LECTURE NOTES

EMT 2411: Spatial Mechanisms II

Lecture notes compiled by: ***T. Mulembo***

Students are reminded not to treat these lecture notes as a comprehensive and solely sufficient material for their studies since the purpose of the notes is not meant to be a substitute for regularly attending classes, reading relevant textbooks and recommended books. The notes are aimed at providing a quick reference and a brief guidance for the students.

Course Outline

Kinematic analysis of spatial mechanisms with multi-degree of freedom; the inverse kinematic problem. Spatial slidercrank mechanism as a case study. Position analysis, velocity analysis and acceleration analysis, position, velocity and acceleration of a point on a spatial mechanism. Modeling of spatial mechanisms. Dynamics of spatial mechanisms. Design case study. Computer simulation.

Reference Textbooks

1. George N Sandor, Arthur G. Erdman, (1984), *Advanced mechanisms design: Analysis and synthesis*, Printice-Hall, inc

2. Edward J. Haug, (1989), *Computer Aided Kinematics and dynamics of mechanical systems*, ALLYN AND BACON

1 Introduction

1.1 Representing orientations: Rotation Matrices

Presentation of orientation is very important and especially in robotics. In order for the robot jaws to grip an object properly, the known position alone is not sufficient. The coordinate frame of the gripper must be aligned to coordinate frame of the object in some way. The concept of moving and frames becomes very important.

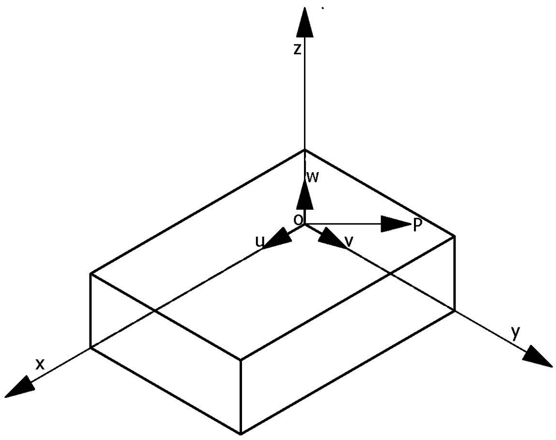


Figure 1:

- $\{x\,y\,z\}$: fixed coordinate frame. [F]
- $\{u\,v\,w\}$: moving coordinate frame. [M]

The orientation of the object can be described by the orientation of the object frame with respective to the reference frame. The orientation of frame $\{B\}$ with respective to the reference frame $\{A\}$ is given by:

$$ {}^B_AR = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} {}^BX_A & {}^BY_A & {}^BZ_A \end{bmatrix} = \begin{bmatrix} {}^AX_B^T \\ {}^AY_B^T \\ {}^AZ_B^T \end{bmatrix} $$

Rotational matrix R is orthonormal i.e.,

$$ {}^B_AR = {}^A_BR^{-1} = {}^A_BR^T $$

The rotation matrix has the following properties:

1. R is normalized: The squares of the elements in any row or column sum to 1.
2. R is orthogonal: The dot product of any pair of row or any pair of columns is 0
3. The rows of R represent the coordinates in the original space of unit vectors along the coordinate axes of the rotated space.

4. The columns of R represent the coordinates in the rotated space of unit vectors along the axes of the original space.

Suppose we know P in some frame $\{B\}$ and we want to describe it in $\{A\}$ where $\{A\}$ and $\{B\}$ have same origin. We first project components onto unit directions and use.

$$^{A}P = {}^{B}_{A}R\,{}^{B}P$$

The rotation mapping changes the description of a point from one coordinate system to another. The point does not change! only its description. The rotation matrix R is a transformation matrix that operate on a position vector and maps coordinates into a rotated coordinate system. It can be shown that det $R = \pm 1$. Since $RR^T = 1$. Therefore (det $R)^2$=det 1=1. Implying det $R = \pm 1$

1.2 Important fundamental rotational matrices

The basic rotation matrices representing rotation about a_x, a_y and a_z axes are given as:

$$R(a_x,\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \quad R(a_y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \quad R(a_z,\psi) = \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.3 Coordinate transformation

If $P_{xyz} = RP_{uvw}$ and $P_{uvw} = QP_{xyz}$, then $Q = R^{-1} = R^T$
Therefore: $QR = R^TR = R^{-1}.R = 1$, but $R_AR_B \neq R_BR_A$

1.4 Composite rotation algorithm

Using composite rotations (multiple rotations), we can establish an arbitrary single orientation.

1. Initialize the rotation matrix to $R = 1$, which corresponds to the orthonormal coordinate frame F(fixed coordinate frame) and M(mobile coordinate frame) being coincident.

2. If the mobile coordinate frame M is to be rotated by an amount φ about the k-th unit vector of the fixed coordinate frame F, then **premultiply** R by $R_k(\varphi)$

3. If the mobile coordinate frame M is to be rotated by an amount φ about its own k-th unit vector, then **postmultiply** R by $R_k(\varphi)$

4. If there are more fundamental rotations to be performed go to step 2, else stop. The resulting composite rotation matrix R maps mobile M coordinates into fixed F coordinates

1.5 Homogeneous coordinates

During coordinate transformations, instead of using both rotation matrix (3X3) and a translation vector (3X1), the homogeneous coordinates system seeks to use a single 4X4 matrix. This way the computation is simplified. For a variety of reasons, it is desirable to keep transformation matrices in square form, either 3X3 or 4X4. For instance, it is much easier to calculate the inverse of square matrices than rectangular matrices and in order to multiply two matrices, their dimensions must match, such that the number of columns of the first matrix must be the same as the number of rows of the second matrix. In four dimensional space of homogeneous coordinates, representing both position and rotation of the frame, the transformation is expressed as:

$$^{A}P = {}^{B}_{A}H^{B}P$$

Matrix H is called homogeneous transformation operator (matrix). A homogeneous transformation matrix is therefore a 4X4 matrix that maps a position vector in homogeneous coordinates from one coordinate system to another. The physical meaning of homogeneous transformations is depicted below.

Example 1.1 *For the following Assembly workspace shown in figure 2, describe point P with respect to the base frame. For given* (x_{4p}, y_{4p}, z_{4p}) *with respect to frame* $\{4\}$*.* Note drawn to scale

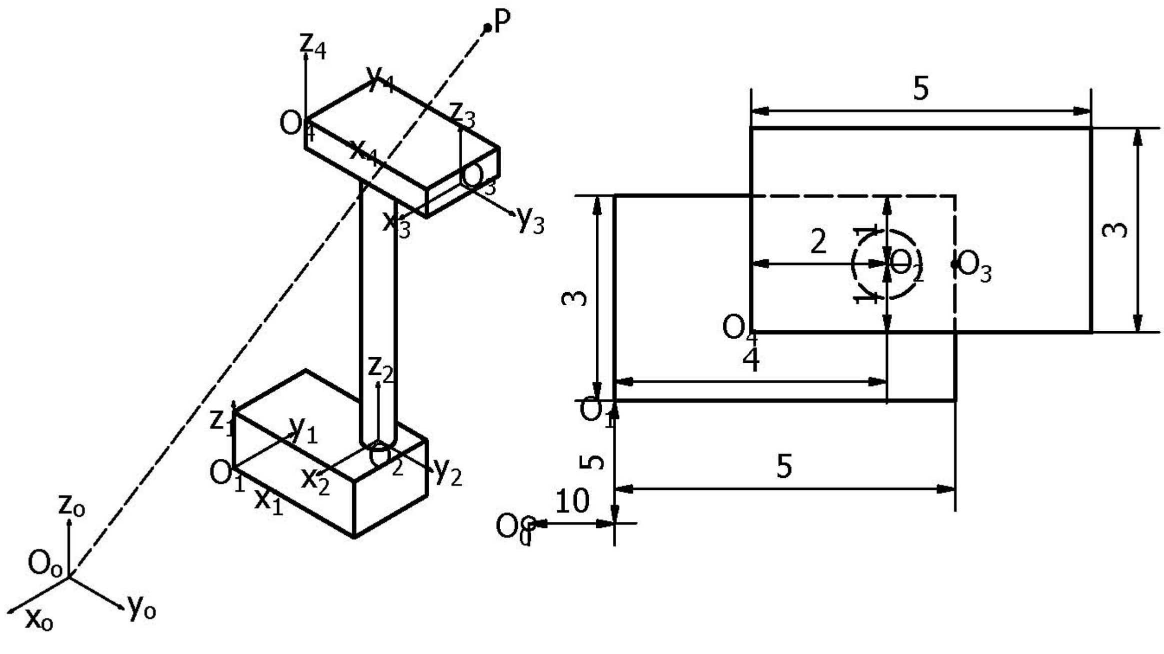


Figure 2:

Solution

To be solved in class.

2 Spatial kinematic analysis

2.1 Forward and inverse kinematic problem

2.1.1 Forward kinematic problem

To understand forward and inverse kinematic problem, will consider a industrial robot with 6 degrees of freedom plus a gripper as shown in figure 3. The first three degrees of freedom are translations followed by three rotations. We wish to determine the six independent joint movements (a, c, b, θ_1, θ_2, θ_3) given the final position of the hand with respect to a "home position" (these variables are zero in the home position). Before modeling this robot using elementary matrices, local coordinate systems on each link are established. Notice that this spatial mechanism is *open loop* mechanism. The last, 7th coordinate system, that of the hand opening, need not be coincident with the first. Each coordinate system x_i, y_i, z_i is defined relative to the x_{i-}, y_{i-1}, z_{i-1} coordinate system by

$$\begin{aligned} S_{01} =& T_{01}(a,\ 0,\ 0) \\ S_{12} =& T_{12}(0,\ 0,\ c) \\ S_{23} =& T_{23}(0,\ L_1 + b,\ 0) \\ S_{34}(\theta_1) =& T_{34}(0,\ 0,\ 0,\ \theta_1) \end{aligned} \tag{1}$$

where θ_1 is a rotation about the y_3 axis,

$$S_{45}(\theta_1) = T_{45}(0,\ 0,\ 0,\ \theta_2)$$

where θ_2 is a rotation about the z_4 axis,

$$S_{56}(\theta_3) = T_{56}(0,\ L_2,\ 0,\ \theta_3)$$

where θ_3 is the hand rotation about the y_5 axis,

$$S_{67} = T_{67}(0,\ 0,\ 0,\ d)$$

where L_l and L_2 are constants, and a, c, b, d, θ_1. θ_2, and θ_3 are variables (d is the gripper opening).

