

If we have a desired torque to generate from the fins we can say:

$$\tau = \sum \tau_i$$

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i + M_i \hat{l}_i$$

$$\vec{r}_i = z\hat{k} + d\hat{l}_i$$

(we only care about the aero component normal to span and z, ie lift, drag does do something, but it over complicates)

$$\vec{F}_i = F_i(\hat{k} \times \hat{l}_i)$$

Where r_i is the arm from COG of rocket to COP of the fin

Where l_i is the span vector (normal to lift, along moment of the fin)

$$\vec{r}_i \times \vec{F}_i = (z\hat{k} + d\hat{l}_i) \times F_i(\hat{k} \times \hat{l}_i) = F_i \left\{ z\hat{k} \times (\hat{k} \times \hat{l}_i) + d\hat{l}_i \times (\hat{k} \times \hat{l}_i) \right\}$$

$$z\hat{k} \times (\hat{k} \times \hat{l}_i) = z \left\{ \hat{k}(\hat{k} \cdot \hat{l}_i) - \hat{l}_i(\hat{k} \cdot \hat{k}) \right\} = -z\hat{l}_i$$

$$d\hat{l}_i \times (\hat{k} \times \hat{l}_i) = d \left\{ \hat{k}(\hat{l}_i \cdot \hat{l}_i) - \hat{l}_i(\hat{k} \cdot \hat{l}_i) \right\} = d\hat{k}$$

$$\vec{\tau}_i = F_i(d\hat{k} - z\hat{l}_i)$$

We'll assume that the lift can only occur in the linear region of the lift curve

$$F_i = \alpha_i Cl_\alpha$$

The desired torque will be zero in z so we can write

$$\sum_{i=0}^N \alpha_i Cl_\alpha (d\hat{k} - z\hat{l}_i) = \begin{bmatrix} \tau_x \\ \tau_y \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sum \alpha_i l_{i,x} \\ \sum \alpha_i l_{i,y} \\ \sum \alpha_i \end{bmatrix} = \begin{bmatrix} -\tau_x / z Cl_\alpha \\ -\tau_y / z Cl_\alpha \\ 0 \end{bmatrix}$$

3 equations and N unknowns (α_i). If $N = 3$ there is only one solution.

For $N = 4$, we can use symmetry such that the problem reduces to 2 equations, $\alpha_i = -\alpha_{i+2}$ (Note that this also implies $z = 0$, therefore only the x and y component equations are used).

For $N > 3$, we can optimize by reducing drag, which can be done by minimizing:

$$\min \sum_{i=0}^N \alpha_i^2$$

For $N = 4$ this also has a closed form solution, using lagrange multipliers.
 We minimize subject to the previous:

$$L(\alpha_i, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=0}^N \alpha_i^2 + \lambda_1 \left(\frac{\tau_x}{zCl_\alpha} - \sum_{i=0}^N l_{i,x} \alpha_i \right) + \lambda_2 \left(\frac{\tau_y}{zCl_\alpha} - \sum_{i=0}^N l_{i,y} \alpha_i \right) + \lambda_3 \sum_{i=0}^N \alpha_i$$

The additional equations for the gradient are:

$$\nabla_{\alpha, \lambda} L(\alpha_i, \lambda) = 0$$

$$\frac{d}{d\alpha_i} L(\alpha_i, \lambda) = 2\alpha_i - \lambda_1 l_{i,x} - \lambda_2 l_{i,y} + \lambda_3$$

$$\frac{d}{d\lambda_i} L(\alpha_i, \lambda) = g_{\tau_i}(\alpha_i)$$

So there are 6 equations and 6 unknowns.

Alternative:

In general one can minimize the fin control input, by prioritizing fins with more moment arm
 So this can easily be done as:

$$a_i = K \left(\vec{\tau} \cdot \hat{l}_i \right)$$

So we just need to solve for the gain K which solves:

$$\vec{\tau} = \sum_{i=0}^N Cl_\alpha K \left(\vec{\tau} \cdot \hat{l}_i \right) \left(d\hat{k} - z\hat{l}_i \right)$$