STAT 457 Homework 04

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Problem 1

Let $y_1, \cdots, y_n$ be an iid sample from the Poisson distribution with parameter λ. Derive Jeffery's (noninformative) prior. This prior corresponds to the gamma distribution with which parameters?

$$
\begin{aligned}
L(\lambda \mid Y) &= \frac{\lambda^y e^{-\lambda}}{y!} \propto \lambda^y e^{-\lambda} \\
\ell(\lambda \mid Y) &= y \log \lambda - \lambda \\
\frac{\partial \ell(\lambda \mid Y)}{\partial \lambda} &= \frac{y}{\lambda} - 1 \\
\frac{\partial^2 \ell(\lambda \mid Y)}{\partial \lambda^2} &= -\frac{y}{\lambda^2} \\
\mathcal{I}(\lambda) &= \mathbb{E}\left[-\frac{\partial^2 \ell(\lambda \mid Y)}{\partial \lambda^2}\right] \\
&= \mathbb{E}\left[\frac{y}{\lambda^2}\right] \\
&= \frac{1}{\lambda} \\
\text{Jeffrey's Prior} \quad p(\lambda) &= \sqrt{\mathcal{I}(\lambda)} = \sqrt{\frac{1}{\lambda}} \approx \text{Gamma}\left(\frac{1}{2}, 0\right)
\end{aligned}
$$

Problem 2

In the multivariate setting, $\theta = (\theta_1, \cdots, \theta_d),\ p(\theta) \propto |J(\theta)|^{1/2}$, provides an invariant prior where the ij^{th} entry of $J(\theta)$ is equal to

$$-\mathbb{E}\left[\frac{\partial^2 \ell(\theta \mid Y)}{\partial \theta_i \partial \theta_j}\right]$$

and $|X|$ is the determinant of the matrix X.

Let $y_1, \cdots, y_n$ be an iid sample from the $\mathcal{N}(\mu, \sigma^2)$ distribution, where μ and σ are both unknown. Derive the invariant prior. How does it compare with the prior $p(\theta, \sigma^2) \propto {}^{1}/_{\sigma^2}$?

$$\begin{aligned}
L(\mu, \sigma^2 \mid \boldsymbol{Y}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(y_i - \mu)^2}{2\sigma^2}\right\} \\
\ell(\mu, \sigma^2 \mid \boldsymbol{Y}) &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(y_i - \mu)^2}{2\sigma^2}\right\}\right) \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{\sum_i (y_i - \mu)^2}{2\sigma^2} \\
\frac{\partial^2 \ell}{\partial \mu^2} &= -\frac{n}{\sigma^2} \\
\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2} &= -\frac{\sum_i (y_i - \mu)^2}{\sigma^4} = \frac{\partial^2 \ell}{\partial \sigma^2 \partial \mu} \\
\frac{\partial^2 \ell}{\partial (\sigma^2)^2} &= \frac{n}{\sigma^2} = \frac{n}{2\sigma^4} - \frac{\sum_i (y_i - \mu)^2}{\sigma^6} \\
p(\mu, \sigma^2) &\propto |J(\mu, \sigma^2)|^{\frac{1}{2}} \\
&= \left(-\frac{1}{n}\det\begin{bmatrix} -{}^{n}/_{\sigma^2} & 0 \\ 0 & -{}^{n}/_{2\sigma^4} \end{bmatrix}\right)^{\frac{1}{2}} \\
&= \frac{1}{\sigma^3}
\end{aligned}$$

