UNIVERSITY EXAMINATIONS 2013/2014
EMT 2411: SPATIAL MECHANISMS II
EXAM SOLUTION

QUESTION ONE - Compulsory

(a) What are the three fundamental rotational matrices? [3 marks]

SOLUTION

The basic rotation matrices representing rotation about a_x, a_y and a_z axes are given as:

$$R(a_x, \phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \quad R(a_y, \theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \quad R(a_z, \psi) = \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Derive an equation that relates the time derivative of any vector A shown in Figure 1a defined from a translating-rotating reference to its time derivative defined from a fixed reference. [7 Marks]

SOLUTION

Consider the x, y, z axes of the moving frame of reference to have an angular velocity Ω which is measured from the fixed X, Y, Z axes, Fig. 1a. Expressing vector A in terms of its i, j, k components,

$$A = A_x i + A_y j + A_z k$$

Taking time change of A with respect to the moving frame, only a change in the magnitudes of the components of A must be accounted for, since the directions of the components do not change with respect to the moving reference.

$$\left(\dot{A}\right)_{xyz} = \dot{A}_x i + \dot{A}_y j + \dot{A}_z k$$

When the time derivative of A is taken with respect to the fixed frame of reference, the directions of i, j, and k change only on account of the rotation Ω of the axes and not their translation. Hence, in general,

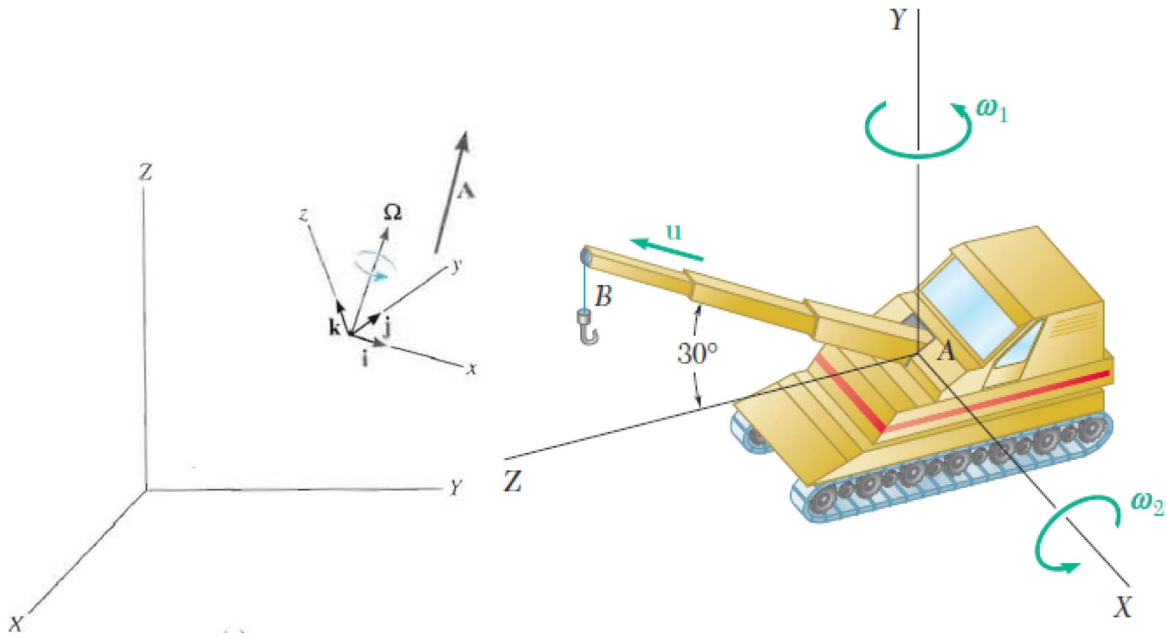
$$\left(\dot{A}\right) = \dot{A}_x i + \dot{A}_y j + \dot{A}_z k + A_x \dot{i} + A_y \dot{j} + A_z \dot{k}$$

But From

$$\dot{A} = \omega \times A \Rightarrow \dot{i} = \Omega \times i, \;\; \dot{j} = \Omega \times j, \;\; \dot{k} = \Omega \times k$$

Substituting in the above equation gives

$$\left(\dot{A}\right) = \dot{A}_{xyz} + \Omega \times A$$



(a) (b)

Figure 1

(c) The crane shown in Figure 1b rotates at the constant rate $\omega_1 = 0.25rad/s$; simultaneously, the telescoping boom is being lowered at the constant rate $\omega_1 = 0.40rad/s$. Knowing that at the instant shown the length of the boom is $20m$ and is increasing at the constant rate $u = 1.5m/s$, determine the velocity and acceleration of point B.

