PROBABILITY AND STATISTICS

CHAPTER 2: RANDOM VARIABLES AND RANDOM VECTORS

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Outline

1. Random variables.

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RANDOM VARIABLE AND ITS NOTATION

DEFINITION 1.1

A variable that associates a number $X(\omega)$ with the outcome ω of a random experiment is called a random variable.

$$X : \Omega \to \mathbb{R}$$
$$\omega \to X(\omega)$$

Random Variable and its Notation

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- Uppercase letters (X, Y, Z): Random variables.
- Lowercase letters (x, y, z): Measured values of random variables (after the experiment is conducted), e.g., $x = 70$ milliamperes.

Random Variable and its Notation

Definition 1.2

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- Number of scratches on a surface.
- Number of defective parts among 100 tested.
- Number of transmitted bits received in error.
- Number of common stock shares traded per day.

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- Electrical current and voltage.
- Physical measurements, e.g., length, weight, time, temperature, pressure.

Probability mass function

Definition 2.1

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n, \ldots$ a probability mass function (p.m.f) is a function such that

$$f(u) = \begin{cases} \mathbb{P}(X = u) & \text{if } u \in \{x_1, \ldots, x_n, \ldots\} \\ 0 & \text{if } u \notin \{x_1, \ldots, x_n, \ldots\} \end{cases}$$

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Proposition 2.1 (Characteristic properties)

A discrete function f is a probability mass function iff

1. $f(u) \geq 0$ for all u.
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2. $\sum_u f(u) = 1$.

Note: For any event A, $\mathbb{P}(X \in A) = \sum_{u \in A} f(u)$.

Example 1

There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X denote the number of bits received in error in 4 bits transmitted. Suppose that the probabilities are

P(X =0)	=	0.6561
P(X =1)	=	0.2916
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P(X =4)	=	0.0001
Total		1.0000

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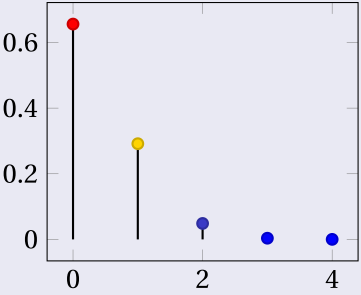

Example 2

Suppose that a random variable X has a discrete distribution with the following p.m.f.

$$f(u) = \begin{cases} cu, & u = 1, \cdots, 5 \\ 0, & \text{otherwise} \end{cases}$$

- (A) Determine the value of the constant c.
- (B) Find $\mathbb{P}(X \in [0,3])$.

EXAMPLE 3

Computers in a shipment of 100 units contain a portable hard drive, solid-state memory, or both, according to the following table:

	Portable Hard Drive	
Solid-state memory	Yes	No
Yes	15	80
No	4	1

Two computers are selected randomly, with replacement, from the the shipment. What is the probability mass function of the number of computers in the sample having both Portable Hard Drive and Solid-state memory?

EXAMPLE 4

A discrete r.v X has the p.m.f as below

X	-1	0	1	2
$\mathbb{P}$	0.1	0.3	0.4	0.2

(A) Find the p.m.f of the r.v $Y = 2X + 3$.

(B) Find the p.m.f of the r.v $Z = X^2$.

Cumulative distribution function

Definition 2.2

The cumulative distribution function (c.d.f) of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = \mathbb{P}(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

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1. $0 \leq F(x) \leq 1$.
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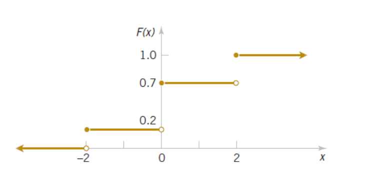
5. $\mathbb{P}(a < X \le b) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ and $a \le b$.

Example 5

Let X be a discrete random variable which is identified by the following c.d.f.

$$F(x) = \begin{cases} 0, & x < -2 \\ 0.2, & -2 \le x < 0 \\ 0.7, & 0 \le x < 2 \\ 1, & 2 \le x \end{cases}$$



- (A) Compute $F(2)$.
- (B) Compute $\mathbb{P}(-2 < X \le 2)$.
- (C) Compute $\mathbb{P}(1.5 \le X \le 3)$.

PROBABILITY DENSITY FUNCTION

DEFINITION 3.1

For a continuous random variable X, a probability density function is a function such that

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x) = 1$
3. $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$ = area under f(x) from a to b.

