Let $d[i] = gas[i] - cost[i]$

$(j-1 \geq i)$

Claim: If $d[i] + d[i+1] \cdots d[j-1] \geq 0$

and $d[i] + d[i+1] \cdots d[k] \geq 0$ for $k \leq j-1$

and if $d[i] + d[i+1] \cdots d[j-1] + d[j] < 0$

Then starting point cannot be in $[i, j]$

Proof: Consider a starting point $l \in [i, j]$

$$\overbrace{d[i] + d[i-1] \cdots}^{\geq 0} d[l] + \cdots d[j] < 0$$

Because $d[i] + d[i-1] \cdots d[l-1] \geq 0$

$\Rightarrow d[l] + \cdots d[j] < 0$

l cannot be a starting point

because $d[l] + \cdots d[j] < 0$

$\Rightarrow [i, j]$ cannot be a starting point, consider $j+1 \cdots$

After this algorithm,

we are sure we will find a i s.t.

$d[i] + d[i+1] \cdots d[n-1] \geq 0$ and $d[i] \cdots d[j < n] \geq 0$

Q: Does this i exist if $\sum_{i=0}^{n-1} d[i] \geq 0$?

A: If $\boxed{\underbrace{d[0]\ d[1]\ d[2]}_{<0} \cdots \underbrace{\cdots}_{<0} \cdots \underbrace{d[n-1]}_{<0}}$

because we break the loop when we encounter sum < 0

$\Rightarrow$ Then $\sum_{i=0}^{n-1} d[i] < 0$ contradicts $\sum_{i=0}^{n-1} d[i] \geq 0$

The remaing part is to prove

$$\left(d[i] + d[i+1] \cdots d[n-1]\right) + \left(d[0] + \cdots d[k]\right) \geq 0$$

for every $k \leq i-1$

Proof: By the algorithm,

$$\underbrace{d[0], d[1], d[2]}_{<0} \cdots \underbrace{\cdots d[i-1]}_{<0}$$

can be split into multiple parts

s.t. each part < 0

By $d[0]+d[1]+d[2] \cdots d[n-1] \geq 0$

$$\Rightarrow (d[i]+d[i+1] \cdots d[n-1]) + (\underbrace{d[0]+ \cdots}_{<0} \underbrace{d[i-1]}_{<0}) \geq 0$$

$$\Rightarrow (d[i]+d[i+1] \cdots d[n-1]) \geq -(d[0]+ \cdots d[i-1]) \quad \text{—— ①}$$

And because each $\smile$ part's prefix sum will be $\geq \underline{\text{sum } \square}$ (<0) by the algorithm

Thus, $d[0]+d[1]+d[2] \cdots d[k]$ (every prefix sum)

$$\geq \text{sum } \square + \text{sum } \square \cdots\cdots$$

$$= d[0]+d[1] \cdots d[i-1] \quad \text{—— ②}$$

Combine ① and ②

$$\Rightarrow (d[i]+d[i+1]+ \cdots d[n-1]) + (d[0]+d[1] \cdots d[k])$$

$$\geq -(d[0]+ \cdots d[i-1]) + (d[0]+ \cdots d[i-1])$$

$$= 0 \ \#$$

$\Rightarrow i \longrightarrow n-1 \longrightarrow k \leq i-1$

's sum will always ≥ 0 #