[20 Marks]

SOLUTION

Given

$$\omega_1 = (0.25j)rad/s \qquad u = 1.5m/s$$
$$\omega_2 = (0.4i)rad/s \qquad \Rightarrow (v_B)_{xyz} = 1.5\sin 30^\circ j + 1.5\cos 30^\circ k$$
$$l = 20m \qquad = (0.75j + 1.3k)m/s$$

The vector, r_B and total angular velocity, ω of point B are given as

$$r_B = (20\sin 30)j + (20\cos 30)k \qquad \omega = \omega_1 + \omega_2$$
$$= (10j + 17.32k)m \qquad = (0.4i + 0.25j)rad/s$$

Therefore linear velocity of point B, v_B is

$$\begin{aligned} v_B = \dot{r}_B &= v_O + \omega \times r_B + (v_B)_{xyz} \\ &= 0 + (0.4i + 0.25j) \times (10j + 17.32k) + (0.75j + 1.3k) \\ &= 4k - 6.93j + 4.33i + 0.75j + 1.3k \\ &= \mathbf{(4.33i - 6.18j + 5.30k)m/s} \end{aligned}$$

To obtain acceleration, then the angular acceleration of point B must be obtained first. To obtain the angular acceleration, individual motions are analyzed separately.

ω_1 has no acceleration wrt the movinf frame xyz. ω_2 doesn't cause a rotation of the cab, hence $\Omega = 0$. Hence

$$\begin{aligned}\dot{\omega}_1 &= (\dot{\omega}_1)_{xyz} + \Omega \times \omega_1 \\ &= 0 + 0 = 0\end{aligned}$$

Likewise, ω_2 has no acceleration wrt moving frame xyz. However, ω_1 is causing a change in direction of the telescoping boom, hence the moving frame. Therefore, $\Omega = \omega_1$.

$$\begin{aligned}\dot{\omega}_2 &= (\dot{\omega}_2)_{xyz} + \Omega \times \omega_2 \\ &= (\dot{\omega}_2)_{xyz} + \omega_1 \times \omega_2 \\ &= 0 + 0.25j \times 0.40i \\ &= (-0.1k)rad/s\end{aligned}$$

Therefore the total angular acceleration, α of point B is

$$\begin{aligned}\alpha = \dot{\omega} &= \dot{\omega}_1 + \dot{\omega}_2 \\ &= 0 - 0.1k \\ &= (-0.1k)rad/s^2\end{aligned}$$

Therefore, the linear velocity of point B, a_B is

$$\begin{aligned}a_B = \ddot{r}_B =& a_O + (\alpha \times r_B) + \omega \times (\omega \times r_B) + 2\omega \times (v_B)_{xyz} + (a_B)_{xyz} \\ =& 0 + [-0.1k \times (10j + 17.32k)] + (0.4i + 0.25j) \times [(0.4i + 0.25j) \times (10j + 17.32k)] \\ & 2(0.4i + 0.25j) \times (0.45j + 1.3k) + 0 \\ =& -1i + (0.4i + 0.25j) \times (4k - 6.93j + 4.33i) + (0.8i + 0.5j) \times (0.75j + 1.3k) \\ =& -1i - 1.6j - 2.77k + 1i - 1.08k + 0.6k - 1.04j + 0.65i \\ =& (\mathbf{0.65i - 2.64j - 3.25k})\mathbf{m/s}\end{aligned}$$

QUESTION TWO

(a) Explain the following terms as used in robotics [3 Marks]

(i) **Direct kinematics**- The forward transformation equation is solved in order to find the location of the end effector in terms of the angles and displacement between the links.

(ii) **Inverse kinematics**- All sets of joint angles which could be used to attain the given position and orientation of the end effector of the manipulator are calculated.

(iii) **Manipulator workspace**- This is defined as the set of points that can be reached by a robot's end effector. Analyzing workspace characteristics and shape provides a means to evaluate the efficiency and the kinematic performance of the manipulator mechanism.

