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Wave dispersion in gradient elastic solids and structures: A unified treatment

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ABSTRACT

Analytical wave propagation studies in gradient elastic solids and structures are presented. These solids and structures involve an infinite space, a simple axial bar, a Bernoulli–Euler flexural beam and a Kirchhoff flexural plate. In all cases wave dispersion is observed as a result of introducing microstructural effects into the classical elastic material behavior through a simple gradient elasticity theory involving both micro-elastic and micro-inertia characteristics. It is observed that the micro-elastic characteristics are not enough for resulting in realistic dispersion curves and that the micro-inertia characteristics are needed in addition for that purpose for all the cases of solids and structures considered here. It is further observed that there exist similarities between the shear and rotary inertia corrections in the governing equations of motion for bars, beams and plates and the additions of micro-elastic (gradient elastic) and micro-inertia terms in the classical elastic material behavior in order to have wave dispersion in the above structures.

1. Introduction

It has been observed both experimentally and analytically that the mechanical behavior of microstructured linear elastic solids and linear elastic structures with dimensions comparable to the length scale of their material microstructure cannot be adequately described by the classical theory of elasticity and resort should be made to higher order linear elastic theories. These theories, such as strain gradient (Mindlin, 1964) micropolar (Eringen, 1966) or couple stress (Koiter, 1964) elasticity theories, are capable of taking into account macroscopically microstructural effects, such as size effects, elimination of singularities and wave dispersion. Among the review type of articles on these higher order elasticity theories and their applications one can mention the papers of Tiersten and Bleustein (1974), Lakes (1995) and Exadaktylos and Vardoulakis (2001).

In this paper one of the most important microstructural effects, that of wave dispersion in gradient elastic solids and structures is studied analytically. More specifically, the simplest possible version of Mindlin's (Mindlin, 1964) gradient elastic theory of form II with just one elastic constant, the gradient coefficient g^2 with dimensions of (length)^2, in addition to the two classical elastic Lamé constants λ and μ with dimensions of (force) (length)$^{-2}$, also known as dipolar gradient elastic theory, is employed here and wave propagation in an infinitely extended solid, an axial bar, a flexural beam and a flexural plate is studied analytically. In all cases wave dispersion is observed, in contrast to the corresponding classical elastic cases where such dispersion is absent or not realistic.

Wave dispersion in gradient elastic solids and structures has been extensively studied in the past. One can mention here Altan et al. (1996), Tsepoura et al. (2002) and Papargyri-Beskou et al. (2003) who studied wave dispersion in axial bars and flexural beams and observed dispersion curves physically unacceptable as a result of using $-g^2$ in the constitutive equation of the gradient elastic solid. Incidentally, Papargyri-Beskou et al. (2003) presented the correct dispersion curve due to a mistake in the dispersion equation (use of $+g^2$ instead of $-g^2$). Altan and Aifantis (1997) in their one-dimensional wave dispersion studies considered both the $+g^2$ and $-g^2$ cases and observed that the $+g^2$ case provides a dispersion curve in qualitative agreement with the physically acceptable atomic-lattice theory.

Chang and Gao (1995, 1997), Chang et al. (1998) and Suiker et al. (1999, 2001a,b) developed an enhanced gradient elastic material model with more than one non-classical material constants and used it to derive dispersion curves for axial bars, the infinite space and the half space, which were physically acceptable under certain conditions. This was possible because their model was in essence like the one with just one non-classical elastic constant with positive sign ($+g^2$). Askes et al. (2002) also developed an

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enhanced one-dimensional gradient elastic model with two positive non-classical elastic constants and second and fourth derivatives of strain which produced acceptable dispersion curves (for small wavenumbers), while retaining the desirable stability and uniqueness properties.

Vardoulakis and Georgiadis (1997), Georgiadis et al. (2000), Georgiadis and Velgaki (2003) and Georgiadis et al. (2004) in their study of surface waves in a gradient elastic half-space were able to (a) observe that, while stability and uniqueness are satisfied for $-g^2$ in the gradient elastic model, acceptable (i.e., consistent with atomic-lattice theory) dispersion curves are obtained only for $+g^2$; and (b) propose the inclusion of micro-inertia (through the coefficient h with length dimensions) in the model with $-g^2$, which results in acceptable dispersion curves for high frequencies. These findings were restated by Askes and Aifantis (2006) giving results for the one-dimensional case and by Yerofeyev and Sheshenina (2005) giving results for body and surface waves. To be sure, inclusion of micro-inertia in the gradient elastic model with $-g^2$ has been first proposed by Mindlin (1964) and in homogenized composite materials by Wang and Sun (2002). At this point one should also mention the one-dimensional gradient elastic models of Metrikine and Askes (2002) and Chen and Fish (2001) and Fish et al. (2002) constructed on the basis of a discrete microstructure and homogenization of a periodical heterogeneous medium, respectively, which can lead to acceptable dispersion curves. By adding a fourth order derivative with respect to time in the Metrikine and Askes (2002) model, Metrikine (2006) was able to arrive at a causal model.

However, it has been observed experimentally by Lakes (1982, 1995) and Chen and Lakes (1989) in connection with torsional wave propagation in bones and cellular polymeric materials, by Stavropoulou et al. (2003) in connection with surface wave propagation in marble and Aggelis et al. (2004) in connection with body wave propagation in concrete that, unlike the prediction of crystal lattice models, dispersion curves exhibit increase of phase velocity with increasing frequency or wavenumber (at least for low values of them). Use of gradient elastic models with micro-inertia succeeded in verifying these results analytically-numerically (Stavropoulou et al., 2003; Aggelis et al., 2004). Very recently, Thomas et al. (2009) through analytical (gradient elastic and viscoelastic models) and numerical (distinct element method) experiments on body wave propagation in dry sand also observed dispersion exhibiting phase velocity increase with increasing frequency or wavenumber.

On the other hand, experiments by Kondratev (1990) and Savin et al. (1970) have shown that most metals and alloys exhibit lattice type of dispersion showing decrease of phase velocity with frequency or wave number, while experiments by Erofeyev and Rodyushkin (1992) have shown that reinforced and granular composite materials exhibit the opposite type of dispersion. A brief account of these results can also be found in the book of Erofeyev (2003).

