STA255_Mid

Wednesday, March 11, 2020 2:19 PM

University of Toronto

Department of Statistical Sciences

STA255H1-S: Statistical Theory

Midterm Test, Mar 03, 2020

Duration: 1 hour and 40 minutes

SHAHRIAR

Last Name:__________ First Name:__________

Student ID:__________ Signature:__________

Aids allowed: A calculator (No phone calculators are allowed).

All your work must be presented clearly in order to get credit. Answer alone (even though correct) will only qualify for ZERO credit. Show your answer in the space provided on the pages with QR code on the top.

There are **8 pages** excluding this title page. Please check to see if you have all the pages.

Good Luck!

Question	1	2	3	4	5	6	Total (out of 60)
Marks							

1. [10 points] Suppose you have a standard deck of 52 cards (26 black cards and 26 red cards).

a) [3.5 points] In the first exercise, you pick the first card, see what it is, put it back into the deck and then pick another card after shuffling the deck. You repeat the process (this is so called "sampling with replacement"). In this process if you **pick three cards**, what is the probability that **at least one of them is a red card**?

P[Black card in any draw] = $\frac{1}{2}$ = P[Red card in any draw]

P[at least one red card] = 1 - P[no red cards]

$= 1 - \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{7}{8}$

b) [3.5 points] In another exercise, every time you pick a card, you see what it is and do not put it back (this is so called "sampling without replacement"). In this process, if you **pick three cards**, what is the probability that you get the **first red card on the third pick**.

P[first red card on third pick]

= P[black card on the 1st & black card on the 2nd & red card on 3rd]

$= \frac{26}{52} * \frac{25}{51} * \frac{26}{50}$

c) [3 points] The setups that were presented in part (a) and part (b), what **distribution** do you think each of them represent? Give your reasoning.

In part (a) picking cards are indep. 3 cards are picked. So it's more than one trial. Success is defined as whether or not its a red card $\Longrightarrow$ Binomial $dist^n$.

In part (b) even though it sounds like a geometric $dist^n$, the picks are not independent. $\Rightarrow$ It's NOT Geom. $dist^n$.

2. [10 points] The length of stay (measured in terms of days, e.g. 4 days or 4.1 days or 4.25 days etc.) for each patient at a hospital with a certain disease follows an Exponential distribution with $\lambda = 0.25$

a) [5 points] What is the probability that a randomly selected patient with that disease will **stay more than 7 days**?

Let, X = length of stay. $\Rightarrow f(x) = 0.25\, e^{-0.25x}$, $x > 0$

$$P[X > 7] = \int_7^{\infty} 0.25\, e^{-0.25x}\, dx$$

$$= -e^{-0.25x}\Big|_7^{\infty} = e^{-0.25*7} - e^{-\infty}$$

$$= e^{-0.25*7}$$

$$= 0.1738$$

b) [5 points] 10 patients with that disease got admitted into the hospital today. What is the probability that **at least 1 of them** will be **released within a week**?

P[at least 1 being released within a week]

= 1 − P[none of them are released within a week]

= 1 − P[all of them stay more than 7 days]

$= 1 - (0.1738)^{10}$

≈ 1

3. [10 points] Suppose X_1, X_2 and X_3 are three **independent** random variables each following an Exponential distribution with expectation $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$. Let,

$$Y = \frac{X_1 + X_2 + X_3}{3} - \frac{1}{\lambda}$$

Use the properties of expectation and variance to answer these following questions.

a) [3.5 points] Calculate $E(Y)$.

$$E[Y] = E\left[\frac{X_1+X_2+X_3}{3} - \frac{1}{\lambda}\right]$$

$$= \frac{1}{3}E[X_1+X_2+X_3] - E\left[\frac{1}{\lambda}\right]$$

$$= \frac{1}{3}\left\{E[X_1]+E[X_2]+E[X_3]\right\} - \frac{1}{\lambda}$$

$$= \frac{1}{3}\left\{1/\lambda + 1/\lambda + 1/\lambda\right\} - \frac{1}{\lambda} = 0$$

b) [3.5 points] Calculate $V(Y)$

$$V[Y] = V\left[\frac{X_1+X_2+X_3}{3} - \frac{1}{\lambda}\right] = \frac{1}{9}V[X_1+X_2+X_3]$$

$$= \frac{1}{9}\left\{V[X_1]+V[X_2]+V[X_3]\right\}$$

$$= \frac{1}{9}\left\{\frac{1}{\lambda^2}+\frac{1}{\lambda^2}+\frac{1}{\lambda^2}\right\}$$

$$= \frac{1}{3\lambda^2}$$

c) [3 points] Instead of three, suppose we have n number of these variables. For large n, what will be the **distribution** (along with parameters) of

$$\frac{X_1 + X_2 + ... + X_n}{n} - \frac{1}{\lambda} = \bar{X} - \frac{1}{\lambda}$$

? **Justify** your answer.