(b) Figure 2 shows a bent rod OAB which rotates about the vertical axis OB. At the instant considered, its angular velocity and angular acceleration are, respectively, 20 rad/s and 200 rad/s^2, both clockwise when viewed from the positive Y axis. The collar D moves along the rod, and at the instant considered, OD is 8 cm. The velocity and acceleration of the collar relative to the rod are, respectively, 50 cm/s and 600 cm/s^2, both upward. Determine the velocity of the collar and acceleration of the collar. [17 marks]

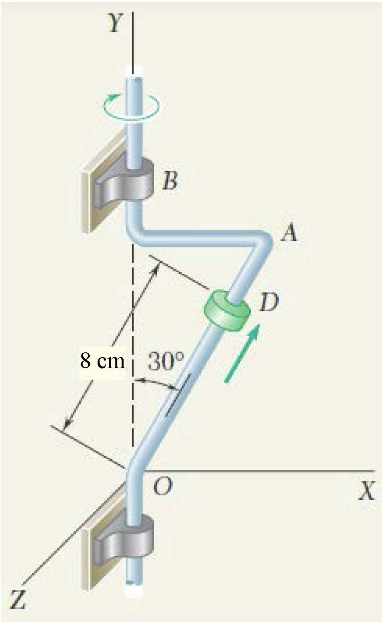


Figure 2

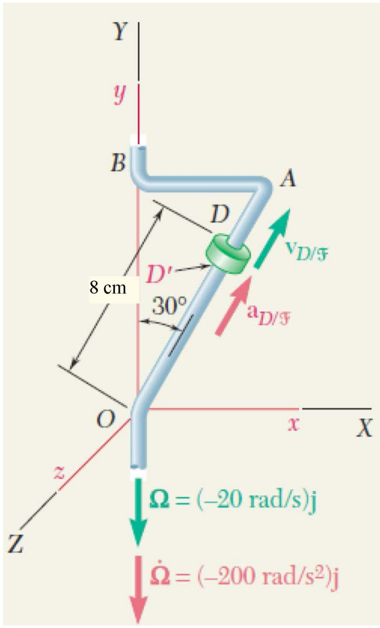


Figure 3

<u>SOLUTION</u>

The frame OXYZ is fixed. Attaching rotating Oxyz frame to the bent rod, its angular velocity and angular acceleration relative to OXYZ are $\Omega = -20j\ rad/s$ and $\dot{\Omega} = -200j\ rad/s^2$ respectively.

The position vector of D is $r = 8(\sin 30°i + \cos 30°j)cm = (4i + 6.93j)cm$

Velocity, $\mathbf{V_D}$

Denoting by D' the point of the rod which coincides with D and by $\Im$ the rotating frame Oxyz, then

$$V_D = V_{D'} + V_{D/\Im}$$

where

$$V_{D'} = \Omega \times r = (-20)j \times [4i + 6.93j] = (80k)cm/s$$
$$V_{D/\Im} = 50 \times (\sin 30°i + \cos 30°j) = (25i + 43.3j)cm/s$$

Substituting the values

$$\mathbf{V_D = (25i + 43.3j + 80k)cm/s}$$

Acceleration $\mathbf{a_D}$

$$a_D = a_{D'} + a_{D/\Im} + a_c$$

Where

$$\begin{aligned}
a_{D'} &= \dot{\Omega} \times r + \Omega \times (\Omega \times r) \\
&= -200j \times (4i + 6.93j) - 20j \times 80k \\
&= (800k - 1600i) cm/s^2 \\
a_{D/\Im} &= 600 \times (\sin 30^\circ i + \cos 30^\circ j) = (300i + 520j) cm/s^2 \\
a_c &= 2\Omega \times V_{D/\Im} \\
&= 2(-20j) \times (25i + 43.3j) = (1000k) cm/s^2
\end{aligned}$$

On substitution,

$$\mathbf{a_D} = (\mathbf{-1300i + 520j + 1800k})\mathbf{cm/s^2}$$

QUESTION THREE

(a) Briefly describe the steps taken in establishing a single orientation using composite rotations [4 marks]

(i) Initialize the rotation matrix to $R = 1$, which corresponds to the orthonormal coordinate frame F(fixed coordinate frame) and M(mobile coordinate frame) being coincident.