In the present study, dispersion is studied on the basis of the $-g^2$ gradient model with micro-inertia and involves the infinite space, the axial bar, the flexural beam and the flexural plate. It is proved that this simple model is flexible enough to successfully simulate all possible physically acceptable cases of dispersion (in granular and polycrystalline solids) by appropriately choosing its two length scale parameters g and h. Thus, a unification of the available and sometimes conflicting information in the literature becomes possible. Some progress towards this unification has been recently presented by Askes and Aifantis (2006). However, the present paper goes further by also considering the relevance of cases with $h < g$ and is wider in scope as it deals not only with the one-dimensional case but also with the infinite space and flexural beams and plates. Furthermore, it is observed in the present paper that there exist similarities between the shear and lateral or rotary inertia corrections in the governing equations of motion for axial bars, flexural beams and plates and the additions of micro-elastic (gradient elastic) and micro-inertia terms in the classical elastic material behavior in order to have wave dispersion in those structures. Metrikine (2006) also observed an analogy between his gradient elastic model and the tensioned Timoshenko beam model. However, his gradient elastic model is one-dimensional and not a gradient elastic beam model as in the present paper. Furthermore, he only observes the analogy, while the present work also demonstrates that by suitable values of g and h the dispersion curves of Timoshenko's beam model can be satisfactorily approximated. In addition, this analogy is not restricted here to beams but covers bars and plates as well.

2. The form II gradient elastic theory of Mindlin and its simplified versions

Mindlin in the form II version of his gradient elastic theory (Mindlin, 1964) considered that the potential energy density $\hat{W}$ is a quadratic form of the strains ε_{ij} and the gradient of strains, $\hat{\kappa}_{ijk}$, i.e.,

$$\hat{W} = \frac{1}{2}\tilde{\lambda}\varepsilon_{ii}\varepsilon_{jj} + \tilde{\mu}\varepsilon_{ij}\varepsilon_{ij} + \hat{\alpha}_1\hat{\kappa}_{iik}\hat{\kappa}_{kjj} + \hat{\alpha}_2\hat{\kappa}_{ijj}\hat{\kappa}_{ikk} + \hat{\alpha}_3\hat{\kappa}_{iik}\hat{\kappa}_{jjk} + \hat{\alpha}_4\hat{\kappa}_{ijk}\hat{\kappa}_{ijk} + \hat{\alpha}_5\hat{\kappa}_{ijk}\hat{\kappa}_{kji} \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad \hat{\kappa}_{ijk} = \partial_i\varepsilon_{jk} = \frac{1}{2}(\partial_i\partial_j u_k + \partial_i\partial_k u_j) = \hat{\kappa}_{ikj} \tag{2}$$

with ∂_i denoting space differentiation, u_i being displacements and $\tilde{\lambda}$, $\tilde{\mu}$ and $\hat{\alpha}_1$–$\hat{\alpha}_5$ being constants explicitly defined in Mindlin (1964). It should be noticed that the constants $\tilde{\lambda}$, $\tilde{\mu}$ are not the same with the corresponding Lamé constants λ,μ of classical elasticity. Thus, this particular case of Mindlin's theory has in total just 7 elastic constants instead of the 18 ones of his general theory.

Strains ε_{ij} and gradient of strains $\hat{\kappa}_{ijk}$ are dual in energy with the Cauchy and double stresses, respectively, defined as

$$\hat{\tau}_{ij} = \frac{\partial\hat{W}}{\partial\varepsilon_{ij}} = \hat{\tau}_{ji} \tag{3}$$

$$\hat{\mu}_{ijk} = \frac{\partial\hat{W}}{\partial\hat{\kappa}_{ijk}} = \hat{\mu}_{ikj} \tag{4}$$

which implies that

$$\hat{\tau}_{pq} = 2\tilde{\mu}\varepsilon_{pq} + \tilde{\lambda}\varepsilon_{ii}\delta_{pq} \tag{5}$$

and

$$\hat{\mu}_{pqr} = \frac{1}{2}\hat{\alpha}_1[\hat{\kappa}_{rii}\delta_{pq} + 2\hat{\kappa}_{iip}\delta_{qr} + \hat{\kappa}_{qii}\delta_{rp}] + 2\hat{\alpha}_2\hat{\kappa}_{pii}\delta_{qr} + \hat{\alpha}_3(\hat{\kappa}_{iir}\delta_{pq} + \hat{\kappa}_{iiq}\delta_{pr}) + 2\hat{\alpha}_4\hat{\kappa}_{pqr} + \hat{\alpha}_5(\hat{\kappa}_{rpq} + \hat{\kappa}_{qrp}) \tag{6}$$

The total stress tensor $\hat{\sigma}_{pq}$ is then defined as

$$\hat{\sigma}_{pq} = \hat{\tau}_{pq} - \partial_r\mu_{rpq} \tag{7}$$

At this point, one could mention that according to Polizzoto (2003), stress $\hat{\tau}$ in Eq. (3) is called Cauchy-like (not Cauchy) stress and that $\hat{\sigma}$ in Eq. (7) is the Cauchy stress.

Extending the idea of non-locality to the inertia of the continuum with microstructure, Mindlin (1964) proposed for the isotropic case a new expression for the kinetic energy density function T, which includes the gradients of the velocities, i.e.,

$$T = \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \frac{1}{6}\rho d^2\partial_i\dot{u}_j\partial_i\dot{u}_j \tag{8}$$

where ρ is the mass density, over dots indicate differentiation with respect to time t and d^2 is another material constant with units of m^2 called velocity gradient coefficient. Taking the variation of strain and kinetic energy, according to the Hamilton's principle, one can obtain the equation of motion of a continuum with microstructure, which in terms of the displacement vector $\mathbf{u}$ is written as

$$\left(\tilde{\lambda}+2\tilde{\mu}\right)\left(1-\hat{l}_1^2\nabla^2\right)\nabla\nabla\cdot\mathbf{u}-\tilde{\mu}(1-\hat{l}_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u}$$
$$=\rho\left(\ddot{\mathbf{u}}-h_1^2\nabla\nabla\cdot\ddot{\mathbf{u}}+h_2^2\nabla\times\nabla\times\mathbf{u}\right) \quad (9)$$

where

$$\hat{l}_1^2=2(\hat{a}_1+\hat{a}_2+\hat{a}_3+\hat{a}_4+\hat{a}_5)\Big/\left(\tilde{\lambda}+2\tilde{\mu}\right)$$
$$\hat{l}_2^2=(\hat{a}_3+2\hat{a}_4+\hat{a}_5)/2\tilde{\mu} \quad (10)$$

$$h_1^2=d^2\left[2\alpha^2+(\alpha+\beta)^2\right]\Big/3$$
$$h_2^2=d^2(1+\beta^2)/6 \quad (11)$$

with α, β two constants defined in Mindlin (1964).