To model this robot, the position vector $\mathbf{p}^i$ of a point P expressed in the ith coordinate system needs to be expressed in terms of coordinate system $x_o y_o z_o$ by

$$\mathbf{P}^o = S_{oi}\mathbf{P}^i \tag{2}$$

The concatenated matrix, S_{oi}, for each link i is

$$S_{01} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

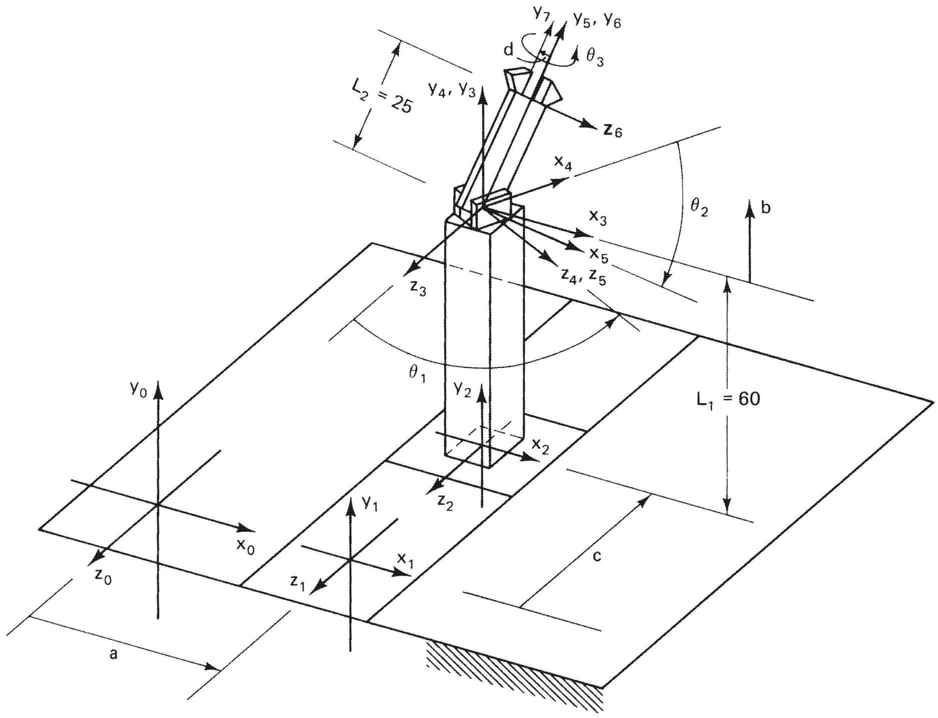


Figure 3: Definition of coordinate systems for the robot: Note that translation c defined with respect to the z_1 axis, is a negative quantity. Similarly, θ_2 , a rotation about the z_4 axis, is a negative angle.

$$S_{02} = S_{01}S_{12} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\begin{aligned} S_{03} &= \{S_{01}S_{12}\}\, S_{23} \\ &= S_{02}S_{23} \\ &= \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & L_1 + b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{5}$$

$$\begin{aligned} S_{04} &= S_{03}S_{34} \\ &= S_{03} \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a \\ 0 & 1 & 0 & b + L_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6}$$

$$
\begin{aligned}
S_{05} &= S_{04}S_{45} \\
&= S_{04}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 & \sin\theta_1 & a \\ \sin\theta_2 & \cos\theta_2 & 0 & b+L_1 \\ -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
S_{06} &= S_{05}S_{56} \\
&= S_{05}\begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ 0 & 1 & 0 & L_2 \\ -\sin\theta_3 & 0 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C\theta_1C\theta_2C\theta_3 - S\theta_1S\theta_3 & -C\theta_1S\theta_2 & C\theta_1C\theta_2S\theta_3 + S\theta_1C\theta_3 & -L_2C\theta_1S\theta_2 + a \\ S\theta_2C\theta_3 & C\theta_2 & S\theta_2S\theta_3 & L_2C\theta_2 + b + L_1 \\ -S\theta_1C\theta_2C\theta_3 - C\theta_1S\theta_3 & S\theta_1S\theta_2 & -S\theta_1C\theta_2S\theta_3 + C\theta_1C\theta_3 & L_2S\theta_1S\theta_2 + c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
S_{07} &= S_{06}S_{67} \\
&= S_{06}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C\theta_1C\theta_2C\theta_3 - S\theta_1S\theta_3 & -C\theta_1S\theta_2 & C\theta_1C\theta_2S\theta_3 + S\theta_1C\theta_3 & \begin{matrix}\{-L_2C\theta_1S\theta_2 + a \\ +d(C\theta_1C\theta_2S\theta_3 + S\theta_1C\theta_3)\}\end{matrix} \\ S\theta_2C\theta_3 & C\theta_2 & S\theta_2S\theta_3 & \begin{matrix}\{L_2C\theta_2 + b + L_1 \\ +d(S\theta_2S\theta_3)\}\end{matrix} \\ -S\theta_1C\theta_2C\theta_3 - C\theta_1S\theta_3 & S\theta_1S\theta_2 & -S\theta_1C\theta_2S\theta_3 + C\theta_1C\theta_3 & \begin{matrix}\{L_2S\theta_1S\theta_2 + c \\ d(S\theta_1C\theta_2S\theta_3 + C\theta_1C\theta_3)\}\end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

When specific values are for the seven degrees freedom: a, b, c, d θ_1, θ_2, and θ_3, they can be substituted into T_{01}, T_{12}, T_{23}, T_{34}, ... [Eq. (1)]. Then the SOi'S can be computed in a sequence. For instance,

$$
\begin{aligned}
S_{05} &= S_{01}S_{12}S_{23}S_{34}S_{45} \\
&= T_{01}T_{12}T_{23}T_{34}T_{45} \qquad (9)
\end{aligned}
$$

We obtain a position vector $\mathbf{P}^5$ of a point expressed in system 5 in terms the fixed-coordinate system $x_oy_oz_0$ as follows:

$$
\begin{bmatrix} x_o \\ x_o \\ z_o \\ 1 \end{bmatrix} = S_{05}\begin{bmatrix} x_5 \\ y_5 \\ z_5 \\ 1 \end{bmatrix} \qquad (10)
$$

2.1.2 Inverse Kinematic problem

This is more complex than the forward kinematics problem because a systematic closed form solution applicable to robots in general is not available. The relative translations and rotations of each degree of freedom (a, c, b, d, θ_1, θ_2 and θ_3) are given with respect to successively translated and rotated coordinate systems, and by concatenation, with respect to a home position. In many cases, the reverse is required.
When programming a robot, the displacements at the individual degrees of freedom are to be determined, given a required robot gripper (*end effector*) position translated and rotated away from the home position. This is sometimes referred to as the "inverse kinematics problem". In many cases, this task results in cumbersome equations with multiple solutions. For this example, however, since there are three translations followed by three rotations, the inverse kinematics problem is more easily solved.
The prescribed position of the gripper in the fixed coordinate system, $x_o y_o z_o$ of Fig. 3, is given by x_g , y_g, z_g, θ_{1g} , θ_{2g} , θ_{3g}. Since the three rotational degrees of freedom are the same as the rotations of the gripper, we only solve for the three translations: a, b, and c.
In order to allow the gripper to pick up the object, the origin of link 6 needs to have the same coordinate (in the fixed coordinate system) and orientation as the object does. The coordinates of the origin of system 6 need to be expressed in the fixed coordinate system by

$$\begin{bmatrix} x_{06} \\ y_{06} \\ z_{06} \\ 1 \end{bmatrix} = S_{06} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{11}$$

where (S_{06}) was defined by Eq. (8). Thus

$$\begin{bmatrix} x_{06} \\ y_{06} \\ z_{06} \\ 1 \end{bmatrix} = \begin{bmatrix} -L_2 \cos\theta_1 \sin\theta_2 + a \\ L_2 \cos\theta_2 + b + L_1 \\ L_2 \sin\theta_1 \sin\theta_2 + c \\ 1 \end{bmatrix} \tag{12}$$

So we can obtain a, b, and c simply by

$$\begin{aligned} a =& x_{06} + (L_2 \cos\theta_1 \sin\theta_2) \\ b =& y_{06} - (L_2 \cos\theta_2 + L_1) \\ c =& z_{06} - (L_2 \sin\theta_1 \sin\theta_2) \end{aligned} \tag{13}$$

Here x_{06}, y_{06}, z_{06}, θ_1 and θ_2 are known, and

$$\begin{aligned} x_{06} &= x_g \\ y_{06} &= y_g \\ z_{06} &= z_g \\ \theta_1 &= \theta_{1g} \\ \theta_2 &= \theta_{1g} \end{aligned}$$

It follows that if $x_g = 20$, $y_g = 80$, $z_g = -80$, $\theta_{1g} = 30^o$, $\theta_{2g} = -30^o$, $\theta_{3g} = 40^o$, then a, b and c can be obtained by

$$\begin{aligned} a &= 20 + 25\cos 30^0 \sin(-30^o) = 9.17 \\ b &= 80 - 25\cos(-30^o) - 60 = -1.65 \\ c &= -80 - 25\sin 30^o \sin(-30^o) = -73.75 \end{aligned}$$

Example 2.1 *Figure 4 shows a 3-DOF articulated robot arm with three revolute joints. The axes of joint 2 and joint 3 are parallel and axis of joint 1 is perpendicular to these two. At the end of the arm, a faceplate is provided to attach the wrist.*

(a) Determine the following transformation matrices T_{01}, T_{12}, T_{23} and T_{34}.

(b) Given that $\theta_1 = 90^o$, $\theta_2 = -30^o$, $\theta_3 = -10^o$, $L_1 = 600$ mm, $L_2 = 400$ mm and $L_3 = 300$ mm. Find the overall transformation matrix T_{04}

(c) Point P^4 position vector with respect to frame 4 axis is (2, 3, 5). Determine P^3, P^2, P^1 and P^o.

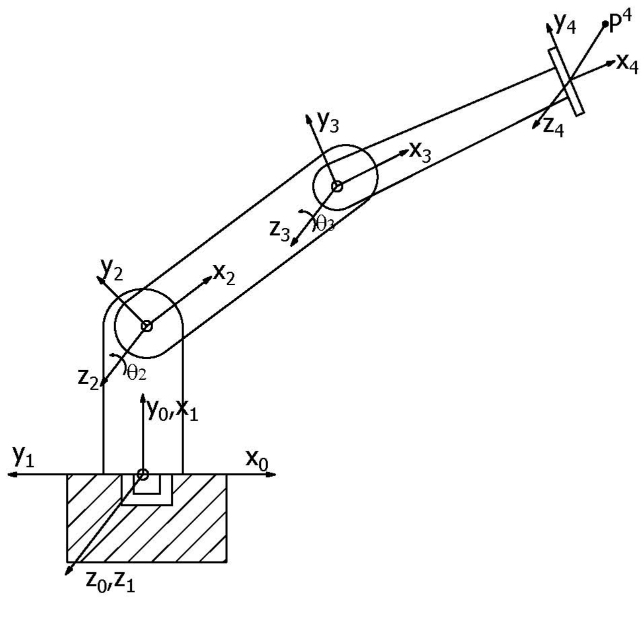


Figure 4:

Solution

Try to solve this on your own: This is application of what you learnt in EMT 2405: Spatial Mechanisms I

2.2 Vector Algebra

Vector: Vector quantity has both magnitude and direction eg. Velocity, acceleration, force
Scalar: Scalar quantity has magnitude only, eg Mass, volume, work.

2.2.1 Vector sum

$$A + B = B + A$$
$$A + (B + C) = (A + B) + C$$
$$A - B = A + (-B)$$

2.2.2 Dot product (Scalar)

$A.B + |A||B| \cos\theta$ where $|A|$ represents the magnitude of a vector A

$$A.(B + C) = A.B + A.C$$

$$A.B = B.A$$

If $A.B = 0$ either one of the vectors A, B is zero or A and B are perpendicular to each other.

2.2.3 Cross product (vector)

$A \times B$, where the magnitude $=|A||B| \sin\theta$ and direction given by the right hand rule.

$$A \times (B + C) = A \times B + A \times C \tag{14}$$
$$A \times B = -B \times A \tag{15}$$

If $A \times B = 0$, either one of the vectors A and B is zero or A and B are parallel.

2.2.4 Unit vectors: i, j, k

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

If $A = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $B = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, then

- $A + B = (a_x + b_x)\mathbf{i} + (a_x + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$
- $A \times B = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- $A.B = a_x b_x + a_y b_y + a_z b_z$

Note

- $A.(A \times B) = B.(A \times B) = 0$
- $A.(B \times C) = B.(C \times A) = C.(A \times B) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$
- $A \times (B \times C) = (A.C)B - (A.B)C$
- $(A \times B) \times C = (A.C)B - (B.C)A$

2.2.5 Vector differentiation

For vector function of time $A(t)$, $B(t)$, $C(t)$, and scalar function of time $m(t)$

- $\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t}$
- $\frac{d}{dt}(A + B) = \frac{dA}{dt} + \frac{dB}{dt}$
- $\frac{d}{dt}(A \times B) = \frac{dA}{dt} \times B + A \times \frac{dB}{dt}$
- $\frac{d}{dt}(A.B) = \frac{dA}{dt}.B + A.\frac{dB}{dt}$
- $\frac{d}{dt}(mA) = \frac{dm}{dt}A + m\frac{dA}{dt}$
- $\frac{d}{dt}[A \times (B \times C)] = \frac{dA}{dt} \times (B \times C) + A \times \left(\frac{dB}{dt} \times C\right) + A \times \left(B \times \frac{dC}{dt}\right)$

2.3 Vector analysis

2.3.1 Motion in the stationary coordinate system i.e $\frac{d}{dt}(\mathbf{i}, \mathbf{j}, \mathbf{k}) = 0$

Consider any point P in space relative to a fixed point O as shown in figure 5. To reach P we must use a straight line $\mathbf{r}$ directed from O to P. This is the position vector. A vector may be expressed conveniently in terms of its components and the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, whose directions are those of the axes of the cartesian frame of reference Ox, Oy, and Oz.

