

# STAT 457 Homework 04

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## Problem 1

Let  $y_1, \dots, y_n$  be an iid sample from the Poisson distribution with parameter  $\lambda$ . Derive Jeffery's (noninformative) prior. This prior corresponds to the gamma distribution with which parameters?

$$L(\lambda | Y) = \frac{\lambda^y e^{-\lambda}}{y!} \propto \lambda^y e^{-\lambda}$$

$$\ell(\lambda | Y) = y \log \lambda - \lambda$$

$$\frac{\partial \ell(\lambda | Y)}{\partial \lambda} = \frac{y}{\lambda} - 1$$

$$\frac{\partial^2 \ell(\lambda | Y)}{\partial \lambda^2} = -\frac{y}{\lambda^2}$$

$$\mathcal{I}(\lambda) = \mathbb{E} \left[ -\frac{\partial^2 \ell(\lambda | Y)}{\partial \lambda^2} \right]$$

$$= \mathbb{E} \left[ \frac{y}{\lambda^2} \right]$$

$$= \frac{1}{\lambda}$$

$$\text{Jeffrey's Prior} \quad p(\lambda) = \sqrt{\mathcal{I}(\lambda)} = \sqrt{\frac{1}{\lambda}} \approx \text{Gamma} \left( \frac{1}{2}, 0 \right)$$

## Problem 2

In the multivariate setting,  $\theta = (\theta_1, \dots, \theta_d)$ ,  $p(\theta) \propto |J(\theta)|^{1/2}$ , provides an invariant prior where the  $ij^{\text{th}}$  entry of  $J(\theta)$  is equal to

$$-\mathbb{E} \left[ \frac{\partial^2 \ell(\theta | Y)}{\partial \theta_i \partial \theta_j} \right]$$

and  $|X|$  is the determinant of the matrix  $X$ .

Let  $y_1, \dots, y_n$  be an iid sample from the  $\mathcal{N}(\mu, \sigma^2)$  distribution, where  $\mu$  and  $\sigma$  are both unknown. Derive the invariant prior. How does it compare with the prior  $p(\theta, \sigma^2) \propto 1/\sigma^2$ ?

$$\begin{aligned} L(\mu, \sigma^2 | \mathbf{Y}) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(y_i - \mu)^2}{2\sigma^2} \right\} \\ \ell(\mu, \sigma^2 | \mathbf{Y}) &= \log \left( \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(y_i - \mu)^2}{2\sigma^2} \right\} \right) \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{\sum_i (y_i - \mu)^2}{2\sigma^2} \\ \frac{\partial^2 \ell}{\partial \mu^2} &= -\frac{n}{\sigma^2} \\ \frac{\partial^2 \ell}{\partial \mu \partial \sigma^2} &= -\frac{\sum_i (y_i - \mu)^2}{\sigma^4} = \frac{\partial^2 \ell}{\partial \sigma^2 \partial \mu} \\ \frac{\partial^2 \ell}{\partial (\sigma^2)^2} &= \frac{n}{\sigma^2} = \frac{n}{2\sigma^4} - \frac{\sum_i (y_i - \mu)^2}{\sigma^6} \\ p(\mu, \sigma^2) &\propto |J(\mu, \sigma^2)|^{\frac{1}{2}} \\ &= \left( -\frac{1}{n} \det \begin{bmatrix} -n/\sigma^2 & 0 \\ 0 & -n/2\sigma^4 \end{bmatrix} \right)^{\frac{1}{2}} \\ &= \frac{1}{\sigma^3} \end{aligned}$$

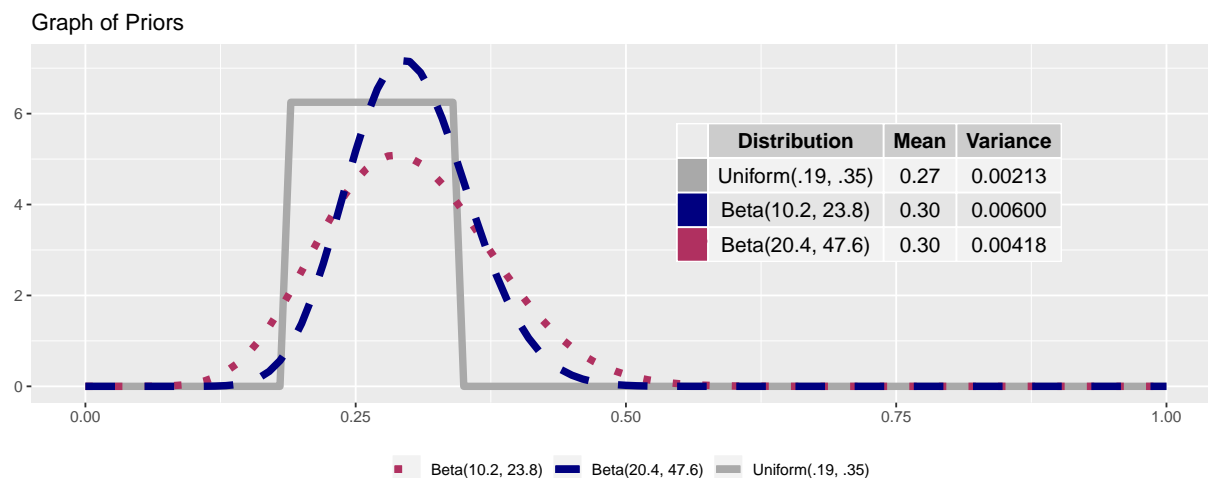
This is different than commonly used prior of  $1/\sigma^2$ . This shows that the Jeffery's prior does not work in every situation.

