Two-loop planar master integrals for $q\bar{q} \to WW$

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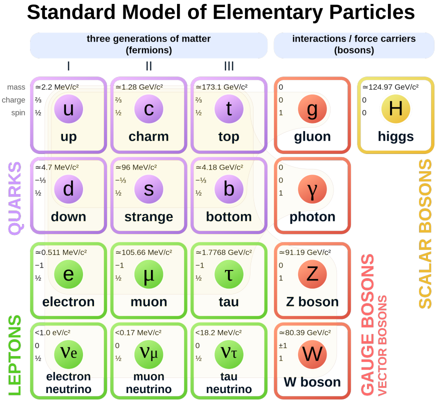
Outline

1. Introduction

2. W-pair production

Introduction



The Standard Model is not perfect.

? New Physics

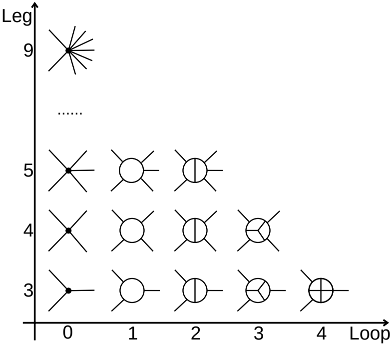
- Dark matter&energy
- Neutrino mass
- Matter–antimatter asymmetry
- ...

High Precise

- Parton shower
- Resummations effects
- Higher-order feynman diagrams
- ...

Feynman Integrals

Feynman diagrams originated from perturbative calculations in quantum field theory and multi-loop Feynman integrals are an important ingredient in computing Feynman diagrams. However, it is not always a trival job.



- Legs
- Loops
- Scales

Feynman Integrals

For a specific process in colliders, the Feynman integrals that need to be dealt with can often number in the hundreds or thousands.

Feynman integrals

$$I_{\alpha_1\ldots\alpha_n} = \int \prod_i^L \frac{d^D l_i}{i\pi^{D/2}} \frac{1}{P_1^{\alpha_1}\cdots P_n^{\alpha_n}}, \quad \text{where} \quad \begin{aligned} P_i &= q_i^2 - m_i^2 \\ q_i &= \sum_k^L a_{ik} l_k + \sum_k^{N_{\mathbf{ext}}} b_{ik} p_k \end{aligned}$$

The Feynman integrals depend solely on kinematic variables,

$$p_i \cdot p_j, \qquad m_i^2.$$

IBP identities provide us with a viable approach to handling the masses.

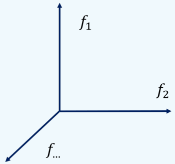
Feynman integrals

$$I_{\alpha_1\ldots\alpha_n} = \int \prod_i^L \frac{d^D l_i}{i\pi^{D/2}} \frac{1}{P_1^{\alpha_1}\cdots P_n^{\alpha_n}}$$

IBP identities : $$0 = \int \prod_i^L \frac{d^D l_i}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{\nu^\mu}{P_1^{\alpha_1}\cdots P_n^{\alpha_n}}$$

Master integrals

$$I_{\alpha_1\ldots\alpha_n} = \sum_{i=1}^n c_i f_i$$



IBP reduction

The master integrals (MIs) constitute a linear space, where these integrals can be viewed as the basis within this linear space. But the complexity of IBP reduction occupy the most time for a scattering amplitude computation.

- Laporta algorithm has been widely used in many packages, such as `Kira`, `FIRE` and `Reduce`.
- Simultaneously, there have been proposed some more efficient IBP methods, intersection theory, syzygy and so on.

In our work, the focus is MIs.

Differential Equations

The derivation of MIs can still be expressed as linear combinations of the MIs via IBP reduction. A closed differential equation system is constructed in the following form:

$$\partial_{x_i}\vec{f}(\boldsymbol{x},\epsilon) = \mathbb{A}_{x_i}(\boldsymbol{x},\epsilon)\vec{f}(\boldsymbol{x},\epsilon)$$

$$\Big\downarrow \quad \vec{g} = \mathbb{T}\vec{f}$$

Canoical equation

$$\partial_{x_i}\vec{g}(\boldsymbol{x},\epsilon) = \epsilon\,\mathbb{A}_{x_i}(\boldsymbol{x})\vec{g}(\boldsymbol{x},\epsilon)$$

Canonical DEs exhibit greater operability when solved through iterative integration methods.