If x, y and z are the directed lengths of the projections of the vector $\mathbf{r}$ on these axes, we then have the following vector equation:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{16}$$

Furthermore, if the particle is moving along the space curve AB then x, y, and z must be functions of time, i.e

$$x = x(t) \quad y = y(t) \quad z = z(t)$$

Hence, the vector equation for the position of the particle at any instant may be written as:

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{17}$$

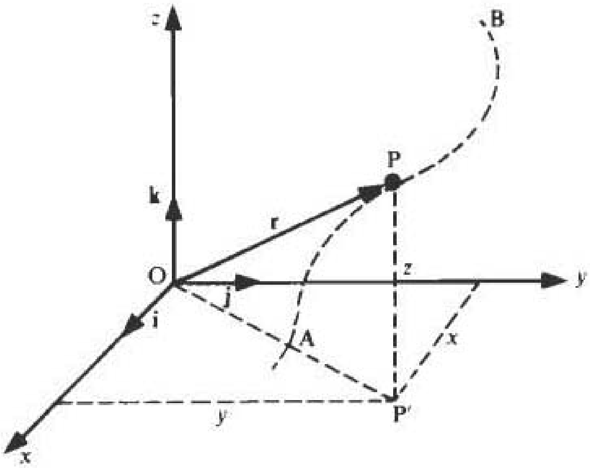


Figure 5: Position vector of a point in Cartesian coordinate

The magnitude of the position vector $\mathbf{r}$, denoted by $|\mathbf{r}|$ or simply r, is

$$r = \sqrt{x^2 + y^2 + z^2}$$

The velocity of the particle is obtained by differentiating Eq. 17 with respect to time:

$$\dot{\mathbf{r}} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$$

$\dot{\mathbf{i}} = \dot{\mathbf{j}} = \dot{\mathbf{k}} = 0$, since the axes are Cartesian axes, fixed in space and of constant magnitude. Similarly, the acceleration of the particle is

$$\ddot{\mathbf{r}} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}$$

2.3.2 Motion of a rigid body about a fixed axis (without translation)

A point, which is fixed in the body rotating with ω about an axis that is fixed in a stationary coordinate system is described by a vector R.
The velocity of P is

$$\begin{aligned}
\dot{R} = v_P &= \omega \times R = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}) \\
&= (\omega_y R_z - \omega_z R_y)\mathbf{i} + (\omega_z R_x - \omega_x R_z)\mathbf{j} + (\omega_x R_y - \omega_y R_x)\mathbf{k}
\end{aligned}$$

Direction: $(thumb \times Index) = third \quad finger$
The acceleration of P is,

$$a = v_P = \dot{\omega} \times R + \omega \times \dot{R} = \alpha \times R + \omega \times (\omega \times R) = a^t + a^n$$

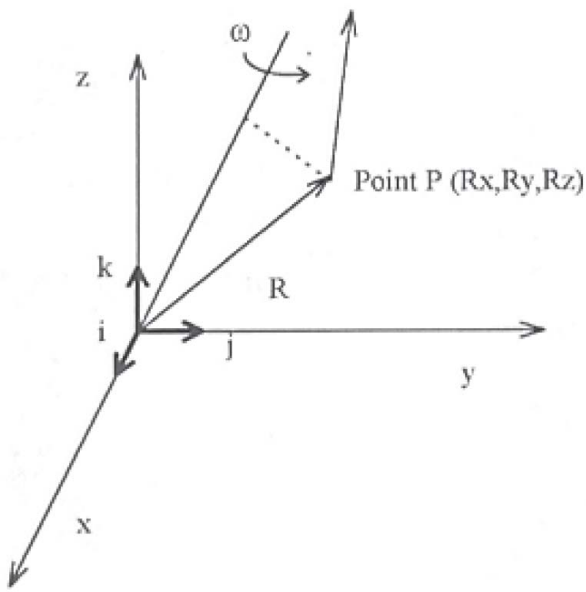


Figure 6:

2.3.3 Moving coordinate systems

The velocity of point P

$$v_P = \dot{R} = \dot{R}_o + \dot{r} = \dot{R}_{ox}\mathbf{I} + \dot{R}_{oy}\mathbf{J} + \dot{R}_{oz}\mathbf{K} + \dot{r}_x\mathbf{i} + \dot{r}_y\mathbf{j} + \dot{r}_z\mathbf{k} + \left(\frac{di}{dt}\right) r_x + \left(\frac{dj}{dt}\right) r_y + \left(\frac{dk}{dt}\right) r_z$$

By using $R = \omega \times R \Rightarrow (di/dt) = \omega \times i$

$$v_P = \dot{R} = \dot{R}_o + \dot{r}_r + \omega \times r$$

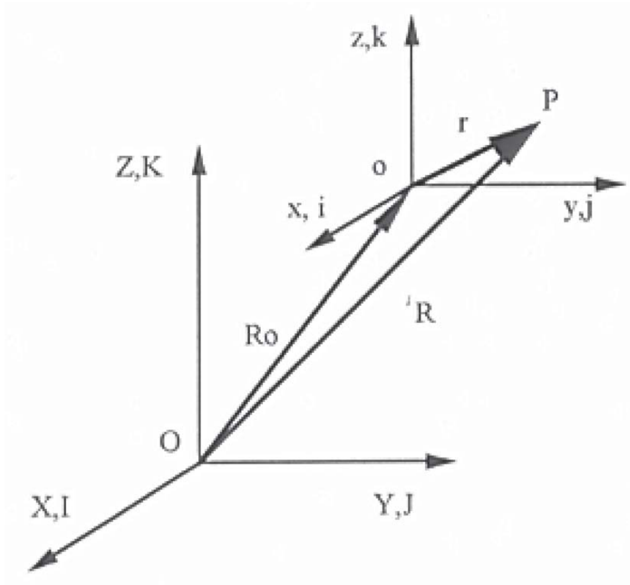
Where:

- ω is the angular velocity of the xyz coordinate system,
- r is the position vector of P in xyz'
- $\dot{R}$ is the absolute velocity of P in XYZ,
- $\dot{R}_o$ is the velocity of the origin o of xyz, and
- $\dot{r}_r$ is the velocity of point P relative to xyz.

The acceleration of point P

$$\begin{aligned} a_P = \ddot{R} &= \ddot{R}_o + \alpha \times r + \omega \times (\omega \times r) + \ddot{r}_r + 2\omega \times \dot{r}_r \\ &= \ddot{R}_o + \ddot{r}_r + a^t + a^n + a^c \end{aligned}$$

Where:

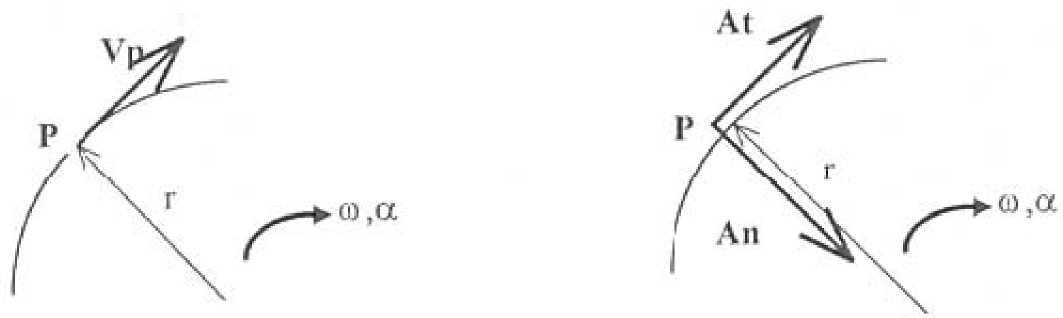


- ω and α are the angular velocity and angular acceleration of the moving system ($oxyz$) in the $OXYZ$ system
- r is the position vector of P in the $oxyz$ system,
- $\dot{r}_r$ is the velocity of point P relative to the $oxyz$ system

Note
If the $oxyz$ system has no translation relative to the $OXYZ$ system ($\dot{R}_o = 0$), the point P is fixed in the body whose coordinate system is $oxyz$ ($\dot{r}_r = 0$), and the body rotates with ω and α in the $OXYZ$ system, the velocity and acceleration of point P become,

$$v_P = \dot{R} = \omega \times r \quad \text{and} \quad a_P = \ddot{R} = \alpha \times r + \omega \times (\omega \times r)$$

For planar motion $\omega = \omega\mathbf{k}$, $\alpha = \alpha\mathbf{k}$, $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j}$ Normal acceleration: $a^n = v^2/R =$



$R\omega^2 = v\omega$ (Direction- Toward the center of curvature)
Tangential acceleration: $a^t = R\alpha$ (Direction- Tangential to the path)
Coriolis acceleration: $a^c = 2v_r\omega$

Example 2.2 *Two sliders A and B are constrained to move in slots at right angles to each other, as shown in Fig. 7, and are connected by the rigid link AB of length $450mm$. At the instant when $\theta = 30^o$ slider A is moving with a velocity of $0.6m/s$ and acceleration of $1.2m/s^2$ in the direction shown. Calculate the velocity and acceleration of the slider B at that instant and the angular velocity and acceleration of the link AB*

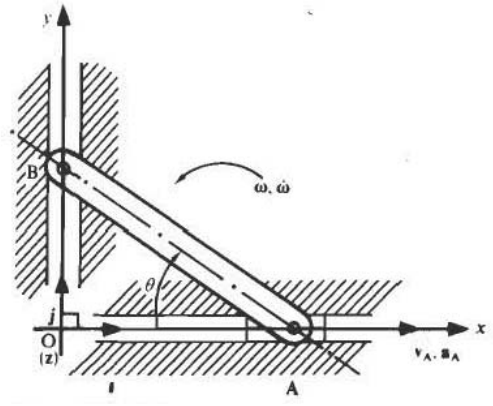


Figure 7:

Solution

$$r_{B/A} = -0.45\cos 30i + 0.45\sin 30j = -0.39i + 0.225j$$
$$v_B = -v_B j \quad v_A = 0.6i \quad \omega = \omega k$$

$$\begin{aligned} v_B &= v_A + \omega \times r_{B/A} \\ -v_B j &= 0.6i + \omega k \times (-0.39i + 0.225j) \\ &= 0.6i - 0.39\omega j - 0.225\omega i \end{aligned}$$

Comparing like terms

$$0.6 - 0.225\omega = 0 \tag{18}$$
$$0.39\omega - v_B = 0 \tag{19}$$

From Eqn. 18

$$\omega = \frac{0.6}{0.225} = 2.67$$
$$\therefore \omega = (2.67k)rad/s$$

And from Eqn. 19

$$v_B = 0.39 \times 2.67 = 1.04$$
$$v_B = (-1.04j)m/s$$

To obtain acceleration

$$a_B = a_A + \alpha \times r_{B/A} + \Omega \times (\Omega \times r_{B/A})$$
$$a_B = -a_B j \quad a_A = 1.2i \quad \alpha = \alpha k$$

$$\begin{aligned}
-a_B j &= 1.2i + \alpha k \times (-0.39i + 0.225j) + 2.67k \times [2.67k \times (-0.39i + 0.225j)] \\
&= 1.2i + -0.39\alpha j - 0.225\alpha i + 2.67k \times [-1.04j - 0.6i] \\
&= 1.2i + -0.39\alpha j - 0.225\alpha i + 2.78i - 1.6j \\
&= (3.98 - 0.225\alpha)i - (1.6 + 0.39\alpha)j + 2.67k
\end{aligned}$$

Comparing the like terms, i.e. i, j and k

$$3.98 - 0.225\alpha = 0 \tag{20}$$

$$1.6 - 0.39\alpha = -a_B \tag{21}$$

From Eqn. 20

$$\alpha = 17.69rad/s^2 \quad (\alpha = 17.69k)rad/s^2$$

And Eqn. 21

$$-a_B = 1.6 + 0.39(17.69) = 8.5 \quad a_B = (-8.5j)m/s^2$$

2.4 Kinematic of a typical four-bar spatial linkage

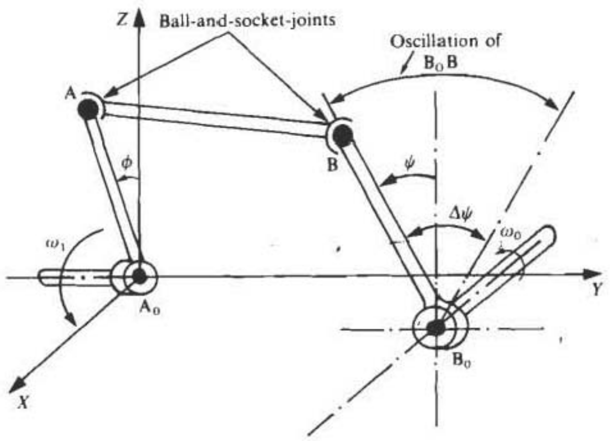


Figure 8: Typical four-bar spatial linkage

Figure 8 shows a four-bar $RSSR$ spatial linkage in which the input crank A_oA rotates about the y-axis so that A moves in a circular path in the $x-z$ plane. The output link B_oB rotates about an axis parallel to the x-axis in the $x-y$ plane and oscillates through an angle $\Delta\psi$ in a plane parallel to the $y-z$ plane giving the coupler AB motion in three dimensions.