### Problem 3

Let  $p$  denote the probability that a specific major league baseball player will get a hit in a particular at bat. Assume that batting averages usually fall in the range .19 to .35.

#### Problem 3a

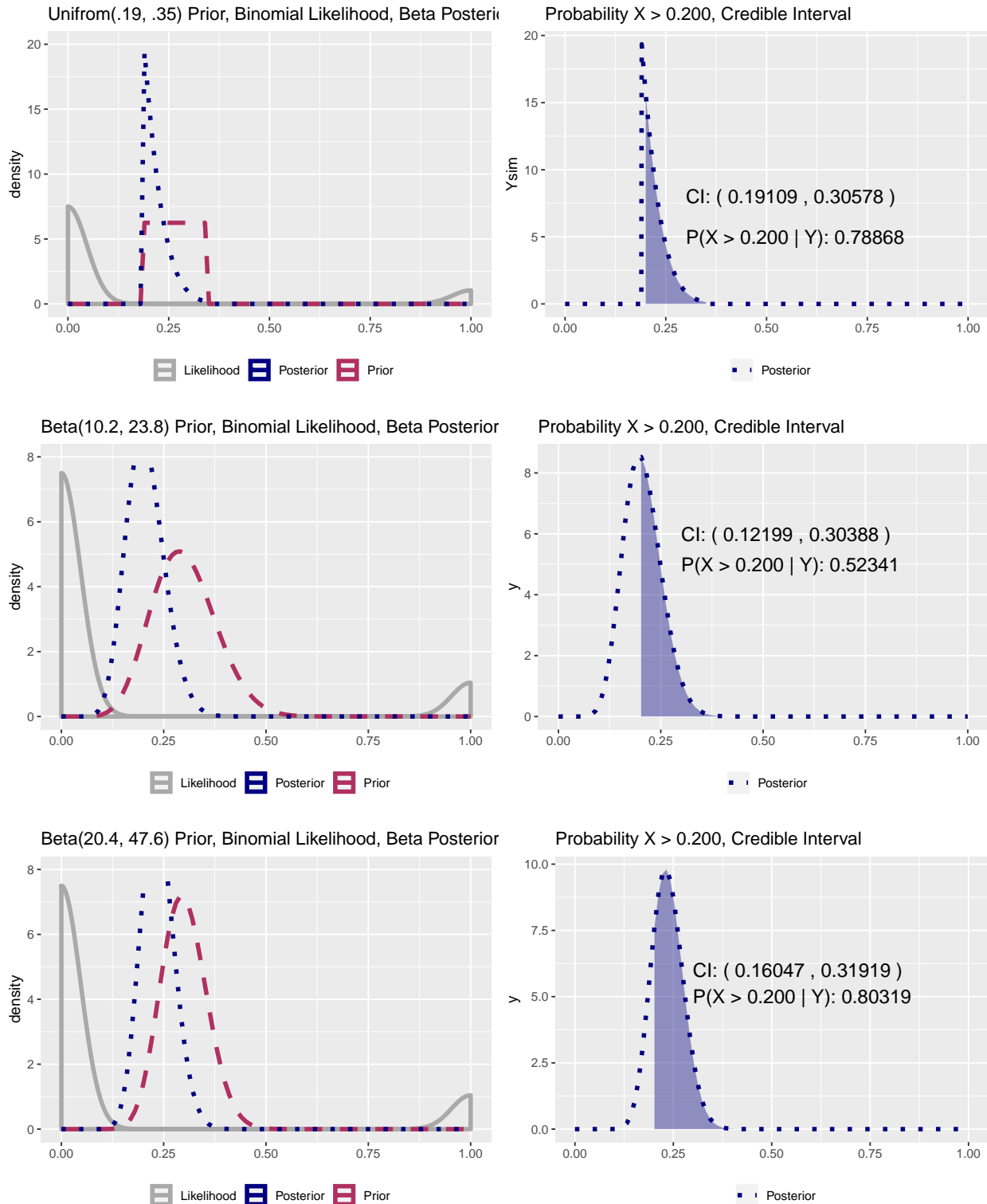
Consider the priors  $\text{Uniform}(.19, .35)$ ;  $\text{Beta}(10.2, 23.8)$ ; and  $\text{Beta}(20.4, 47.6)$ . Plot these priors and discuss each choice.