Positive definiteness of $\hat{W}$ (for reasons of uniqueness and stability) requires that (Mindlin, 1964) $\tilde{\mu}>0$, $\tilde{\lambda}+2\tilde{\mu}>0$, $l_i^2>0$ and $h_i^2>0$. In the simplest possible case where the potential energy density $\hat{W}$ is defined as

$$\hat{W}=\varepsilon_{ij}\tau_{ij}+g^2\partial_i\varepsilon_{jk}\partial_i\tau_{jk} \quad (12)$$

the constants $\hat{\alpha}_1$–$\hat{\alpha}_5$ become $\hat{\alpha}_1=\hat{\alpha}_3=\hat{\alpha}_5=0$, $\hat{\alpha}_2=\lambda g^2/2$, $\hat{\alpha}_4=\mu g^2$ and the constants l_1^2, l_2^2 in (10), $l_1^2\equiv l_2^2=g^2$. Also from the definition of $\tilde{\lambda}$, $\tilde{\mu}$ in Mindlin (1964), it is easy to see one that $\tilde{\lambda}$, $\tilde{\mu}$ become identical to the classical Lamé constants λ, μ when $\alpha = 0$, $\beta = 1$. In this case, as it is apparent from (11), $h_1^2\equiv h_2^2=h^2$. Thus, the above restrictions on the material constants become

$$\mu>0,\quad \lambda+2\mu>0,\quad g^2>0,\quad h^2>0 \quad (13)$$

Under the above simplifications, the equation of motion (9) through the well-known identity $\nabla^2\mathbf{u} = \nabla\ \nabla\cdot\mathbf{u} - \nabla\times\nabla\times\mathbf{u}$ obtains the simple form

$$(1-g^2\nabla^2)\left[\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}\right]=\rho(\ddot{\mathbf{u}}-h^2\nabla^2\ddot{\mathbf{u}}) \quad (14)$$

where $g^2\nabla^2[\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{u}]$ and $\rho h^2\nabla^2\ddot{\mathbf{u}}$ are the microstructural and the micro-inertia terms, respectively, and the operator ∇^2 is the Laplacian. The micro-inertia term is, in general, important when the rate of loading and hence the rate of strain is high. Likewise, under the above simplifications, the stresses (5)–(7) become

$$\tau_{ij}=2\mu\varepsilon_{ij}+\lambda\varepsilon_{ii}\delta_{ij} \quad (15)$$
$$\mu_{ijk}=g^2\partial_i\tau_{jk} \quad (16)$$
$$\sigma_{ij}=\tau_{ij}-g^2\nabla^2\tau_{ij} \quad (17)$$

Eq. (17) is the same with the one proposed by Aifantis (1992), Altan and Aifantis (1992) and Ru and Aifantis (1993) for gradient elastostatics.

It should be emphasized that the above necessary conditions (13) refer to the gradient elastic model with a minus sign in front of the term $g^2\nabla^2\tilde{\tau}$ exactly as in Eq. (17) of the present model. Other models, such as those of Chang and Gao (1997) and Chang et al. (1998) with a plus sign in front of the term $g^2\nabla^2\tilde{\tau}$, do not satisfy uniqueness and stability when $h = 0$ but they result in dispersion relations in agreement with crystal lattice models, which is not the case with the present model when $h = 0$. More on these aspects will be discussed in subsequent sections of this paper.

3. Wave dispersion in the infinite space

Consider the infinite three-dimensional gradient elastic space with its equation of motion described by Eq. (14). The Helmholtz vector decomposition implies that the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ with $\mathbf{x}$ being the position vector, can be written as a sum of irrotational and solenoidal fields according to the relation

$$u=\nabla\phi+\nabla\times A \quad (18)$$

where the vectors $\nabla\phi$ and $\nabla\times A$ with ϕ and $\mathbf{A}$ being a scalar and a vector, respectively, and $(\times)$ the symbol of the cross-product, denote volumetric and shape with constant volume changes, respectively. In terms of wave propagation this means that $\nabla\phi$ corresponds to longitudinal waves, while $\nabla\times A$ represents shear waves propagating through the medium, i.e.,

$$\nabla\varphi=\hat{\mathbf{k}}e^{i\left(k_p\hat{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)}$$
$$\nabla\times\mathbf{A}=\hat{\mathbf{b}}e^{i\left(k_s\hat{\mathbf{k}}\cdot\mathbf{r}-\omega t\right)} \quad (19)$$

where $\hat{k}$ represents the direction of incidence, $\hat{b}$ is the polarization vector for the shear wave, $\mathbf{r}$ stands for the position vector, k_p and k_s are the wave numbers of the longitudinal and shear disturbances, respectively, ω is the frequency of the propagating waves and $i=\sqrt{-1}$. Representing by C_p, C_s the classical phase velocities of longitudinal (P) and shear (S) waves, respectively, and inserting Eqs. (18) and (19) into Eq. (14) one obtains the relation

$$\omega^2=C_p^2\frac{k_p^2(1+g^2k_p^2)}{1+h^2k_p^2},\quad C_p^2=\frac{\lambda+2\mu}{\rho} \quad (20)$$

for longitudinal waves and the relation

$$\omega^2=C_s^2\frac{k_s^2(1+g^2k_s^2)}{1+h^2k_s^2},\quad C_s^2=\frac{\mu}{\rho} \quad (21)$$

for shear waves. These relations can also be obtained from those of Mindlin (1964) for the much more general gradient elastic model of Eq. (9).