To analyse such a linkage we replace the links by the vectors a, b, c and d as shown in fig. 9 Writing the vector or loop equation for the linkage we have

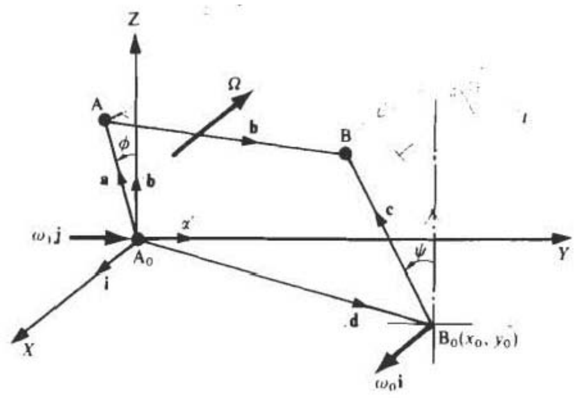


Figure 9: Vector representation of the linkage in Fig 8

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} \tag{22}$$

Which upon differentiating becomes

$$\dot{\mathbf{a}} + \dot{\mathbf{b}} = \dot{\mathbf{c}}$$

since d is a vector of constant magnitude and direction. If ω_1 is the input angular velocity of the link A_oA, then in vector form we have

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{j}$$

Similarly, the output angular velocity of the link B_oB is

$$\boldsymbol{\omega}_o = \omega_o \mathbf{i}$$

Let v_A be the velocity of point A as a point on A_oA, then,

$$v_A = \boldsymbol{\omega}_1 \times \mathbf{a} = \omega_1 \mathbf{j} \times \mathbf{a}$$

Also, if $\mathbf{v}_B$ is the velocity of point B as a point on B_oB, then

$$v_B = \boldsymbol{\omega_o} \times \mathbf{c} = \omega_o \mathbf{i} \times \mathbf{c}$$

But the velocity of v_B of point B on the coupler AB is given by

$$v_B = v_A + v_{BA} = v_A + \Omega \times \mathbf{b}$$

where Ω is the angular velocity of the coupler AB which can be expressed in terms of its components ω_x, ω_y, and ω_z, i.e

$$\Omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

Hence,

$$\Omega \times \mathbf{b} = v_B - v_A = \omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{j} \times \mathbf{a} \tag{23}$$

Since v_{BA}, i.e., the velocity of B on AB as seen by an observer at A, is perpendicular to the coupler AB then by taking the 'dot' (scalar) product of v_{BA} with $\mathbf{b}$ we must have

$$v_{BA}.\mathbf{b} = 0$$

Substituting for v_{BA} yields

$$(\omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{a}).\mathbf{b} = 0 \tag{24}$$

To solve for ω_o we need to express $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the input angle ϕ and the output angle ψ. Referring to figure 9

$$\begin{aligned} \mathbf{a} &= a\sin\phi\mathbf{i} + a\cos\phi\mathbf{k} \\ \mathbf{c} &= -c\sin\psi\mathbf{j} + c\cos\psi\mathbf{k} \\ \mathbf{d} &= x_o\mathbf{i} + y_o\mathbf{j} \end{aligned} \tag{25}$$

x_o and y_o are the coordinates of B_o.

Solving for $\mathbf{b}$ in equation 22 we get,

$$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a}$$

and substituting for $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ from eq. 25 yields

$$b = (x_o - a\sin\phi)\mathbf{i} + (y_o - c\sin\psi)\mathbf{j} + (c\cos\psi - a\cos\phi)\mathbf{k} \tag{26}$$

Substituting Eq. 26 in Eq. 23 and performing the dot product will yield an expression for the output angular velocity ω_o.
An alternative approach is to consider the length $b = AB$ of the coupler and make use of the fact that since the link is rigid $\dot{b} = 0$, as follows. From Eq. 26,

$$b^2 = (x_o - a\sin\phi)^2 + (y_o - c\sin\psi)^2 + (c\cos\psi - a\cos\phi)^2$$

Differentiating yields

$$\begin{aligned} 0 = {} & (x_o - a\sin\phi)(-a\cos\phi\dot{\phi}) + (y_o - c\sin\psi)(-c\cos\psi)\dot{\psi} \\ & +(c\cos\psi - a\cos\phi)(-c\sin\psi\dot{\psi} + a\sin\phi\dot{\phi}) \end{aligned}$$

Expanding and collecting terms the angular velocity ω_o of the output is given by

$$\omega_o = \frac{a(x_o\cos\phi - c\sin\phi\cos\psi)}{c(a\cos\phi\sin\psi - y_o\cos\psi)} \tag{27}$$

Example 2.3 *Figure 10 shows a crank A_oA rotating in the $x - z$ plane at a constant angular velocity $\omega_1 = 10$ rad/s and driving the slider B on the rod PQ by means of the link AB. The rod is in the $y - z$ plane and parallel to A_oy. Calculate the velocity of the slider and the angular velocity of the link AB when $\phi = 90^o$.*

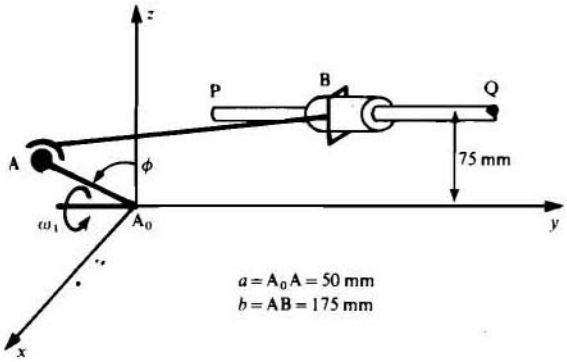


Figure 10:

Solution

When $\phi = 90^o$ the mechanism is in the position shown in Fig. 11

$$\begin{aligned}
\mathbf{v_B} &= v_{Bj} \\
\mathbf{v_A} &= \omega_1 \mathbf{j} \times \mathbf{a} \\
&= 10\mathbf{j} \times 50\mathbf{i} = -500\mathbf{k}
\end{aligned}$$

- Velocity of the slider: Let Ω be the angular velocity of the link AB, then we have

$$v_B = v_A + v_{BA} = v_A + \Omega \times \mathbf{b}$$

where $\mathbf{b} = \mathbf{R_B} - \mathbf{a} = 150j + 75k - 50i$
Hence,

$$v_B j = -500k + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ -50 & 150 & 75 \end{vmatrix}$$

$$v_B\mathbf{j} = (75\omega_x - 150\omega_z)\mathbf{i} + (-75\omega_x - 50\omega_z)\mathbf{j} + (150\omega_x + 50\omega_y)\mathbf{k} - 500\mathbf{k}$$

Equating the coefficients of the unit vectors, we get

$$\begin{aligned}
75\omega_y - 150\omega_z &= 0 && (28) \\
-75\omega_x - 50\omega_z &= v_B && (29) \\
150\omega_x + 50\omega_y &= 500 && (30)
\end{aligned}$$

Solving the above equations simultaneously, we get,

$$v_B = -250 \text{ mm/s}$$

That is, the collar is moving towards P at that particular instant.

- Angular velocity of link AB: From observation of mechanism, the slider cannot rotate about $A'B = \mathbf{R}$, so that the projection Ω on $\mathbf{R}$ must be zero, i.e., $\Omega.\mathbf{R} = 0$. From Fig. 11 we see that

$$\mathbf{R} + \mathbf{R_B} = \mathbf{d}$$

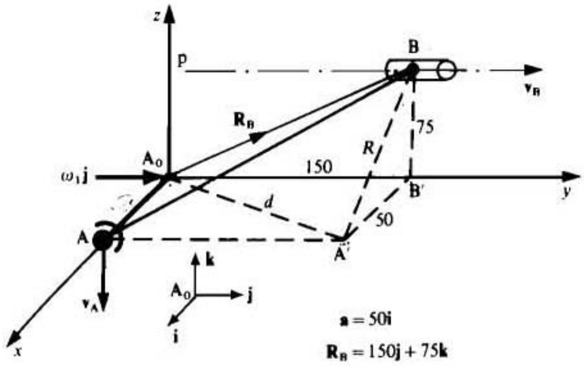


Figure 11:

Hence

$$R = 50\mathbf{i} + 150\mathbf{j} - 150\mathbf{j} - 75\mathbf{k} = 50\mathbf{i} - 75\mathbf{k}$$

and

$$(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}).(50\mathbf{i} - 75\mathbf{k}) = 0$$

i.e.,

$$50\omega_x - 75\omega_z = 0 \tag{31}$$

Solving Eqs. 28 to 31 yields

$$v_B = -250 \text{ mm/s} \quad \omega_x = 57.7 \text{ rad/s} \quad \omega_y = 76.9 \text{ rad/s} \quad \omega_z = 38.5 \text{ rad/s}$$

Example 2.4 *Figure 12 shows a four-bar spatial linkage where the input and output shafts are at* 90^o *to each other. Double revolutes are used at* A *and* B*. The input shaft at* A_o *has a constant angular velocity* ω_1 *in a clockwise direction looking along the shaft towards* A_o*. Links dimensions are;* $A_oA = a = 102$ *mm;* $AB = b = 381$ *mm;* $B_oB = c = 254$ *mm;* $A_oB_o = d = 315$ *mm.*

(a) Obtain expression for angular velocity ratio and for the acceleration.