The Uniform has a mean of 0.27 and both beta distributions have a mean of 0.30. The uniform distribution has the lowest variance and a block shape. The Beta(10.2,23.8) has the highest variance and a lower peak than the Beta(20.4,47.6).

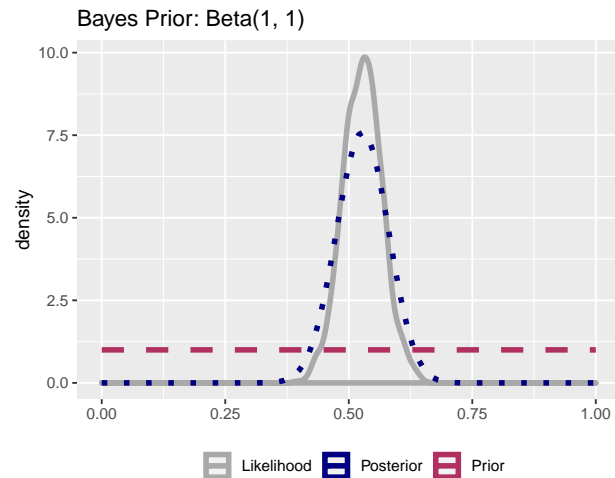
### Problem 3b

Suppose a player gets 5 hits in 40 at-bats. For each of the above priors: plot the likelihood, posterior and prior; compute the probability that he player is better than a .200 hitter; compute your best guess as to the batting average of the player; compute a 95% credible interval for  $p$ .

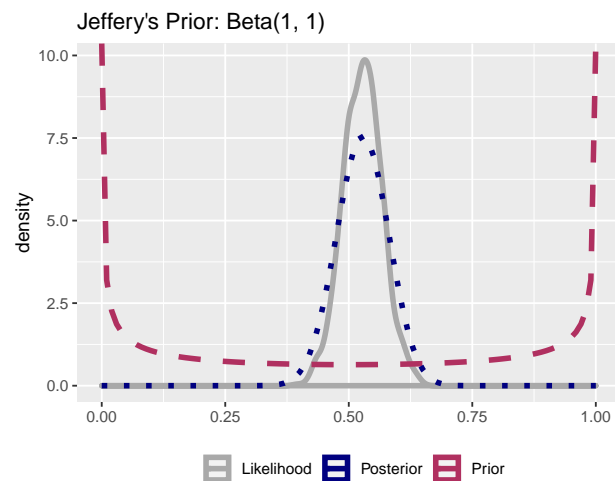


### Problem 3c

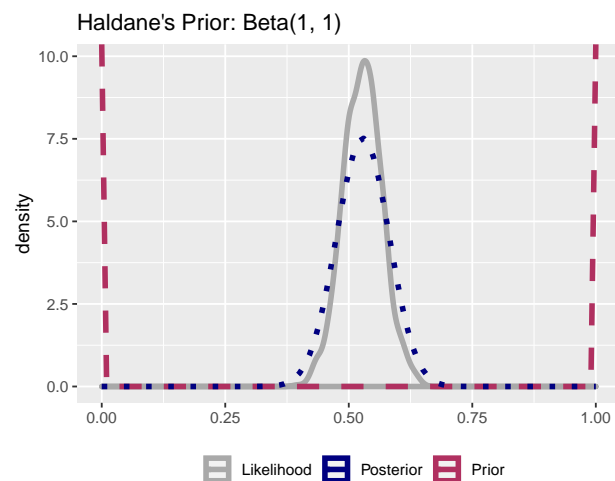
Look at the Cubs during the 2019 MLB season. As of the allstar break on 7/7/2019 the Cubs had win/loss record of 47/42 (played 89 games). In the remaining 73 games, i.e. the rest of the regular season, predict how many games the cubs will win. Note: the actual final win/loss record for the 2019 Cubs was 84/78