Thus, using the above relations (20) and (21), one can obtain expressions for the phase velocities V_p and V_s of the longitudinal and shear waves, respectively, of the form

$$V_{p,s}=\frac{\omega}{k_{p,s}}=C_{p,s}\sqrt{\frac{1+g^2k_{p,s}^2}{1+h^2k_{p,s}^2}} \quad (22)$$

Eq. (22) reveal that, unlike the classical elastic case characterized by constant velocities of longitudinal and shear waves and hence non-dispersive wave propagation, the gradient elastic is characterized by phase velocities for longitudinal and shear waves, which are functions of the wave number, indicating wave dispersion. This dispersion is entirely due to the presence of the two microstructural material constants g^2 and h^2. It is easy to see that by letting $g = h = 0$ or just $g = h$ in (22) one finds that $V_{p,s} = C_{p,s}$, i.e., he recovers the classical elastic case with constant wave speeds and hence no dispersion.

For the case without the micro-inertia term ($h = 0$), the dispersion relation (22) becomes

$$V_{p,s}=C_{p,s}\sqrt{1+g^2k_{p,s}^2} \quad (23)$$

which is physically unacceptable as predicting unbounded values of phase velocities for very large values (going to infinity) of the wave numbers. On the other hand, for the case without the gradient elastic term ($g = 0$), the dispersion relation (22) becomes

$$V_{p,s}=C_{p,s}\sqrt{\frac{1}{1+h^2k_{p,s}^2}} \quad (24)$$

which is physically acceptable as being bounded for large values of $k_{p,s}$. The presence of the two length scale parameters in (22) allows one to investigate for which relation between g and h, the disper-

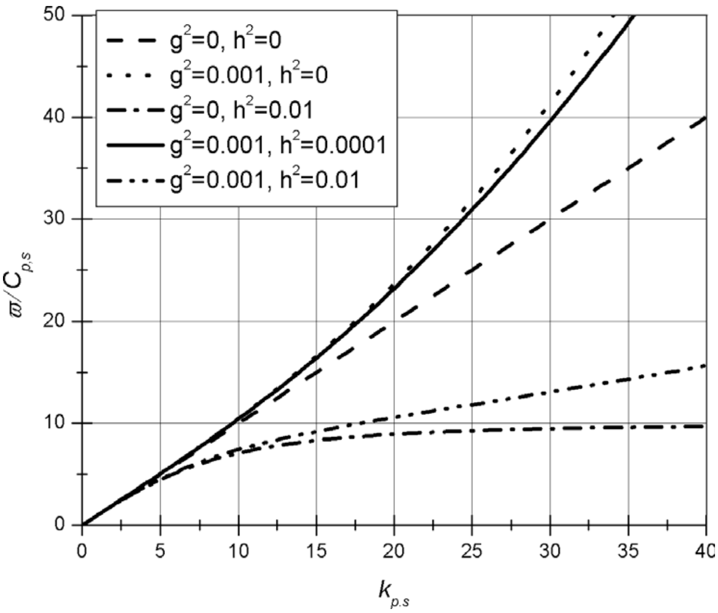

Fig. 1. Dispersion curves of the $\omega/C_{p,\,s}$ versus $k_{p,s}$ type for elastic medium with microstructure obeying equation of motion (14).

sion relation (22) will be physically acceptable. For example, for $V_{p,\,s} \leqslant C_{p,\,s}$, Eq. (22) implies that $g \leqslant h$, while for $V_{p,\,s} \geqslant C_{p,s}$ that $g \geqslant h$. One should also mention that Eq. (22) are also valid for the two-dimensional case, provided that the shear waves (S) are replaced by the vertically polarized shear waves (SV).

Figs. 1 and 2 provide the dispersion curves for longitudinal (P) and shear (S) waves propagating in an infinitely extended gradient elastic medium for various combinations of g and h as they correspond to the $\omega/C_{p,s}$ versus $k_{p,s}$ and the $V_{p,s}/C_{p,s}$ versus $k_{p,s}$ relations, respectively, obtained from Eq. (22).

On the basis of the results of Figs. 1 and 2, one can conclude that:

(i) For $h = g$ or $h = g = 0$ there is no dispersion and $V_{p,s} = C_{p,s}$.
(ii) For $h > g$ there is dispersion, $V_{p,\,s} \leqslant C_{p,s}$ and $V_{p,s}$ decreases with increasing $k_{p,s}$. This is a physically acceptable case in agreement with results of crystal lattice theories for the two-dimensional space (Yim and Sohn, 2000; Suiker et al., 2001a,b) and the two-dimensional half-space (Gazis et al., 1960). The relation $h > g$ was first found to lead to results in agreement with lattice theories during the numerical studies of Georgiadis et al. (2004) for wave dispersion in the half-plane. This case is also in agreement with experimental results on metals and alloys (Kondratev, 1990; Savin et al., 1970; Erofeyev, 2003), something to be expected as these materials have a polycrystalline type of structure which can be ideally modeled by crystal lattice theories.
(iii) For $g = 0$ one observes the same type of behavior as in the previous case ($h > g$) with $V_{p,s}$ decreasing now faster with increasing $k_{p,s}$. However, $V_{p,s}$ remains bounded for large $k_{p,s}$ and hence this case is physically acceptable.
(iv) For $h < g$ there is dispersion, $V_{p,\,s} \geqslant C_{p,s}$ and $V_{p,s}$ increases with increasing $k_{p,s}$ in agreement with experimental results on granular type of materials, such as marble, sand, concrete,