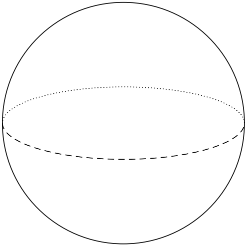
Solution

The solutions are expressed as series expansion in $\epsilon = \frac{4-d}{2}$ which the coefficients are iterated integrals.

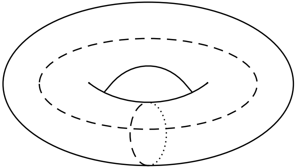
$$\vec{g}(\boldsymbol{x}, \epsilon) = \sum_i \vec{g}^{(i)} \epsilon^i, \quad \text{where} \quad \vec{g}^{(i)}(\boldsymbol{x}) = \int_0^x d\mathbb{A}\, \vec{g}^{(i-1)}(\boldsymbol{x}) + \vec{g}_0^{(i)}.$$

Large of class Feynman integrals are described by Multiple polylogarithms (MPLs). Several elliptic generalizations of MPLs (eMPLs) show up at two-loop level.

MPLs:



eMPLs:



The two functions live in different topologies.

MPLs

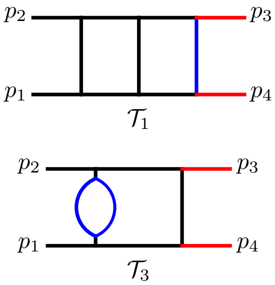
The MPLs are defined as iterated integrals by

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \quad \text{and} \quad G(; z) = 1.$$

The eMPLs is defined differently on elliptic curves (E_4 functions) and torus ($\tilde{\Gamma}$ functions).

$u\bar{u} \to W^- W^+$

At two-loop level, we present the topologies appearing in this process.



Three systems of the differential equation are constructed for the $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. The topology corresponds to a set which includes all the Feynman integrals belonging to the topology or sub-topology.

Conventions

The MIs in this work depend on four scales s, t, m_t^2, m_W^2, where

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2, \quad \text{and} \quad s + t + u = 2m_W^2.$$

We choose m_t^2 to establish the dimensionless Feynman integrals,

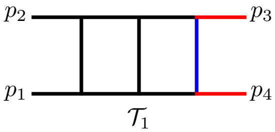
$$F_{\alpha_1 \ldots \alpha_n} = (m_t^2)^{\alpha - d} I_{\alpha_1 \ldots \alpha_n}.$$

All the MIs only rely on three dimensionless variables,

$$x = -\frac{s}{m_t^2}, \qquad y = -\frac{t}{m_t^2}, \qquad z = -\frac{m_W^2}{m_t^2}.$$

Topology 1

A complete set of the propagators describes a family integral.

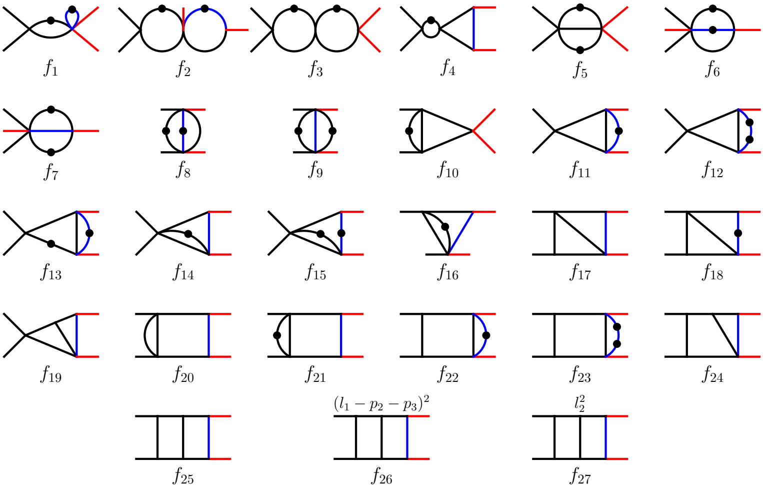


$$
\begin{aligned}
&P_1 = l_1^2, && P_2 = (l_1 + p_1)^2, && P_3 = (l_1 - p_2)^2, \\
&P_4 = l_2^2, && P_5 = (l_2 + p_1)^2, && P_6 = (l_2 - p_2)^2, \\
&P_7 = (l_1 - l_2)^2, && P_8 = (l_1 - p_2 - p_3)^2, && P_9 = (l_2 - p_2 - p_3)^2 - m_t^2.
\end{aligned}
$$

There are 27 MIs for this topology. The 27 dimensional DEs is relatively easy to handle.

