# STAT 457 Homework 04

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## Problem 1

Let $y_1, \cdots, y_n$ be an iid sample from the Poisson distribution with parameter $\lambda$. Derive Jeffery's (noninformative) prior. This prior corresponds to the gamma distribution with which parameters?

$$
\begin{aligned}
L(\lambda \mid Y) &= \frac{\lambda^y e^{-\lambda}}{y!} \propto \lambda^y e^{-\lambda} \\
\ell(\lambda \mid Y) &= y \log \lambda - \lambda \\
\frac{\partial \ell(\lambda \mid Y)}{\partial \lambda} &= \frac{y}{\lambda} - 1 \\
\frac{\partial^2 \ell(\lambda \mid Y)}{\partial \lambda^2} &= -\frac{y}{\lambda^2} \\
\mathcal{I}(\lambda) &= \mathbb{E}\left[-\frac{\partial^2 \ell(\lambda \mid Y)}{\partial \lambda^2}\right] \\
&= \mathbb{E}\left[\frac{y}{\lambda^2}\right] \\
&= \frac{1}{\lambda} \\
\text{Jeffrey's Prior} \quad p(\lambda) &= \sqrt{\mathcal{I}(\lambda)} = \sqrt{\frac{1}{\lambda}} \approx \text{Gamma}\left(\frac{1}{2}, 0\right)
\end{aligned}
$$

## Problem 2

In the multivariate setting, $\theta = (\theta_1, \cdots, \theta_d),\ p(\theta) \propto |J(\theta)|^{1/2}$, provides an invariant prior where the $ij^{\text{th}}$ entry of $J(\theta)$ is equal to

$$-\mathbb{E}\left[\frac{\partial^2 \ell(\theta \mid Y)}{\partial\theta_i \partial\theta_j}\right]$$

and $|X|$ is the determinant of the matrix $X$.

Let $y_1, \cdots, y_n$ be an iid sample from the $\mathcal{N}(\mu, \sigma^2)$ distribution, where $\mu$ and $\sigma$ are both unknown. Derive the invariant prior. How does it compare with the prior $p(\theta, \sigma^2) \propto {}^1\!/\!_{\sigma^2}$?

$$\begin{aligned}
L(\mu, \sigma^2 \mid \boldsymbol{Y}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(y_i-\mu)^2}{2\sigma^2}\right\} \\
\ell(\mu, \sigma^2 \mid \boldsymbol{Y}) &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(y_i-\mu)^2}{2\sigma^2}\right\}\right) \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{\sum_i (y_i-\mu)^2}{2\sigma^2} \\
\frac{\partial^2 \ell}{\partial\mu^2} &= -\frac{n}{\sigma^2} \\
\frac{\partial^2 \ell}{\partial\mu\partial\sigma^2} &= -\frac{\sum_i (y_i-\mu)^2}{\sigma^4} = \frac{\partial^2 \ell}{\partial\sigma^2\partial\mu} \\
\frac{\partial^2 \ell}{\partial(\sigma^2)^2} &= \frac{n}{\sigma^2} = \frac{n}{2\sigma^4} - \frac{\sum_i (y_i-\mu)^2}{\sigma^6} \\
p(\mu, \sigma^2) &\propto |J(\mu, \sigma^2)|^{\frac{1}{2}} \\
&= \left(-\frac{1}{n}\det\begin{bmatrix} -{}^n\!/\!_{\sigma^2} & 0 \\ 0 & -{}^n\!/\!_{2\sigma^4} \end{bmatrix}\right)^{\frac{1}{2}} \\
&= \frac{1}{\sigma^3}
\end{aligned}$$

