Probability and Statistics

Chapter 5: One sample testing

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Outline

OUTLINE

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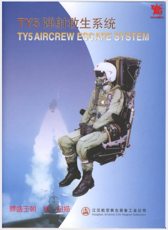
INTRODUCTION

EXAMPLE 1

Suppose that an engineer is designing an air crew escape system that consists of an ejection seat and a rocket motor that powers the seat. The rocket motor contains a propellant, and for the ejection seat to function properly, the propellant should have a **mean burning rate of 50 cm/sec**. If the burning rate is too low, the ejection seat may not function properly, leading to an unsafe ejection and possible injury of the pilot. Higher burning rates may imply instability in the propellant or an ejection seat that is too powerful, again leading to possible pilot injury.



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$$50.1, 50.3, 49.2, 48.5, 50.1, 49.9, 48.3, 51.2, 51.4, 50.6$$

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1. Does the data give enough evidence to confirm that the mean burning rate of the propellant equal 50 cm/sec?

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1. Does the data give enough evidence to confirm that the mean burning rate of the propellant equal 50 cm/sec?
2. Does the data give enough evidence to confirm that the mean burning rate of the propellant is greater than 50 cm/sec?

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1. Does the data give enough evidence to confirm that the mean burning rate of the propellant equal 50 cm/sec?
2. Does the data give enough evidence to confirm that the mean burning rate of the propellant is greater than 50 cm/sec?
3. Does the data give enough evidence to confirm that the mean burning rate of the propellant is less than 50 cm/sec?

Test of hypotheses

- A statistical hypothesis, or just hypothesis, is a claim or assertion either about the value of a single parameter, about the values of several parameters, or about the form of an entire probability distribution.
- The null hypothesis, denoted by H_0, is the claim that is initially assumed to be true.
- The alternative hypothesis, denoted by H_1, is the assertion that is contradictory to H_0.
- A test of hypotheses is a method for using sample data to decide whether the null hypothesis should be rejected.

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  If the test rejects H0 $\Rightarrow$ we have enough evidence to confirm H1 is true.

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- rejecting H0 is always a strong conclusion.
- put the statement about which it is important to make a strong conclusion in the alternative hypothesis (H1).
- Only make conclusion of rejecting or not on H0:
  If the test rejects H0 $\Rightarrow$ we have enough evidence to confirm H1 is true.
  If the test fail to reject H0 $\Rightarrow$ we **don't have enough evidence** to confirm H1 is true (it means we don't known whether H1 is true or not).

Test of hypotheses

- Two-sided Alternative Hypotheses

$$H_0 : \mu = 50\ cm/s \text{ and } H_1 : \mu \neq 50\ cm/s$$

- One-sided Alternative Hypotheses

$$H_0 : \mu = 50\ cm/s \text{ and } H_1 : \mu > 50\ cm/s$$

or

$$H_0 : \mu = 50\ cm/s \text{ and } H_1 : \mu < 50\ cm/s$$

TEST OF HYPOTHESES

EXAMPLE 2

Write the claim as a mathematical statement. State the null and alternative hypotheses in words and in symbols, and identify which represents the claim. Then determine whether the hypothesis test is a left-tailed test, right-tailed test, or two-tailed test. How should you interpret your decision if you reject H_0? If you fail to reject H_0?

1. A school publicizes that the proportion of its students who are involved in at least one extracurricular activity is 61%.
2. A car dealership announces that the mean time for an oil change is less than 15 minutes.
3. A company advertises that the mean life of its furnaces is more than 18 years.

Test of a Hypothesis and errors

- A procedure leading to a decision about a particular hypothesis
- The key is H0 is assumed to be **TRue**.
- The two possible conclusions: reject H_0 or fail to reject H_0.

	Retain null	Reject null
H0 true	$\surd$	type I error
H1 true	type II error	$\surd$

Definition 1.1

- A type I error consists of rejecting H_0 when it is true.
- A type II error involves not rejecting H_0 when H_0 is false.