Topology 1

The 27 MIs f_i's are defined as



Topology 1

The canonical basis of g_i's is defined as a linear combination of f_i's.

$$g_1 = \epsilon^2 f_1 x, \qquad g_2 = \epsilon^2 f_2 x z,$$
$$g_3 = \epsilon^2 f_3 x^2, \qquad g_4 = \epsilon^3 f_4 x r,$$
$$g_5 = \epsilon^2 f_5 x, \qquad g_6 = \epsilon^2 f_6 z,$$
$$g_7 = \epsilon^2 f_7 (z+1) + 2\epsilon^2 f_6, \qquad g_8 = \epsilon^2 f_8 y,$$
$$g_9 = \epsilon^2 f_9 (y+1) + 2\epsilon^2 f_8, \qquad g_{10} = \epsilon^3 f_{10} x,$$
$$g_{11} = \epsilon^3 f_{11} r, \qquad g_{12} = \epsilon^2 f_{12} r,$$
$$g_{13} = \epsilon^2 f_{13} x z + \epsilon^2 f_{12} x + 3/2\,\epsilon^3 f_{11} x, \qquad g_{14} = \epsilon^3 f_{14} r,$$
$$g_{15} = \epsilon^2 f_{15} \left[x - (z+1)^2\right] + 3/2\,\epsilon^3 f_{14} (x - 2z - 2), \qquad g_{16} = \epsilon^3 f_{16} (y - z),$$
$$g_{17} = \epsilon^4 f_{17} (x + y - z), \qquad g_{18} = \epsilon^3 f_{18} \left[x + (z+1)(y+z)\right],$$
$$g_{19} = \epsilon^4 f_{19} r, \qquad g_{20} = \epsilon^3 (1 - 2\epsilon) f_{20} r,$$
$$g_{21} = \epsilon^3 f_{21} x (y+1), \qquad g_{22} = \epsilon^3 f_{22} x y,$$
$$g_{23} = \epsilon^2 f_{23} x (y+1) + \epsilon^3 f_{22} x, \qquad g_{24} = \epsilon^4 f_{24} x (y - z),$$
$$g_{25} = \epsilon^4 f_{25} x^2 (y+1), \qquad g_{26} = \epsilon^4 f_{26} x r,$$
$$g_{27} = \sum_{i=1}^{27} c_i f_i$$

Topology 1

The g_i's satisfy the form of canonical DEs

$$\partial_i \boldsymbol{g}(\boldsymbol{x};\epsilon) = \epsilon\, \mathbb{A}_i(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{x};\epsilon),$$

and we can write the system of differential equation into $d\log$ form,

$$d\boldsymbol{g}(\boldsymbol{x};\epsilon) = \epsilon\, d\mathbb{A}(\boldsymbol{x})\, \boldsymbol{g}(\boldsymbol{x};\epsilon), \quad \text{where} \quad d\mathbb{A} = \sum_{i=1}^{n} \mathbb{C}_i\, d\log\, \omega_i(\boldsymbol{x}).$$

The ω_i's are called letters,

$$\begin{aligned}
\omega_1 &= x, & \omega_2 &= y, & \omega_3 &= z, \\
\omega_4 &= 1+z, & \omega_5 &= 1+y, & \omega_6 &= z-y, \\
\omega_7 &= x-4z, & \omega_8 &= x+y-z, & \omega_9 &= x-(z+1)^2, \\
\omega_{10} &= x+(z+1)(y-z), & \omega_{11} &= x\,y+(y-z)^2, & \omega_{12} &= \frac{x-r}{x+r}, \\
\omega_{13} &= \frac{x-2z-2-r}{x-2z-2+r}, & \omega_{14} &= \frac{x+2y-2z-r}{x+2y-2z+r}.
\end{aligned}$$