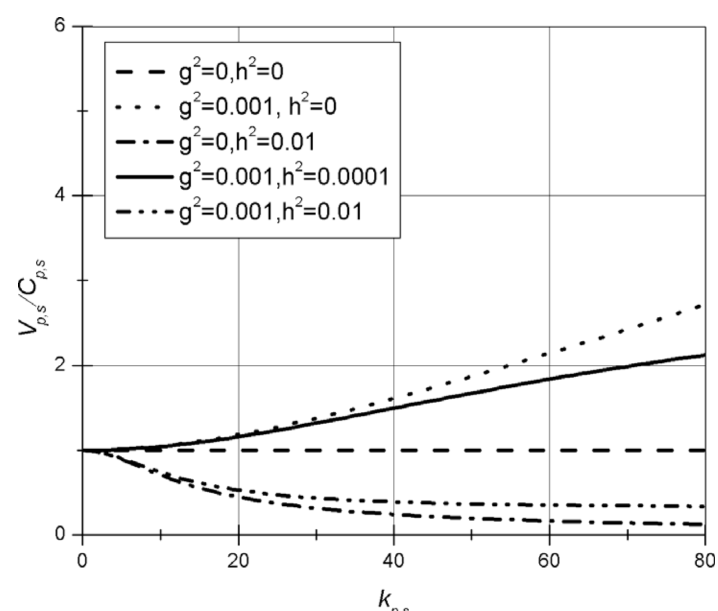

Fig. 2. Dispersion curves of the $V_{p,s}/C_{p,s}$ versus $k_{p,\,s}$ type for elastic medium with microstructure obeying equation of motion (14).

granular composites, bones and cellular materials (Lakes, 1982; Chen and Lakes, 1989; Stavropoulou et al., 2003; Aggelis et al., 2004; Erofeyev and Rodyushkin, 1992; Erofeyev, 2003) as well as with numerical experiments (distinct element method) modeling sand (Thomas et al., 2009).

(v) For $h = 0$ one observes the same type of behavior as in the previous case ($h < g$) with $V_{p,s}$ increasing now faster with increasing $k_{p,s}$ and going to infinity for very large $k_{p,s}$. Thus, this case is not acceptable.

4. Wave dispersion in an axial bar

Consider the one-dimensional wave propagation in an axial bar of gradient elastic material. For this case, one can easily obtain from Eqs. (14), (15) and (17) the equations

$$Eu'' - g^2 Eu^{IV} = \rho \ddot{u} - \rho h^2 \ddot{u}'' \tag{25}$$

$$\sigma = E\varepsilon - g^2 E\varepsilon'' \tag{26}$$

where primes indicate differentiation with respect to the x-coordinate along the bar axis and $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is the classical elastic modulus.

Assuming an axial wave propagation of the form

$$u = u(x,t) = Ue^{i(kx - \omega t)} \tag{27}$$

where ω is the circular frequency, k the wave number and U the wave amplitude, one can obtain from (25) the dispersion relation

$$V_{gh} = \frac{\omega}{k} = \sqrt{\frac{E}{\rho}}\sqrt{\frac{1+g^2k^2}{1+h^2k^2}} = V_c\sqrt{\frac{1+g^2k^2}{1+h^2k^2}} \tag{28}$$

between the phase velocity V_{gh} and the wavenumber k. It is observed that for $k \to \infty, V_{gh}/V_c = g/h$. For the case of $g = h = 0$ or just $g = h$ one has from (28) that $V_{gh} \equiv V_c$, the classical case with no dispersion, while for case of $g = 0$ and $h \neq 0$ dispersion relation (28) is essentially the same with that of Chen and Fish (2001) and Fish et al. (2002). Furthermore, dispersion relation (28) with $g \neq 0$ and $h = 0$, which is the same as the one presented in Altan et al. (1996) and Tsepoura et al. (2002), is not physically acceptable as predicting unbounded velocities for $k \to \infty$. In order to rectify this, Chang and Gao (1997), Chang et al. (1998) and Suiker et al. (1999) assumed a plus sign in front of $g^2 E\varepsilon''$ of Eq. (26) which led to the dispersion relation

$$V'_g = V_c\sqrt{1 - g^2k^2} \tag{29}$$

This relation is physically acceptable only when $gk < 1$. However, the employed material model violates the necessary condition for uniqueness and stability.

The presence of the term $1 + h^2k^2$ due to micro-inertia in Eq. (28) allows one to investigate for which relation between g and h, the dispersion relation (28) will be physically acceptable in the sense of being in agreement with crystal lattice theories (Brillouin, 1953), i.e., $V_{gh} \leqslant V_c$. This implies $g \leqslant h$, which is the same relation derived for the general case of body waves propagating in an infinite gradient elastic solid of the previous section. The dispersion relation (28) is physically acceptable even when $g = 0$ and $h \neq 0$ indicating the importance of micro-inertia in the dispersional character of the wave propagation. One should note here that the gradient elastic model of Metrikine and Askes (2002) constructed on the basis of its discrete microstructure is mathematically the same with the one described by (25) and leads to essentially the same results.

At this point it is interesting to discuss the similarity existing between the above defined gradient elastic beam velocity and the one provided by the classical equation of motion enriched with lateral inertia effects, known as Love's equation (Graff, 1975), i.e.,

$$Eu'' = \rho \ddot{u} - \rho \nu^2 \gamma^2 \ddot{u}'' \tag{30}$$

where ν is Poisson's ratio and γ is the polar radius of gyration of the cross-section of the bar defined as $\gamma^2 = J/A$, with J being the polar moment of inertia of the cross-section A. Considering axial wave propagation of the form of Eq. (27), one can easily derive the dispersion relation pertaining to Eq. (30) as

$$V_L = V_c\frac{1}{\sqrt{1 + \nu^2\gamma^2k^2}} \tag{31}$$

which, is physically acceptable as predicting bounded velocity for very large wavenumbers.

