STA255 Week-1

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Learning goals

- Definition of *sample space*, *outcomes*, *events*
- Intuition behind probability
- Probability function
- Probability calculation using counting method
- Conditional Probability
- Law of total probability
- Bayes rule
- Independence

Outcomes, Sample Space and Events

- An ***experiment*** , in literal sense, is a scientific test that involves testing something under controlled environment.
- In this course, we will use it in a general sense where we are doing a task and the outcomes of that task are random (what will be the outcome of the test is not known beforehand)
  - For example, tossing a coin, rolling a dice, picking a card from a deck, checking the blood pressure of a patient, recording the stock prices, recording temperatures etc.
- ***Sample space*** (denoted by S) represents the *collection of all the possible outcomes* of an experiment.
  - for tossing a coin, $S = \{H, T\}$
  - for rolling a dice, $S = \{1, 2, 3, ..., 6\}$
- ***Events*** are defined as any subset of the sample space
  - Let A represent an event where the outcome of a dice is even $\implies A = \{2, 4, 6\}$
  - Let B represent an event where the outcome of a dice is a number $\implies B = S$

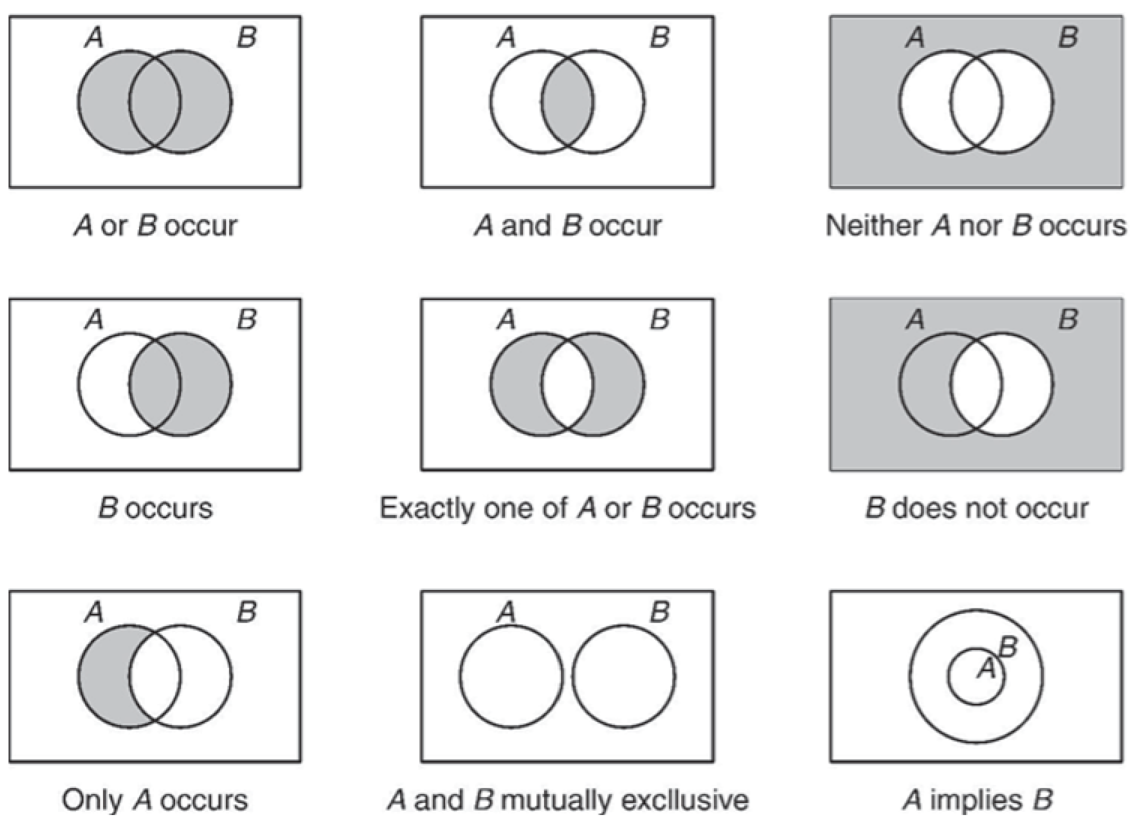
Intuition behind probability

- Formally, *probability* is a number between 0 and 1 that has certain mathematical properties (we will talk about these in the next section)

- What do we mean when we say that *the probability that event A will occur is x*? For example,
  - while tossing a coin, the probability of a $\{H\}$ is 0.5
  - There is a 30% chance that it will rain tomorrow
- Our *intuition* is that keeping everything similar if we observe today's conditions over and over again, 30% of the 'tomorrows' will result in a rain.
- Intuitively we think of *relative frequency* (how many times something happens out of the total) as the probability of some event.
- This is not based on rigorous mathematical theory.