Topology 1

- A square root r appear in the DEs,

$$r^2 = x(x+4).$$

We can adopt a changing of variables to rationalizing the square root so that MPLs can be easily used to express the solutions.

$$x = \frac{(x_1 + z)^2}{x_1}, \quad \text{and then} \quad r^2 = \frac{\left(x_1^2 - z^2\right)^2}{x_1^2}.$$

- The solutions with undetermined constants obtained from the DEs are regular in some limits which means there are no singularties. So the terms that brings divergence in the solutions are required to be cancel. For example, we know the results are finite in the limit $y \to 0$ via other methods.

$$I = (c_1 + c_2) \log y + c_3 \log y + 1 + \ldots$$

The $\log y$ constraints the constants $c_1 + c_2 = 0$.

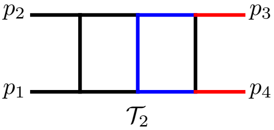
Topology 1

The results are combination of MPLs of x_1, y, z. For example, the order ϵ^2 of g_{23} is represented as

$$
\begin{aligned}
g_{23}^{(2)} = & -G(0;x_1)\,G(\text{-}1;y) - G(0;x_1)\,G(\text{-}x_1;z) + 1/2\,G(0;x_1)\,G(\text{-}\sqrt{x_1};z) \\
& + 1/2\,G(0;x_1)\,G(\sqrt{x_1};z) - 1/2\,G(0,0;x_1) + 1/2\,G(\text{--}1,0;x_1) \\
& + 2\,G(\text{-}1;y)\,G(\text{-}1;z) - 2\,G(\text{-}1;y)\,G(\text{-}x_1;z) - 2\,G(z;y)\,G(\text{-}1;z) \\
& + 2\,G(0,\text{-}1;y) - 4\,G(\text{-}1,\text{-}1;y) + 2\,G(z,\text{-}1;y) - 4\,G(0,\text{-}1;z) \\
& + 4\,G(\text{-}1,\text{-}1;z) + 2\,G(\text{-}x_1,\text{-}1;z) - 2\,G(\text{-}x_1,\text{-}x_1;z) - G(\text{-}\sqrt{x_1},\text{-}1;z) \\
& + G(\text{-}\sqrt{x_1},\text{-}x_1;z) - G(\sqrt{x_1},\text{-}1;z) + G(\sqrt{x_1},\text{-}x_1;z) + \pi^2/12.
\end{aligned}
$$

All MIs are expressed as MPLs.

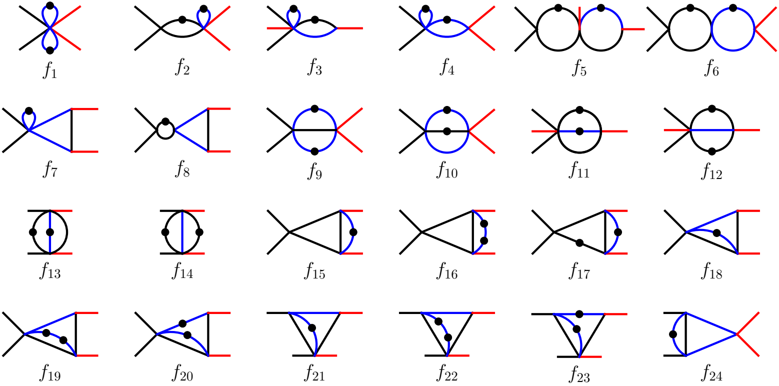
Topology 2



$$P_1 = l_1^2, \quad P_2 = (l_1 + p_1)^2, \quad P_3 = (l_1 - p_2)^2,$$
$$P_4 = l_2^2 - m_t^2, \quad P_5 = (l_2 + p_1)^2 - m_t^2, \quad P_6 = (l_2 - p_2)^2 - m_t^2,$$
$$P_7 = (l_1 - l_2)^2 - m_t^2, \quad P_8 = (l_1 - p_2 - p_3)^2, \quad P_9 = (l_2 - p_2 - p_3)^2.$$

Topology 2

The system of differential equation is constructed by 45 MIs.