(ii) If the mobile coordinate frame M is to be rotated by an amount φ about the k-th unit vector of the fixed coordinate frame F, then **premultiply** R by $R_k(\varphi)$

(iii) If the mobile coordinate frame M is to be rotated by an amount φ about its own k-th unit vector, then **postmultiply** R by $R_k(\varphi)$

(iv) If there are more fundamental rotations to be performed go to step 2, else stop. The resulting composite rotation matrix R maps mobile M coordinates into fixed F coordinates

(b) The crane shown in Figure 4 rotates with a constant angular velocity ω_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity ω_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12m$, determine:

(i.) the angular velocity ω of the boom, [2 marks]

(ii.) the angular acceleration α of the boom, [6 marks]

(iii.) the velocity v of the tip of the boom, [3 marks]

(iv.) the acceleration a of the tip of the boom. [5 marks]

<u>**SOLUTION**</u>

Angular Velocity of Boom

At the instant shown, the angular velocity of the boom is $\omega = \omega_1 + \omega_2$ giving

$$(\omega = \mathbf{0.30j + 0.50k})\mathbf{rad/s}$$

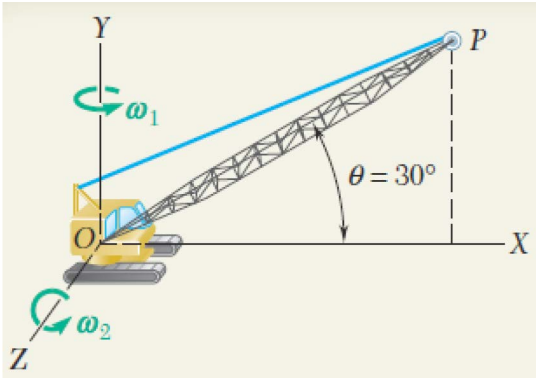


Figure 4

Angular Acceleration of Boom

The angular acceleration α of the boom is obtained by differentiating ω. Since the vector ω_1 is constant in‘magnitude and direction, then

$$\alpha = \dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2 = 0 + \dot{\omega}_2$$

It is more convenient to use a frame Oxyz attached to the cab and rotating with it, since the vector ω_2 also rotates with the cab, it has zero rate of change with respect to that frame. The rate of change $\dot{\omega}_2$ is to be computed with respect to the fixed frame OXYZ. In this case, the the axis is rotating at $\Omega = \omega_1$ such that

$$(\dot{\omega}_2)_{OXYZ} = (\dot{\omega}_2)_{Oxyz} + \Omega \times \omega_2$$
$$(\dot{\omega}_2)_{OXYZ} = (\dot{\omega}_2)_{Oxyz} + \omega_1 \times \omega_2$$
$$(\dot{\omega}_2)_{OXYZ} = 0 + 0.30j \times 0.50k = 0.15i$$
$$\text{Hence } \alpha = (\dot{\omega}_2)_{OXYZ} = \mathbf{(0.15i)rad/s^2}$$

Velocity of Tip of Boom

The position vector of point P is $r = (10.39i + 6j)m$ Hence

$$v = \omega \times r = \begin{vmatrix} i & j & k \\ 0 & 0.30 & 0.50 \\ 10.39 & 6 & 0 \end{vmatrix}$$
$$= \mathbf{(3i + 5.20j - 3.12k)m/s}$$

Acceleration of Tip of Boom

$$a = \alpha \times r + \omega \times (\omega \times r) = \alpha \times r + \omega \times v$$
$$= \begin{vmatrix} i & j & k \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$
$$= 0.90k - 0.94i - 2.60i - 1.50j + 0.90k$$
$$= \mathbf{(-3.54i - 1.50j + 1.80k)m/s^2}$$