(b) When the input speed is 240 *rev/min, calculate, the angular velocity of the coupler* AB *for* $\phi = 60^o$*.*

(c) Calculate the angular acceleration of the coupler AB *when the input is rotating at* 1200 *rev/min and accelerating at* 3000 *rev/min/s in the sense of* ω_1 *at the instant when* $\phi = 0$*.*

Solution

- Output velocity and acceleration:
  Figure 13 shows the linkage with the links expressed as vectors given by

$$\begin{aligned}
\mathbf{a} &= -a\cos\phi\mathbf{i} + \sin\phi\mathbf{k} \\
\mathbf{b} &= -c\cos\psi\mathbf{i} + c\sin\psi\mathbf{j} \\
\mathbf{c} &= -d\mathbf{i}
\end{aligned}$$

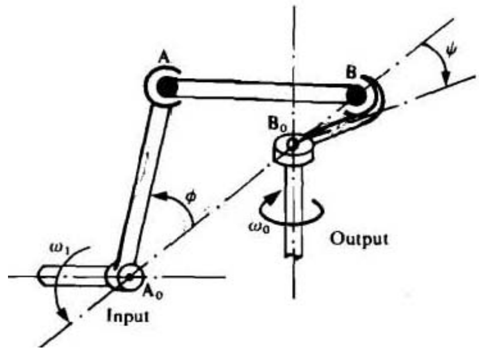


Figure 12:

Since $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$, then

$$\mathbf{b}=\mathbf{c}+\mathbf{d}-\mathbf{a}$$

Substituting for $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{a}$ we get

$$\mathbf{b}=(-c\cos\psi-d+a\cos\phi)\mathbf{i}+(c\sin\psi)\mathbf{j}-(a\sin\phi)\mathbf{k} \tag{32}$$

For the velocity of B we have $v_B=v_A+v_{BA}$

$$\begin{aligned}\therefore v_{BA} &= v_B-v_A=-\omega_o\times\mathbf{c}-\omega_1\times\mathbf{a}\\ &= -(\omega_o\mathbf{k}\times\mathbf{c}+\omega_1\mathbf{j}\times\mathbf{a})\end{aligned}$$

where ω_o is the angular velocity of the output link BB_o.
Since v_{BA} is perpendicular to AB then

$$v_{BA}.\mathbf{b}=0$$

Hence

$$(\omega_o\mathbf{k}\times\mathbf{c}+\omega_1\mathbf{j}\times\mathbf{a}).\mathbf{b}=0$$

but $v_B=-\omega_o\mathbf{k}\times\mathbf{c}=-\omega_o\mathbf{k}\times(c\cos\psi\mathbf{i}+c\sin\psi\mathbf{j})=\omega_o c\cos\psi\mathbf{j}+\omega_o c\sin\psi\mathbf{i}$
and $v_A=\omega_1\mathbf{j}\times\mathbf{a}=\omega_1\mathbf{j}\times(-a\cos\phi\mathbf{i}+a\sin\phi\mathbf{k})=\omega a\cos\phi\mathbf{k}+\omega_1 a\sin\phi\mathbf{i}$
It follows that

$$v_{BA}=(\omega_o c\sin\psi-\omega_1 a\sin\phi)\mathbf{i}+\omega_o c\cos\psi\mathbf{j}-\omega_1 a\cos\phi\mathbf{k}$$

Upon performing the dot product with $\mathbf{b}$ we get

$$-\omega_o c^2\sin\psi\cos\psi+(\omega_1 a\sin\phi-\omega_o c\sin\psi)(a\cos\phi-c\cos\psi-d)-\omega_1 a^2\sin\phi\cos\phi=0$$

which upon expanding and solving for velocity ratio yields

$$\frac{\omega_o}{\omega_1}=\frac{a\sin\phi(d+c\cos\psi)}{c\sin\psi(d-a\cos\phi)} \tag{33}$$

In order to solve for this ratio we need the value of the output angle ψ. From Eq. 32 we have

$$b^2 = (-c\cos\psi + a\cos\phi - d)^2 + c^2\sin^2\psi + a^2\sin^2\phi \tag{34}$$

Expanding and solving for $\cos\psi$ yields

$$\cos\psi = \frac{b^2 - a^2 - c^2 - d^2 + 2ad\cos\phi}{2c(d - a\cos\phi)} \tag{35}$$

To obtain an expression for the acceleration α_o of the output BB_o it is best to differentiate Eq. 33
Let

$$\begin{aligned} u &= a\sin\phi(d + c\cos\psi) \\ v &= c\sin\psi(d - a\cos\phi) \end{aligned}$$

then

$$\omega_0 = \frac{u}{v}\omega_1 \tag{36}$$

Differentiating (36) with respect to time, we get

$$\alpha_o = \frac{vu' - uv'}{v^2}\omega_1^2 + \frac{u}{v}\alpha_1 \tag{37}$$

We also note that

$$\omega_o = \frac{d\psi}{dt} = \frac{d\psi}{d\phi}\frac{d\phi}{dt} = \frac{u}{v}\omega_1$$

Hence

$$\begin{aligned} u' &= a\cos\phi(d + c\cos\psi) - ac\sin\phi\sin\psi\left(\frac{u}{v}\right) \\ v' &= c\cos\psi(d - a\cos\phi)\left(\frac{u}{v}\right) + ac\sin\phi\sin\psi \end{aligned}$$

- Angular velocity of the coupler
  There are two situation to consider

  (a) <u>The magnitude only of the angular velocity of the coupler</u>

$$\begin{aligned} v_{BA} &= v_B - v_A \\ &= -\omega_o \times \mathbf{c} - \omega \times \mathbf{a} \\ &= (\omega_o c\sin\psi - \omega_1 a\sin\phi)\mathbf{i} + \omega_o c\cos\psi\mathbf{j} - \omega_1 a\cos\phi\mathbf{k} \end{aligned}$$

The magnitude v_{BA} is given by

$$v_{BA} = \sqrt{(\omega_o c\sin\psi - \omega_1 a\sin\phi)^2 + (\omega_o c\cos\psi)^2 + (\omega_1 a\cos\phi)^2}$$

If Ω is the magnitude of the angular velocity of the coupler then

$$\Omega = \frac{v_{BA}}{b}$$

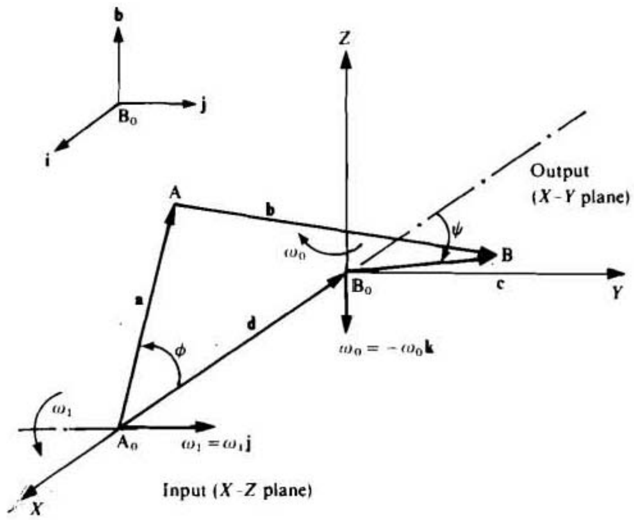


Figure 13:

Since we already have expressions for ψ and ω_o we can readily calculate Ω. When $\phi = 60^o$, $\psi = 88.42^o$, $\omega_o = 10.7$ rad/s, $b = 0.381$ m.
Substitution in the above equation yields

$$\Omega = \frac{1}{0.381}\sqrt{\begin{array}{l}(10.7 \times 0.254 \sin 88.42 - 25.1 \times 0.102 \sin 60^o)^2 + \\ (10.7 \times 0.254 \cos 88.42)^2 + (25.1 \times 0.102 \cos 60)^2\end{array}}$$

Hence $\Omega = 3.61$ rad/s

(b) The components of the angular velocity of the coupler parallel to the axes We have

$$\Omega \times \mathbf{b} = v_B - v_A \tag{38}$$

Substituting for $\mathbf{b}$, v_B and v_A we get

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ (a\cos\phi - c\cos\psi - d) & (c\sin\psi) & (-a\sin\phi) \end{vmatrix} = \begin{array}{c}(\omega_o c\sin\psi - \omega_1 a\sin\phi)\mathbf{i} \\ +\omega_o c\cos\psi\mathbf{j} - \omega_1 a\cos\phi\mathbf{k}\end{array}$$

Expanding the determinant and equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ yields the following equation

$$\begin{bmatrix} 0 & -a\sin\phi & -c\sin\psi \\ a\sin\phi & 0 & (a\cos\phi - c\cos\psi - d) \\ c\sin\psi & -(a\cos\phi - c\cos\psi - d) & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_o c\sin\psi - \omega_1 a\sin\phi \\ \omega_o c\cos\psi \\ -\omega_1 a\cos\phi \end{bmatrix} \tag{39}$$

i.e.,$A\Omega = B$

An examination of these equations reveals that the matrix A is singular, i.e the determinant of the matrix is zero, hence this equation has either an infinite number of solutions or none at all. To obtain the angular velocity Ω we can proceed as follows. Let us consider Eq. 38. ie

$$\Omega \times \mathbf{b} = v_{BA}$$

If we pre-multiply by $\mathbf{b}$ we get

$$\mathbf{b} \times (\Omega \times \mathbf{b}) = b \times v_{BA}$$

$$\mathbf{b} \times (\Omega \times \mathbf{b}) = (\mathbf{b}.\mathbf{b})\Omega - (\mathbf{b}.\Omega)\mathbf{b} \tag{40}$$

The first term of Eq. (40) equals $b^2\Omega$ and the second term is zero because the components of Ω taken along body axes i.e axes fixed to the coupler AB at A (Fig. 14) are perpendicular to $\mathbf{b}$, except for ω'_y which is along AB and may be equated to zero since any spin of AB about its own axis does not affect the output motion of the mechanisms. Thus we have

$$b^2\Omega = \mathbf{b} \times v_{BA} = \mathbf{b} \times (v_B - v_A)$$

Substituting for Ω, $\mathbf{b}$, v_B and v_B and solving for Ω yields

$$(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) = \frac{1}{b^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (a\cos\phi - c\cos\psi - d) & c\sin\psi & -a\sin\phi \\ (\omega_o c\sin\psi - \omega_1 a\sin\phi) & \omega_o c\cos\psi & -\omega_1 a\cos\phi \end{vmatrix}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and using the numerical values of case (a) we find
$\omega_x = -2.239$ rad/s, $\omega_y = -2.685$ rad/s, and $\omega_z = -1.013$ rad/s
Hence $\Omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{2.239^2 + 2.685^2 + 1.013^2} = 3.64$ rad/s
Which is close to that obtained for case (a)

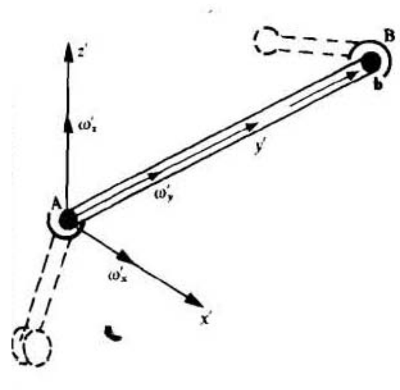


Figure 14:

- For acceleration of a point P on a link AP, we have

$$\mathbf{a}_P = \mathbf{a}_A + \alpha \times \mathbf{R} + \omega \times (\omega \times \mathbf{R})$$

where
a_A=acceleration of link A
α=angular acceleration of the link
ω=angular velocity of the link
$\mathbf{R}$=length of the link

– *Point A on link A_oA*

$$\begin{aligned}\mathbf{a} &= -a\cos\phi\mathbf{i} + a\sin\phi\mathbf{k}\\ &= -0.102\mathbf{i} \quad \text{since} \quad \phi = 0\end{aligned}$$

$$\begin{aligned}v_A &= \omega_1\mathbf{j}\times\mathbf{a}\\ &= 125.7\mathbf{j}\times(-0.102\mathbf{i})\\ &= 12.82\mathbf{k}\end{aligned}$$

$$\mathbf{a}_A = \underbrace{\alpha_1\mathbf{j}\times\mathbf{a}}_{\text{Tangential}} + \underbrace{\omega_1\mathbf{j}\times(\omega_1\mathbf{j}\times\mathbf{a})}_{\text{Centripetal}}$$

But $\omega_1\mathbf{j}\times\mathbf{a} = v_A$
We have $\omega_1\mathbf{j}\times v_A = 125.7\mathbf{j}\times 12.82\mathbf{k} = 1611\mathbf{i}$
Hence $a_A = 1611\mathbf{i} + 32.05\mathbf{k}$

– *Position B on link B_oB*

$$\begin{aligned}\mathbf{c} &= -c\cos\psi\mathbf{i} + c\sin\psi\mathbf{j}\\ &= -0.254\cos 70.64\mathbf{i} + 0.254\sin 70.64\mathbf{j}\\ &= -0.0842\mathbf{i} + 0.240\mathbf{j}\end{aligned}$$

Since 70.64^o when $\phi = 0$

$$\begin{aligned}v_B &= -\omega_o\mathbf{k}\times\mathbf{c}\\ &= 0 \quad \textit{since} \quad \omega_o/\omega_1 = 0 \quad \textit{when} \quad \phi = 0\end{aligned}$$

$$\begin{aligned}a_B &= -\alpha_o\mathbf{k}\times\mathbf{c} - \omega_o\mathbf{k}\times v_B = -\alpha_o\mathbf{k}\times\mathbf{c}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & -\alpha_o\\ -0.0842 & 0.240 & 0\end{vmatrix} = 0.240\alpha_o\mathbf{i} + 0.0842\alpha_o\mathbf{j}\end{aligned} \tag{41}$$

