CSC410 Assignment 5

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1 Question One

1.1 (a) Construct a model of this problem in NuSMV and then reduce the question ?can all soldiers cross the bridge in 60 minutes or less? to model checking problem by stating the question as an LTL property.

```
In command console
        check_ltlspec -p "F (total_time < 60 & all_crossed = TRUE)"
Equivalent to command in NuSMV file
    LTLSPEC
        !F (total_time < 60 & all_crossed = TRUE)
```

1.2 (b) Can all soldiers eventually cross the bridge? Note that this is a yes/no question.

Yes

1.3 (c) Can you rephrase the first property so that you get the model checker to tell you the step-by-step scenario under which all soldiers can cross the bridge?

```
In command console
        check_ltlspec -p "!F (all_crossed = TRUE)"
Equivalent to command in NuSMV file
    LTLSPEC
        !F (all_crossed = TRUE)
```

1.4 (d) Is there a scenario in which only one soldier is left at the enemy side of the bridge? (yes/no question)

Yes

1.5 (e) Can you rephrase property (c) so that you get the model checker to tell you how to get all soldiers across the bridge within 60 minutes?

```
In command console
        check_ltlspec -p "!F (total_time <= 60 & all_crossed = TRUE)"
Equivalent to command in NuSMV file
    LTLSPEC
        !F (total_time <= 60 & all_crossed = TRUE)
```

2 Question Two

2.1 (a) $\phi_1 = \Diamond\Box c$ = Eventually always c

No
$\pi = S_1 S_3 S_4 S_3 S_4 ...$ is a counter example
Because, $S_4 = b$ the state will always circulate between b and c.

2.2 (b) $\phi_2 = \Box\Diamond c$ = Always eventually c

Yes
If the path starts at S_2
$\implies$ always reach c

If the path starts at S_1
case 1: Continuous to S_3
$\implies$ reach c

case 2: Continous to S_4
$\implies$ all successor of $S_4(S_2, S_3, S_5)$ reach c

2.3 (c) $\phi_3 = \bigcirc\neg c \implies \bigcirc\bigcirc c$ = Next not c $\implies$ next next c Eventually c

Yes
S_1, S_4 are the only states that doesn't contain c. S_1 is not a successor of any states, and all states next to S_4 contains c, thus, it will always reach c after S_4

2.4 (d) $\phi_4 = a \cup \Box(b \vee c)$ = a until always (b or c)

No
If the path starts at S_2, then the initial state doesn't contain a

2.5 (e) $\phi_5 = (\bigcirc\bigcirc b) \cup (b \vee c)$ = (next next b) until (b or c)

No
$\pi = S_2 S_4 S_2 S_4 ...$ is a counter example.
After two next step, the path reaches S_2, which doesn't contain b.

3 Question Three

3.1 (a) $\bigcirc\Diamond\phi \equiv \Diamond\bigcirc\phi$

Transform the left hand side formula as follows.

$$\bigcirc\Diamond\phi = \bigcirc(True \bigcup \phi) = \bigcirc True \bigcup \bigcirc\phi = True \bigcup \bigcirc\phi$$

The right hand side formula can be transform as follow

$$\Diamond\bigcirc\phi = True \bigcup \bigcirc\phi$$

From here, it is trivia to see that they are indeed equivalent.

3.2 (b) $\Box\Diamond\phi \implies \Box\Diamond\psi \equiv \Box(\phi \implies \Diamond\psi)$

The counter example is as follows.
Let there be a path $s_0 s_1 s_1 s_1 s_1 s_1 s_1 ...$, such that $s_0 = \phi \wedge \neg\psi$, $s_1 = \neg\phi \wedge \neg\psi$
Notice in this example, ψ always does not hold.

However, the left hand side formula holds, since $\Box\Diamond\phi$ does not hold (because once the path reaches S_1, it will never arrive at any other states, and ϕ no longer holds for all future states), making it vacuously true. However, the right hand side formula does not hold. The path starts at S_0 which ϕ holds, making the precondition of the implication true, however ψ never holds, making the postcondition false.

3.3 (c) $(\Diamond\Box\phi_1) \land (\Diamond\Box\phi_2) \equiv \Diamond(\Box\phi_1 \land \Box\phi_2) \equiv \Diamond\Box(\phi_1 \land \phi_2)$

Assume:
$\exists j_1 \geq 0, \forall i \geq j_1, \sigma[i...] \models \phi_1$
and
$\exists j_2 \geq 0, \forall i \geq j_2, \sigma[i...] \models \phi_2$
Thus, let $k = max(j_1, j_2)$
Then, $k \geq j_1$ and $k \geq j_2$
Then, $\forall i \geq k, \sigma[i...] \models \phi_1$ and $\forall i \geq k, \sigma[i...] \models \phi_2$
Thus, $\exists k \geq 0, \forall i \geq k, \sigma[i...] \models \phi_1 \land \phi_2$ which is the right hand side.