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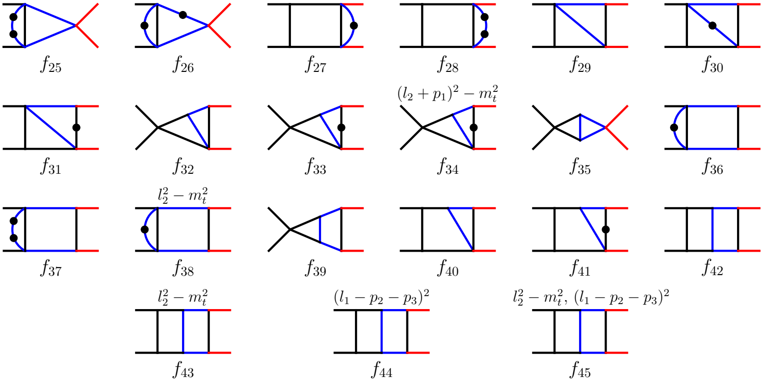

Topology 2

The canonical DEs for $\mathcal{T}_2$ involve four square roots r_i's,

$$r_1^2 = x\,(4+x), \qquad r_2^2 = x\,(x-4z),$$
$$r_3^2 = x\left[\,x\,(1+y)^2 + 4\,(y-z)^2\,\right], \qquad r_4^2 = x\,y\left[\,4\,(z+1) - (x+4)\,y\,\right].$$

The letter can be categorized into three forms which containing 0, 1 and 2 square roots,

$$\omega_i = P_i, \quad \omega_i = \frac{P_i - P_j\,R_i}{P_i - P_j\,R_i}, \quad \omega_i = \frac{P_i\,R_i - P_j\,R_j}{P_i\,R_i - P_j\,R_j}$$

- Rationalizing the roots simultaneously is not achieved.
- Solutions are expressed as the combination of one-fold integrals and MPLs.

Boundary

Reserving a square root makes analyzing the asymptotic behavior less convenient. We can solve the expanded DEs to obtain the results of the MIs at a regular point, regard it as the initial point for a path integral, and subsequently derive the value at the arbitrary point.

In this work, we choose the initial point $(y, z) = (0, 0)$ where all the MIs are regular. The expanded DEs are written as

$$\begin{cases} \mathbb{A}_x(x, y, z) = \sum_{n=0,m=0}^{+\infty} \mathbb{A}_{x,n,m}(x)\, y^n z^m & \text{(Taylor expansion)} \\ \mathbb{A}_y(x, y, z) = \dfrac{\mathbb{A}_{y,-1,0}}{y} + \sum_{n=0,m=0}^{+\infty} \mathbb{A}_{y,n,m}(x)\, y^n z^m & \\ & \text{(Laurent expansion)} \\ \mathbb{A}_z(x, y, z) = \dfrac{\mathbb{A}_{z,0,-1}}{z} + \sum_{n=0,m=0}^{+\infty} \mathbb{A}_{z,n,m}(x)\, y^n z^m & \end{cases}$$

The value of $\boldsymbol{g}(x, 0, 0)$ are obtained through solving the lowest-approximated equations. $\mathbb{A}_{x,n,m}$, $\mathbb{A}_{y,-1,0}$, $\mathbb{A}_{z,0,-1}$ contributes the finite values around $(y, z) = (0, 0)$.

Boundary

Following the process in topology 1, regularities fix the constants of the solutions and $\boldsymbol{g}(x,0,0)$ is specific.

- Rationalizing:

$$x = \frac{x_2^2}{x_2+1}, \qquad z = -\frac{x_2^2\,(z_2+1)}{(x_2+1)\,z_2^2}$$

$$y = \frac{x_2\left[x_2(z_2+1)(1+x_2-y_2^2)+(x_2+1)y_2z_2^2\right]}{(x_2+1)(y_2+1)(y_2-x_2-1)\,z_2^2}.$$

- Integration path:

$$x_2(\kappa) = x_2, \qquad y_2(\kappa) = \kappa\, y_2, \qquad z_2(\kappa) = \kappa\,(z_2+1)-1$$