– *Point B on link AB*
$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a} = (-c\cos\psi - d + a\cos\phi)\mathbf{i} + c\sin\psi\mathbf{j} - a\sin\phi\mathbf{k}$
Substituting values we get

$$\begin{aligned}\mathbf{b} &= (-0.254\cos 70.64 - 0.314 + 0.102)\mathbf{i} + 0.254\sin 70.64\mathbf{j}\\ &= -0.296\mathbf{i} + 0.240\mathbf{j}\\ \mathbf{a}_{BA} &= \alpha_c\times\mathbf{b} + \omega_c\times(\omega_c\times\mathbf{b}) = \alpha_c\times\mathbf{b} + \omega_c\times v_{BA}\end{aligned}$$

where

$$\begin{aligned}\omega_c &= \text{angular velocity of the coupler} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}\\ \alpha_c &= \text{angular acceleration of the coupler} = \alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k}\end{aligned}$$

$$\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.296 & 0.24 & 0 \\ 0 & 0 & -12.82 \end{vmatrix}$$

Hence $\omega_x = -21.2$, $\omega_y = -26.13$, and $\omega_z = 0$

$$v_{BA} = -\omega_1 a\cos\phi\mathbf{k} = -125.7\times 0.102\mathbf{k} = -12.82\mathbf{k}$$

$$\omega_c \times v_{BA} = \begin{vmatrix} i & j & k \\ -12.2 & -26.13 & 0 \\ 0 & 0 & -12.82 \end{vmatrix} = 335\mathbf{i} - 272\mathbf{j} \tag{42}$$

$$\alpha_c\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_x & \alpha_y & \alpha_z \\ -0.296 & 0.240 & 0 \end{vmatrix} = -0.240\alpha_z\mathbf{i}-0.296\alpha_z\mathbf{j}+(0.240\alpha_x+0.296\alpha_y)\mathbf{k} \tag{43}$$

Writing the vector equation for the acceleration of the point B on AB we have

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA}$$

i.e.,

$$\mathbf{a}_B^n + \mathbf{a}_B^t = \mathbf{a}_A^n + \mathbf{a}_A^t + \mathbf{a}_{BA}^n + \mathbf{a}_{BA}^t \tag{44}$$

Before we can solve for α_c we need to introduce the condition that $\alpha_c.\mathbf{b} = 0$ to eliminate any possible spin of the coupler about its axis. This leads to the following equation:

$$\begin{aligned} (\alpha_x\mathbf{i}+\alpha_y\mathbf{j}+\alpha_z\mathbf{k}).(-0.296\mathbf{i}+0.240\mathbf{j}) &= 0 \\ -0.296\alpha_x + 0.240\alpha_y &= 0 \end{aligned} \tag{45}$$

Substituting Eqs.(41) and (43) into Eq. (44), we get

$$0.240\alpha_o\mathbf{i} + 0.0842\alpha_o\mathbf{j} =$$
$$1611\mathbf{i} + 32.05\mathbf{k} - 0.240\alpha_z\mathbf{i} - 0.296\alpha_z\mathbf{j} + 0.2400\alpha_x\mathbf{k} + 0.296\alpha_y\mathbf{k} + 335\mathbf{i} - 272\mathbf{j}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, yields

$$0.240\alpha_o + 0.240\alpha_z = 1611 + 335 = 1946 \tag{46}$$
$$0.0842\alpha_o + 0.296\alpha_z = -292 \tag{47}$$
$$0.240\alpha_x + 0.296\alpha_y = -32.05 \tag{48}$$
$$-0.296\alpha_x + 0.240\alpha_y = 0 \tag{49}$$

Solving Eqs. (46) to (49), we have;

$$\alpha_o = 12.607 \text{ rad/s}, \quad \alpha_z = -4497 \text{ rad/s}, \quad \alpha_x = -53 \text{ rad/s}^2$$

Hence at the instant when $\phi = 0$, the angular acceleration of the output $\alpha_o = -12,607$ rad/s^2, and that of the coupler $\alpha_c = -53\mathbf{i} - 65.3\mathbf{j} - 4497\mathbf{k}$ rad/s^2

The output acceleration is given by

3 Static forces in spatial mechanisms

For a particle to be in equilibrium requires that

$$\sum \mathbf{F} = 0 \tag{50}$$

If the forces acting on the particle are resolved into their respective **i**, **j**, **k** components, we can then write

$$\sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = 0$$

To ensure that Eqn. 50 is satisfied, we must therefore require that the following three scalar component equations be satisfied:

$$\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum F_z &= 0 \end{aligned}$$

These equations represent the algebraic sums of the x, y, z force components acting on the particle. Using them we can solve for at most three unknowns, generally represented as angles or magnitudes of forces shown on the particle's free-body diagram.

3.1 Procedure for analysis

The following procedure provides a method for solving three dimensional for equilibrium problem.

- **Free-Body Diagram:** Draw a free body diagram of the particle and label all the known and unknown forces forces on this diagram.
- **Equation of Equilibrium:** Establish the x, y, z coordinate axes with origin located at the particles and apply the equations of equilibrium. Use the above three scalar equations in cases where it is easy to resolve each force acting on the particle into its x, y, z components. By setting the respective i, j k components equal to zero, the three scalar equations can be generated. If more than three unknowns exist and the problem involves a spring, consider using $F = ks$ to relate the spring force to the deformation s of the spring.

Example 3.1 *The 100-kg box shown in figure 15 is supported by three cords, one of which is connected to a spring. Determine the tension in each cord and the stretch of the spring.*

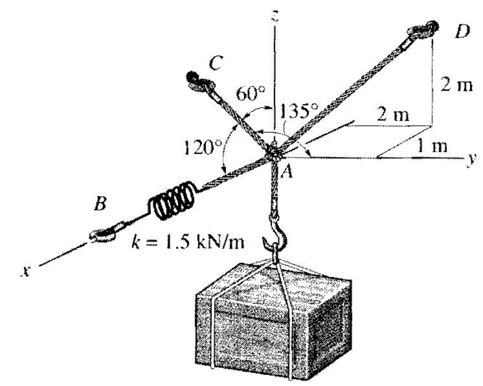


Figure 15:

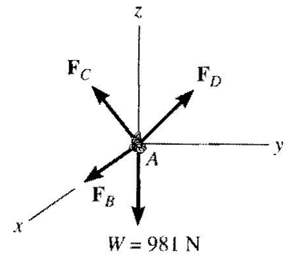


Figure 16:

solution

- ***Free-Body Diagram***, The force in each of the cords can be determined by investigating the equilibrium of point A. The free-body diagram is shown in Fig. 16. The weight of the cylinder is $W = 100(9.81)$=981 N.

- ***Equation of Equilibrium***, Each vector on the free-body diagram is first expressed in cartesian vector form. For F_c and noting point D(-1 m, 2 m, 2 m) for F_D, we have

$$\begin{aligned}
\mathbf{F}_B &= F_B\mathbf{i} \\
\mathbf{F}_c &= F_c\cos 120^o\mathbf{i} + F_c\cos 135^0\mathbf{j} + F_c\cos 60^0\mathbf{k} \\
&= 0.5F_c\mathbf{i} - 0.707F_c\mathbf{j} + 0.5F_c\mathbf{k} \\
\mathbf{F}_D &= F_D\left[\frac{-1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(-1)^2 + (2)^2 + (2)^2}}\right] \\
&= -0.333F_D\mathbf{i} + 0.667F_D\mathbf{j} + 0.667F_D\mathbf{k} \\
\mathbf{W} &= -981\mathbf{k}\text{ N}
\end{aligned}$$

Equilibrium requires

$$\sum \mathbf{F} = 0;\ \Rightarrow \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = 0$$

i.e

$$F_B\mathbf{i} - 0.5F_c\mathbf{i} - 0.707F_c\mathbf{j} + 0.5F_c\mathbf{k} - 0.333F_D\mathbf{i} + 0.667F_D\mathbf{j} + 0.667F_D\mathbf{k} - 981\mathbf{k} = 0$$

Equating the respective **i**, **j**, **k**, components to zero,

$$\begin{aligned}
\sum F_x &= 0 \quad F_B - 0.5F_c - 0.333F_D = 0 \\
\sum F_y &= 0 \quad -0.707F_c + 0.667F_D = 0 \\
\sum F_z &= 0 \quad 0.5F_c + 0.667F_D - 981 = 0
\end{aligned}$$

Solving the above sets of simultaneous equations, we have.

$$\begin{aligned}
F_c &= 813 \text{ N} \\
F_D &= 862 \text{ N} \\
F_B &= 693.7 \text{ N}
\end{aligned}$$

The stretch of the spring is therefore

$$F = ks \Rightarrow 693.7 = 150s \Rightarrow s = 0.462 \text{ m}$$

4 Dynamics of spatial mechanisms

4.1 Moment of inertia

Consider the rigid body shown in Fig 17. The moment of inertia for a differential element dm of the body about any one of the three coordinate axes is defined as the product of the mass of the element and the square of the shortest distance from the axis to the element. For example in the figure, $r_x = \sqrt{y^2 + z^2}$, so that the mass moment of inertia of dm about x axis is

$$dI_{xx} = r_x^2 dm = (y^2 + z^2)dm$$

The moment of inertia I_{xx} for the body is determined by integrating this expression over the entire mass of the body. Hence, for each of the axes, we may write.

$$\begin{aligned}
I_{xx} &= \int_m r_x^2 dm = \int_m (y^2 + z^2)dm \\
I_{yy} &= \int_m r_y^2 dm = \int_m (x^2 + z^2)dm \\
I_{zz} &= \int_m r_z^2 dm = \int_m (x^2 + y^2)dm
\end{aligned} \tag{51}$$

Hence it is seen that the moment of inertia is always a positive quantity, since it is the summation of the product of the mass dm, which is always positive, and the distance squared.

4.2 Product of Inertia

The product of inertia for a differential element dm is defined with respect to a set of two orthogonal planes as the product of the mass of the element and the perpendicular (or shortest) distances from the planes to the element. For instance, with respect to $y-z$ and $x-z$ planes, the product of inertia dI_{xy} for the element dm shown in Fig. 17 is

$$dI_{xy} = xydm$$

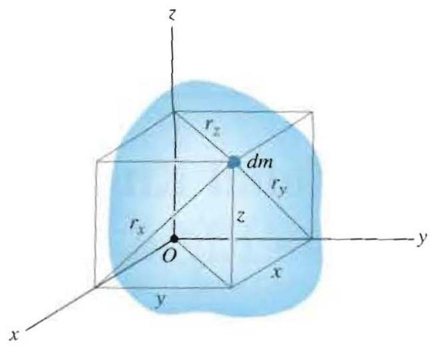


Figure 17:

Note also that $dI_{yx} = dI_{xy}$. By integrating over the entire mass, the product of inertia of the body for each combination of planes may be expressed as

$$\begin{aligned} I_{xy} &= I_{yx} = \int_m xydm \\ I_{zy} &= I_{zy} = \int_m yzdm \\ I_{zx} &= I_{zx} = \int_m xzdm \end{aligned} \quad (52)$$

Unlike the moment of inertia, which is always positive, the product of inertia may be positive, negative, or zero. The results depends on the sign of the two defining coordinates, which vary independently from one another. In particular, if either one or both of the orthogonal planes are planes of symmetry for the mass, the product of inertia with respect to these planes will be zero.

