Probability and Statistics

Chapter 8: Simple Linear Regression and Correlation

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HCM city — 2021.

Outline

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6. Use R software to fit simple linear regression models and interpret the output.

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⇒ the relationship between those variables are not deterministic
⇒ The collection of statistical tools that are used to model and explore relationships between variables that are related in a nondeterministic manner is called regression analysis.

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Suppose that for a specified value X of the independent variable the value of the response variable Y can be expressed as

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- ε is called the random error and assumed to be normally distributed with $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}ar(\varepsilon) = \sigma^2$.

A SIMPLE LINEAR REGRESSION MODEL

A simple linear regression model given in the equation (1) states that mean of the random variable Y is related to x by the following straight-line relationship:

$$\mathbb{E}[Y|x] = \beta_0 + \beta_1 x,$$

where β_0 and β_1 are respectively the intercept and the slope of the straight-line.

Assumptions of the error term

Given n pairs of observations (x_1, y_1), (x_2, y_2),... , (x_n, y_n) which are collected from a random sample of size n, the equation (1) indicates

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i,\ i = 1,\, 2,\, \ldots, n$$

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- The error terms ε_i are mutually independent.
- $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ or $Y \sim \mathcal{N}(\beta_0 + \beta_1 x, \sigma^2)$.

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	speed	dist
1	4.00	2.00
2	4.00	10.00
3	7.00	4.00
4	7.00	22.00
5	8.00	16.00
...	...	...
48	24.00	93.00
49	24.00	120.00
50	25.00	85.00

A scatter diagram for pair data

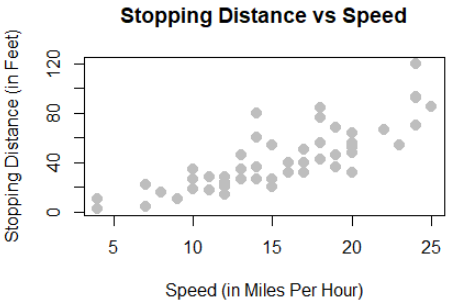

FIGURE 1: The scatter diagram of the cars dataset.

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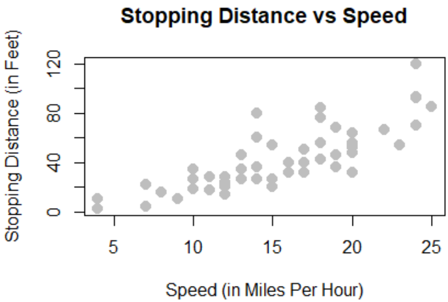

FIGURE 1: The scatter diagram of the `cars` dataset.

⇒ A scatter diagram of the observed dataset can give us an suggestion of a linear regression model.

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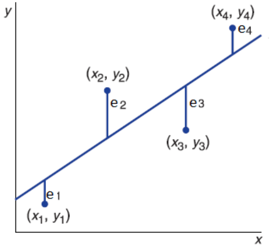

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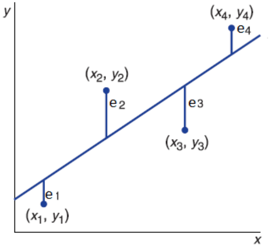

The residual $e_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) = y_i - \hat{y}_i$ describes the error in the fit of the model to the ith observation y_i.

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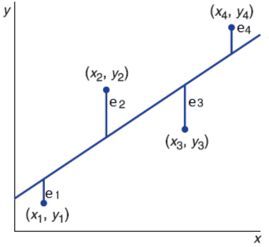

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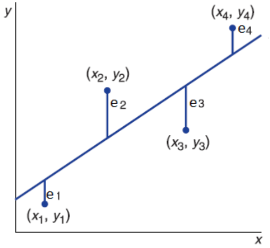

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- The key concept: An optimized fitted regression line should be "close to the observed data".
- $\hat{\beta}_0$ and $\hat{\beta}_1$ will be found by the least-square method.

ESTIMATING THE REGRESSION PARAMETERS

DEFINITION

For a dataset of n observations (x_1, y_1), ..., (x_n, y_n), the sum of squares for errors is defined by

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)]^2$$

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The least-square method aims to find the estimates $\hat{\beta}_0$, and $\hat{\beta}_1$ by minimizing SSE. Those estimates are called least squares estimates.