|                  | Mean    | Mode    | Est..Median |
|------------------|---------|---------|-------------|
| Posterior Value  | 0.52747 | 0.52809 | 0.52768     |
| Est Games Won    | 85.451  | 85.551  | 85.483      |
| Actual Games Won | 84      | 84      | 84          |
| Difference       | -1.451  | -1.551  | -1.483      |



|                  | Mean    | Mode    | Est..Median |
|------------------|---------|---------|-------------|
| Posterior Value  | 0.52778 | 0.52841 | 0.52799     |
| Est Games Won    | 85.500  | 85.602  | 85.534      |
| Actual Games Won | 84      | 84      | 84          |
| Difference       | -1.500  | -1.602  | -1.534      |



|                  | Mean    | Mode    | Est..Median |
|------------------|---------|---------|-------------|
| Posterior Value  | 0.52809 | 0.52874 | 0.52830     |
| Est Games Won    | 85.551  | 85.655  | 85.585      |
| Actual Games Won | 84      | 84      | 84          |
| Difference       | -1.551  | -1.655  | -1.585      |

## Problem 4

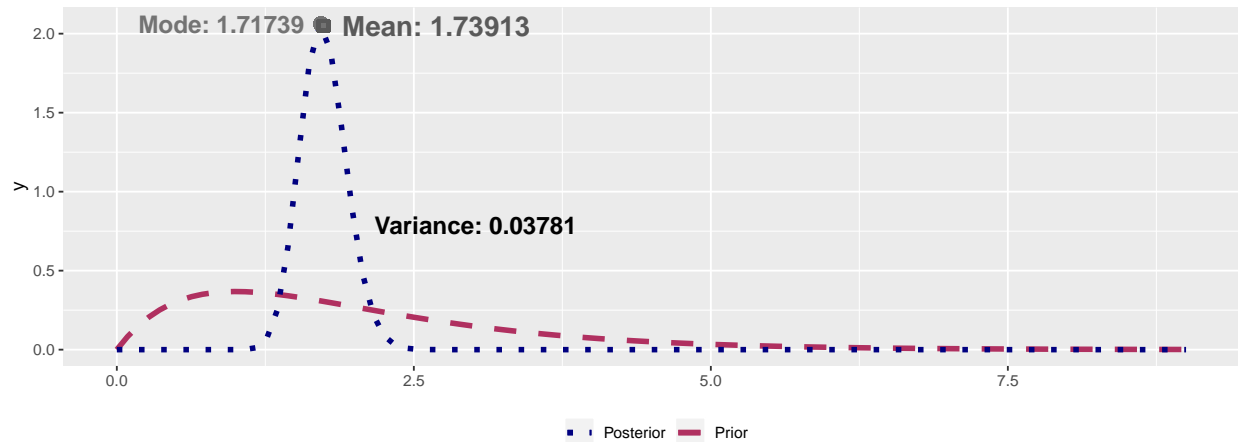
The following data represents the number of arrivals for 45 time intervals of length 2 minutes at a cashier's desk at a supermarket and are taken from Andersen (1980):

```
Arrival <- c(rep(0,6), rep(1,18), rep(2,9), rep(3,7), rep(4, 4), rep(5, 1))
```

### Problem 4a

For a Gamma(2,1) prior, obtain the posterior distribution under a Poisson ( $\lambda$ ) model for the data. Draw the prior and the posterior. Note on your plot the mean, variance and mode of the posterior.

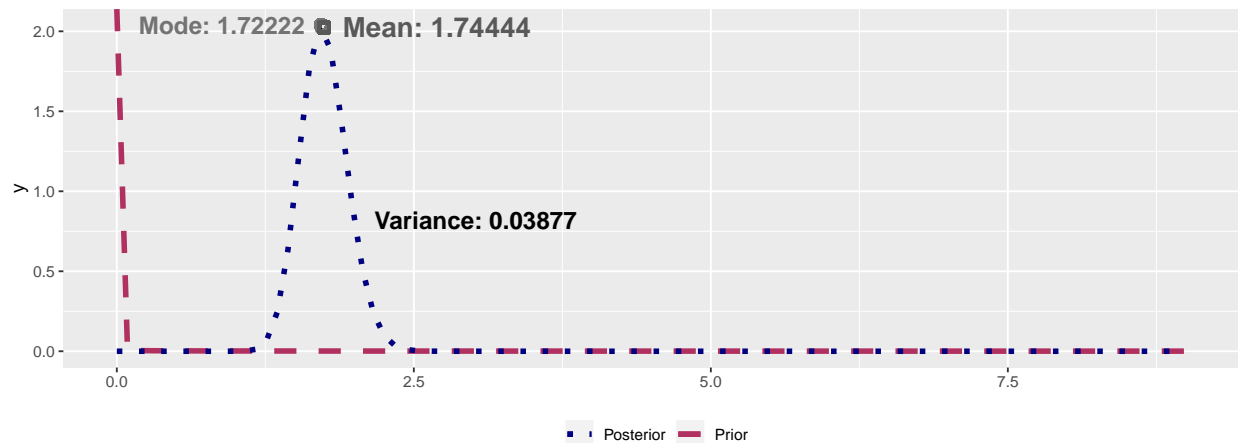
Gamma(2,1) Prior, Poisson Likelihood, Gamma Posterior