QUESTION FOUR

Figure 5 shows a four bar RSSR spatial linkage in which the input crank AB rotates about y-axis so that B moves in a circular path in the $x-z$ plane. The output link CD rotates about an axis parallel to the x-axis in the $x-y$ plane and oscillates through an angle $\Delta\psi$ in a plane giving a coupler BC, motion in three dimensional. Show that the output angular velocity ω_o is given by [20 Marks]

$$\omega_o = \left(\frac{a(x_0\cos\phi - c\sin\phi\cos\psi)}{c(a\cos\phi\sin\psi - y_o\cos\psi)}\right)\omega_i$$

where, $AB = a$, $BC = b$, $CD = c$

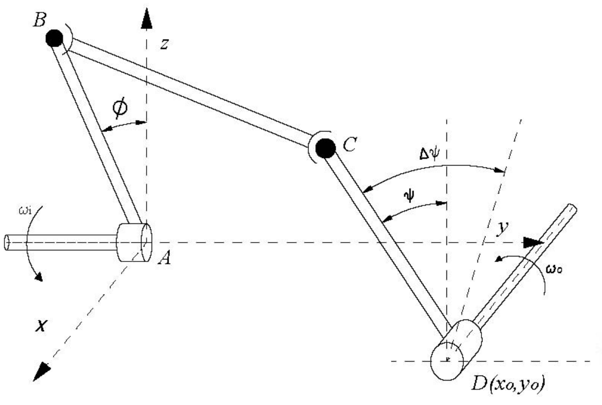


Figure 5

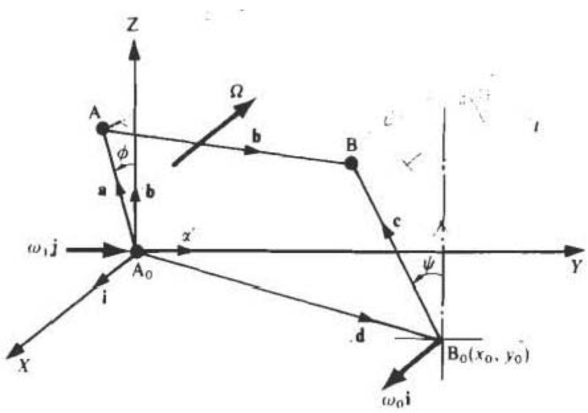


Figure 6: Vector representation

SOLUTION

Writing the vector or loop equation for the linkage we have

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} \tag{1}$$

Which upon differentiating becomes

$$\dot{\mathbf{a}} + \dot{\mathbf{b}} = \dot{\mathbf{c}}$$

since d is a vector of constant magnitude and direction. If ω_1 is the input angular velocity of the link A_oA, then in vector form we have

$$\boldsymbol{\omega}_1 = \omega_1\mathbf{j}$$

Similarly, the output angular velocity of the link B_oB is

$$\boldsymbol{\omega}_o = \omega_o\mathbf{i}$$

Let v_A be the velocity of point A as a point on A_oA, then,

$$v_A = \boldsymbol{\omega}_1 \times \mathbf{a} = \omega_1\mathbf{j} \times \mathbf{a}$$

Also, if $\mathbf{v}_B$ is the velocity of point B as a point on B_oB, then

$$v_B = \boldsymbol{\omega_o} \times \mathbf{c} = \omega_o \mathbf{i} \times \mathbf{c}$$

But the velocity of v_B of point B on the coupler AB is given by

$$v_B = v_A + v_{BA} = v_A + \Omega \times \mathbf{b}$$

where Ω is the angular velocity of the coupler AB which can be expressed in terms of its components ω_x, ω_y, and ω_z, i.e

$$\Omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

Hence,

$$\Omega \times \mathbf{b} = v_B - v_A = \omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{j} \times \mathbf{a} \tag{2}$$

Since v_{BA}, i.e., the velocity of B on AB as seen by an observer at A, is perpendicular to the coupler AB then by taking the 'dot' (scalar) product of v_{BA} with $\mathbf{b}$ we must have

$$v_{BA}.\mathbf{b} = 0$$

Substituting for v_{BA} yields

$$(\omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{a}).\mathbf{b} = 0 \tag{3}$$

To solve for ω_o we need to express $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the input angle ϕ and the output angle ψ. Referring to figure 6

$$\begin{aligned} \mathbf{a} &= a \sin\phi \mathbf{i} + a \cos\phi \mathbf{k} \\ \mathbf{c} &= -c \sin\psi \mathbf{j} + c \cos\psi \mathbf{k} \\ \mathbf{d} &= x_o \mathbf{i} + y_o \mathbf{j} \end{aligned} \tag{4}$$

x_o and y_o are the coordinates of B_o.