A comparison of Eq. (28) with $g = 0$ and Eq. (31) indicates that one can go from the first to the second by simply replacing h by $\nu\gamma$. Fig. 3 depicts the dispersion curves described by Eq. (28) for ax-

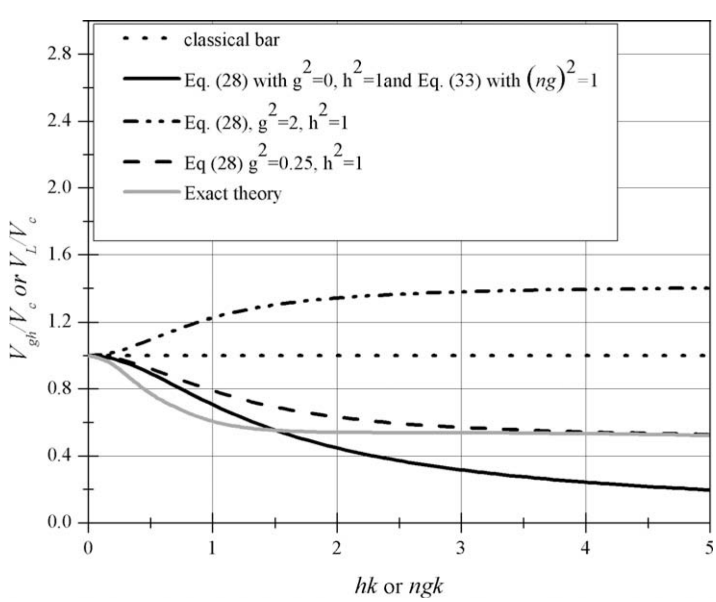

Fig. 3. Dispersion curves for wave propagation in an axial bar.

ial wave propagation in a gradient elastic bar as well as those of the classical one-dimensional bar, the classical three-dimensional (exact) elastic case taken from Graff (1975) and the classical bar with lateral inertia effects.

One can observe pictorially all the kinds of behavior previously described in this section and also the fact that the classical exact case exhibits dispersion only for small values of wavenumber k, while the close to it gradient elastic case with micro-inertia (Eq. (28)) for a much larger range of values of k. An interesting result is that for $g^2 = 0.25$, $h^2 = 1$ and $hk > 3$. Eq. (28) agrees with the three-dimensional exact solution of a bar, which means that the one-dimensional version of the gradient elastic theory with both micro-inertia and micro-structural effects is able to capture wave propagation behavior of a three-dimensional classical elasticity model.

5. Wave dispersion in flexural plates and beams

Consider the flexural free motion of a gradient elastic Kirchhoff plate (Papargyri-Beskou and Beskos, 2008), which with the addition of the micro-inertia term $\rho s h^2 \nabla^2 \ddot{w}$ takes the form

$$D\nabla^4 w - g^2 D\nabla^6 w + \rho s \ddot{w} - \rho s h^2 \nabla^2 \ddot{w} = 0 \tag{32}$$

where $D = Es^3/12(1-\nu^2)$ is the flexural rigidity of the plate with s being its thickness and ν the Poisson's ratio of its material, ρ is the mass density of the material, $w = w(x, y, t)$ is the lateral deflection of the plate with the axes x and y defining its middle surface plane and $\nabla^4 w = \nabla^2(\nabla^2 w)$, $\nabla^6 w = \nabla^2(\nabla^4 w)$. The total stresses σ_{ij} of the plate for a plane stress state are given by Eqs. (17) and (15), i.e., with a minus sign in front of g^2.

In order to investigate under which conditions harmonic plane waves may propagate in an infinite gradient elastic plate, a plane wave of the form

$$w = We^{i(k\mathbf{n}\cdot\mathbf{r} - \omega t)} \tag{33}$$

is considered, where W is the wave amplitude, k the wave number, ω the circular frequency, $\mathbf{r}$ the position vector to a point on the plane wave front and $\mathbf{n}$ the normal to that plane.

Substituting expression (33) for w in the equation of motion (32) one can obtain the frequency-wavenumber relation as

$$k^4 + g^2 k^6 - (\omega^2/\alpha^2)\left(1 + h^2 k^2\right) = 0 \tag{34}$$

where

$$\alpha^2 = \frac{D}{\rho s} = \frac{s^2}{6(1-\nu)}\left(\frac{\mu}{\rho}\right) = q^2 C_S^2 \tag{35}$$

Introducing the dimensionless phase velocity $\overline{V}_{gh} = V_{gh}/C_S$ with $V_{gh} = \omega/k$, the dimensionless wavenumber $\bar{k} = qk$ and the dimensionless microstructural parameters $\bar{g} = g/q$ and $\bar{h} = h/q$, the dispersion relation (34) takes the form

$$\overline{V}_{gh} = \bar{k}\sqrt{\frac{1 + \bar{g}^2\bar{k}^2}{1 + \bar{h}^2\bar{k}^2}} \tag{36}$$

For the classical case ($g = h = 0$) one has from (36) that $\overline{V}_{gh} = \bar{k}$ predicting unbounded wave velocity for very-high wavenumbers or frequencies. This is physically unacceptable. For the case without the micro-inertia term ($h = 0$), Eq. (36) yields $\overline{V}_g = \bar{k}\sqrt{1 + \bar{g}^2\bar{k}^2}$ which, for the same reason as that mentioned for the classical case, is physically unacceptable.

As in the previous cases, the presence of the term $1 + \bar{h}^2\bar{k}^2$ in Eq. (36) due to micro-inertia allows one to find out the relation between g and h for the dispersion relation (36) to be physically acceptable, i.e., in agreement with higher order plate theories, which have been verified experimentally, such as that of Mindlin (Graff, 1975). This is possible due to the existence of a similarity between the shear and rotary inertia corrections and the micro-elastic (or gradient elastic) and micro-inertia ones, respectively, in the Kirchhoff theory of elastic plates as related to wave dispersion.

Inclusion of shear deformation and rotary inertia in the Kirchhoff flexural plate motion model leads to the Mindlin corresponding model described by equation (Graff, 1975)

$$D\nabla^4 w - \left(\frac{\rho s^3}{12} + \frac{\rho D}{\mu K^2}\right)\nabla^2\ddot{w} + \frac{\rho^2 s^3}{12\mu K^2}\ddot{\ddot{w}} + \rho s\ddot{w} = 0 \tag{37}$$

where K is the shear correction factor for plates. The resulting dispersion relation for Eq. (37) taken from Graff (1975) can be solved for the dimensionless phase velocity $\overline{V}_M = V_M/C_S$ reading

$$\overline{V}_M = \sqrt{\frac{\left(\frac{1-\nu}{2} + \frac{1}{K^2}\right)\bar{k}^2 + 1 \pm \sqrt{\left[\left(\frac{1-\nu}{2} + \frac{1}{K^2}\right)\bar{k}^2 + 1\right]^2 - 2(1-\nu)\frac{1}{K^2}\bar{k}^4}}{(1-\nu)\frac{1}{K^2}\bar{k}^2}} \tag{38}$$

where $\bar{k} = qk$ with $q^2 = s^2/6(1-\nu)$.