Estimating the regression parameters

Theorem

The least squares estimates of the intercept and slope in the simple linear regression model are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - \dfrac{\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n}}{\sum_{i=1}^n x_i^2 - \dfrac{\left(\sum_{i=1}^n x_i\right)^2}{n}} = \frac{S_{xy}}{S_{xx}}, \qquad \text{and} \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

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where S_{xx} and S_{xy} are defined by

$$S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n}, \qquad \bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

$$S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^n x_i y_i - \frac{\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n}, \qquad \bar{y} = \frac{\sum_{i=1}^n y_i}{n}$$

Estimating the regression parameters

Example 1

A large midwestern bank is planning on introducing a new word processing system to its secretarial staff. To learn about the amount of training that is needed to effectively implement the new system, the bank chose eight employees of roughly equal skill. These workers were trained for different amounts of time and were then individually put to work on a given project. The following data indicate the training times and the resulting times (both in hours) that it took each worker to complete the project.

Training time($= x$)	22	18	30	16	25	20	10	14
Time to complete project ($= Y$)	18.4	19.2	14.5	19.0	16.6	17.7	24.4	21.0

ESTIMATING THE REGRESSION PARAMETERS

EXAMPLE 1 (CONTINUED)

(A) What is the estimated regression line?

(B) Predict the amount of time it would take a worker who receives 28 hours of training to complete the project.

(C) Find the residual e_i of an observation $(x_i, y_i) = (22, 18.4)$.

Solution:

Analysis of variance

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- $SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 = S_{yy}$ is the total sum of squares. SST measure the total variation of y_i which is the variation of y_i compared to the average value $\bar{y}$.

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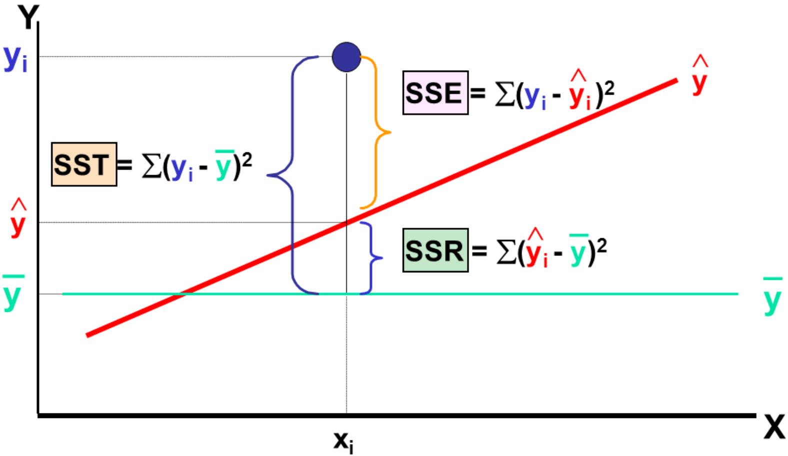
Thus, we have a fundamental identity

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \tag{2}$$

$$SST = SSR + SSE$$

Estimating the variance



COEFFICIENT OF DETERMINATION

DEFINITION

The coefficient of determination is the proportion of variation in the response variables that is explained by the different values of independent variable compared to the total variation. That is computed by

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} \tag{3}$$

Note that $0 \leq R^2 \leq 1$.

Coefficient of determination

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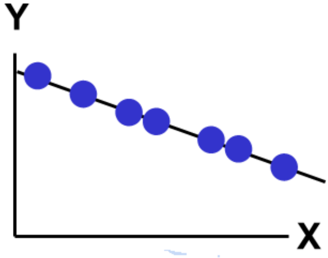

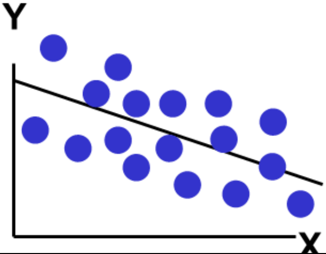

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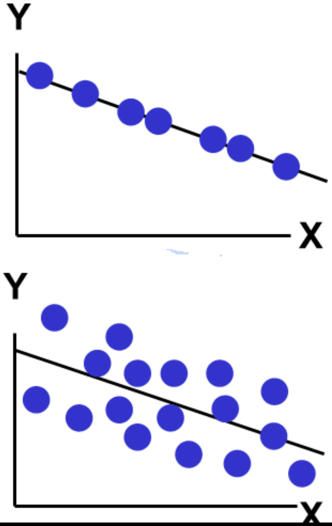

A value of R^2 near 1 indicates that most of the variation of the response data is explained by the different values of independent variable. In other word, a the linear regression model is explaining well the relationship between Y and x.

