varepsilon}{2}\right) \geq \frac{\varepsilon}{2}.$$

(b) Let $X \geq 0$ be a random variable. Prove that

$$\Pr(X = 0) \leq \frac{\mathrm{Var}(X)}{\left(\mathbb{E}(X)\right)^2}.$$

Solution. (a)

$$\begin{aligned}\varepsilon = \mathbb{E}(X) &\leq \Pr\left(X < \frac{\varepsilon}{2}\right) \cdot \frac{\varepsilon}{2} + \Pr\left(X \geq \frac{\varepsilon}{2}\right) \cdot 1 \\ &\leq \frac{\epsilon}{2} + \Pr\left(X \geq \frac{\varepsilon}{2}\right)\end{aligned}$$

rearranging gives us the inequality.

(b) Using Chebyshev inequality

$$\Pr(X = 0) \leq \Pr(|X - \mathbb{E}(X)| \geq \mathbb{E}(X)) \leq \frac{\mathrm{Var}(X)}{\left(\mathbb{E}(X)\right)^2}.$$

Problem 2. Let RandomSign(n) be the distribution of vectors of n entries where each entry is independently chosen to be ± 1 with probability $\frac{1}{2}$. Sample m vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)} \sim \text{RandomSign}(n)$. Define the normalized vectors $\mathbf{w}^{(i)} = \mathbf{v}^{(i)}/\sqrt{n}$ so that $\left\|\mathbf{w}^{(i)}\right\| = 1$ for all $i = 1, 2, \ldots, m$. Prove the following claims:

(a) For all $i \neq j$, the inner product $\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle = \sum_k \mathbf{w}^i_k \mathbf{w}^j_k$ is small with high probability. That is,

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \geq \delta\right) \leq \exp\left(-\Omega\left(\delta^2 n\right)\right).$$

(b) There exists some $m = \exp(\Omega(\delta^2 n))$ such that the m vectors are pairwise almost-orthogonal with high probability. More precisely,

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \leq \delta \text{ for all pairs } i \neq j\right) \geq 0.99.$$

(Note: By probabilistic method, this proves that there are exponentially many almost-orthogonal unit vectors in $\mathbb{R}^n$ even though there are at most n exactly orthogonal vectors.)

Solution. (a) Let $X_k = \mathbf{w}_k^i \mathbf{w}_k^j$, we can check that $X_k \in \{-1/n, 1/n\}$. By Chernoff bound, we have that

$$\Pr\left(|\sum_{k=1}^{n} X_k| \geq \delta\right) \leq 2\exp\Bigl(-\frac{2\delta^2}{n(1/n)^2}\Bigr) = 2\exp(-2\delta^2 n).$$

(b) Consider the complement event, i.e. $\exists i, j, i \neq j \left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| > \delta$. By union bound,

$$\Pr\left(\exists i, j, i \neq j \left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| > \delta\right) \leq 2\binom{m}{2}\exp(-2\delta^2 n)$$

We can set $m = 0.01\exp(\delta^2 n)$, thus the right hand side of the inequality is smaller than 0.01, satisfying our requirement.

Problem 3. Let RandomGraph(n, p) be the distribution of random graphs of n vertices where, for each pair of vertices u, v, $\{u, v\}$ is chosen as an edge of the graph independently with probability p. Prove the following for such a random graph $G \sim$ RandomGraph(n, p).

(a) If $p = o(n^{-2/3})$,

$$\lim_{n\to\infty} \Pr(G \text{ constains a 4-clique}) = 0.$$

(b) If $p = \omega(n^{-2/3})$,

$$\lim_{n\to\infty} \Pr(G \text{ does not constain a 4-clique}) = 0.$$

(Hint: Use Part (b) of Problem 1 and you need to carefully calculate the probability of 4-cliques occurring simultaneously on vertex sets A and B when $|A \cap B| \geq 2$.)

Solution. (a) For any four vertices, the probability that they form a 4 clique is p^6, thus by union bound

$$\Pr(G \text{ constains a 4-clique}) \leq \binom{n}{4}p^6 \leq n^4 p^6,$$

if $p = o(n^{-2/3})$, we can see that $\lim_{n\to\infty} \Pr(G \text{ constains a 4-clique}) = 0$.

(b) Let I_A be the indicator for 4-vertices set A that forms a 4 clique. Let $X = \sum_{A\subset V, |A|=4} I_A$.

By linearity of expectation, we have $\mathbb{E}(X) = \binom{n}{4}p^6$. Now we turn to estimate $\mathbb{E}(X^2)$.

$$\mathbb{E}(X^2) = \sum_{|A|=4} \mathbb{E}[I_A] + \left(\sum_{|A|=4,|B|=4,|A\cap B|\le 1} \mathbb{E}[I_A I_B] + \sum_{|A|=4,|B|=4,|A\cap B|=2} \mathbb{E}[I_A I_B] + \sum_{|A|=4,|B|=4,|A\cap B|=3} \mathbb{E}[I_A I_B] \right)$$

Discuss each case separately. First note that we have about $\binom{n}{4}^2 = \frac{n^8}{24^2} + \Theta(n^7)$ possible pairs of (A, B) in total. We can count the size of different A, B pairs according to their intersection size:

- $|A \cap B| = 1$: Choose 7 points, of order $\Theta(n^7)$.
- $|A \cap B| = 2$: Choose 6 points, of order $\Theta(n^6)$.
- $|A \cap B| = 3$: Choose 5 points, of order $\Theta(n^5)$.
- $|A \cap B| = 0$: Subtracting the above three cases, of order $\frac{n^8}{24^2} + \Theta(n^7)$.

We now calculate $\mathbb{E}(I_A I_B)$ in each case.

- $|A \cap B| \le 1$, I_A and I_B is independent, $\mathbb{E}(I_A I_B) = p^{12}$.
- $|A \cap B| = 2$, share a common edge, $\mathbb{E}(I_A I_B) = p^{11}$.
- $|A \cap B| = 2$, share a common triangle, $\mathbb{E}(I_A I_B) = p^9$.

Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \Theta(n^4p^6) + \Theta(n^7p^{12}) + \Theta(n^6p^{11}) + \Theta(n^5p^9)$. Divided by $\mathbb{E}(X)^2 = \Theta(n^8p^{12})$, we can check that $n^{-4}p^{-6} = o(1)$, $n^{-2}p^{-1} = o(n^{-4/3})$, $n^{-3}p^{-3} = o(n^{-1})$, Thus as $n \to \infty$, $\mathrm{Var}(X)/(\mathbb{E}(X))^2 \to 0$.

The main point is to show the n^8p^{12} order term cancel out in $\mathrm{Var}(X)$.