4.3 Parallel-Axis and Parallel-plane Theorem

Parallel-axis theorem is often used to transfer the moment of inertia of a body from an axis passing through its mass center G to a parallel axis passing through some other point. Consider Fig. 18 where center of mass G has coordinates x_G, y_G, z_G defined from x, y z axes, then the parallel-axis equations used to calculate the moments of inertia about the x, y, z axes are.

$$\begin{aligned} I_{xx} &= (I_{x'x'})_G + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{y'y'})_G + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{z'z'})_G + m(x_G^2 + y_G^2) \end{aligned} \quad (53)$$

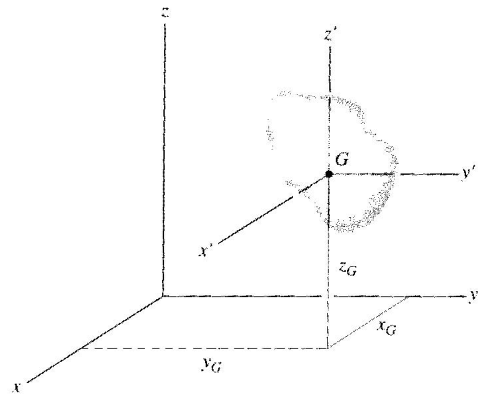


Figure 18:

Parallel-plane theorem is used to transfer the products of inertia of the body from one set of three orthogonal planes passing through the body's mass center to a corresponding set of three parallel planes passing through some other point O. Defining the perpendicular distance between the planes as x_G, y_G and z_G, Fig, 18, the parallel-plane equations can be written as.

$$\begin{aligned} I_{xy} &= (I_{x'y'})_G + m x_G y_G \\ I_{yz} &= (I_{y'z'})_G + m y_G z_G \\ I_{zx} &= (I_{z'x'})_G + m z_G x_G \end{aligned} \tag{54}$$

4.4 Inertia Tensor

The inertial properties of a body are completely characterized by nine terms, six of which are independent of one another. This set of terms is defined using Eqs. 51 and 52 and can be written as

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

This array is called an inertia tensor. In general, for point O we can specify a unique axes inclination for which the product of inertia for the body are zero when computed with respect to these axes. When this is done, the inertia tensor is said to be "diagonalized" and may be written in the simplified form

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Here, $I_x = I_{xx}$, $I_y = I_{yy}$ and $I_z = I_{zz}$ are termed as the *principal moments of inertia* for the body, which are computed from the principal axes of inertia. Of of these three principal moments of inertia, one will be maximum and another a minimum of the body's moment of inertia.

4.5 Moment of Inertia about an arbitrary axis

Consider the body shown in Fig. 19, where the nine elements of the inertia tensor have been computed for the x, y, z axes having an origin at O. Here we wish to determine the moment of inertia of the body about the Oa axis, for which the direction is defined by the unit vector $\mathbf{u}_a$.

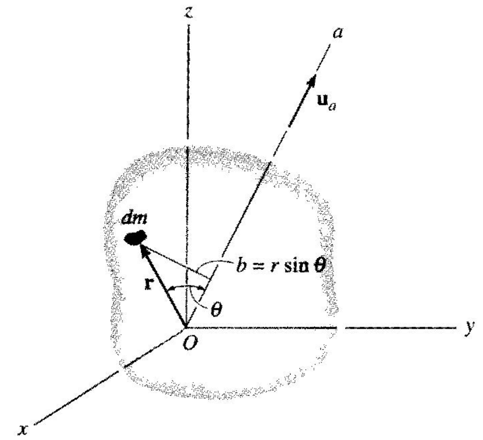


Figure 19:

By definition $I_{Oa} = \int b^2 dm$, where b is the perpendicular distance from dm to Oa. If the position of dm is located using $\mathbf{r}$, then $b = r \sin\theta$, which represents the magnitude of the cross product $\mathbf{u}_a \times \mathbf{r}$. Hence, the moment of inertia can be expressed as

$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r}).(\mathbf{u}_a \times \mathbf{r}) dm$$

Provided $\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so that $\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$, then, after substituting and performing the dot-product operation, we can write the moment of inertia as

$$\begin{aligned} I_{Oa} &= \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2] dm \\ &= u_x^2 \int_m (y^2 + z^2) dm + u_y^2 \int_m (z^2 + x^2) dm + u_z^2 \int_m (x^2 + y^2) dm \\ & \quad -2u_x u_y \int_m xy dm - 2u_y u_z \int_m yz dm - 2u_z u_x \int_m zx dm \end{aligned}$$

Recognizing the integrals to be the moments and products of inertia of the body. Eqs. 51 and 52, we have

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x \tag{55}$$

Thus, if the inertia tensor is specified for the x, y, z axes, the moment of inertia of the body about the inclined Oa axis can be found by using Eq. 55

4.6 Angular Momentum

In this section the necessary equations to be used to determine the angular momentum of a rigid body about an arbitrary point will be developed. Consider the rigid body in Fig. 20, which has a mass m and center of mass at G. The X, Y, Z coordinate system represents an inertial frame of reference, and hence, its axes are fixed or translate with a constant velocity. The angular momentum as measured from this reference will be computed relative to the arbitrary point A. The position vectors $\mathbf{r}_A$ and $\boldsymbol{\rho}_A$ are drawn from the origin of coordinates to point A and from A to the ith particle of the body. If the particle's mass is m_i, the angular momentum about point A is

$$(\mathbf{H}_A)_i = \boldsymbol{\rho}_A \times m_i \mathbf{v}_i$$

where $\mathbf{v}_i$ represents the particle's velocity measured from the X, Y, Z coordinate system. If the body has an angular velocity $\boldsymbol{\omega}$ at the instant considered, v_i may be related to the velocity of A by

$$\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$$

Thus,

$$\begin{aligned}(\mathbf{H}_A)_i &= \boldsymbol{\rho}_A \times m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= (\boldsymbol{\rho}_A m_i) \times \mathbf{v}_A + \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i\end{aligned}$$

Summing all the particles of the body requires an integration, and since $m_i \to dm$, we have

$$\mathbf{H}_A = \left(\int_m \boldsymbol{\rho}_A dm \right) \times \mathbf{v}_A + \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \tag{56}$$

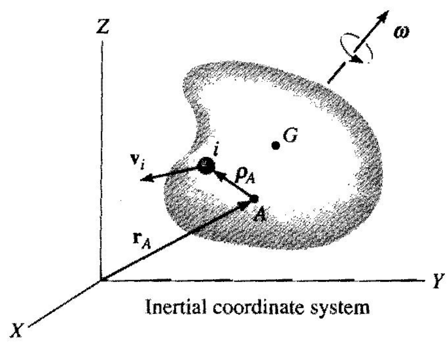
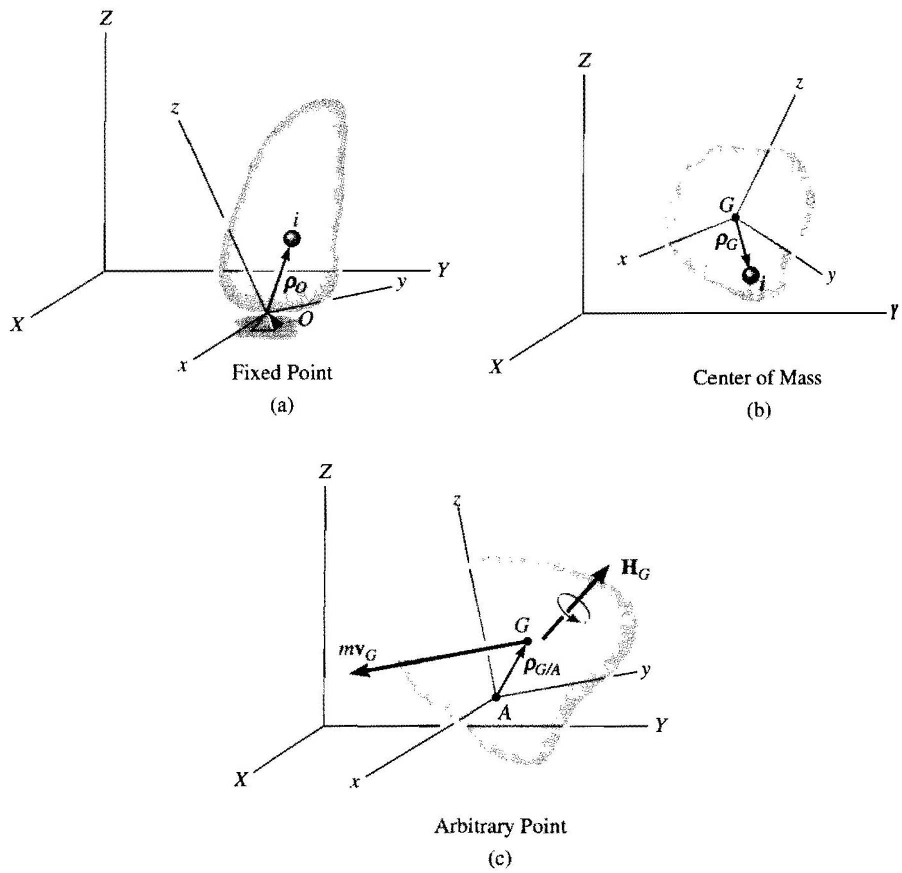


Figure 20:

- *Fixed point O*. If A becomes a fixed point O in the body. Fig. 4.6a, then $\mathbf{v}_A = 0$ and Eq. 56 reduces to

$$\mathbf{H}_o = \int_m \boldsymbol{\rho}_o \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_o) dm \tag{57}$$



Fixed Point
(a)

Center of Mass
(b)

Arbitrary Point
(c)

- *Center of mass G*. If A is located at the center of mass G of the body, Fig. 4.6b, then $\int_m \boldsymbol{\rho}_A dm = 0$ and

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G) dm \tag{58}$$

- *Arbitrary point A*. In general, A may be some point other than O or G, Fig. 4.6c, in which case Eq. 56 may be simplified to the following form

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G \tag{59}$$

Here the angular momentum consists of two parts-the moment of linear momentum $m\mathbf{v}_G$ of the body about point A added vectorially to the angular momentum $\mathbf{H}_G$.

4.7 Rectangular Components of H

To make practical use of Eqs. 57 through 59, the angular momentum must be expressed in terms of its scalar components. For this purpose, it is convenient to choose a second set of x, y, z axes having an arbitrary orientation relative to X, Y, Z axes, Fig. 57. For general formulation, Eqs. 57 and 58 have the form

$$\mathbf{H} = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) dm$$

Expressing $\mathbf{H}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ in terms of x, y and z components we have

$$H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = \int_m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times [(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})]dm$$

Expanding the cross products and combining terms yields

$$\begin{aligned} H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} &= \left[\omega_x \int_m (x^2 + z^2)dm - \omega_y \int_m xydm - \omega_z \int_m xzdm\right]\mathbf{i} \\ &= +\left[-\omega_x \int_m xydm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yzdm\right]\mathbf{j} \\ &= +\left[-\omega_x \int_m zxdm - \omega_y \int_m yzdm + \omega_z \int_m (x^2 + y^2)dm\right]\mathbf{k} \end{aligned}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and recognizing that the integral represent the moments and the product of inertia, we obtain

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \tag{60}$$

These three equations represent the scalar form of the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components of $\mathbf{H}_o$ or $\mathbf{H}_G$. Eq. 60 may be simplified further if the x, y, z coordinate are oriented such that they become principal axes of inertia for the body at the point.When these axes are used, the product of inertia $I_{xy} = I_{yz} = I_{zx} = 0$, and if the principal moments of inertia about the x, y, z axes are represented as $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$, the three components of angular momentum become

$$H_x = I_x\omega_x \quad H_y = I_y\omega_y \quad H_z = I_z\omega_z \tag{61}$$

4.8 Kinetic Energy

In this section, the expression for kinetic energy of a general rigid body motion is formulated. Consider the rigid body shown in Fig. 21, which has a mass m and center of mass G. Kinetic energy of the ith particle of the body having a mass m_i and velocity $\mathbf{v}_i$, measured relative to the inertial X, Y, Z frame of reference, is

$$T_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(\mathbf{v}_i.\mathbf{v}_i)$$

Provided the velocity of an arbitrary point A in the body is known, $\mathbf{v}_i$ may be related to $\mathbf{v}_A$ by equation $\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$, where $\boldsymbol{\omega}$ is the angular velocity of the body, measured from the X, Y, Z coordinate system, and $\boldsymbol{\rho}_A$ is a position vector drawn from A to i. Using this expression for $\mathbf{v}_i$, the kinetic energy for the particle may be written as

$$\begin{aligned} T_i &= \frac{1}{2}m_1(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A).(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{\mathbf{A}}) \\ &= \frac{1}{2}(\mathbf{v}_A.\mathbf{v}_A)m_i + \mathbf{v}_A.(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)m_i + \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho}_A).(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)m_i \end{aligned}$$