Boundary

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- Rationalizing:

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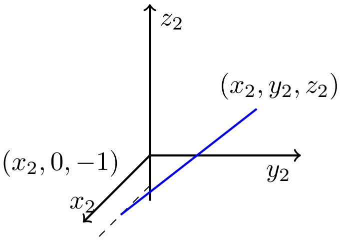
- Integration path:



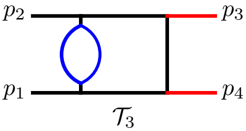
Topology 2

The form of one-fold integrals is expressed as follows

$$\int_{\gamma_\kappa} d\log\omega_i\, G(a_1,\ldots,a_n;\kappa)+\ldots$$

And the one-fold integrals appear in the order ϵ^3 of g_{36} and the order ϵ^4 of $g_{36,37,38,44,45}$.

Topology 3



$$P_1 = l_1^2, \quad P_2 = (l_1 + p_1)^2, \quad P_3 = (l_1 - p_2)^2,$$
$$P_4 = l_2^2 - m_t^2, \quad P_5 = (l_2 + p_1)^2 - m_t^2, \quad P_6 = (l_2 - p_2)^2 - m_t^2,$$
$$P_7 = (l_1 - l_2)^2 - m_t^2, \quad P_8 = (l_1 - p_2 - p_3)^2, \quad P_9 = (l_2 - p_2 - p_3)^2,$$

Topology 3

The maximal cut of Feynman integrals reveals the properties to some extent. It gives the homogeneous solutions of DEs. The maximal cut of $F_{01120110}$ is as follows

$$\text{Maxcut}(f_{13}) = \frac{1}{4\,\pi^3\sqrt{x\,(x-4\,z)}}\int\frac{d\xi}{\sqrt{(\xi-\xi_1)\,(\xi-\xi_2)\,(\xi-\xi_3)\,(\xi-\xi_4)}}+\mathcal{O}(\epsilon).$$

The integral corresponds to an elliptic curve,

$$\vartheta(\xi)^2 = (\xi-\xi_1)\,(\xi-\xi_2)\,(\xi-\xi_3)\,(\xi-\xi_4),$$

where

$$\xi_1 = -\frac{y\,(x-2\,z)+2\,z^2+2\,z\sqrt{x\,y+(y-z)^2}}{x-4\,z},$$

$$\xi_2 = -\frac{y\,(x-2\,z)+2\,z^2-2\,z\sqrt{x\,y+(y-z)^2}}{x-4\,z},$$

$$\xi_3 = 0,$$

$$\xi_4 = 4.$$

Topology 3

The canonical DEs are not achieved in the topology 3, where the ϵ^0-parts $\mathbb{B}_i^{(0)}(\boldsymbol{x})$ are strictly lower triangular matrices.

- The periods of elliptic curve

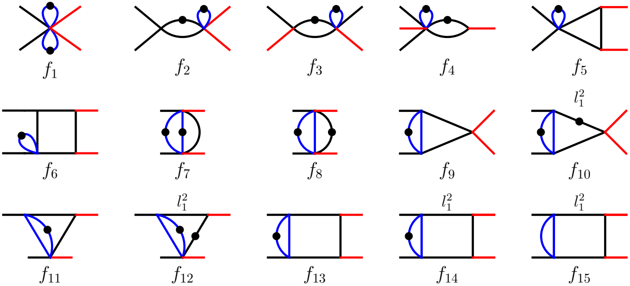
$$\Psi_1 = \frac{4\,K(k)}{U_3^{1/2}}, \qquad \Psi_2 = \frac{4\,i\,K(1-k)}{U_3^{1/2}}.$$

- Linear form of DEs

$$\partial_i \boldsymbol{g}(\boldsymbol{x};\epsilon) = \left(\mathbb{B}_i^{(0)}(\boldsymbol{x}) + \epsilon\,\mathbb{B}_i^{(1)}(\boldsymbol{x}) \right) \boldsymbol{g}(\boldsymbol{x};\epsilon),$$

By introducing the period Ψ_1, the linear form of DEs is constructed.

The system of differential equation is constructed by 15 MIs.