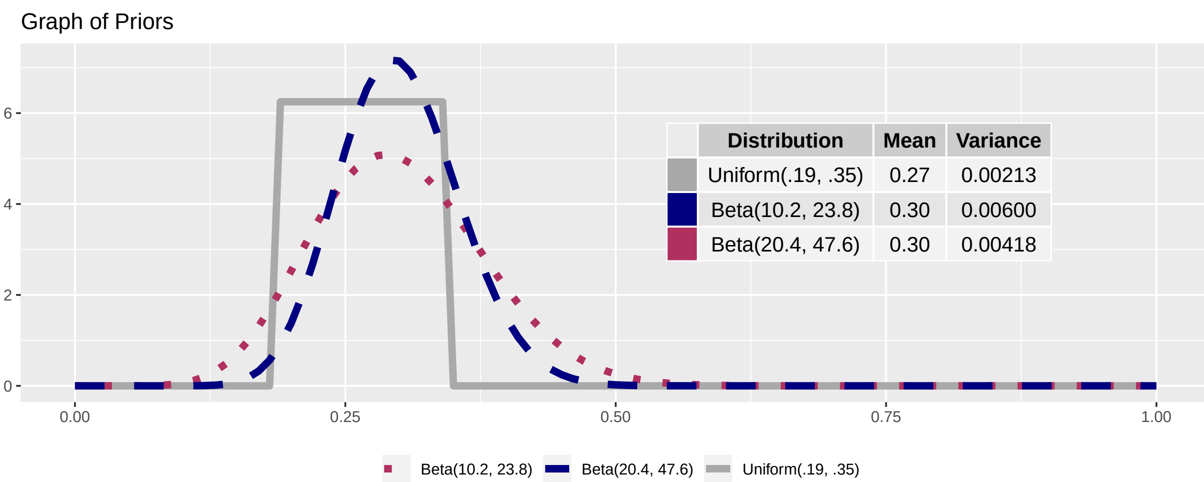
This is different than commonly used prior of ${}^1\!/\!_{\sigma^2}$. This shows that the Jeffery's prior does not work in every situation.

## Problem 3

Let $p$ denote the probability that a specific major league baseball player will get a hit in a particular at bat. Assume that batting averages usually fall in the range .19 to .35.

### Problem 3a

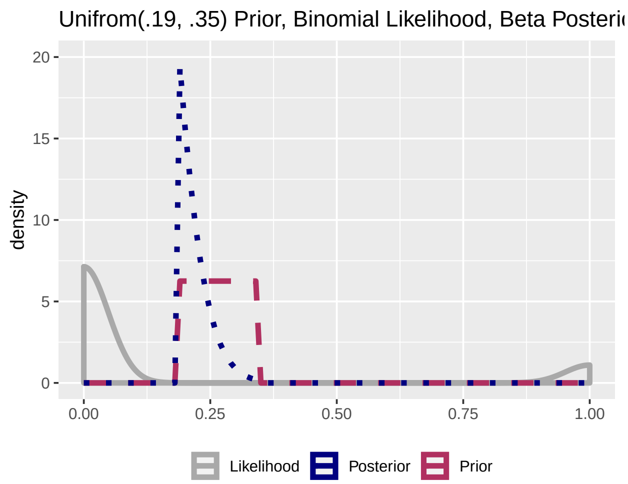
Consider the priors Uniform(.19, .35); Beta(10.2, 23.8); and Beta(20.4,47.6). Plot these priors and discuss each choice.