COEFFICIENT OF DETERMINATION

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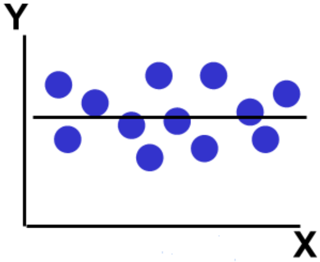

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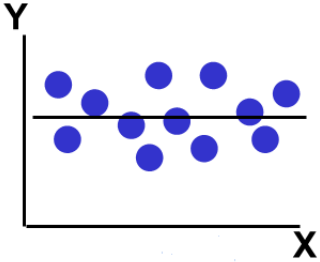

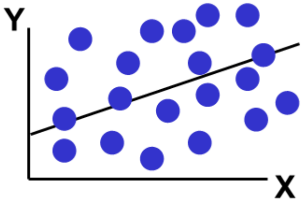

COEFFICIENT OF DETERMINATION

EXAMPLE 4

A new-car dealer is interested in the relationship between the number of salespeople working on a weekend and the number of cars sold. Data were gathered for six consecutive Sundays:

Number of salespeople	5	7	4	2	4	8
Number of cars sold	22	20	15	9	17	25

(A) Determine the estimated regression line.

(B) What is the coefficient of determination?

(C) How much of the variation in the number of automobiles sold is explained by the number of salespeople?

(D) Test the null hypothesis that the mean number of sales does not depend on the number of salespeople working.

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THEOREM

The mean squares error (MSE) of a simple linear regression is defined by

$$MSE = \frac{SSE}{n-2}.$$

The mean squares error is an unbiased estimate of σ^2, that is

$$\hat{\sigma}^2 = MSE = \frac{SSE}{n-2}$$

Proof:

ESTIMATING THE VARIANCE

Estimating the variance

- A more convenient computing formula for SSE is

$$SSE = SST - \hat{\beta}_1 S_{xy}.$$

- The standard error of $\hat{\sigma}^2$ is

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- $SE(\hat{\sigma}^2)$ indicates the variation of the observed data y_i compared to the fitted linear regression line.

ESTIMATING THE VARIANCE

EXERCISE 2

The following data give, for certain years between 1982 and 2002, the percentages of British women who were cigarette smokers.

Year	1982	1984	1988	1990	1994	1996	1998	2000	2002
Percentage	33.1	31.8	30.4	24.3	26.3	27.7	26.3	25.3	24.8

Treat these data as coming from a linear regression model, with the input being the year and the response being the percentage. Take 1982 as the base year, so 1982 has input value $x = 0$, 1986 has input value $x = 4$, and so on.

(A) Estimate the value of σ^2.

(B) Predict the percentage of British women who smoked in 1997.

Hypothesis tests in Simple Linear Regression

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- Hypothesis tests of β_1 includes the following cases:

$$(a)\begin{cases} H_0 : \beta_1 = b_1 \\ H_1 : \beta_1 \neq b_1 \end{cases} \qquad (b)\begin{cases} H_0 : \beta_1 = b_1 \\ H_1 : \beta_1 < b_1 \end{cases} \qquad (c)\begin{cases} H_0 : \beta_1 = b_1 \\ H_1 : \beta_1 > b_1 \end{cases}$$

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  where b_1 and a confident level α are given.
- An important hypothesis is $\beta_1=0$. Its importance lies in the fact that it is equivalent to stating that a response does not linearly depend on the value of the input; or, in other words, there is no regression on the input value.

HYPOTHESIS TESTS IN SIMPLE LINEAR REGRESSION

THEOREM

Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ be a simple linear regression model for a dataset of n independent observations where $\varepsilon_i \sim \mathcal{N}(0,1)$. Considering $\hat{\beta}_0$ and $\hat{\beta}_1$ are respectively the least-square estimates of β_0 and β_1, then

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1. $\hat{\beta}_0$ and $\hat{\beta}_1$ follow normal distribution.