Here, $\frac{X_1+X_2+\cdots+X_n}{n} = \bar{X}$

Using CLT, $\bar{X} \longrightarrow N\left(E[X], \frac{V[X]}{n}\right)$

$$\Rightarrow \bar{X} \longrightarrow N\left(\frac{1}{\lambda}, \frac{1}{n\lambda^2}\right)$$

$$\Rightarrow \bar{X} - \frac{1}{\lambda} \longrightarrow N\left(0, \frac{1}{n\lambda^2}\right)$$

4. [10 points] For part(a), evaluate the actual expectation and do not use any memorized formula like $E[X] = V[X] = \lambda$.

a) [7 points] Suppose $X \sim Pois(\lambda)$. **Show (in details) that** $E[X(X-1)] = \lambda^2$

$X \sim Pois(\lambda) \Rightarrow p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$

$\therefore E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda}\lambda^x}{x!}$

$= 0(0-1) \cdot \frac{e^{-\lambda}\lambda^0}{0!} + 1(1-1) \cdot \frac{e^{-\lambda}\lambda^1}{1!} + \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$

$= 0 + 0 + \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!}$

$= \lambda^2 * \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{(x-2)}}{(x-2)!}$

$= \lambda^2 * \underbrace{\sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}}_{\text{pmf of } Pois(\lambda)}$ [where, $y = x-2$]

$= \lambda^2 * 1$

$= \lambda^2$

b) [3 points] If we randomly take 25 observations from a $Pois(\lambda = 5)$ distribution, what will be the **distribution**(write down the parameters as well) **of the sample mean?** (hint: for a $Pois(\lambda)$, mean=variance=λ)

$X \sim Pois(\lambda) \Rightarrow E[X] = V[X] = \lambda$

Also, $n = 25$.

Applying CLT, $\bar{X} \longrightarrow N\left(\lambda, \frac{\lambda}{25}\right)$

5. [10 points] Suppose the joint pdf of two random variables X and Y is

$$f(x,y) = \frac{1}{210}(2x+y); \quad 2 < x < 6,\ 0 < y < 5$$

a)[6 points] Find the **conditional distribution of** X **for a given** Y (Show detailed calculation).

Cond. distn of X given Y = $\dfrac{\text{joint distn of X and Y}}{\text{marginal distn of Y}}$

$$\Rightarrow f(x|y) = \frac{f(x,y)}{f(y)}$$

Here,
$$f(y) = \int_2^6 \frac{1}{210}(2x+y)\,dx$$
$$= \frac{1}{210}\left(\frac{2x^2}{2} + yx\right)\Big|_2^6$$
$$= \frac{1}{210}\left(x^2 + yx\right)\Big|_2^6$$
$$= \frac{1}{210}\left\{(36-4) + y(6-2)\right\}$$
$$= \frac{1}{210}\left\{32 + 4y\right\}, \quad 0 < y < 5$$

$$\therefore f(x|y) = \frac{\frac{1}{210}(2x+y)}{\frac{1}{210}(32+4y)} = \frac{2x+y}{32+4y}, \quad 2 < x < 6$$

b)[4 points] Find the **median** of X, given $Y = 2$

let m be the median.

at $Y=2$, conditional dist^n of X $\Rightarrow f(x|2) = \frac{2x+2}{32+4*2}$

$$= \frac{2(x+1)}{40}$$

$$= \frac{x+1}{20}$$

$$\therefore \int_2^m \frac{x+1}{20}\,dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{20}\left(\frac{x^2}{2} + x\right)\Big|_2^m = \frac{1}{2}$$

$$\Rightarrow \frac{1}{20}\left\{\frac{m^2-2^2}{2} + (m-2)\right\} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{20}\left\{\frac{m^2-4+2m-4}{2}\right\} = \frac{1}{2}$$

$$\Rightarrow m^2+2m-8 = 20$$

$$\Rightarrow m^2+2m-28 = 0$$

6. [10points] Suppose $X \sim N(\mu = 10, \sigma = 2)$, $Y \sim N(\mu = 15, \sigma = 3)$ and $Z \sim N(\mu = 90, \sigma = 4)$. Furthermore, they are independent.

a) [6 points] Calculate $P[3X + Y < \frac{Z}{2}]$

$P[3X+Y < \frac{Z}{2}] = P[3X+Y - \frac{Z}{2} < 0]$

let, $W = 3X+Y-\frac{Z}{2}$

W is a linear combination of X, Y and Z

$\therefore$ $W \sim$ Normal with $E[W] = 3*10+15-\frac{90}{2} = 0$

and $V[W] = 3^2*2^2+3^2+\frac{1}{4}*4^2 = 49$

$\therefore W \sim N(0, 49)$

$\therefore P[W<0] = \frac{1}{2}$ [We are looking at half of the density]

b) [4 points] If the standard deviations of X, Y *and* Z are not given to you, would you be able to calculate the probability in part(a)? Explain briefly.

