

1)

i.

$\text{Sat}(\phi_1) = \{s_0, s_1, s_2, s_3, s_4\}$

$\text{TS} \models \phi$

ii.

$\text{Sat}(\phi_2) = \{s_4\}$

$\neg \text{TS} \models \phi$

iii.

$\text{Sat}(\phi_1) = \{s_0, s_1, s_2, s_3, s_4\}$

$\text{TS} \models \phi$

2)

i.

$\phi_1 = \exists \diamond \forall \Box c$

Subformulas:  $\forall \Box c, c$

$\text{Sat}(c) = \{s_2, s_3, s_4\}$

$\text{Sat}(\forall \Box c) = \{s_2, s_3, s_4\}$

$\text{Sat}(\exists \diamond \forall \Box c) = \{s_0, s_1, s_2, s_3, s_4\}$

Therefore  $\text{TS} \models \phi_1$

ii.

$\phi_2 = \forall (a \cup \forall \diamond c)$

Subformulas:  $\forall \diamond c, c$

$\text{Sat}(c) = \{s_2, s_3, s_4\}$

$\text{Sat}(\forall \diamond c) = \{s_0, s_1, s_2, s_3, s_4\}$

$\text{Sat}(\forall (a \cup \forall \diamond c)) = \{s_0, s_1, s_2, s_3, s_4\}$

Or alternatively normalizing the formula for the algorithm:  $\forall (a \cup \neg \exists \Box \neg c)$

Subformulas:  $c, \neg c, \exists \Box \neg c, \neg \exists \Box \neg c$

$\text{Sat}(c) = \{s_2, s_3, s_4\}$

$\text{Sat}(\neg c) = \{s_0, s_1\}$

$\text{Sat}(\exists \Box \neg c) = \{\}$

$\text{Sat}(\neg \exists \Box \neg c) = \{s_0, s_1, s_2, s_3, s_4\}$

And we get the same result.

Therefore  $\text{TS} \models \phi_1$

3)  $(TS \models \exists(\phi U \psi)) = (TS' \models \exists \diamond \psi)$ . Theorem

Since TS has more or equal transitions than TS' and everything else the same,  
for any formula  $\phi$  we have that

$(TS' \models \phi) \Rightarrow (TS \models \phi)$  ----- assump\_0

I formalize the removal of outgoing transitions

$\forall \pi: TS'. \forall i. \forall j. j < i \Rightarrow (\pi_j \models \neg(\neg \phi \vee \psi) \vee \pi_i = \pi_j)$  ---- assump\_1

Proving  $(TS \models \exists(\phi U \psi)) \Rightarrow (TS' \models \exists \diamond \psi)$ :

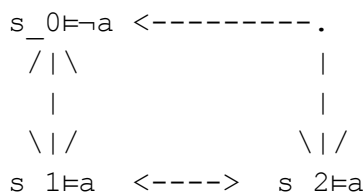
$TS \models \exists \diamond \psi$       def of  $\diamond$   
 $= TS \models \exists(TU\psi)$     base law  
 $\Leftarrow TS \models \exists(\phi U \psi)$     by assump\_1, this path is unchanged until  $\psi$  hold.  
 $= TS' \models \exists(\phi U \psi)$

Proving  $(TS' \models \exists \diamond \psi) \Rightarrow (TS \models \exists(\phi U \psi))$ :

$TS' \models \exists \diamond \psi$   
 $= TS' \models \exists \pi: Paths(TS). \exists i. \pi_i \models \psi$   
 $= TS' \models \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge T$   
 $= TS' \models \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge (\forall j. j < i \Rightarrow (\pi_j \models \neg(\neg \phi \vee \psi) \vee \pi_i = \pi_j))$   
 $\Rightarrow TS' \models \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge ((\forall j. j < i \Rightarrow \pi_j \models \neg(\neg \phi \vee \psi)) \vee (\forall j. j < i \Rightarrow \pi_i = \pi_j))$   
 $= TS' \models \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge \forall j. j < i \Rightarrow \pi_j \models \neg(\neg \phi \vee \psi)$   
 $\vee \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge \forall j. j < i \Rightarrow \pi_i = \pi_j$   
 $= TS' \models \exists(\phi U \psi) \vee \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge \forall j. j < i \Rightarrow \pi_i = \pi_j$   
 $\Rightarrow TS' \models \exists(\phi U \psi) \vee \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge \forall j. j < i \Rightarrow \pi_i \models \psi = \pi_j \models \psi$   
 $= TS' \models \exists(\phi U \psi) \vee \exists \pi: Paths(TS). \exists i. \pi_i \models \psi \wedge \forall j. j < i \Rightarrow \pi_j \models \psi$   
 $= TS' \models \exists(\phi U \psi) \vee \exists \pi: Paths(TS). \exists i. \forall j. j \leq i \Rightarrow \pi_j \models \psi$   
 $\Rightarrow TS' \models \exists(\phi U \psi) \vee \exists(FU\psi)$   
 $\Rightarrow TS' \models \exists(\phi U \psi) \vee \exists(\phi U \psi)$   
 $= TS' \models \exists(\phi U \psi)$   
 $\Rightarrow TS \models \exists(\phi U \psi)$

4)

i.  $\forall \diamond \forall \diamond \phi = \forall \diamond \forall \diamond \phi$ . Not a theorem. Example TS:



ii.  $\exists \circ \exists \diamond \phi = \exists \diamond \exists \circ \phi$ . Theorem.

$\exists \circ \exists \diamond \phi = \exists \diamond \exists \circ \phi$  double negation  
 $= \exists \circ \exists \diamond \neg \neg \phi = \exists \diamond \exists \circ \neg \neg \phi$  duality laws  
 $= \exists \circ \neg \forall \square \neg \phi = \exists \diamond \neg \forall \circ \neg \phi$  duality laws  
 $= \neg \forall \circ \forall \square \neg \phi = \neg \forall \square \forall \circ \neg \phi$  eliminate double negation in equality  
 $= \forall \circ \forall \square \neg \phi = \forall \square \forall \circ \neg \phi$  theorem iii. from below  
 $= T$

iii.  $\forall \circ \forall \square \phi = \forall \square \forall \circ \phi$ . Theorem

Informally: The right hand side says that for all paths starting at this state, it is always true that in any path starting from the next state  $\phi$  is always true. In other words, starting from the next state,  $\phi$  always holds. This is the same as what the left hand side says.

More formally it is easier to first convert to an equivalent LTL formula and prove  $\circ \square \phi = \square \circ \phi$ .

$\circ \square \phi$   
 $= \pi_1 \models \square \phi$   
 $= \forall i. \pi_{i+1} \models \phi$   
 $= \forall i. \pi_i \models \circ \phi$   
 $= \square \circ \phi$

iv.  $\exists \circ \exists \square \phi = \exists \square \exists \circ \phi$ . Not a theorem. Example TS:



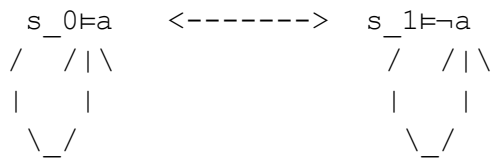
Bonus

Informally:

CTL formula:  $\forall \diamond \exists \circ \forall \diamond \neg a$

To get a LTL possibly equivalent formula we remove the quantifiers:  $\diamond \circ \neg a$

Consider this transition system:



The CTL formula holds. On the path that starts and stays at s\_0 the LTL formula does not hold.