Following Graff (1975) the following interesting remarks for relation (38) can be made:

(i) Considering the ($-$) sign in Eq. (38) one has $\lim_{\bar{k}\to\infty}\overline{V}_M = K$ or $\lim_{\bar{k}\to\infty}V_M = KC_S$. However, according to the exact plate theory, for large $\bar{k}$, velocity V_M should approach the Rayleigh velocity C_R and this leads to $K = C_R/C_S$. It is apparent that for $K = 1$ and large $\bar{k}$, velocity V_M becomes identical to the shear velocity $C_S = \sqrt{\frac{\mu}{\rho}}$.

(ii) For large $\bar{k}$, the (+) sign in Eq. (38) leads to $\lim_{\bar{k}\to\infty}\overline{V}_M = \sqrt{\frac{2}{1-\nu}}$ or $\lim_{\bar{k}\to\infty}V_M = C_S\sqrt{\frac{2}{1-\nu}}$. This value of V_M is the same with the one predicted for large $\bar{k}$ by the classical plate theory enhanced with only rotary inertia correction terms.

Fig. 4 portrays the dispersion curves described by Eq. (36) for flexural wave propagation in a gradient elastic plate, by Eq. (38) for the Mindlin plate model as well as the classical Kirchhoff plate model all of them with a Poisson's ratio $\nu = 0.2$ implying a shear correction coefficient $K = 0.913$. The three-dimensional classical elastic (exact) model is very close to that of Mindlin (Graff, 1975). One can observe from Fig. 4 that the gradient elastic model with $g = 0$ and $\bar{h} = 1/K$ or $h = q/K$ is capable of simulating almost exactly the behavior of the Mindlin plate model. For the particular case of $\bar{g} = 0$, $\bar{h} = \sqrt{(1-\nu)/2}$ the dispersion curve of the gradient model becomes identical to that of the Rayleigh type of Mindlin's plate model ($K \to \infty$).

Consider the flexural free motion of a gradient elastic Bernoulli–Euler beam (Papargyri-Beskou et al., 2003), which with the addition of the micro-inertia term $mh^2\ddot{w}''$ takes the form

$$EIw^{IV} - g^2 EIw^{VI} + m\ddot{w} - mh^2\ddot{w}'' = 0 \tag{39}$$

where EI is the flexural rigidity of the beam with I being the cross-sectional moment of inertia, $m = \rho A$ is the mass per unit length of the beam with A being its cross-sectional area and $w = w(x, t)$ is the lateral deflection of the beam with x being the coordinate along the beam axis.

Observing that one can go from gradient elastic flexural plates (Eq. (32)) to gradient elastic flexural beams (Eq. (39)) by replacing ∇^2 by $\frac{\partial^2}{\partial x^2}$, ρs by ρA and D by EI, it is easily seen that the dispersion relation (36) for gradient elastic plates can also be used for gradient elastic beams provided that $q^2 = I/A$. Thus one can easily conclude that for the classical case of beams ($g = h = 0$) and the

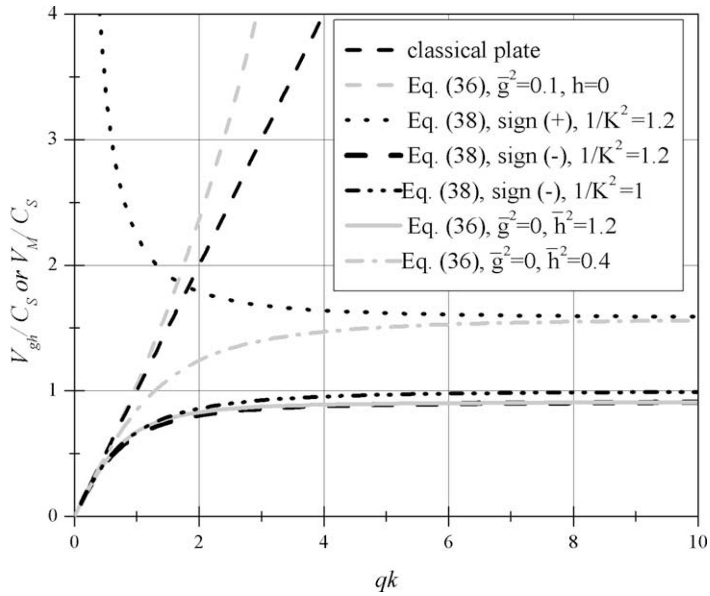

Fig. 4. Dispersion curves for wave propagation in a flexural plate.

gradient elastic beam cases with $g = h$ or $g \neq h$, $h = 0$, the phase velocity increases without limit for very high frequencies, which is physically unacceptable.

At this point it should be mentioned that the expression $\overline{V'}_g = \bar{k}\sqrt{1-\bar{g}^2\bar{k}^2}$ in Papargyri-Beskou et al. (2003) is not correct and should be replaced by the correct one $\overline{V}_g = \bar{k}\sqrt{1+\bar{g}^2\bar{k}^2}$, i.e., the one resulting from (36) with $g \neq h$, $h = 0$, which, as just mentioned, is physically unacceptable.

As in the previous section, the presence of the term $1+\bar{h}^2\bar{k}^2$ in Eq. (36) due to micro-inertia allows one to investigate for which relation between g and h, the dispersion relation (36) will be physically acceptable, i.e., in agreement with higher order beam theories, which have been verified experimentally, such as that of Timoshenko (Graff, 1975). This is possible due to the existence of a similarity between the shear and rotary inertia corrections and the micro-elastic (or gradient elastic) and micro-inertia ones, respectively, in the Bernoulli–Euler theory of elastic beams as related to wave dispersion.