Since, W follows Normal with mean zero.
And we are interested in calculating area to the left of the mean, we can say it will be $\frac{1}{2}$ irrespective of the standard deviations.

Formula Sheet

- Demorgans Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; $P(A^c) = 1 - P(A)$; $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- For a series of k disjoint events $A_1, A_2, ..., A_k$ and an event B
  - Law of total probability, $P(B) = P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + ... + P(B|A_k) * P(A_k)$
  - Bayes rule, $P(A_j|B) = \frac{P(B|A_j)*P(A_j)}{P(B|A_1)*P(A_1)+P(B|A_2)*P(A_2)+...+P(B|A_k)*P(A_k)}$
- A and B are independent if $P(A|B) = P(A)$ *or* $P(B|A) = P(B)$ *or* $P(A \cap B) = P(A) * P(B)$
- $Bern(p)$: $p_X(x) = p^x(1-p)^{(1-x)}$ where, x =0, 1
- $Bin(n,p)$: $p_Y(x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}$ where, $x = 0, 1, 2, ..., n$; $\begin{pmatrix} n \\ x \end{pmatrix} = \frac{n!}{x!(n-x)!}$
- $Geom(p)$: $p_X(x) = (1-p)^x p$ where $x = 0, 1, 2, ...$
- $Pois(\lambda)$: $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ where, $x = 0, 1, 2, ...$
- $NB(r,p)$: $p_X(x) = \begin{pmatrix} x+r-1 \\ r-1 \end{pmatrix} (1-p)^x p^r$ where, $x = 0, 1, 2, ...$
- $Unif(\alpha, \beta)$: $f(x) = \frac{1}{\beta - \alpha}$ *for* $\alpha \le x \le \beta$
- $Exp(\lambda)$: $f(x) = \lambda e^{-\lambda x}$ *for* $x \ge 0 and \lambda > 0$ another way to write, $Exp(\beta)$: $f(x) = \frac{1}{\beta} e^{-x/\beta}$ *for* $x \ge 0 and \beta > 0$
- $Gamma(\alpha, \lambda)$: $f(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}$ *for* $x \ge 0$, $\alpha > 0$ *and* $\lambda > 0$; if α is integer, $\Gamma(\alpha) = (\alpha - 1)!$
- $N(\mu, \sigma^2)$: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ *for* $-\infty < x < \infty$; $\sigma^2 > 0$;
- $X \sim N(\mu, \sigma^2)$; $Z = \frac{X-\mu}{\sigma}$, Then $Z \sim N(0,1)$
- $\eta(p)$ is called the $100p^{th}$ percentile of a distribution if $F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(x)dx = p$
- Expectation:
  - Discrete, $E(X) = \sum_i x_i * P(X = x_i) = \sum_i x_i * p(x_i)$; Continuous, $E(X) = \int_{-\infty}^{\infty} x f(x) dx$
  - Joint discrete, $E[h(X,Y)] = \sum_x \sum_y h(x,y) * P(X = x, Y = y)$; joint continuous $E[h(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f(x,y) dx dy$
  - $E(aX + bY + c) = aE(X) + bE(Y) + c$
- MGF:
  - $M_X(t) = E[e^{tx}]$; $E[X^r] = M^{(r)}(0) = \frac{d^r}{dt^r} M_X(t)\big|_{t=0}$
  - $Y = aX + b \implies M_Y(t) = e^{bt} M_X(at)$
  - X and Y are indep. and $Z = X + Y \implies M_Z(t) = M_X(t) * M_Y(t)$
- Variance:
  - $V(X) = E[(X - E(X))^2] = E[X^2] - (E[X])^2$
  - X and Y are indep. $\implies V[aX + bY + c] = a^2 V[X] + b^2 V[Y]$
- Covariance: $cov(X,Y) = E[XY] - E[X]E[Y]$; Correlation: $\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}}$
- Conditional distribution,
  - Discrete: $p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{\text{join pmf of x and y}}{\text{marginal pmf of x}}$
  - $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\text{join pdf of x and y}}{\text{marginal pdf of x}}$
- CLT: as $n \to \infty$; $\bar{X} \xrightarrow{D} N(\mu, \frac{\sigma^2}{n})$