HYPOTHESIS TESTS IN SIMPLE LINEAR REGRESSION

THEOREM

Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ be a simple linear regression model for a dataset of n independent observations where $\varepsilon_i \sim \mathcal{N}(0,1)$. Considering $\hat{\beta}_0$ and $\hat{\beta}_1$ are respectively the least-square estimates of β_0 and β_1, then

1. $\hat{\beta}_0$ and $\hat{\beta}_1$ follow normal distribution.
2. The expectation and variance of $\hat{\beta}_0$ and $\hat{\beta}_1$ are respectively

$$\mathbb{E}(\hat{\beta}_0) = \beta_0, \ \mathbb{V}ar(\hat{\beta}_0) = \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\sigma^2, \tag{4}$$

$$\mathbb{E}(\hat{\beta}_1) = \beta_1, \ \mathbb{V}ar(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}} \tag{5}$$

Proof:

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3. Compute the test statistic:

$$T_{\beta_1} = \frac{\hat{\beta}_1 - b_1}{SE(\hat{\beta}_1)} \sim t(n-2)$$

where

$$SE(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}}$$

Hypothesis tests in Simple Linear Regression

4. Determine the rejected range or compute p-value:

Alternative hypothesis	**rejected range**	p **- value**
$H_1 : \beta_1 \neq b_1$	$\lvert t_{\beta_1} \rvert > t^{n-2}_{\alpha/2}$	$p = 2\mathbb{P}(T_{n-2} \geq \lvert t_{\beta_0} \rvert)$
$H_1 : \beta_1 < b_1$	$t_{\beta_1} < -t^{n-2}_{\alpha}$	$p = \mathbb{P}(T_{n-2} \leq t_{\beta_0})$
$H_1 : \beta_1 > b_1$	$t_{\beta_1} > t^{n-2}_{\alpha}$	$p = \mathbb{P}(T_{n-2} \geq t_{\beta_0})$

5. Conclude whether H_0 is rejected or not.

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CONFIDENCE INTERVALS ON PARAMETERS

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Under the assumption that the observations are normally and independently distributed, a $100(1-\alpha)\%$ confidence interval on the slope β_1 in simple linear regression is

$$\hat{\beta}_1 - t_{\alpha/2}^{n-2}\sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \leq \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2}^{n-2}\sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \tag{6}$$

Confidence intervals on parameters

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Similarly, a $100(1-\alpha)\%$ confidence interval on the intercept β_0 is

$$\hat{\beta}_0 - t_{\alpha/2}^{n-2}\sqrt{\left(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right)\hat{\sigma}^2} \leq \beta_0 \leq \hat{\beta}_0 + t_{\alpha/2}^{n-2}\sqrt{\left(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right)\hat{\sigma}^2} \tag{7}$$

CONFIDENCE INTERVALS ON PARAMETERS

EXERCISE 3

The following table relates the number of sunspots that appeared each year from 1970 to 1980 to the number of automobile accident deaths during that year. The data for automobile accident deaths are in units of 1000 deaths.

Year	Sunspots	Automobile deaths
70	165	54.6
71	89	53.3
72	55	56.3
73	34	49.6
74	9	47.1
75	30	45.9
76	59	48.5
77	83	50.1
78	109	52.4
79	127	52.5
80	153	53.2

Test the hypothesis that the number of automobile accident deaths is not linearly related to the number of sunspots. Use the 5 percent level of significance.

SAMPLE CORRELATION COEFFICIENT

DEFINITION

Considering a sample of n observations: $(X_i, Y_i), i = 1, \ldots, n$. The sample correlation coefficient r_{XY}, is defined by

$$r_{XY} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}} = \frac{S_{XY}}{\sqrt{S_{XX}SST}} \quad (8)$$

SAMPLE CORRELATION COEFFICIENT

Note that

$$\hat{\beta}_1 = \sqrt{\frac{SST}{S_{XX}}} r_{XY}$$

thus

$$r_{XY}^2 = \hat{\beta}_1^{\,2} \frac{S_{XX}}{SST} = \hat{\beta}_1 \frac{S_{XY}}{SST} = \frac{SSR}{SST}$$

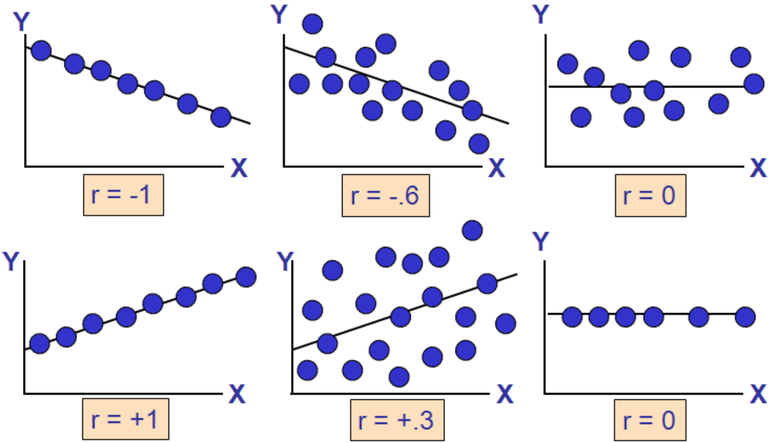
• The coefficient of determination R^2 in a simple linear regression model equals to the square of the sample correlation coefficient.