Topology 3

The DEs constructed by g_i's satisfy the linear form.

$$
\begin{aligned}
g_1 &= \epsilon^2 f_1, & g_2 &= \epsilon^2 f_2\, x, \\
g_3 &= \epsilon^2 f_3\, y, & g_4 &= \epsilon^2 f_4\, z, \\
g_5 &= \epsilon^3 f_5\, r'_3, & g_6 &= \epsilon^3 f_6\, x\, y, \\
g_7 &= \epsilon^2 f_7\, y, & g_8 &= \epsilon^2 f_8 r'_2 + 1/2\, f_7\, r'_2, \\
g_9 &= \epsilon^3 f_9\, x, & g_{10} &= \epsilon^2 f_{10}\, r'_1 - \epsilon^3 f_9\, r'_1, \\
g_{11} &= \epsilon^3 f_{11}\, (y-z), & g_{12} &= \epsilon^2 [\, f_{12}\, z + \epsilon\, f_{11}\, (y-z) - f_8\, y\,]\, r'_4/(y-z), \\
g_{13} &= \pi\, \epsilon^3 f_{13}\, r'_3/\Psi_1, & g_{14} &= \epsilon^3 f_{14}\, r'_3, \\
g_{15} &= \frac{1}{\epsilon}\frac{\Psi_1^2}{2\, i\, \pi\, W_y}\frac{\partial g_{13}}{\partial y}, & &
\end{aligned}
$$

where the roots r_i''s are

$$
\begin{aligned}
{r'_1}^2 &= x\,(x-4), & {r'_2}^2 &= y\,(y+4), \\
{r'_3}^2 &= x\,(x-4z), & {r'_4}^2 &= (z-y)\,(y-z+4).
\end{aligned}
$$

Boundary

We evaluate the values of all the master integrals at $y = 0,\ z = 1$. Only ϵ^1-parts of the DEs are left when we employ the same process to expand the matrices around $y = 0,\ z = 1$. Combining the regular condition, the $g(x, 0, 1)$, as the initial values, are specified. The period and its derivation contributes a new square root,

$$\Psi_1\big|_{y=0,z=1} = \frac{\pi\sqrt{x-4}}{2}, \qquad \partial_y \Psi_1\big|_{y=0,z=1} = \frac{\pi(2-x)\sqrt{x-4}}{8},$$

and r_1' remains in the expansion.

They can be rationalized by

$$x = \frac{(1+x_3^2)^2}{x_3^2}.$$

The results are expressed in terms of iterated integrals by following the direct integration path.

$$\mathsf{I}(f_1, f_2, \ldots, f_n) = \int_0^1 d\kappa_1\, f_1(\kappa_1) \int_0^{\kappa_1} d\kappa_2\, f_2(\kappa_2) \ldots \int_0^{\kappa_{n-1}} d\kappa_n\, f_n(\kappa_n),$$

Conclusion

- We introduced the basic properties of Feynman integrals and the method of differential equation.
- For the process $u\bar{u} \to W^-W^+$, we establish three closed and complete system of differential equation.
  - Set 1: One square root $\to$ GPLs.
  - Set 2: Four square roots $\to$ GPLs and one-fold integrals.
  - Set 3: Elliptic sector $\to$ iterated integrals.
- Nonplanar master integrals is limited by IBP reduction. Maybe a better computer can assist me with future work.
- And the QCD corrections to W-pair production are likewise significant for precise prediction.

Feynman diagrams for $u\bar{u} \to W^-W^+$

101 diagrams

- Not involve top quark
  - 88 diagrams

- Involve top quark
  - 13 diagrams

Nonplanar

- Factorizable 1
- Non-factorizable 1

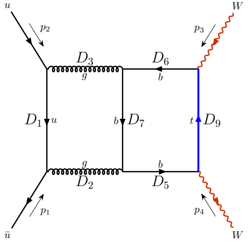
Planar

- Factorizable 3
- Non-factorizable 3
  - Family 1
  - Family 2

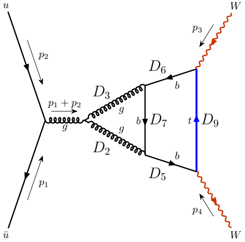
Family 1 (Set 1)

$$
\begin{aligned}
D_1 &= l_1^2, & D_2 &= (l_1+p_1)^2, & D_3 &= (l_1-p_2)^2, \\
D_4 &= l_2^2, & D_5 &= (l_2+p_1)^2, & D_6 &= (l_2-p_2)^2, \\
D_7 &= (l_1-l_2)^2, & D_8 &= (l_1-p_2-p_3)^2, & D_9 &= (l_2-p_2-p_3)^2 - m_t^2.
\end{aligned}
$$

