E: __/__/__

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DEPT: COMPUTER SCIENCE

SUBJ: MULTIVARIATE CALCULUS

ASSIGNMENT #01"

DATE : ___/___/______

Q1:- Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a) Find the area of the region.

b) Find the volume of the solid formed by revolving this region about x-axis.

(Hint: for area see 6 of Ex. 8.2 and for volume use volume formula:

$$V = \int_{x=a}^{x=b} A(x)\, dx = \int_a^b \pi r^2 dx.$$

a)

$$\int_0^e \ln x\, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

then, $uv - \int v\, du$

$$\Rightarrow \int_0^e \ln x\, dx = x\ln x - \int x \frac{1}{x} dx$$

$$= x\ln x \Big|_0^e - \int 1\, dx \Big|_0^e$$

$$= x\ln x \Big|_0^e - x \Big|_0^e$$

$= ~~e+0~~$

$$= \left| x\ln x - x \right|_0^e$$

$$= (e - e) - (0 - 1)$$

$$= 1$$

DATE : ___/___/______

$$\int_1^e A(x)\,dx$$

$$A(x) = \pi \ln^2 x$$

And so

$$V = \pi \int_1^e \ln^2 x\,dx$$

Integerate by pards

$u = \ln^2 x$, $dv = dx$

$du = \frac{2\ln x}{x}\,dx$, $v = x$

$$= \left[x \ln^2 x\right]_1^e - 2\int_1^e \ln x\,dx$$

We know that $\int_1^e \ln x\,dx = 1$, and so

$$= \left[x \ln^2 x\right]_1^e - 2$$

$$= \left[e \ln^2 e - 0\right] - 2$$

$$= e - 4$$

and therefore

$$V = \pi(e - 2)$$

b)

$$V = 2\pi \int_1^e (x+2)\ln x\,dx$$

$$= 2\pi \int_1^e (x\ln x + 2\ln x)\,dx$$

$$= 2\pi\left[\int_1^e x\ln x\,dx + 2\int_1^e \ln x\,dx\right]$$

DATE : ___/___/______

$$= 2\pi\left[\int_1^e x\ln x\,dx + 2(1)\right]$$

$$= 2\pi\int_1^e x\ln x\,dx + 4\pi$$

we've used the result $\int_1^e \ln x\,dx = 1$

$u = \ln x \qquad v = \frac{1}{2}x^2$

$du = \frac{dx}{x} \qquad dv = x\,dx$

$$= \left[\ln x\left(\frac{1}{2}x^2\right)\right]_1^e - \int_1^e \left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\right)dx$$

$$= \left[\frac{1}{2}x^2\ln x\right]_1^e - \frac{1}{2}\int_1^e x\,dx$$

$$= \left[\frac{1}{2}e^2\ln e - 0\right] - \frac{1}{4}\left[e^2 - 1\right]$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$V = 2\pi\left(\frac{1}{4}e^2 + \frac{1}{4}\right) + 4\pi$$

$$= \frac{1}{2}\pi e^2 + \frac{1}{2}\pi + 4\pi$$

$$= \frac{1}{2}\pi e^2 + \frac{9}{2}\pi$$

$$= \frac{1}{2}\pi(e^2 + 9)$$

$$= \frac{1}{2}\pi(e^2 + 9)$$

DATE: __/__/____

We already have

$$\int_1^e x\ln x \, dx = \frac{1}{4}\left(e^2+1\right)$$

$$\int_1^e \ln^2 x \, dx = e-2$$

$$\Rightarrow \left(\frac{1}{4}(e+1), \frac{1}{2}(e-2)\right)$$

Q2:- Evaluate $\int x^3\sqrt{1-x^2}\, dx$ using

a) Integration by parts
b) a u-substitution
c) a trigonometric substitution.

a) Integration by parts.

$$\int x^3\sqrt{1-x^2}\, dx \text{ us.}$$

$u=\sqrt{1-x^2}$ $\quad dv = x^3 dx$

$du = \frac{-1}{2\sqrt{1-x^2}}\, 2x\, dx$ $\quad v = \frac{x^4}{4}$

$$\Rightarrow \frac{x^4}{4}\sqrt{1-x^2} + \int \frac{x^4}{4}\cdot\frac{1}{2\sqrt{1-x^2}}\, 2x\, dx$$

$$\Rightarrow \frac{x^4}{4}\sqrt{1-x^2} + \frac{1}{4}\int x^4 \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

DATE : ___/___/______

$$\Rightarrow \frac{1}{4}\int \frac{\sin^4\theta}{\sqrt{1-\cos^2\theta}}\, d\theta$$

$$\Rightarrow \frac{1}{4}\int \frac{\sin^4\theta\, d\theta}{\sin\theta}$$

$$\Rightarrow \left[\frac{1}{4}\int \sin^3\theta\, d\theta\right]$$

Now,

$$\Rightarrow \frac{1}{4}\int \sin^2\theta \cdot \sin\theta\, d\theta$$

$$\Rightarrow \frac{1}{4}\int (1-\cos^2\theta)\sin\theta\, d\theta$$

$$u = \cos\theta$$
$$du = -\sin\theta\, d\theta$$

$$\Rightarrow \frac{1}{4}\int (1-u^2)\, du$$

$$\Rightarrow \frac{1}{4}\left[u - \frac{u^3}{3}\right]$$

$$\Rightarrow \frac{x^4}{4}\sqrt{1-x^2} + \frac{1}{4}\frac{\cos^3\theta}{3} - \cos\theta + C$$