$$R^2 = r_{XY}^2$$

Sample correlation coefficient

- The range of r_{XY}: $-1 \leq r_{XY} \leq 1$,
- $-1 \leq r_{XY} < 0$: negative correlation. r_{XY} is closer to -1 indicating a stronger negative correlation between X and Y.
- $0 < r_{XY} \leq 1$: positive correlation. r_{XY} is closer to 1 indicating a stronger positive correlation between X and Y.
- r_{XY} is closer to 0 indicating a weak correlation between X and Y. $r_{XY} = 0$: indicating linearly independent between X and Y.

Sample correlation coefficient



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  - The linear relationship of Y and x: $Y = \beta_0 + \beta_1 x + \epsilon$ where β_0 and β_1 are the regression coefficients such that given x we have $\mathbb{E}(Y|x) = \beta_0 + \beta_1 x$.

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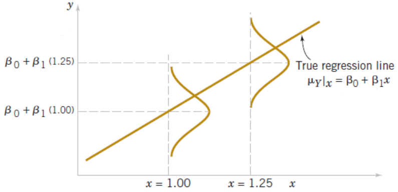
  - Constant variation: The variance σ^2 of Y is invariant for all value of x, e.i. $Var(Y|x) = \sigma^2$.



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$\Rightarrow$ to test the normality, we use the Normal probability plot (Q-Q plot) of the residuals or the standardized residuals.

$\Rightarrow$ to test the linearity, independence, and constant variances we use the scatter plot of the residuals or the standardized residuals.

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- A Q–Q plot is a plot of the quantiles of two distributions against each other, or a plot based on estimates of the quantiles. The pattern of points in the plot is used to compare the two distributions.

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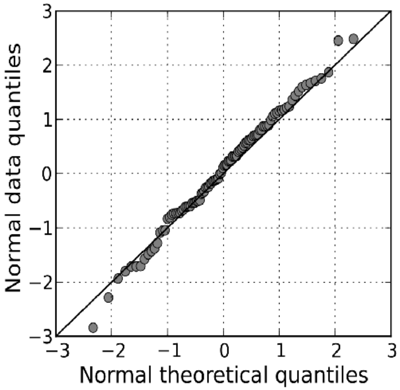
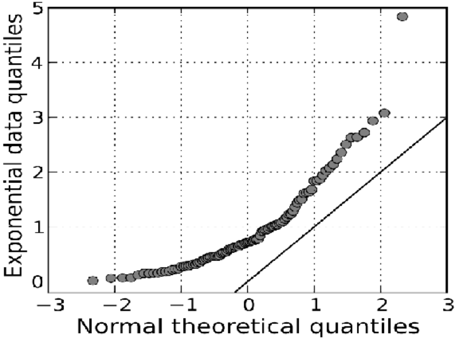
- A Q–Q plot is a plot of the quantiles of two distributions against each other, or a plot based on estimates of the quantiles. The pattern of points in the plot is used to compare the two distributions.
- The points plotted in a Q–Q plot are always non-decreasing when viewed from left to right. If the two distributions being compared are identical, the Q–Q plot follows the 45 deg line $y = x$

Analysis of residuals: assessing the model.





ANALYSIS OF RESIDUALS: ASSESSING THE MODEL.

STANDARDIZED RESIDUALS

The standardized residuals are defined as

$$E_i = \frac{Y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)}{\sqrt{SSE/(n-2)}},\ i = 1, 2, \ldots, n$$

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- their scatter plot should not indicate any distinct pattern.