ERROR

DEFINITION 1.2

- The probability of a type I error is called the significance level of the test and is usually denoted by α.

$$\alpha = P(\text{ type I error }) = P(H_0 \text{ is rejected when it is true})$$

- The probability that the null hypothesis is rejected when it is false is called the power of the test, and equals $1-\beta$

$$\beta = P(\text{type II error}) = P(H_0 \text{ is not rejected when it is false})$$

- A type I error is usually more serious than a type II error.

Test procedure

Definition 1.3

- A test statistic, denoted by TS, is a function of the sample data on which the decision (reject H_0 or do not reject H_0) is to be based.
- A rejection region, the set of all test statistic values for which H_0 will be rejected. The complement of the rejection region is called the accepted region denoted by AR.

$$RR = \{ts : \; H0 \text{ is rejected}\}$$

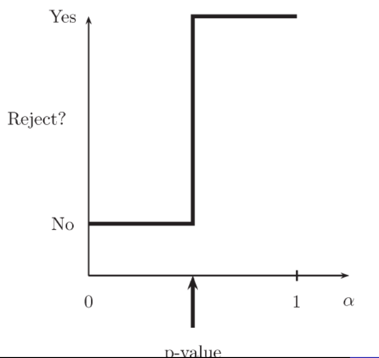
The rejection region corresponding to a significant level α is denoted by RR and determined by

$$\mathbb{P}(TS \in RR) = \alpha$$

Test procedure

P-value

- The p-value is the smallest significance level α at which the null hypothesis can be rejected. Because of this, the P-value is alternatively referred to as the observed significance level (OSL) for the data.



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- Reject H0 $\Leftrightarrow$ $p_v < p$.
- Smaller α $\Leftrightarrow$ stronger conclusion.

TEST PROCEDURE

P-VALUE

- Reject H0 $\Leftrightarrow p_v < p$.
- Smaller $\alpha \Leftrightarrow$ stronger conclusion.
- P-value provides a measure of the credibility of the null hypothesis.

p-value	evidence
<0.01	very strong evidence against H0
0.01 - 0.05	strong evidence against H0
0.05 - 0.1	weak evidence against H0
>0.1	little or no evidence against H0

TEST PROCEDURE

A hypothesis testing follows this procedure:

- STEP 1: State the null hypothesis H0 and the alternative hypothesis H1.
- STEP 2: Compute the test statistic value ts based on the sampled data and based on the assumption that $H0$ **is TRUE**.
- STEP 3: Determine the rejection region W_α or the p-value p_v based on the distribution of TS.
- STEP 4: Decide reject the null hypothesis H0 if $ts \in W_\alpha$ or $p_v < \alpha$.

Normal Population + Known σ

1. Compute a test statistic: $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$
2. Determine the rejection region W_α or p_v basing on the following decision rule: (z-test)

Alternative hypothesis	Rejection region	P_value
$H1 : \mu \neq \mu_0$	$W_\alpha = \{z_0 : \|z_0\| > z_{\alpha/2}\}$	$p_v = 2\,[1 - \Phi(\|z_0\|)]$
$H1 : \mu > \mu_0$	$W_\alpha = \{z_0 : z_0 > z_\alpha\}$	$p_v = 1 - \Phi(z_0)$
$H1 : \mu < \mu_0$	$W_\alpha = \{z_0 : z_0 < -z_\alpha\}$	$p_v = \Phi(z_0)$

Normal Population + Known σ

Example 3

50 smokers were questioned about the number of hours they sleep each day. It was found that the sample mean is 7.5 hours, and suppose that the population has normal distribution with standard deviation 0.5 hours. Test at the 5% significance level the hypothesis that the smokers need less sleep than the general public which needs an average of 7.7 hours of sleep.