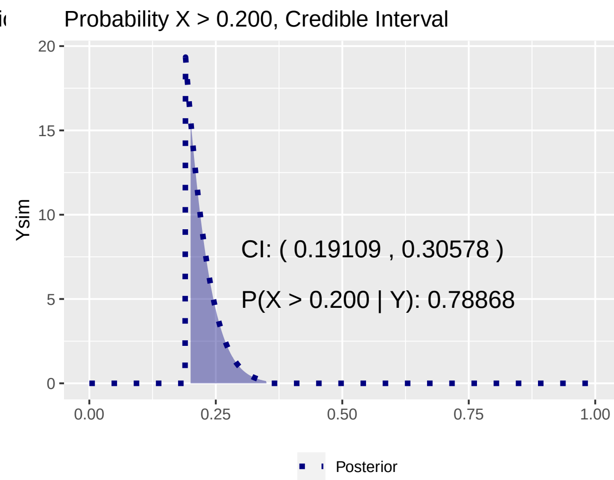


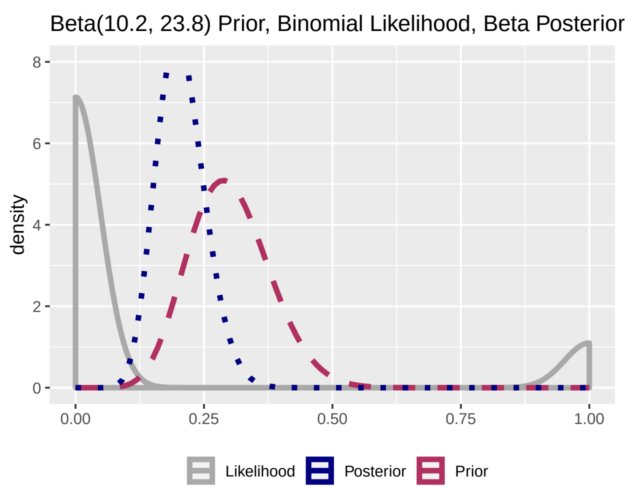

The Uniform has a mean of 0.27 and both beta distributions have a mean of 0.30. The uniform distribution has the lowest variance and a block shape. The Beta(10.2,23.8) has the highest variance and a lower peak than the Beta(20.4,47.6).

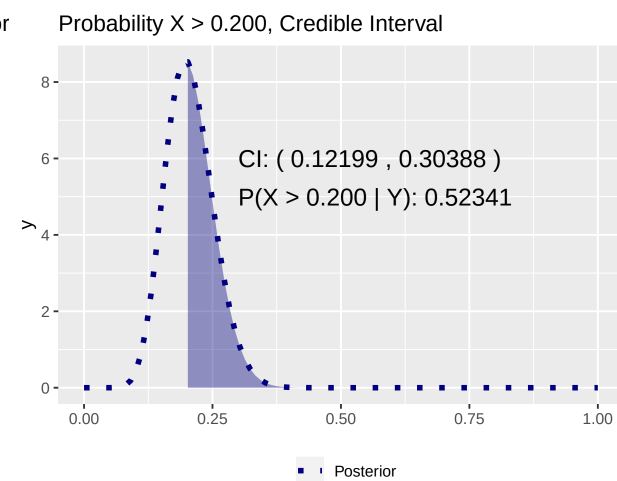
### Problem 3b

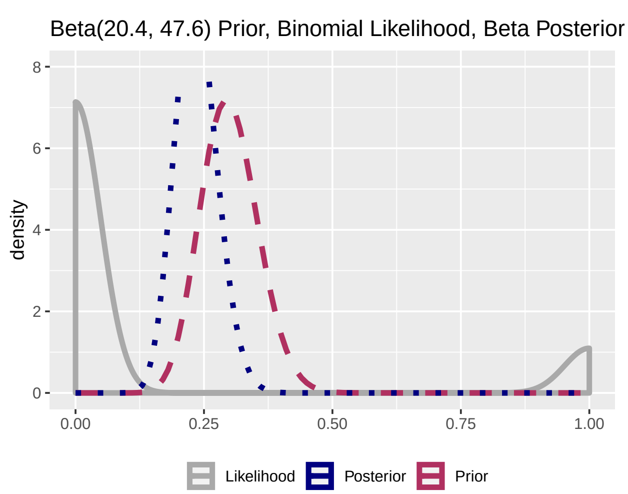
Suppose a player gets 5 hits in 40 at-bats. For each of the above priors: plot the likelihood, posterior and prior; compute the probability that he player is better than a .200 hitter; compute your best guess as to the batting average of the player; compute a 95% credible interval for $p$.