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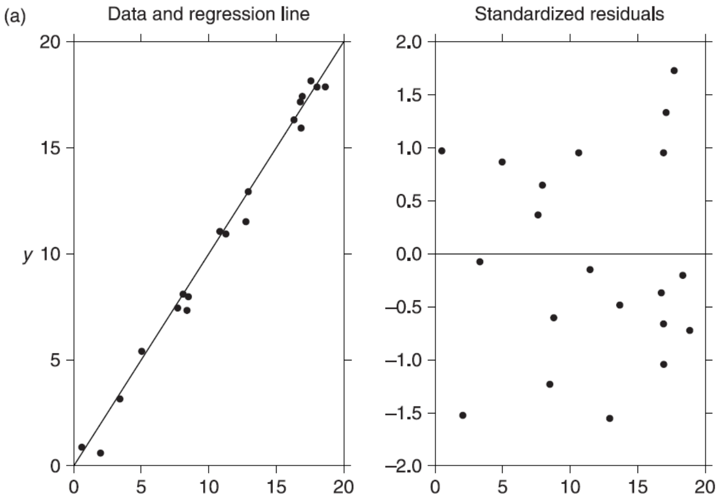

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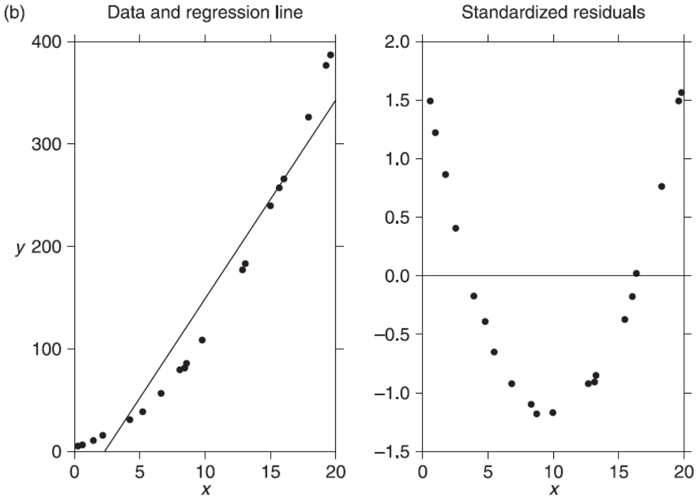

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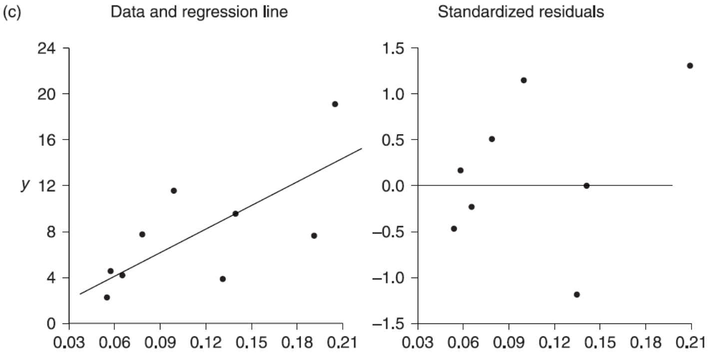

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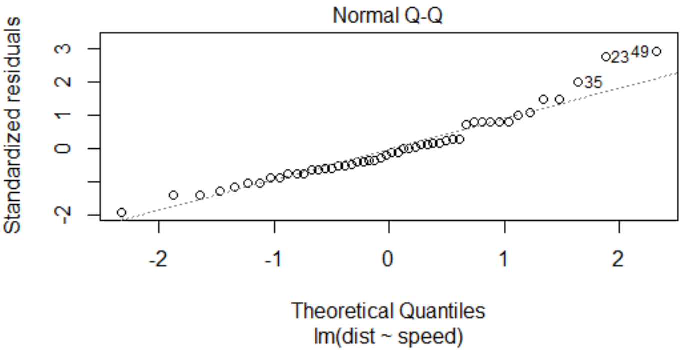
- Regression relationships are valid for values of the regression variable only within the range of the original data. ⇒ be careful with extrapolates.
- It's hard to define what level of R^2 is appropriate to claim the model fits well. Essentially, it will vary with the application and the domain studied.

Interpreting R results

```
> M<- lm(dist ~ speed, data = cars)
> summary(M)

Call:
lm(formula = dist ~ speed, data = cars)

Residuals:
    Min      1Q  Median      3Q     Max
-29.069  -9.525  -2.272   9.215  43.201

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5791     6.7584  -2.601   0.0123 *
speed         3.9324     0.4155   9.464 1.49e-12 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 15.38 on 48 degrees of freedom
Multiple R-squared:  0.6511,	Adjusted R-squared:  0.6438
F-statistic: 89.57 on 1 and 48 DF,  p-value: 1.49e-12
```

INTERPRETING R RESULTS



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