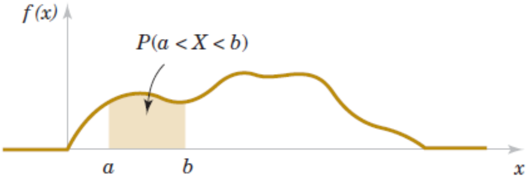


FIGURE 4-2 Probability determined from the

Continuous Distribution

Note: $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$

Continuous Distribution

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Example 6

Suppose that the p.d.f. of X is as

$$f(u) = \begin{cases} cu, & 0 < u < 4 \\ 0, & \text{otherwise} \end{cases}$$

Find c. Then determine $P(1 \le X \le 2)$ and $P(X > 2)$.

Continuous Distribution

Example 7

The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function

$$f(u) = \begin{cases} \lambda e^{-u/100}, & u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Compute the probability that a computer will function

- A. between 50 and 150 hours before breaking down?
- B. for fewer than 100 hours?

CONTINUOUS DISTRIBUTION

DEFINITION 3.2

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

for all $-\infty \leq x \leq \infty$.

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1. $\mathbb{P}(a < X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b) = F(b) - F(a)$

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$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$$

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2. $$F'(x) = \frac{d}{dx}F(x) = f(x)$$

CONTINUOUS DISTRIBUTION

EXAMPLE 8

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[4.9, 5.1]$ mA, and assume that the probability density function of X is $f(x) = 5$ for $4.9 \leq x \leq 5.1$.

(A) Find the the c.d.f of X.

(B) What is the probability that a current measurement is less than 5 milliamperes?

CONTINUOUS DISTRIBUTION

EXAMPLE 9

Assume that the lifespan of a new released product (hours) follows the given c.d.f

$$F(x) = \begin{cases} \frac{a}{100} - \frac{a}{x} & \text{if } x \geq 100 \\ 0 & \text{if } x < 100 \end{cases}$$

where $a \in \mathbb{R}$.

(A) Find the p.d.f. of X.

(B) Determine the value of a such that $f(x)$ is defined.

(C) The product will be classified as type A if its lifespan is at least 400 hours. Find the portion of type A.

EXPECTATION

DEFINITION 4.1

The expected value (mean) of a random variable X is

$$\mathbb{E}(X) = \sum_i x_i f(x_i) = \sum_i x_i \mathbb{P}(X = x_i) \qquad \text{(discrete)}$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx \qquad \text{(continuous)}.$$

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Other names: Expected value, Mean, Mean value, Average value.
Other notation: $\mu = \mathbb{E}(X)$

EXPECTATION

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- A weighted average of the possible values of X, with each value weighted by the probability that X assumes it. ==> $\mathbb{E}(X)$ describes the "center" or the "balance" point of the distribution.

Expectation

Interpretation of Mean

- A weighted average of the possible values of X, with each value weighted by the probability that X assumes it. ==> $\mathbb{E}(X)$ describes the "center" or the "balance" point of the distribution.
- The average value of X over a large number of replications of the experiment is approximately $\mathbb{E}[X]$.

EXPECTATION

PROPOSITION 4.1

Properties of Expectation

1. $\mathbb{E}(C) = C$.
2. Expectation is linear: $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$ and $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
3. If X and Y are two independent variables, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

Expectation

Example 10

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet \$1 that an odd number comes up, you win or lose \$1 according to whether that event occurs. If X denotes your net gain, find $\mathbb{E}(X)$.

EXPECTATION

EXAMPLE 11

Find $\mathbb{E}[X]$ when the density function of X is

$$f(u) = \begin{cases} 2u, & 0 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$$

THE SECOND MOMENT OF RANDOM VARIABLES

DEFINITION 4.2

The expected value of a random variable X^2 is called the second moment of X that is computed by

$$\mathbb{E}(X^2) = \sum_u u^2 f(u) \qquad \text{(discrete)}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} u^2 f(u)\,du \qquad \text{(continuous)}.$$

THE SECOND MOMENT OF RANDOM VARIABLES

EXAMPLE 12

Let X denote a random variable that takes on any of the values -1, 0, and 1 with respective probabilities

$$P(X=-1)=0.2, \qquad P(X=0)=0.5, \qquad P(X=1)=0.3$$

Compute $\mathbb{E}(X^2)$.

The second moment of random variables

Example 12

Let X denote a random variable that takes on any of the values -1, 0, and 1 with respective probabilities

$$P(X=-1)=0.2, \qquad P(X=0)=0.5, \qquad P(X=1)=0.3$$

Compute $\mathbb{E}(X^2)$.