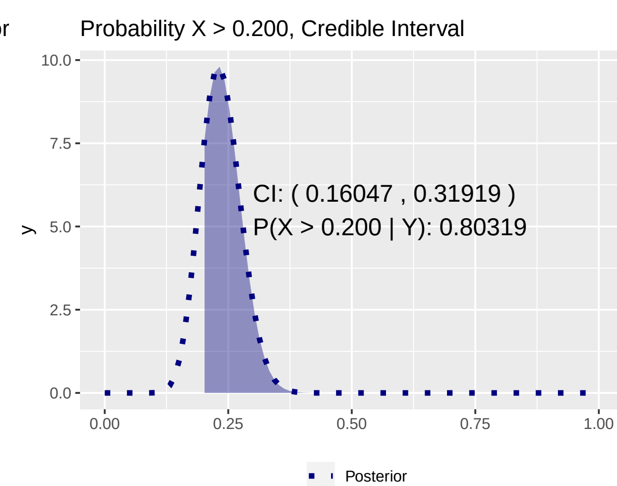


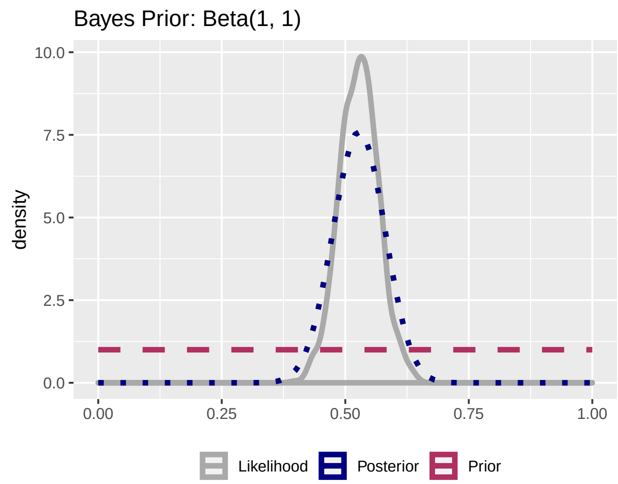



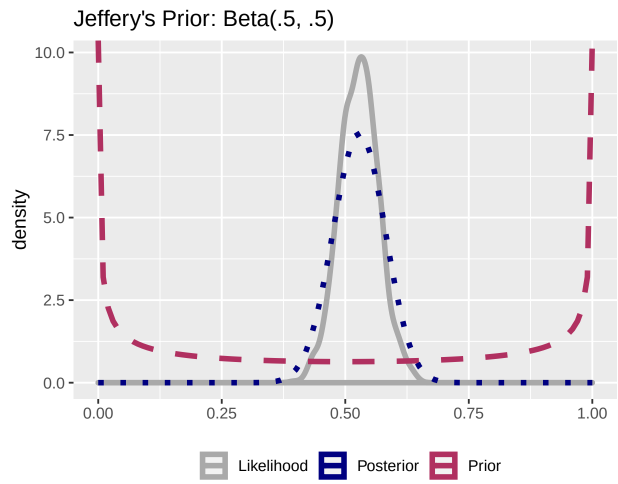



### Problem 3c

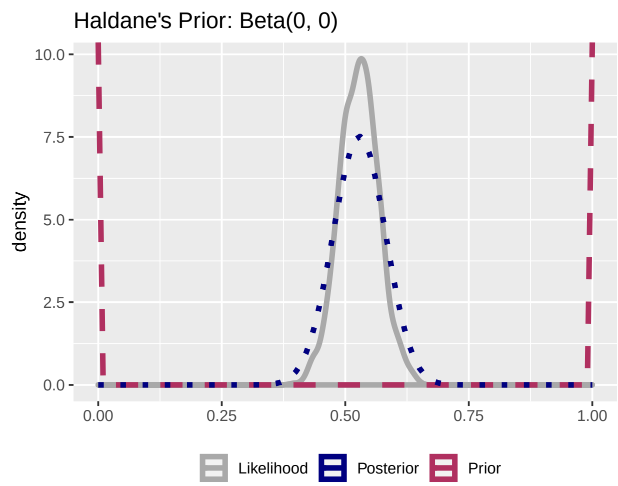
Look at the Cubs during the 2019 MLB season. As of the allstar break on 7/7/2019 the Cubs had win/loss record of 47/42 (played 89 games). In the remaining 73 games, i.e. the rest of the regular season, predict how many games the cubs will win. Note: the actual final win/loss record for the 2019 Cubs was 84/78




| | Mean | Mode | Est..Median |
|---|---|---|---|
| *Posterior Value* | 0.52747 | 0.52809 | 0.52768 |
| *Est Games Won* | 85.451 | 85.551 | 85.483 |
| *Actual Games Won* | 84 | 84 | 84 |
| *Difference* | −1.451 | −1.551 | −1.483 |




| | Mean | Mode | Est..Median |
|---|---|---|---|
| *Posterior Value* | 0.52778 | 0.52841 | 0.52799 |
| *Est Games Won* | 85.500 | 85.602 | 85.534 |
| *Actual Games Won* | 84 | 84 | 84 |
| *Difference* | −1.500 | −1.602 | −1.534 |




| | Mean | Mode | Est..Median |
|---|---|---|---|
| *Posterior Value* | 0.52809 | 0.52874 | 0.52830 |
| *Est Games Won* | 85.551 | 85.655 | 85.585 |
| *Actual Games Won* | 84 | 84 | 84 |
| *Difference* | −1.551 | −1.655 | −1.585 |