DATE : ___/___/______

b) By Substitution:-

$$\int x^3 \sqrt{1-x^2}\, dx$$

$u = 1-x^2$
$du = -2x\, dx$
$x^2 = 1-u$

$$\Rightarrow -\frac{1}{2}\left[\int (1-u)\sqrt{u}\, du\right]$$

$$\Rightarrow -\frac{1}{2}\left[\int u^{1/2} - u^{3/2}\, du\right]$$

$$\Rightarrow -\frac{1}{2}\int u^{1/2} du + \frac{1}{2}\int u^{3/2} du$$

$$\Rightarrow -\frac{1}{2}\frac{u^{3/2}}{3/2} + \frac{1}{2}\frac{u^{5/2}}{5/2}$$

$$\Rightarrow \frac{-u^{3/2}}{3} + \frac{u^{5/2}}{5}$$

$$\Rightarrow -\frac{1}{15}\left(u^{3/2} - u^{5/2}\right) + C \text{ Ans.}$$

c) By Trignometry

$$\int x^3 \sqrt{1-x^2}\, dx$$

$x = a\sin\theta$

Now,

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

DATE : ___/___/______

$\Rightarrow \int \sin^{3}\theta \sqrt{1-\sin^2\theta}^{\,2} \cos\theta \, d\theta$

$\Rightarrow \int \sin^{3}\theta \cdot \cos^{2}\theta \, d\theta$

$\Rightarrow \int \sin^{3}\theta \left(1-\sin^{2}\theta\right) d\theta$

$\Rightarrow \int \sin^{3}\theta \, d\theta - \int \sin^{5}\theta \, d\theta$

then

$\Rightarrow \int \sin^{2}\theta \cdot \sin\theta \, d\theta - \int \sin^{4}\theta \cdot \sin\theta \, d\theta$

$\Rightarrow \int \left(1-\cos^{2}\theta\right) \sin\theta \, d\theta - \int \left(1-\sin^{2}\theta\right)^{2} \sin\theta \, d\theta$

$\Rightarrow u = \cos\theta \qquad , \quad u = \cos\theta$

$du = -\sin\theta \, d\theta \qquad du = -\sin\theta \, d\theta$

$\Rightarrow d\theta = \frac{-du}{\sin\theta} \quad , \quad d\theta = \frac{-du}{\sin\theta}$

$\Rightarrow \int \left(1-u^{2}\right) \cdot \cancel{\sin\theta} \cdot \frac{1}{\cancel{\sin\theta}} \, du + \int \left(1-u^{2}\right)^{2} \cancel{\sin\theta} \frac{1}{\cancel{\sin\theta}} \, du$

$\Rightarrow \int \left(1-u^{2}\right) du + \int \left(1-u^{2}\right)^{2}$

$\Rightarrow -u - u^{3/3} + u - u^{5/5}$

$\Rightarrow -\cos\theta + \frac{\cos^{3}\theta}{3} + \cos\theta - \frac{\cos^{5}\theta}{5} + C$

DATE: __/__/____

Q3:- Perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

$$\int \frac{y^4+y^2-1}{y^3-y}\,dy$$

$$\begin{array}{r} y \\ y^3-y \overline{\big) y^4+y^2-1} \\ \underline{-y^4 \mp y^2} \\ 2y^2-1 \end{array}$$

So, that

$$\int \frac{2y^2-1}{y^3-y}\,dy = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}$$

$$= A(y^2-1) + By(y+1) + Cy(y-1)$$

$$\int \frac{2y^2-1}{y^3-y}\,dy = Ay^2 - A + By^2 + By + Cy^2 - Cy$$

Comparing coefficient of y^2

$A+B+C=2 \rightarrow (1)$

Now, coefficient of y

$B-C=0 \rightarrow (2)$

then,

$-A=-1 \rightarrow (3)$

$A=1$

put $A=1$ in eq (1)

DATE : ___/___/_____

So, $B = \frac{1}{2}$

then, put $C - B$ in eq → (1)

$C = \frac{1}{2}$

Now,

$$\int \frac{2y^2 - 1}{y^3 - y}\, dy = \frac{1}{y} + \frac{1/2}{y-1} + \frac{1/2}{y+1}$$

$$\Rightarrow \ln|y^3 - y| = \int \frac{1}{y}\, dy + \frac{1}{2}\int \frac{1}{y-1}\, dy + \frac{1}{2}\int \frac{1}{y+1}\, dy$$

$$\Rightarrow \ln|y| + \frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y+1| + C$$

"The End"