Example 13

Let X be the current measured in mA. The PDF is $f(x)=0.05$ for $0\le x\le 20$. What is the expected value of power when the resistance is 100 ohms? Use the result that power in watts $P=10^{-6}RI^2$, where I is the current in milliamperes and R is the resistance in ohms.

Expectation of function of a random variable

Proposition 4.2

In general, for any function $g(u)$ and random variable X:

$$\mathbb{E}[g(X)] = \sum_x g(x) f(x) \qquad \text{(discrete)}$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \qquad \text{(continuous).}$$

EXPECTATION OF FUNCTION OF A RANDOM VARIABLE

EXAMPLE 14

There is a chance that a bit transmitted through a digital transmission channel is received in error. X is the number of bits received in error of the next 4 transmitted. The probabilities are

$$P(X=0)=0.6561, \quad P(X=2)=0.0486, \quad P(X=4)=0.0001,$$
$$P(X=1)=0.2916, \quad P(X=3)=0.0036,$$

What is the expected value of the cube of the number of bits in error?

Expectation of function of a random variable

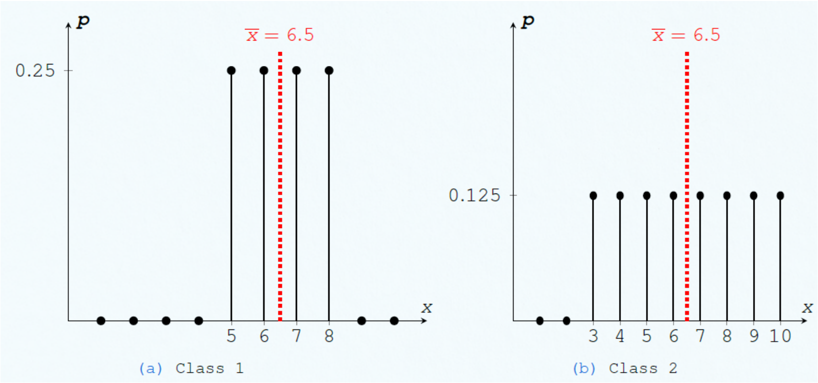
Example 15

The time, in hours, it takes to locate and repair an electrical breakdown in a certain factory is a random variable X whose density function is given by

$$f(u) = \begin{cases} 1, & \text{if } 0 < u < 1 \\ 0, & \text{otherwise} \end{cases}$$

If the cost involved in a breakdown of duration X is X^3, what is the expected cost of such a breakdown?

VARIANCE AND STANDARD DEVIATION



Variance and standard deviation

Definition 4.3 (Variance)

The variance of a r.v X is $\mathbb{V}(X)$, defined by

$$\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \tag{1}$$

Variance and standard deviation

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Definition 4.4 (Standard deviation)

The standard deviation of a r.v X is $\sigma(X)$, defined by $\mathbb{V}(X)$.

$$\sigma(X) = \sqrt{\mathbb{V}(X)}$$

VARIANCE AND STANDARD DEVIATION

PROPOSITION 4.3

If X is a discrete r.v, the variance of X is:

$$\mathbb{V}(X) = \sum_{i=1}^{\infty} (x_i - \mathbb{E}(X))^2 f(x_i) = \left(\sum_{i=1}^{\infty} x_i^2 f(x_i) \right) - \mathbb{E}(X)^2$$

If X is a continuous r.v, the variance of X is:

$$\mathbb{V}(X) = \int_{-\infty}^{+\infty} (x - \mathbb{E}(X))^2 f(x) dx = \int_{-\infty}^{+\infty} x^2 f(x) dx - \mathbb{E}(X)^2$$

PROPOSITION 4.4

Properties of Variance

1. $\mathbb{V}(c) = 0$ where c is a constant.

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Properties of Variance

1. $\mathbb{V}(c) = 0$ where c is a constant.
2. Variance and standard deviation are not linear

$$\mathbb{V}(aX+b) = a^2\mathbb{V}(X) \qquad and \quad \sigma(aX+b) = a\sigma(X)$$

PROPOSITION 4.4

Properties of Variance

1. $\mathbb{V}(c) = 0$ where c is a constant.
2. Variance and standard deviation are not linear

$$\mathbb{V}(aX + b) = a^2\mathbb{V}(X) \qquad and \quad \sigma(aX + b) = a\sigma(X)$$

3. If X and Y are independent then

$$\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y)$$

MEDIAN

DEFINITION 4.5

For a continuous r.v, X, defined by the p.d.f f, the median $md(X) = m$ can be found by solving

$$\int_{-\infty}^{m} f(x)dx = 0.5$$

.