Solving for $\mathbf{b}$ in equation 1 we get,

$$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a}$$

and substituting for $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ from eq. 4 yields

$$b = (x_o - a \sin\phi)\mathbf{i} + (y_o - c \sin\psi)\mathbf{j} + (c \cos\psi - a \cos\phi)\mathbf{k} \tag{5}$$

Substituting Eq. 5 in Eq. 2 and performing the dot product will yield an expression for the output angular velocity ω_o.
An alternative approach is to consider the length $b = AB$ of the coupler and make use of the fact that since the link is rigid $\dot{b} = 0$, as follows. From Eq. 5,

$$b^2 = (x_o - a \sin\phi)^2 + (y_o - c \sin\psi)^2 + (c \cos\psi - a \cos\phi)^2$$

Differentiating yields

$$0 = (x_o - a\sin\phi)(-a\cos\phi\dot{\phi}) + (y_o - c\sin\psi)(-c\cos\psi)\dot{\psi}$$
$$+(c\cos\psi - a\cos\phi)(-c\sin\psi\dot{\psi} + a\sin\phi\dot{\phi})$$

Expanding and collecting terms the angular velocity ω_o of the output is given by

$$\omega_o = \frac{a(x_o\cos\phi - c\sin\phi\cos\psi)}{c(a\cos\phi\sin\psi - y_o\cos\psi)}\omega_i \tag{6}$$

QUESTION FIVE

(a) At the instant $\theta = 60^o$, the gyrotop shown in figure 7 has three components of angular motion directed as shown and having the magnitudes defined as
spin: ω_s=10 rad/s, increasing at the rate of 6 rad/s^2
nutation: ω_n=3 rad/s, increasing at the rate of 2 rad/s^2
precession: ω_p=5 rad/s, increasing at the rate of 4 rad/s^2.
Determine the angular velocity and angular acceleration of the top. [15 Marks]

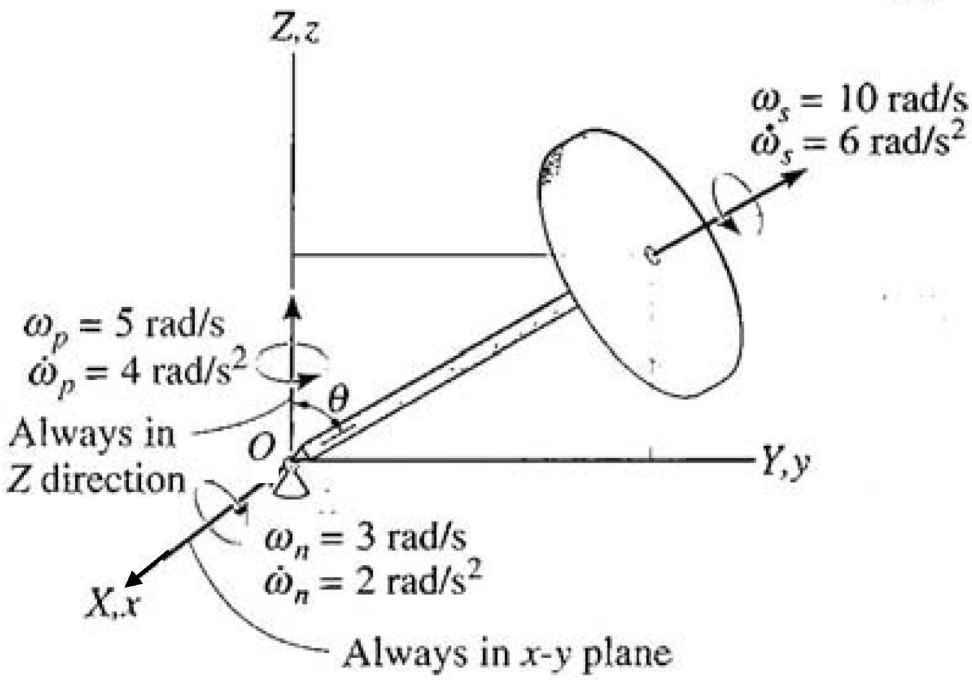


Figure 7

SOLUTION

Angular Velocity

The top is rotating about the fixed point 0. If the fixed and rotating frames are coincident at the instant shown, then the angular velocity can be expressed in terms of i, j, k components, appropriate to the x, y, Z frame; i.e.,

$$\begin{aligned}\omega &= \omega_n i + (\omega_s \sin\theta)j + (\omega_p + \omega_s \cos\theta)k \\ &= -3i + 10\sin 60^\circ j + (5 + 10\cos 60^\circ)k \\ &\{-3i + 8.66j + 10k\} rad/s\end{aligned}$$