Inclusion of shear deformation and rotary inertia in the Bernoulli–Euler flexural beam motion model leads to the Timoshenko corresponding model described by the equation (Graff, 1975)

$$\frac{EI}{\rho A}w^{IV} - \frac{I}{A}\left(1+\frac{E}{\mu K}\right)\ddot{w}'' + \ddot{w} + \frac{\rho I}{\mu A K}\ddot{\ddot{w}} = 0 \tag{40}$$

where K is the shear correction coefficient, which depends on the cross-sectional shape and is different than the corresponding one used in Mindlin's plates. When there is only the rotary inertia correction, the above model becomes the Rayleigh one and Eq. (40) for $K \to \infty$ reduces to

$$\frac{EI}{\rho A}w^{IV} - \frac{I}{A}\ddot{w}'' + \ddot{w} = 0 \tag{41}$$

Observing that one can go from Mindlin flexural plates (Eq. (37)) to Timoshenko flexural beams (Eq. (40)) by replacing ∇^2 by $\partial^2/\partial x^2$, K^2 by K, ν by 0 and s by $\sqrt{12I/A}$, it is easily seen that the dispersion relation (38) for Mindlin's plates can also be used for Timoshenko beams provided that $\overline{V}_M$ is replaced by $\overline{V}_T = V_T/C_S$ and $q^2 = I/A$. Because it is customary in the literature (e.g. Graff, 1975) to normalize V_T by $V_C = \sqrt{E/\rho}$ rather than C_S the following equation provides an explicit expression for $\overline{V}_T = V_T/V_C$

$$\overline{V}_T = \sqrt{\frac{\left(1+\frac{E}{\mu K}\right)\bar{k}^2 + 1 \pm \sqrt{\left[\left(1+\frac{E}{\mu K}\right)\bar{k}^2+1\right]^2 - 4\frac{E}{\mu K}\bar{k}^4}}{2\frac{E}{\mu K}\bar{k}^2}} \tag{42}$$

Similarly for the Rayleigh model of Eq. (41) the dimensionless phase velocity $\overline{V}_R = V_R/V_C$ is

$$\overline{V}_R = \frac{\bar{k}}{\sqrt{1+\bar{k}^2}} \tag{43}$$

It can be easily shown from (42) that for $\bar{k} \to \infty$, the resulting two wave speeds V_T are bounded by the values $\sqrt{E/\rho}$ and $\sqrt{\mu K/\rho}$ which correspond to the rotary inertia and shear type of corrections, respectively. Also, one can easily find that for $\bar{k} \to \infty$ the wave speed V_R is bounded by the value $\sqrt{E/\rho}$, in agreement with the results of the Timoshenko model.

Fig. 5 shows the dispersion curves for flexural wave propagation in a gradient elastic beam, the Rayleigh beam model and the Timoshenko beam model with $E/\mu K = 2.5$ as well as that for the classical Bernoulli–Euler beam model. The three-dimensional classical elastic (exact) model is very close to that of Timoshenko's (Graff, 1975). One can observe from Fig. 5 that the gradient elastic model: (i) when $g = 0$ is capable of simulating exactly the behavior of the Rayleigh beam model for the whole range of $\bar{k}'$ s provided that $g = 0$ and $h = q = \sqrt{I/A}$, (ii) when $g = 0$ and $h = q\sqrt{E/\mu K} = \sqrt{EI/A\mu K}$ is capable of simulating almost exactly the behavior of the Timoshenko beam model.

6. Conclusions

On the basis of the results of the present study one can draw the following conclusions:

1. Harmonic wave propagation in the infinite space, the axial bar, the flexural beam and the flexural plate made of gradient elastic material has been studied analytically. The material behavior is characterized by the two elastic constants g and h associated with micro-elastic and micro-inertia effects, respectively, and the two classical elastic constants λ and μ of Lamé as well as the mass density ρ.

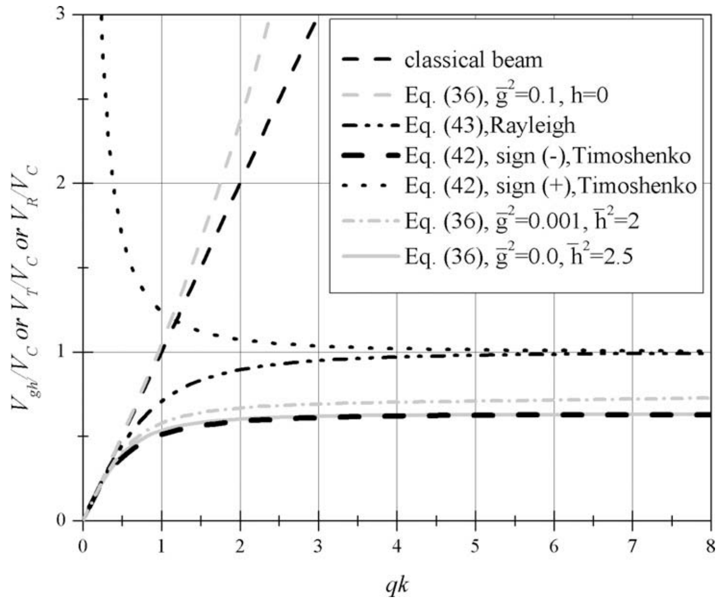

Fig. 5. Dispersion curves for wave propagation in a flexural beam.

2. For all the above gradient elastic solids and structures wave dispersion has been observed in contrast to the case of the classical elastic infinite space and the simple axial bar characterized by non-dispersion and the classical elastic simple flexural beams and plates characterized by non-acceptable dispersion.
3. It has been found that the presence of only the elastic constant g in the material model leads to physically non-acceptable dispersion for all the cases considered here. This dispersion can become physically acceptable with the presence in the material model of both the elastic constants g and h or just the constant h.
4. It has been observed that there exist similarities between the shear and rotary inertia corrections in the governing equations of motion for bars, beams and plates and the addition of micro-elastic and micro-inertia terms in the classical elastic material behavior in order to have wave dispersion in the above structures. These similarities imply an almost complete agreement of the corresponding dispersion curves when the material model has $g = 0$ and $h = v\gamma$ for bars, $h = \sqrt{\frac{EI}{A\mu K}}$ for beams and $h = \frac{s}{K\sqrt{6(1-v)}}$ for plates.

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