For a discrete r.v, X, the median $md(X) = m$ is a value such that

$$\mathbb{P}(X \leq m) \geq 0.5 \qquad \text{and} \qquad \mathbb{P}(X \geq m) \geq 0.5$$

- Median measures center tendency of a distribution.
- In case of continuous distribution, median is a value that divide the pdf into two equal regions.

EXAMPLE 16

Let consider X be a discrete random variable with the given distribution.

X	a	0.1	0.3	0.4	2
$\mathbb{P}$	0.3	0.2	0.2	b	0.1

(A) Find a and b given that $\mathbb{E}(X) = 0.3$.

(B) Find $\mathbb{V}(X)$ and $\sigma(X)$ at values of a and b found in question (a).

(C) Find $\mathbb{P}(0.1 \leq X \leq 1)$.

(D) Find $md(X)$.

VARIANCE AND STANDARD DEVIATION

EXAMPLE 17

Suppose X has the following pdf, where c is a constant to be determined

$$f(u) = \begin{cases} c(1-u^2), & -1 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}(X), \mathbb{V}(X)$, and $md(X)$.

Joint distribution

In many random experiments, more than one quantity is measured, meaning that there is more than one random variable.

Example 5.1 (Cell phone flash unit)

A flash unit is chosen randomly from a production line; its recharge time X (seconds) and flash intensity Y (watt-seconds) are measured.

To make probability statements about several random variables, we need their joint probability distribution.

JOINT PROBABILITY MASS FUNCTION

DEFINITION 5.1

The joint probability mass function of the discrete random variables X and Y denoted as $f_{XY}(u, v)$ satisfies

$$f_{XY}(u, v) = P(X = u, Y = v).$$

Joint Probability Mass Function

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Proposition 5.1 (Characteristic properties)

1. $f_{XY}(x,y) \geq 0$ for all x, y.
2. $\sum_x \sum_y f_{XY}(x,y) = 1$

Discrete random vectors

A joint probability mass function of two discrete random variables X và Y can be illustrated by a table as below

Y/X	x_1	x_2	$\dots$	x_n	sum of rows
y_1	$\mathbb{P}(X=x_1,Y=y_1)$	$\mathbb{P}(X=x_2,Y=y_1)$	$\dots$	$\mathbb{P}(X=x_n,Y=y_1)$	$\mathbb{P}(Y=y_1)$
y_2	$\mathbb{P}(X=x_1,Y=y_2)$	$\mathbb{P}(X=x_2,Y=y_2)$	$\dots$	$\mathbb{P}(X=x_n,Y=y_n)$	$\mathbb{P}(Y=y_2)$
$\dots$					
y_n	$\mathbb{P}(X=x_1,Y=y_n)$	$\mathbb{P}(X=x_2,Y=y_n)$	$\dots$	$\mathbb{P}(X=x_n,Y=y_n)$	$\mathbb{P}(Y=y_n)$
Sum of columns	$\mathbb{P}(X=x_1)$	$\mathbb{P}(X=x_2)$	$\dots$	$\mathbb{P}(X=x_n)$	1

Joint Probability Mass Function

Example 18

A mobile web site is accessed from a smart phone; X is the signal strength, in number of bars, and Y is response time, to the nearest second.

y = Response time (nearest second)	x = Number of Bars of Signal Strength			
	1	2	3	**Total**
1	0.01	0.02	0.25	0.28
2	0.02	0.03	0.20	0.25
3	0.02	0.10	0.05	0.17
4	0.15	0.10	0.05	0.30
Total	0.20	0.25	0.55	1.00

Determine

- A $P(X < 3, Y \leq 2)$.
- B $P(X < 3 | Y \leq 2)$.
- C $P(Y \leq 2 | X < 3)$.

Marginal Probability Distributions (discrete)

The marginal probability distribution for X

$$
\begin{aligned}
f_X(u) &= P(X = u) \\
&= \sum_{v} P(X = u, Y = v) \\
&= \sum_{v} f_{XY}(u, v)
\end{aligned}
$$

Marginal Probability Distributions (discrete)

The marginal probability distribution for X

$$\begin{aligned} f_X(u) &= P(X = u) \\ &= \sum_v P(X = u, Y = v) \\ &= \sum_v f_{XY}(u, v) \end{aligned}$$

Themarginal probability distribution for Y

$$f_Y(v) = \sum_u f_{XY}(u, v).$$

Definition 5.2

Considering a random vector (X, Y), if X has the marginal probability distribution $f_X(x)$, then

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} x_i f_X(x_i)$$

and

$$\mathbb{V}(X) = \sum_{i=1}^{\infty} (x_i - \mathbb{E}(X))^2 f_X(x_i) = \left(\sum_{i=1}^{\infty} x_i^2 f_X(x_i) \right) - \mathbb{E}(X)^2$$

Similar formulas for Y.