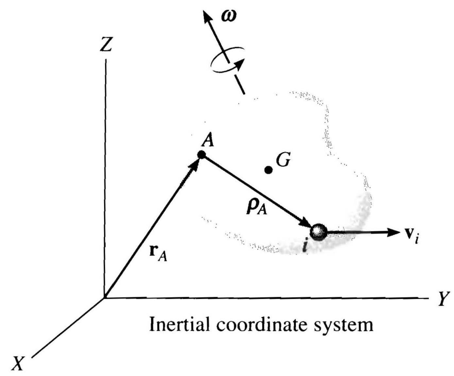


Figure 21:

The kinetic energy from the entire body is obtained by summing the kinetic energies of all the particles of the body. This require an integration, and since $m_i \to dm$, we get.

$$T = \frac{1}{2}m(\mathbf{v}_A.\mathbf{v}_A) + \mathbf{v}_A.\left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \frac{1}{2}\int_m (\boldsymbol{\omega} \times \boldsymbol{\rho}_A).(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)dm$$

The last term on the right may be rewritten using the vector identity $\mathbf{a} \times \mathbf{b}.\mathbf{c} = \mathbf{a}.\mathbf{b} \times \mathbf{c}$, where $\mathbf{a} = \boldsymbol{\omega}$, $\mathbf{b} = \boldsymbol{\rho}_A$, and $\mathbf{c} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. The final result is

$$T = \frac{1}{2}m(\mathbf{v}_A.\mathbf{v}_A) + \mathbf{v}_A.\left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \frac{1}{2}\boldsymbol{\omega}.\int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)dm \tag{62}$$

This equation is rarely used because of the computations involving the integrals. Simplification occurs, however, if the reference point A is either a fixed point O or the center of mass G

- *Fixed point O.* If A is a fixed point O in the body, Fig. 4.6a, then $v_A = 0$, and using Eq. 57, we can express Eq. 62 as

  $$T = \frac{1}{2}\boldsymbol{\omega}.\mathbf{H}_o$$

  If the x, y, z axes represent the principal axes of inertia for the body, then $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ and $\mathbf{H}_o = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$. Substituting into the above equation and performing the dot product operations yields

  $$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 \tag{63}$$

- *Center of mass G.* If point A is located at the center of mass G of the body, Fig. 4.6b, then $\int \boldsymbol{\rho}_A dm = 0$ and, using Eq. 58, we can write Eq. 62 as

  $$T = \frac{1}{2}mv_G^2 + \frac{1}{2}\boldsymbol{\omega}.\mathbf{H}_o \tag{64}$$

In a manner similar to that for a fixed point, the last term on the right side may be represented in scalar form, in which case

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 \tag{65}$$

Kinetic energy consists of two parts; namely, the translational kinetic energy of the mass center, $1/2mv_G^2$, and the body's rotational kinetic energy.

4.9 Principal of work and energy

The principal of work and energy is applied to solve kinetics problems which involve force, velocity, and displacement.

$$T_1 + \sum U_{1-2} = T_2 \tag{66}$$

Example 4.1 *An example on K.E*

4.10 Equations of motion

4.10.1 Equation of translational Motion

The translational motion of a body is defined in terms of the acceleration of the body's mass center, which is measured from an initial X, Y, Z reference. The equation of translational motion for the body can be written in vector form as

$$\sum \mathbf{F} = m\mathbf{a}_G \tag{67}$$

or by the three scalar equations

$$\begin{aligned} \sum F_x &= m(a_G)_x \\ \sum F_y &= m(a_G)_y \\ \sum F_z &= m(a_G)_z \end{aligned} \tag{68}$$

Here, $\sum \mathbf{F} = \sum F_x\mathbf{i} + \sum F_y\mathbf{j} + \sum F_z\mathbf{k}$ represents the sum of all the external forces acting on the body

4.10.2 Equation of Rotational Motion

Equation $\sum \mathbf{M}_o = \dot{\mathbf{H}}_o$, states that the sum of the moments about a fixed point O of all the external forces acting on a system of particles (contained in a rigid body) is equal to the time rate of change of the total angular momentum of the body about point O. When moments of the external forces acting on the particles are summed about

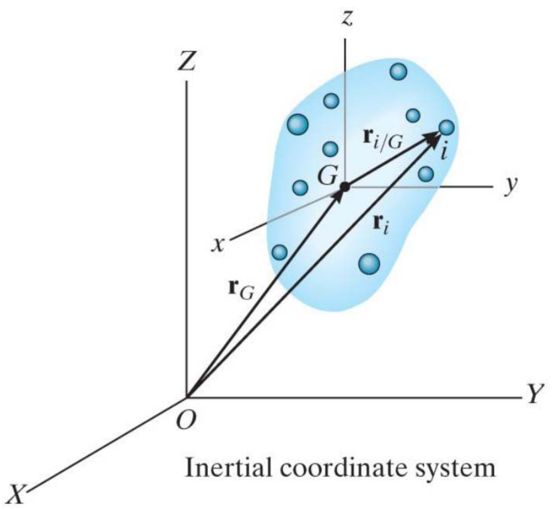


Figure 22:

the system's mass center G, the same simple form of equation is obtained relating the moment summation $\sum \mathbf{M}_G$ to the angular momentum $\mathbf{H}_G$.

To show this, consider the system of particles in Fig. 22, where X, Y, Z represents an internal frame of reference and x, y, z axes, with origin at G, translating with respect to this frame. In general, G is the acceleration, so by definition the translating frame is not an inertial reference. The angular momentum of the ith particle with respect this frame is

$$(\mathbf{H}_i)_G = \mathbf{r}_{i/G} \times m_i \mathbf{v}_{i/G}$$

where $\mathbf{r}_{i/G}$ and $\mathbf{v}_{i/G}$ represent the relative position and relative velocity of the ith particle with respect to G. Taking the time derivative we have

$$(\dot{\mathbf{H}}_i)_G = \dot{\mathbf{r}}_{i/G} \times m_i \mathbf{v}_{i/G} + \mathbf{r}_{i/G} \times m_i \dot{\mathbf{v}}_{i/G}$$

By definition, $\mathbf{v}_{i/G} = \dot{\mathbf{r}}_{i/G}$. Thus, the first term on the right side is zero since the cross product of equal vectors is zero. Also $\mathbf{a}_{i/G} = \dot{\mathbf{v}}_{i/G}$, so that

$$(\dot{\mathbf{H}}_i)_G = (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Similar expressions can be written for the other particles of the body. When the results are summed, we get

$$\dot{\mathbf{H}}_G = \sum (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

The relative acceleration for the ith particle is defined by the equation $\mathbf{a}_{i/G} = \mathbf{a}_i - \mathbf{a}_G$, where a_i and a_G represent, respectively, the accelerations of the ith particle and point G measured with respect to the inertial frame of reference. Substituting and expanding, using the distributive property of the vector cross product, yields

$$\dot{\mathbf{H}}_G = \sum (\mathbf{r}_{i/G} \times m_i \mathbf{a}_i) - (\sum m_i \mathbf{r}_{i/G}) \times \mathbf{a}_G$$

By definition of the mass center, the sum $(\sum m_i \mathbf{r}_{i/G}) = (\sum m_i)\bar{\mathbf{r}}$ is equal to zero, since the position vector $\bar{\mathbf{r}}$ relative to G is zero. Hence, the last term in the above equation is zero. Using the equation of motion, the product $m_i\mathbf{a}_i$ may be replaced by the resultant external force $\mathbf{F}_i$ acting on the ith particle. Denoting $\sum \mathbf{M}_G = (\sum \mathbf{r}_{i/G} \times \mathbf{F}_i),$, the final result may be written as

$$\sum \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{69}$$

When these components are computed about x, y, z axes that are rotating with an angular velocity $\boldsymbol{\Omega}$, which may be different from the body's angular velocity $\boldsymbol{\omega}$, the time derivative $\dot{\mathbf{H}} = d\mathbf{H}/dt$ must account for the rotation of the x, y, z axes as measured from inertial X, Y, Z axes. Hence Eq. 69 become

$$\begin{aligned} \sum \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \\ \sum \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \end{aligned} \tag{70}$$

Here $(\dot{\mathbf{H}})_{xyz}$ is the time rate of change of H measured from the x, y, z reference. There three ways in which one can define the motion of the x, y, z axes.

- x, y, z *Axes having motion* $\Omega = 0$: When $\Omega = 0$ Eq. 70 simplifies to

$$\begin{aligned} \sum M_O &= (\dot{\mathbf{H}}_O)_{xyz} \\ \sum M_G &= (\dot{\mathbf{H}}_G)_{xyz} \end{aligned}$$

- x, y, z *Axes having motion* $\Omega = \omega$: Since $\Omega = \omega$, Eq. 70 become

$$\begin{aligned} \sum \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ \sum \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_G \end{aligned} \tag{71}$$

- x, y, z *Axes having motion* $\Omega \neq \omega$:

ASSIGNMENT

At a given instant (Figure 23), the satellite dish has an angular motion $\omega_1 = 6rad/s$ and $\dot{\omega}_1 = 3rad/s^2$ about the z axis.At this same instant $\theta = 25°$, the angular motion about the x axis is $\omega_2 = 2rad/s$, and $\dot{\omega}_2 = 1.5rad/s^2$. Determine the velocity and acceleration of the signal horn A at this instant.

Solution

Angular Velocity- Coordinates of the fixed frame and the rotating frame are coincident at this instant, thus expressing the angular velocity in terms of unit vectors i,j and k.

$$\omega = \omega_1 + \omega_2 = (2i + 6k)rad/s$$

Angular Acceleration- ω_2 is observed to have a constant direction from the rotating xyz frame but has a change in direction wrt the fixed XYZ, rotating at $\Omega = \omega_1 = 6k$ rad/s

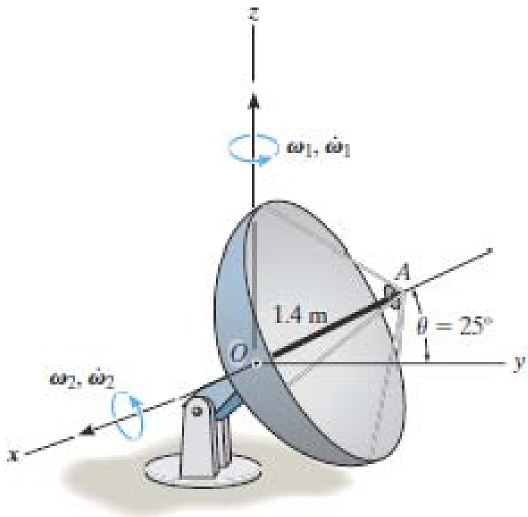


Figure 23:

Using

$$\dot{A} = (\dot{A})_{xyz} + \Omega \times A$$
$$\dot{\omega}_2 = (\dot{\omega}_2)_{xyz} + \omega_1 \times \omega_2$$
$$= 1.5i + 6k \times 2i = (1.5i + 12j)rad/s^2$$

Since ω_1 is always directed along the Z-axis ($\Omega = 0$) then

$$\dot{\omega}_1 = (\dot{\omega}_1)_{xyz} + 0 \times \omega_1$$
$$= (3k)rad/s^2$$

The angular acceleration of the satellite is $\therefore$

$$\alpha = \dot{\omega}_1 + \dot{\omega}_2$$
$$= (1.5i + 12j + 3k)rad/s^2$$

Signal Horn A-Vector r_A is given as

$$r_A = OA\cos\theta j + OA\sin\theta k$$
$$= 1.4\cos 25j + 1.4\sin 25k$$
$$= (1.27j + 0.59k)m$$

The velocity of A is given as

$$v_A = v_0 + \omega \times r_A$$
$$= 0 + (2i + 6k) \times (1.27j + 0.59k)$$
$$= \mathbf{(-7.6i - 1.18j + 2.54k)m/s}$$

And Acceleration of horn A

$$a_A = a_0 + \alpha \times r_A + \omega \times (\omega \times r_A)$$
$$= (1.5i + 12j + 3k) \times (1.27j + 0.59k) + (2i + 6k) \times [(2i + 6k) \times (1.27j + 0.59k)]$$
$$= \mathbf{(10.4i - 51.6j - 0.46k)m/s^2}$$