### Problem 4b

For the noninformative prior, i.e. a Gamma(?, ?), repeat part a.

Gamma(.5, .00001) Prior, Poisson Likelihood, Gamma Posterior



### Problem 5

197 animals are distributed into four categories:  $Y = (y_1, y_2, y_3, y_4)$  according to the genetic linkage model  $\left(\frac{2+\theta}{4}, \frac{1-\theta}{4}, \frac{1-\theta}{4}, \frac{\theta}{4}\right)$

$$\begin{aligned}
 L(\theta | \mathbf{Y}) &= \frac{(y_1 + y_2 + y_3 + y_4)!}{y_1! y_2! y_3! y_4!} \left(\frac{2+\theta}{4}\right)^{y_1} \left(\frac{1-\theta}{4}\right)^{y_2} \left(\frac{1-\theta}{4}\right)^{y_3} \left(\frac{\theta}{4}\right)^{y_4} \\
 &\propto (2+\theta)^{y_1} \cdot (1-\theta)^{y_2+y_3} \cdot (\theta)^{y_4} \\
 \ell(\theta | \mathbf{Y}) &\propto y_1 \log(2+\theta) + (y_2 + y_3) \log(1-\theta) + y_4 \log(\theta) \\
 \frac{\partial \ell}{\partial \theta} &= \frac{y_1}{2+\theta} - \frac{y_2 + y_3}{1-\theta} + \frac{y_4}{\theta} \\
 \frac{\partial^2 \ell}{\partial \theta^2} &= -\frac{y_1}{(2+\theta)^2} - \frac{y_2 + y_3}{(1-\theta)^2} - \frac{y_4}{\theta^2} \\
 \theta^{(i+1)} &= \theta^{(i)} - \frac{\frac{y_1}{2+\theta^{(i)}} - \frac{y_2+y_3}{1-\theta^{(i)}} + \frac{y_4}{\theta^{(i)}}}{-\frac{y_1}{(2+\theta^{(i)})^2} - \frac{y_2+y_3}{(1-\theta^{(i)})^2} - \frac{y_4}{(\theta^{(i)})^2}}
 \end{aligned}$$

#### Problem 5a

$$L(\theta | \mathbf{Y} = (125, 18, 20, 34)) \propto (2+\theta)^{125} \cdot (1-\theta)^{38} \cdot (\theta)^{34}$$

#### Problem 5b

$$L(\theta | \mathbf{Y} = (14, 0, 1, 5)) \propto (2+\theta)^{14} \cdot (1-\theta)^1 \cdot (\theta)^5$$

#### Problem 5c

Use the Newton-Raphson to obtain the MLE ( $\hat{\theta}$ ) for  $Y = (125, 18, 20, 34)$ . Start the algorithm at  $\theta = .1, .2, .3, .4, .6, .8$ .

```
## [1] "Start at 0.1 : Root approximation is 0.626821497870988 with 6 iterations"
## [1] "Start at 0.2 : Root approximation is 0.626821497871005 with 5 iterations"
## [1] "Start at 0.3 : Root approximation is 0.626821497870983 with 5 iterations"
## [1] "Start at 0.4 : Root approximation is 0.626821497870984 with 4 iterations"
## [1] "Start at 0.6 : Root approximation is 0.626821497874505 with 3 iterations"
## [1] "Start at 0.8 : Root approximation is 0.626821497870986 with 5 iterations"
```

#### Problem 5d

Repeat 5c for  $Y = (14, 0, 1, 15)$ .

```
## [1] "Start at 0.1 : Root approximation is 0.903440114240028 with 8 iterations"
## [1] "Start at 0.2 : Root approximation is 0.903440114216673 with 6 iterations"
## [1] "Start at 0.3 : diverges"
## [1] "Start at 0.4 : Root approximation is 0.903440114216814 with 6 iterations"
## [1] "Start at 0.6 : diverges"
## [1] "Start at 0.8 : Root approximation is 0.903440114216679 with 8 iterations"
```