## Problem 4

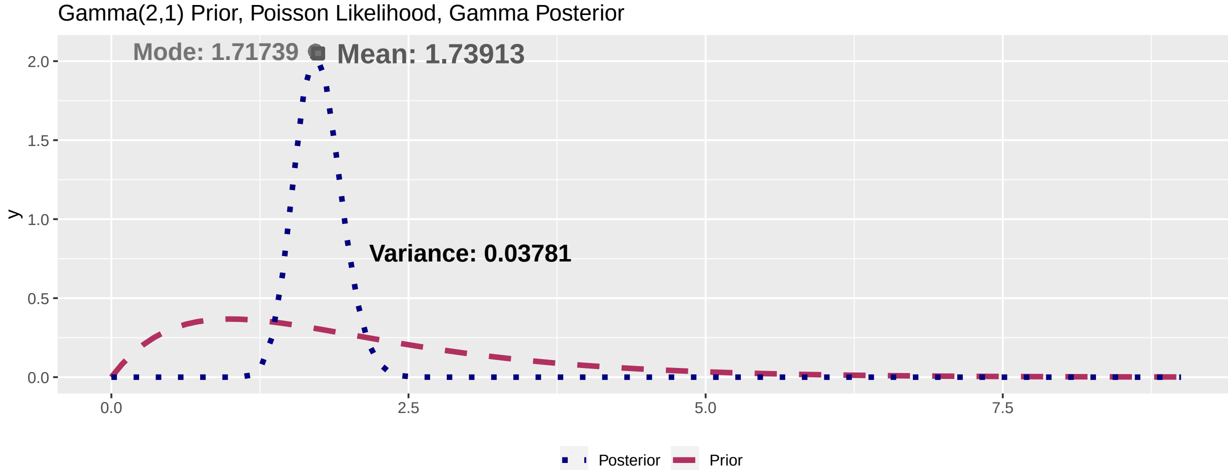
The following data represents the number of arrivals for 45 time intervals of length 2 minutes at a cashier's desk at a supermarket and are taken from Andersen (1980):

```
Arrival <- c(rep(0,6), rep(1,18), rep(2,9), rep(3,7), rep(4, 4), rep(5, 1))
```

### Problem 4a

For a Gamma(2,1) prior, obtain the posterior distribution under a Poisson ($\lambda$) model for the data. Draw the prior and the posterior. Note on your plot the mean, variance and mode of the posterior.

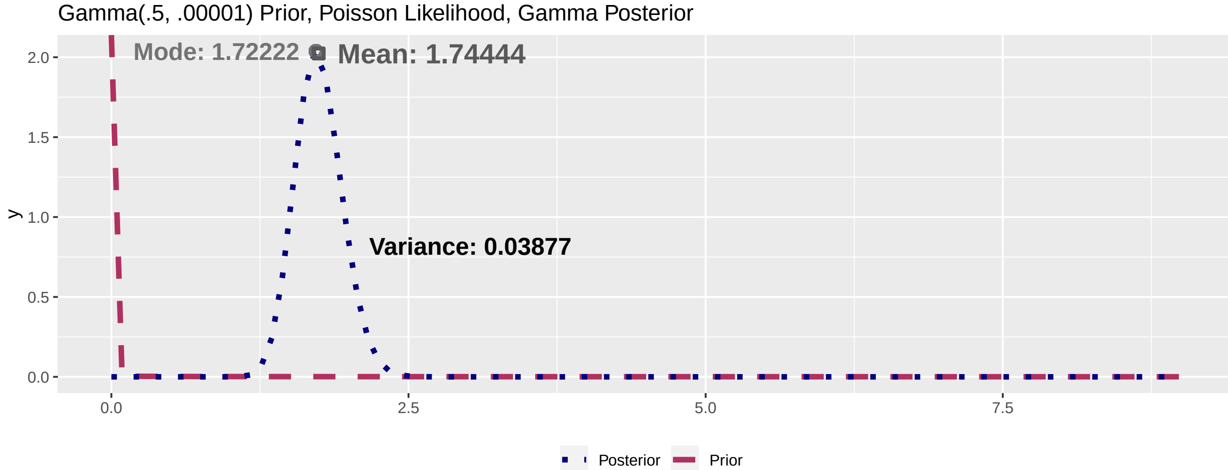
Gamma(2,1) Prior, Poisson Likelihood, Gamma Posterior




### Problem 4b

For the noninformative prior, i.e. a Gamma(?, ?), repeat part a.