- We will learn the mathematical definition of probability but before that we will learn/re-visit some set theory.

Venn diagrams and some useful formulas



- Complement of event A: $A' = S \setminus A$
- Demorgans Laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
- A and B are disjoint or mutually exclusive $\implies A \cap B = \phi$

Probability

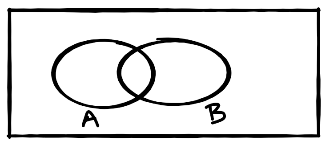
A *probability function* P defined on a finite sample space S assigns to each event A in S a number P(A) such that

- $P(A) \geq 0$
- $P(S) = 1$
- $A_1, A_2, ...$ is an infinite collection of disjoint events, then

  $P(A_1 \cup A_2 \cup ...) = \sum_{i=1}^{\infty} P(A_i)$ when A and B are disjoint.

Some useful formulas

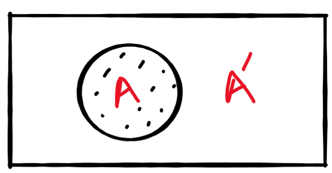
- The probability of a union: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A \cup B = A \cup B \setminus (A \cap B)$

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

- The probability of a complement: $P(A') = 1 - P(A)$



$A \cup A' = S$

A & A' are disjoint

$\Rightarrow P[A \cup A'] = P[S]$

$\Rightarrow P[A] + P[A'] = 1$

$\Rightarrow P[A'] = 1 - P[A]$

Example: Let $P(A) = 1/3, P(B) = 1/2, P(A \cup B) = 3/4$. Find

- $P(A \cap B) =?$
- $P(A' \cup B') =?$

ANS:

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1/3 + 1/2 - 3/4 = 1/12$

$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B) = 1 - 1/12 = 11/12$

Calculating probability by counting outcomes from the sample space (S)

- We can use this method only if
  - all the outcomes of the sample space are equally likely (every outcome has the same chance of happening) and
  - S is finite.
- Let A be our event defined on the sample space S
- Then probability of A can be calculated as

$$P(A) = \frac{\text{number of outcomes satisfying event A}}{\text{Total number of outcomes in } S}$$

Example-1

What is the probability of getting a head if a fair coin is tossed?

- The sample space, $S = \{H, T\}$
- Event: we are looking for a head.
- Out of the two outcomes, one satisfy our condition
- Hence, $P(H) = 1/2$

Example-2

We want to know what is the probability of getting an even number if a fair dice is rolled?

- The sample space, $S = \{1, 2, 3, 4, 5, 6\}$
- We are looking for 2,4 or 6
- Out of the six outcomes, three satisfy our condition
- Hence, theoretically $P(\text{the number is even}) = 3/6 = 1/2$

Question:

- If the coin is not fair or the dice is not fair, can we use this method?

Example-3

We are rolling a fair dice twice. What is the probability that the sum of the two numbers are 11?

Using the first method (counting outcomes)

- The sample space is:

		second dice					
		1	2	3	4	5	6
first dice	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

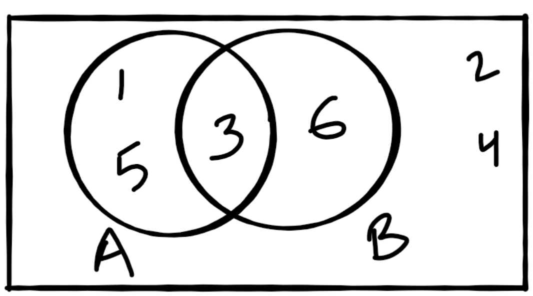
- Two of the 36 outcomes are (5,6) and (6,5) which will give us a sum of 11.

$$P(\text{sum of the two numbers is 11}) = 2/36 = 1/18$$

Conditional Probability

Let take a look at an example first

- We are rolling a fair dice.
- We have two events
  - Event A: it's an odd number
  - Event B: the number is divisible by 3
- we have this following Venn diagram



- By using the counting method
  - $P(A) = 3/6$
  - $P(B) = 2/6$
  - $P(A \cap B) = 1/6$

Idea of conditional probability

- Question: what is the probability that it's an odd number?
  - It's asking for $P(A)$

- Imagine that **"somehow" we know it's a number divisible by 3**. So we know event B has happened.

- Under this condition if some one asks "what is the probability that it's an odd number?"
  - It's still asking for $P(A)$ but there is a extra bit of information.
  - We rephrase this as "What is the probability that it's an odd number *given* it's a number divisible by 3"?
  - This is also phrased as "what is the probability of event A *conditioning* on event B?"
  - In notation this is written as $P(A|B)$
  - The condition (extra info) goes after the "|" sign.