On the other hand assume:
$\exists k \geq 0, \forall i \geq k, \sigma[i...] \models \phi_1 \land \phi_2$
Then, $\exists j_1 = k \geq 0, \forall i \geq j_1, \sigma[i...] \models \phi_1$
and
$\exists j_2 = k \geq 0, \forall i \geq j_2, \sigma[i...] \models \phi_2$

3.4 (d) $\Diamond\Box\phi \implies \Box\Diamond\psi \equiv \Box(\phi \bigcup (\psi \lor \neg\phi))$

We transform the left hand side formula as follows.

$$\Diamond\Box\phi \implies \Box\Diamond\phi = \neg\Diamond\Box\phi \lor \Box\Diamond\psi = \Box\Diamond\neg\phi \lor \Box\Diamond\psi$$

We already proved, that

$$\Box\phi \implies \Diamond\psi \equiv \phi \bigcup (\psi \lor \neg\phi)$$

Hence we can transform the right hand side formula as follows

$$\Box(\phi \bigcup (\psi \lor \neg\phi)) = \Box(\Box\phi \implies \Diamond\psi) = \Box(\Diamond\neg\phi \lor \Diamond\psi)$$

Next, prove

$$\Box\Diamond\neg\phi \lor \Box\Diamond\psi \equiv \Box(\Diamond\neg\phi \lor \Diamond\psi)$$

Assume $\Box\Diamond\neg\phi \lor \Box\Diamond\psi$
Then, $\forall j \geq 0, \exists i \geq j, \sigma[i...] \models \neg\phi$
or
$\forall j \geq 0, \exists i \geq j, \sigma[i...] \models \psi$
Thus, $\forall j \geq 0, \exists i \geq j, \sigma[i...] \models \neg\phi \lor \psi$
Thus, $\Box\Diamond(\neg\phi \lor \psi) \equiv \Box(\Diamond\neg\phi \lor \Diamond\psi)$

Then assume $\Box(\Diamond\neg\phi \lor \Diamond\psi) \equiv \Box\Diamond(\neg\phi \lor \psi)$
Then, $\forall j \geq 0, \exists i \geq j, \sigma[i...] \models \neg\phi \lor \psi$
Assume (proof by contradiction), $\neg(\Box\Diamond\neg\phi \lor \Box\Diamond\psi) \equiv (\Diamond\Box\phi \land \Diamond\Box\neg\psi)$
using part c):
$(\Diamond\Box\phi \land \Diamond\Box\neg\psi) \equiv \Diamond\Box(\phi \land \neg\psi)$
Then $\exists k \geq 0, \forall i \geq k, \sigma[i...] \models (\phi \land \neg\psi)$
but our assumption assumed that: $\exists i \geq k, \sigma[i...] \models \neg\phi \lor \psi \equiv \neg(\phi \land \neg\psi)$
Thus, contradition.
Thus $\Box\Diamond\neg\phi \lor \Box\Diamond\psi$

3.5 (e) $\Box\phi \implies \Diamond\psi \equiv \phi \bigcup (\psi \vee \neg\phi)$

From the left hand side:

$$\Box\phi \implies \Diamond\psi \equiv \neg\Box\phi \vee \Diamond\psi = \Diamond\neg\phi \vee \Diamond\psi = \Diamond(\neg\phi \vee \psi) = True \bigcup (\neg\phi \vee \psi)$$

Assume $\phi \bigcup (\psi \vee \neg\phi)$ Then $\exists j \geq 0, \sigma[j...] \vDash (\psi \vee \neg\phi)$ and $\sigma[i...] \vDash \phi$, for all $0 \leq i < j$
Let k = j, then $\exists k \geq 0, \sigma[k...] \vDash (\psi \vee \neg\phi)$ and $\sigma[i...] \vDash$ True, for all $0 \leq i < k$

Thus, $True \bigcup (\neg\phi \vee \psi)$
Now, need to show the other direction.

Here, the True states will always be one of the following four cases: $(\psi \wedge \phi) \vee (\neg\psi \wedge \phi) \vee (\psi \wedge \neg\phi) \vee (\neg\psi \wedge \neg\phi)$

If the path starts with $(\psi \wedge \phi) \vee (\psi \wedge \neg\phi) \vee (\neg\psi \wedge \neg\phi)$, $\phi \bigcup (\psi \vee \neg\phi)$ holds as either ψ or $\neg\phi$ is true. (i.e. the initial state satifies the right hand side of the until. Let $j = 0, \sigma[j...] \vDash (\psi \vee \neg\phi)$ and there is no i between 0 and j)

If the path starts with $(\neg\psi \wedge \phi)$, then:
$\exists j \geq 0, \sigma[j...] \vDash (\psi \vee \neg\phi)$ and $\sigma[i...] \vDash (\neg\psi \wedge \phi)$, for all $0 \leq i < j$
Then, $\exists j \geq 0, \sigma[j...] \vDash (\psi \vee \neg\phi)$ and $\sigma[i...] \vDash \phi$, for all $0 \leq i < j$
Thus, $\phi \bigcup (\psi \vee \neg\phi)$
Hence, the equivalent holds.

4 Question Four

4.1 (a) "At next" ($\phi N \psi$): at the next point in time where ψ holds (is true), ϕ also holds.

$$\bigcirc((\neg\phi \bigcup (\phi \wedge \psi)) \vee \Box\neg\psi)$$

4.2 (b)"While" ($\phi W \psi$): ϕ holds at least as long as ψ does.

$$(\phi \bigcup \neg\psi) \vee \Box\phi$$

4.3 (c) "Before" ($\phi B \psi$): ψ holds at some point in the future, ϕ holds before.

$$\Diamond\phi(\bigcirc\Diamond\psi) \vee \Box\neg\psi$$
$$(\Diamond\psi) \implies \Diamond\phi(\bigcirc\Diamond\psi)$$