Angular Acceleration

The angular acceleration, α is determined by examining separately the time rate of change of each angular velocity components as observed from fixed XYZ reference frame. Taking Ω as the angular velocity of the xyz frame as viewed from the XYZ reference frame, then

ω_s has a constant direction relative to xyz frame, therefore, these axes rotate at $\Omega = \omega_n + \omega_p$. Thus,

$$\begin{aligned}\dot{\omega}_s &= (\dot{\omega}_s)_{xyz} + \Omega \times \omega_s \\ &= (\dot{\omega}_s)_{xyz} + (\omega_n + \omega_p) \times \omega_s \\ &= (6\sin 60^\circ j + 6\cos 60^\circ k) + (-3i + 5k) \times (10\sin 60^\circ j + 10\cos 60^\circ k) \\ &= \{-43.30i + 20.20j - 22.98k\} rad/s^2\end{aligned}$$

Since ω_n always lies in the x-y plane, this vector has a *constant direction* if the motion is viewed from axes xyz having a rotation of $\Omega = \omega_p$ (not $\Omega = \omega_n + \omega_p$). Thus,

$$\begin{aligned}\dot{\omega}_n &= (\dot{\omega}_n)_{xyz} + \omega_p \times \omega_n \\ &= -2i + (5k) \times (-3i) \\ &= \{-2i - 15j\} rad/s^2\end{aligned}$$

Finally, the component ω_p is always directed along the fixed Z axis so that here it is not necessary to think of xyz as rotating, i.e., $\Omega = O$. Expressing the data in terms of the i, j, k components,

$$\begin{aligned}\dot{\omega}_p &= (\dot{\omega}_p)_{xyz} + 0 \times \omega_p \\ &= \{4k\} rad/s^2\end{aligned}$$

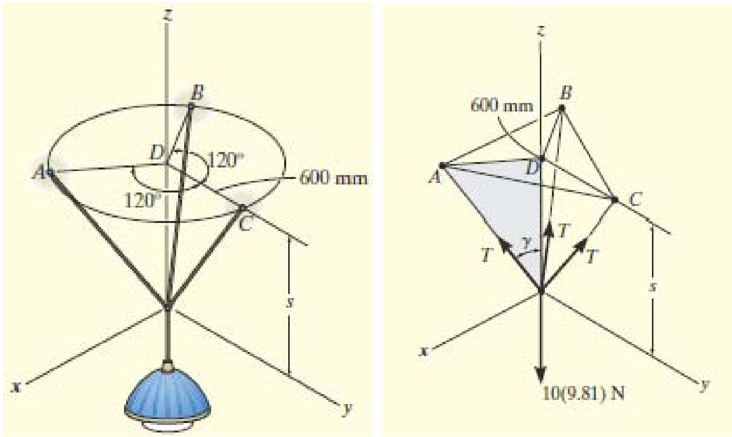
Thus, the angular acceleration of the top is

$$\begin{aligned}\alpha &= \dot{\omega}_s + \dot{\omega}_n + \dot{\omega}_p \\ &= \{-45.3i + 5.20j - 19.0k\} rad/s^2\end{aligned}$$

(b) The 10-kg lamp in Fig. 8a is suspended from the three equal-length cords. Determine its smallest vertical distance s from the ceiling if the force developed in any cord is not allowed to exceed 50 N. [5 Marks]

SOLUTION

The free body diagram will be as in Fig. 8b. Due to symmetry, the distance $DA = DB = DC = 600mm$. It follows that from $\sum F_x = 0$ and $\sum F_y = 0$, the tension T in each cord will be the same. Also, the angle between each cord and the axis is γ.



(a) (b)

Figure 8: Question Five

Applying the equilibrium equation along the z-axis, with $T = 50$ N then,

$$\sum F_z = 0; \qquad 3[50\cos\gamma] - 10 \times 9.81 = 0$$

$$\gamma = \cos^{-1}\frac{98.1}{150} = 49.16^\circ$$

From the shaded triangle shown in Fig. 8b,

$$\tan 49.16^\circ = \frac{600}{s}$$

$$s = 519mm$$