Linear Algebra (in English)
2020 Fall
Final Exam
2020.12.23
Exam Duration: 3 Hours

Name: ____________________

Student ID: ____________________

This exam includes 6 pages (including this page) and 5 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, but do not use anything with an internet connection. Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the provided **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers or write down a process, unless the problem states otherwise.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 60. If your total score exceeds 60 (there are 64 points in total), it will be recorded as 60.

Problem	Points	Score
1	13	
2	14	
3	16	
4	12	
5	9	
Total:	64	

1. Let $A = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) (4 points) Find the Jordan canonical form J for A, and find all eigenvalues, all algebraic multiplicities and all geometric multiplicities for A.

   (b) (2 points) Find the characteristic polynomial and the minimal polynomial for A. (You don't need to expand the polynomial. Factor forms are OK.)

   (c) (4 points) Find a matrix X such that $A = XJX^{-1}$.

   (d) (3 points) List all invariant 2-dimensional subspaces of J. No proof required. (Note that we want J, not A, so that the computations are easier.)

2. For any real number $t \in \mathbb{R}$, let $R_t = \mathrm{e}^{tA}$, where $A = \begin{bmatrix} 0 & \frac{\pi}{3} & -\frac{\pi}{6} \\ -\frac{\pi}{3} & 0 & \frac{\pi}{3} \\ \frac{\pi}{6} & -\frac{\pi}{3} & 0 \end{bmatrix}$.

   (a) (2 points) Calculate $R_t R_t^{\mathrm{T}}$ and $\det(R_t)$ for all $t \in \mathbb{R}$.

   (b) (2 points) Find a non-zero vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $R_t\boldsymbol{v} = \boldsymbol{v}$ for all $t \in \mathbb{R}$.

   (c) (4 points) Suppose $\boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$, calculate the dot product of $\boldsymbol{w}$ and $R_t\boldsymbol{w}$ for all $t \in \mathbb{R}$.

   (d) (2 points) Write a differential equation whose solutions are exactly linear combinations of columns of R_t.

   (e) (4 points) Let $\mathcal{M}$ be the collection of all 3×3 real matrices X such that $XR_t = R_tX$ for all $t \in \mathbb{R}$, show that $\mathcal{M}$ is a subspace of the matrix space $\mathbb{M}_{3\times 3}(\mathbb{R})$ and find $\dim \mathcal{M}$.

3. Let V be the real vector space spanned by the functions $\sin(x), \cos(x), \sin(2x), \cos(2x)$.

   (a) (3 points) Show that $\sin(x), \cos(x), \sin(2x), \cos(2x)$ are linearly independent.

   (b) (5 points) Show that the dual vectors $\int_0^{\pi/2}, \int_0^{\pi}, \int_{\pi/2}^{3\pi/2}, \int_{\pi/4}^{3\pi/4}$ are linearly independent.

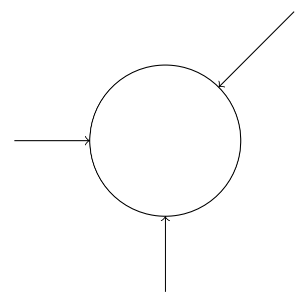
   (c) (4 points) Find the dual basis for $\{\int_0^{\pi/2}, \int_0^{\pi}, \int_{\pi/2}^{3\pi/2}, \int_{\pi/4}^{3\pi/4}\}$.

   (d) (4 points) Write the dual vector $f \mapsto f'(0)$ as the unique linear combination of the dual vectors $\int_0^{\pi/2}, \int_0^{\pi}, \int_{\pi/2}^{3\pi/2}, \int_{\pi/4}^{3\pi/4}$.

4. (The traditional problem is like this: 10 forces are applied on a ping pong ball. Where would the ping pong ball most likely crack? We hereby reduced the problem to a 2-dim version with only three forces to simplify calculation.)

   If we apply a force $\boldsymbol{f} \in \mathbb{R}^2$ on the unit circle at point $\boldsymbol{p}$, then we record this information as $\boldsymbol{p} \otimes \boldsymbol{f} \in \mathbb{R}^2 \otimes \mathbb{R}^2$.

   Suppose we apply three forces $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{v}$ where $\boldsymbol{v} = -\boldsymbol{e}_1 - \boldsymbol{e}_2$, and the forces are applied to the circle perpendicularly, as shown in the graph below.



   (a) (4 points) The three forces are recorded via three tensors. Calculate their sum T in terms of the standard basis $\boldsymbol{e}_{11}, \boldsymbol{e}_{12}, \boldsymbol{e}_{21}, \boldsymbol{e}_{22} \in \mathbb{R}^2 \otimes \mathbb{R}^2$.

   (b) (4 points) Find two mutually orthogonal unit vector $\boldsymbol{x}, \boldsymbol{y}$ such that T is a linear combination of $\boldsymbol{x} \otimes \boldsymbol{x}$ and $\boldsymbol{y} \otimes \boldsymbol{y}$.

   (c) (2 points) Let us say that the circle is squeezed such that its shape has changed a tiny bit, into an ellipse. Which directions would the long axis and short axis be?

   (d) (2 points) Find a tensor $T \in \mathbb{R}^2 \otimes \mathbb{R}^2$ that cannot be obtained via squeezing the circle perpendicularly, with no matter how many forces. Here squeezing means the forces are always inward. Explain why squeezing cannot produce your tensor. (Hint: Suppose T is a sum of tensors for perpendicular squeezing forces. Write T^{ij} into a matrix. Is it positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite?)

5. All the tensor below is over $\mathbb{R}^3$. We take standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ for $\mathbb{R}^3$ and the corresponding standard basis for all the tensor spaces. Compute the following:

   (a) (1 point) $\boldsymbol{e}^1 \otimes \boldsymbol{e}^2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$;

   (b) (1 point) $\boldsymbol{e}^1 \otimes \boldsymbol{e}^2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_3)$;

   (c) (1 point) $\mathrm{Alt}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$;

   (d) (1 point) $\boldsymbol{e}^1 \otimes \boldsymbol{e}^2(\mathrm{Alt}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1))$;

   (e) (1 point) $\mathrm{Alt}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2)(\mathrm{Alt}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1))$;

   (f) (1 point) $\boldsymbol{e}^1 \otimes \boldsymbol{e}^2(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$;

   (g) (1 point) $\boldsymbol{e}^1 \wedge \boldsymbol{e}^2(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$;

   (h) (2 points) $\boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3(\begin{bmatrix}1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\2\\3\end{bmatrix} \otimes \begin{bmatrix}1\\4\\9\end{bmatrix})$.