Normal Population + Unknown σ

1. Compute a test statistic:

$$t_0 = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$$

2. Determine the rejection region W_α or p_v basing on the following decision rule: (t-test)

Alternative hypothesis	Rejection region	P_value
$H1 : \mu \neq \mu_0$	$W_\alpha = \left\{t_0 : \vert t_0\vert > t_{n-1,\alpha/2}\right\}$	$p_v = 2P(T > \vert t_0\vert)$
$H1 : \mu > \mu_0$	$W_\alpha = \left\{t_0 : t_0 > t_{n-1,\alpha}\right\}$	$p_v = P(T > t_0)$
$H1 : \mu < \mu_0$	$W_\alpha = \left\{t_0 : t_0 < -z_{n-1,\alpha}\right\}$	$p_v = P(T < t_0)$

Normal Population + Unknown σ

Example 4

In a random sample of 20 components taken from a production line, the mean length of each component in this sample is 108.6 millimeters with a standard deviation of 6.3 millimeters. Given that each component should measure 105 millimeters and that the population distribution is normal, is there enough statistical evidence to show that the production line is producing components that are of an incorrect length? Test at 5 percent level of significance.

Any Distribution - Large Sample Size

1. Compute a test statistic:

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}, \quad \text{if } \sigma^2 \quad \text{is given}$$

or

$$z_0 = \frac{\overline{x} - \mu_0}{s / \sqrt{n}}, \quad \text{if } \sigma^2 \quad \text{is not given.}$$

2. Determine the rejection region W_α or p_v basing on the following decision rule: (z-test)

Alternative hypothesis	Rejection region	P_value
$H1 : \mu \neq \mu_0$	$W_\alpha = \{z_0 : \lvert z_0 \rvert > z_{\alpha/2}\}$	$p_v = 2\,[1 - \Phi(\lvert z_0 \rvert)]$
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Any Distribution - Large Sample Size

Example 5

A dynamic cone penetrometer (DCP) is used for measuring material resistance to penetration as a cone is driven into pavement or subgrade. Suppose that for a particular application it is required that the true average DCP value for a certain type of pavement be less than 30. The pavement will not be used unless there is conclusive evidence that the specification has been met. Let's state and test the appropriate hypotheses (with $\alpha = 0.05$) using the following data

14.1	14.5	17.8	18.1	20.8	20.8	30.0	31.6	36.7	40.0
55.0	57.0	15.5	16.0	18.2	18.3	21.0	21.5	31.7	31.7
40.0	41.3	16.0	16.7	16.9	18.3	19.0	19.2	23.5	27.5
27.5	32.5	33.5	33.9	41.7	47.5	50.0	17.1	17.5	17.8
19.4	20.0	20.0	28.0	28.3	30.0	35.0	35.0	35.0	51.0
51.8	54.4								

Population Proportion - Large Sample Size

- Let p the proportion of event $\mathscr{A}$ in a population.
- We investigate a random sample of size n, let

$$Y_i = \begin{cases} 1, & \text{if observation } i \text{ showing } \mathscr{A}, \\ 0, & \text{otherwise,} \end{cases}$$

  Then, $Y_i \sim B(1, p)$, $i = 1, \ldots, n$.
- Let $X = Y_1 + \ldots + Y_n = \sum_{i=1}^{n} Y_i$, then X = the number of event $\mathscr{A}$ in a sample of n observations and $X \sim B(n, p)$.
- The sample proportion is

$$\hat{P} = \frac{X}{n}.$$

Population Proportion - Large Sample Size

1. Compute a test statistic:

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

2. Determine the rejection region W_α or p_v basing on the following decision rule: (z-test)

Alternative hypothesis	Rejection region	P_value
$H1: p \neq p_0$	$W_\alpha = \{z_0 : \lvert z_0\rvert > z_{\alpha/2}\}$	$p_v = 2\left[1 - \Phi(\lvert z_0\rvert)\right]$
$H1: p > p_0$	$W_\alpha = \{z_0 : z_0 > z_\alpha\}$	$p_v = 1 - \Phi(z_0)$
$H1: p < p_0$	$W_\alpha = \{z_0 : z_0 < -z_\alpha\}$	$p_v = \Phi(z_0)$

Population Proportion - Large Sample Size

Example 6

Natural cork in wine bottles is subject to deterioration, and as a result wine in such bottles may experience contamination. An article reported that, in a tasting of commercial chardonnays, 16 of 91 bottles were considered spoiled to some extent by cork-associated characteristics. Does this data provide strong evidence for concluding that more than 15% of all such bottles are contaminated in this way? ($\alpha = 5\%$)