Definition 5.3

If a random vector (X, Y) has the joint probability mass function $f_{XY}(x, y)$, then

$$\mathbb{E}(XY) = \sum_i \sum_j x_i y_j f_{XY}(x_i, y_j)$$

và

$$\mathbb{E}(h(X,Y)) = \sum_i \sum_j h(x_i, y_j) f_{XY}(x_i, y_j)$$

Covariance of (X, Y):

$$\mathbb{C}ov(X,Y) = \sigma_{XY} = \mathbb{E}\left[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right] = \mathbb{E}(XY) - E(X)E(Y)$$

INTERPRETION OF COVARIANCE:

Covariance is a measure of linear relationship between the random variables.

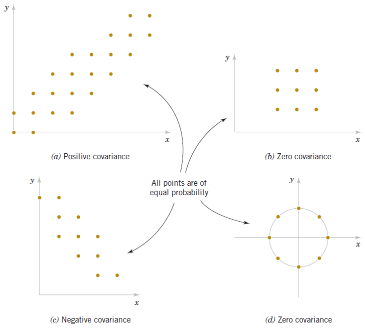

FIGURE 5-12 Joint probability distributions and the sign of covariance between X and Y.

PROPERTIES OF COVARIANCE:

1. $\mathbb{C}ov(X,X) = \mathbb{V}(X)$
2. $\mathbb{C}ov(X, aY+b) = a\mathbb{C}ov(X,Y)$
3. $\mathbb{C}ov(X, Y+Z) = \mathbb{C}ov(X,Y) + \mathbb{C}ov(X,Z)$
4. If X and Y independent, $\mathbb{C}ov(X,Y) = 0$.

DEFINITION 5.4

Correlation between X and Y is

$$\mathbb{C}or(X,Y) = \rho_{XY} = \frac{\mathbb{C}ov(X,Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Note: $-1 \le \rho_{XY} \le 1$. The correlation ρ_{XY} is a measure of linear relationship between X and Y.

Definition 5.5 (Conditional probability distribution)

Given random variables X and Y with the joint probability distribution $f_{XY}(x,y)$, a conditional probability distribution of X at $Y = y_i$ is defined as

$$f_{X|y_i} = P(X = x|Y = y_i) = \frac{P(X = x, Y = y_i)}{P(Y = y_i)} = \frac{f_{XY}(x, y_i)}{f_Y(y_i)}$$

Similarly,

$$f_{Y|x_i} = P(Y = y|X = x_i) = \frac{P(X = x_i, Y = y)}{P(X = x_i)} = \frac{f_{XY}(x_i, y)}{f_X(x_i)}.$$

Hence, the conditional expectation is

$$\mathbb{E}_{X|y_i}(X) = \sum_{i=1}^{\infty} x_i f_X|y_i(x_i).$$

Definition 5.6

For random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are independent.

1. $f_{XY}(x,y) = f_X(x)f_Y(y)$ for all x and y.
2. $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$.
3. $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$.
4. $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively.

EXAMPLE 19

Given random variables X and Y with the joint probability distribution $f_{XY}(x,y)$ as below

	$X = x$		
$Y = y$	1	2	3
0	0.1	0.15	a
1	0.2	b	0.4

(A) Find a and b, given that $\mathbb{E}(Y) = 0.7$.

(B) Find $\mathbb{E}(XY)$.

(C) Find $\mathbb{C}ov(X,Y)$ and $\mathbb{C}or(X,Y)$. Give a comment about relationship between X and Y.

(D) Find $\mathbb{E}_{X|Y=0}$.

(E) Are X and Y independent?

Marginal Probability Distributions (discrete)

Example 20

A mobile web site is accessed from a smart phone; X is the signal strength, in number of bars, and Y is response time, to the nearest second.

y = Response time (nearest second)	x = Number of Bars of Signal Strength		
	1	2	3
1	0.01	0.02	0.25
2	0.02	0.03	0.20
3	0.02	0.10	0.05
4	0.15	0.10	0.05

(A) Compute the mean and the variance of X.

(B) Compute $\mathbb{E}(XY)$.

(C) Are X and Y independent?