Gamma(.5, .00001) Prior, Poisson Likelihood, Gamma Posterior



## Problem 5

197 animals are distributed into four categories: $Y = (y_1, y_2, y_3, y_4)$ according ot the genetic linkage model $\left(\frac{2+\theta}{4}, \frac{1-\theta}{4}, \frac{1-\theta}{4}, \frac{\theta}{4}\right)$

$$
\begin{aligned}
L(\theta \mid \boldsymbol{Y}) &= \frac{(y_1+y_2+y_3+y_4)!}{y_1!y_2!y_2!y_4!}\left(\frac{2+\theta}{4}\right)^{y_1}\left(\frac{1-\theta}{4}\right)^{y_2}\left(\frac{1-\theta}{4}\right)^{y_3}\left(\frac{\theta}{4}\right)^{y_4} \\
&\propto (2+\theta)^{y_1}\cdot(1-\theta)^{y_2+y_3}\cdot(\theta)^{y_4} \\
\ell(\theta \mid \boldsymbol{Y}) &\propto y_1\log(2+\theta) + (y_2+y_3)\log(1-\theta) + y_4\log(\theta) \\
\frac{\partial \ell}{\partial \theta} &= \frac{y_1}{2+\theta} - \frac{y_2+y_3}{1-\theta} + \frac{y_4}{\theta} \\
\frac{\partial^2 \ell}{\partial \theta^2} &= -\frac{y_1}{(2+\theta)^2} - \frac{y_2+y_3}{(1-\theta)^2} - \frac{y_4}{\theta^2} \\
\theta^{(i+1)} &= \theta^{(i)} - \frac{\frac{y_1}{2+\theta^{(i)}} - \frac{y_2+y_3}{1-\theta^{(i)}} + \frac{y_4}{\theta^{(i)}}}{-\frac{y_1}{(2+\theta^{(i)})^2} - \frac{y_2+y_3}{(1-\theta^{(i)})^2} - \frac{y_4}{(\theta^{(i)})^2}}
\end{aligned}
$$

### Problem 5a

$$L(\theta \mid \boldsymbol{Y} = (125, 18, 20, 34)) \propto (2+\theta)^{125}\cdot(1-\theta)^{38}\cdot(\theta)^{34}$$

### Problem 5b

$$L(\theta \mid \boldsymbol{Y} = (14, 0, 1, 5)) \propto (2+\theta)^{14}\cdot(1-\theta)^{1}\cdot(\theta)^{5}$$

### Problem 5c

Use th Newton-Raphson to obtain the MLE $(\hat{\theta})$ for $Y = (125, 18, 20, 34)$. Start the algorithm at $\theta = .1, .2, .3, .4, .6, .8$.

```
## [1] "Start at 0.1 : Root approximation is 0.626821497870988 with 6 iterations"

## [1] "Start at 0.2 : Root approximation is 0.626821497871005 with 5 iterations"

## [1] "Start at 0.3 : Root approximation is 0.626821497870983 with 5 iterations"

## [1] "Start at 0.4 : Root approximation is 0.626821497870984 with 4 iterations"

## [1] "Start at 0.6 : Root approximation is 0.626821497874505 with 3 iterations"

## [1] "Start at 0.8 : Root approximation is 0.626821497870986 with 5 iterations"
```

### Problem 5d

Repeat 5c for $Y = (14, 0, 1, 15)$.

```
## [1] "Start at 0.1 : Root approximation is 0.903440114240028 with 8 iterations"

## [1] "Start at 0.2 : Root approximation is 0.903440114216673 with 6 iterations"

## [1] "Start at 0.3 : diverges"

## [1] "Start at 0.4 : Root approximation is 0.903440114216814 with 6 iterations"

## [1] "Start at 0.6 : diverges"

## [1] "Start at 0.8 : Root approximation is 0.903440114216679 with 8 iterations"
```