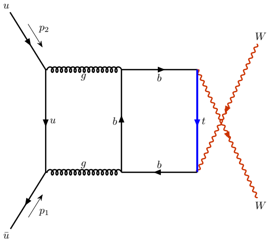
[1, 1, 1, 0, 1, 1, 1, 0, 1]



[0, 1, 1, 0, 1, 1, 1, 0, 1]

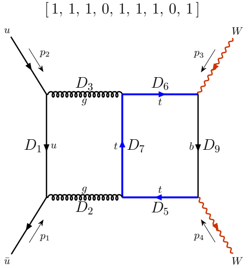


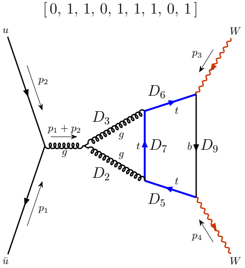
[1, 1, 1, 0, 1, 1, 1, 0, 1] with $p_1 \leftrightarrow p_2$

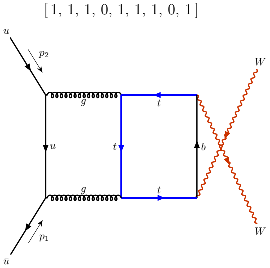


Family 2 (Set 2)

$$
\begin{aligned}
&D_1 = l_1^2, && D_2 = (l_1 + p_1)^2, && D_3 = (l_1 - p_2)^2, \\
&D_4 = l_2^2 - m_t^2, && D_5 = (l_2 + p_1)^2 - m_t^2, && D_6 = (l_2 - p_2)^2 - m_t^2, \\
&D_7 = (l_1 - l_2)^2 - m_t^2, && D_8 = (l_1 - p_2 - p_3)^2, && D_9 = (l_2 - p_2 - p_3)^2.
\end{aligned}
$$

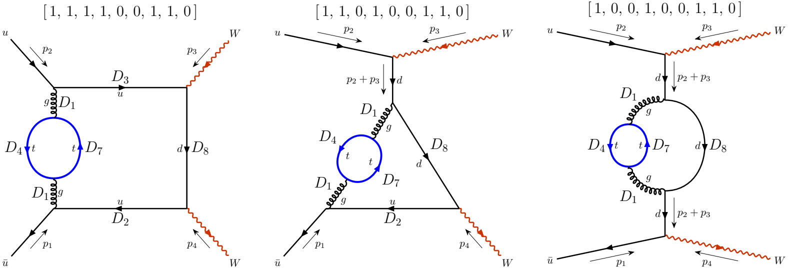






Family 2 (Set 3)

$$
\begin{aligned}
D_1 &= l_1^2, & D_2 &= (l_1+p_1)^2, & D_3 &= (l_1-p_2)^2, \\
D_4 &= l_2^2 - m_t^2, & D_5 &= (l_2+p_1)^2 - m_t^2, & D_6 &= (l_2-p_2)^2 - m_t^2, \\
D_7 &= (l_1-l_2)^2 - m_t^2, & D_8 &= (l_1-p_2-p_3)^2, & D_9 &= (l_2-p_2-p_3)^2.
\end{aligned}
$$



Family 2 (Set 3)

$$
\begin{aligned}
D_1 &= l_1^2, & D_2 &= (l_1+p_1)^2, & D_3 &= (l_1-p_2)^2, \\
D_4 &= l_2^2 - m_t^2, & D_5 &= (l_2+p_1)^2 - m_t^2, & D_6 &= (l_2-p_2)^2 - m_t^2, \\
D_7 &= (l_1-l_2)^2 - m_t^2, & D_8 &= (l_1-p_2-p_3)^2, & D_9 &= (l_2-p_2-p_3)^2.
\end{aligned}
$$