Formula of calculating conditional probability

The *conditional probability* of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

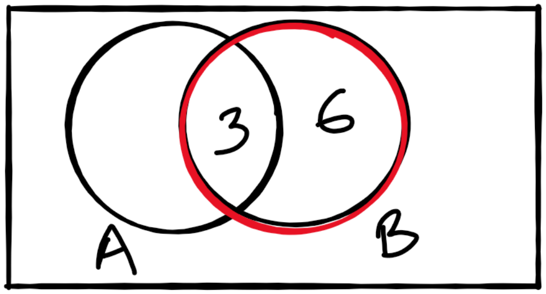
where $P(B) > 0$

For our example (using numbers from the previous page),

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{2/6} = 1/2$$

A different way to look at conditional probability

- If we know event B has occurred, our Venn Diagram will be this



- Since we know that event B has occurred our sample space (S) is now *reduced*.
- We are considering only the outcomes that satisfy our condition B.
- We are left with two outcomes in the sample space and only one of them satisfy event A.
- Therefore, $P(A|B) = 1/2$

The multiplication rule

Just by re-arranging (multiplying both sides by P(B)) the formula of the conditional probability we can write

$$P(A \cap B) = P(A|B) * P(B)$$

The law of total probability

- Suppose we have a series of k disjoint events $A_1, A_2, A_3, ..., A_k$
- They are exhaustive events $\implies A_1 \cup A_2 \cup A_3 \cup ... \cup A_k = S$
- We are interested in calculating the probability of event B.
- Using the multiplication rule We can write,

$$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2) + ... + P(B \cap A_k) \\ &= P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + ... + P(B|A_k) * P(A_k) \end{aligned}$$

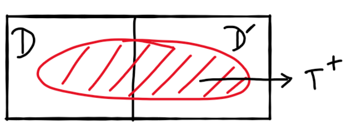
Bayes Rule (Inverting a conditional probability)

- Suppose we have a series of k disjoint events $A_1, A_2, A_3, ..., A_k$
- Suppose $P(A_1), P(A_2)...P(A_k)$ are known and $P(B|A_1), P(B|A_2), ..., P(B|A_k)$ are also known.
- The conditional probability of A_j where j=1,2,. . . ,k given an event B,

$$\begin{aligned} P(A_j|B) &= \frac{P(A_j \cap B)}{P(B)} \\ &= \frac{P(B|A_j) * P(A_j)}{P(B)} \\ &= \frac{P(B|A_j) * P(A_j)}{P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + ... + P(B|A_k) * P(A_k)} \end{aligned}$$

Comment: Used "multiplication rule" in the numerator and "law of total probability" in the denominator.

Example 2.30 (page 80)



- $P[D] = \frac{1}{1000} = 0.001$

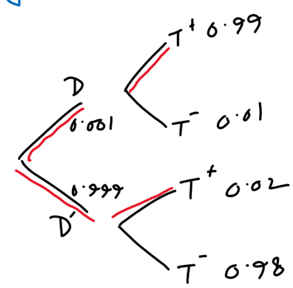
  $\Rightarrow P[D'] = 0.999$

- $P[T^+|D] = 0.99$
- $P[T^+|D'] = 0.02$

$$\begin{aligned} P[T^+] &= P[T^+ \cap D] + P[T^+ \cap D'] \\ &= P[T^+|D] \cdot P[D] + P[T^+|D'] \cdot P[D'] \\ &= 0.99 \cdot 0.001 + 0.02 \cdot 0.999 \end{aligned}$$

$$\begin{aligned} P[D|T^+] &= \frac{P[D \cap T^+]}{P[T^+]} \\ &= \frac{P[T^+|D] \cdot P[D]}{P[T^+]} \\ &= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.02 \cdot 0.999} \end{aligned}$$

Using tree diag.



$$P[D|T^+] = \frac{0.001 \cdot 0.99}{0.001 \cdot 0.99 + 0.999 \cdot 0.02}$$

Independence

An event A is called *independent* of another event B if

$$P(A|B) = P(A)$$

The intuition is:

- On the right hand side, we are calculating $P(A)$ over the entire sample space (S)
- On the left hand side we are calculating $P(A|B)$
  - here, we have the extra information B
- After calculating these two probabilities if we see the numbers are the same, this means
  - the extra information didn't add any value.
  - whether A will happen or not doesn't depend on B
  - we say event **A is independent of B**.
- To show independence we can prove any of the followings

$$P(A|B) = P(A) \;\; or \;\; P(B|A) = P(B) \;\; or \;\; P(A \cap B) = P(A) * P(B)$$

Homework

From the exercise

1, 3, 6, 10, 11, 13, 15, 22, 23, 26, 27,

35, 37, 38, 39,

45, 47, 50, 54, 58, 59, 61, 63,

66, 67, 75, 83, 87, 91, 105, 106, 109