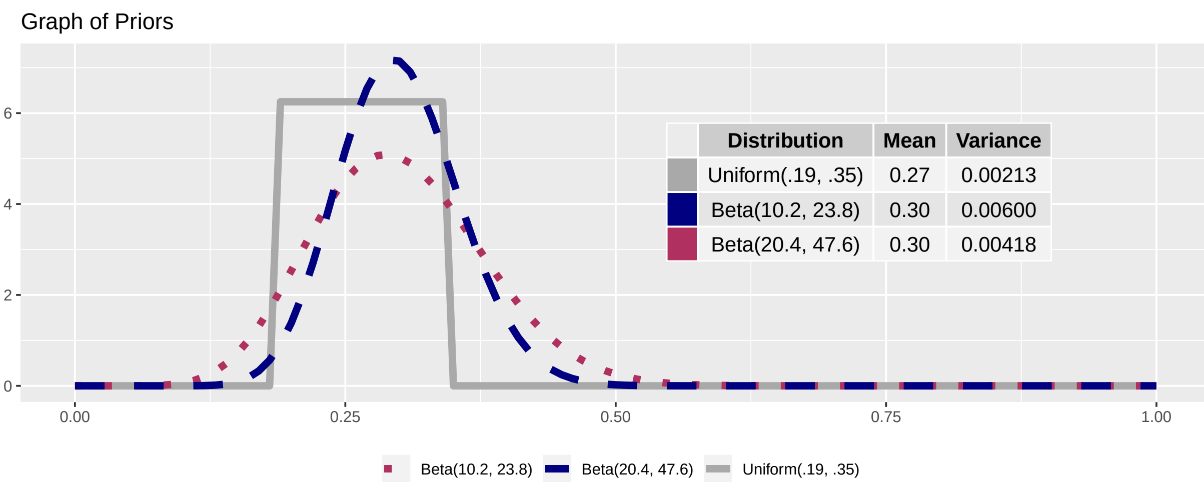
This is different than commonly used prior of ${}^{1}/_{\sigma^2}$. This shows that the Jeffery's prior does not work in every situation.

Problem 3

Let p denote the probability that a specific major league baseball player will get a hit in a particular at bat. Assume that batting averages usually fall in the range .19 to .35.

Problem 3a

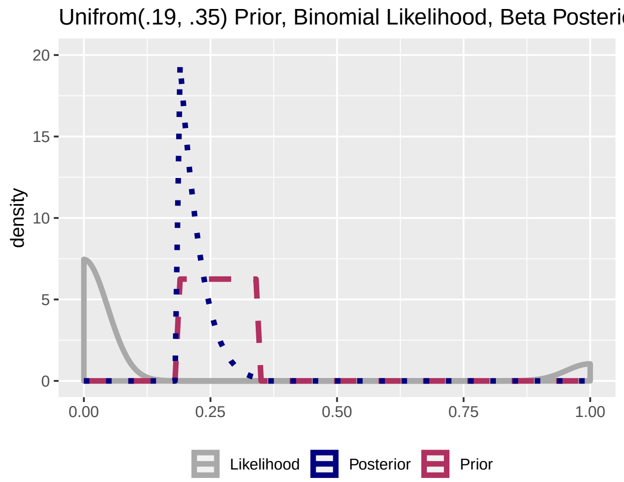
Consider the priors Uniform(.19, .35); Beta(10.2, 23.8); and Beta(20.4,47.6). Plot these priors and discuss each choice.

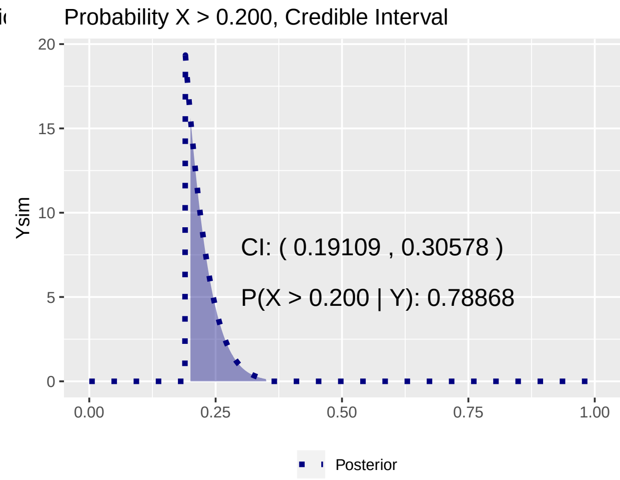


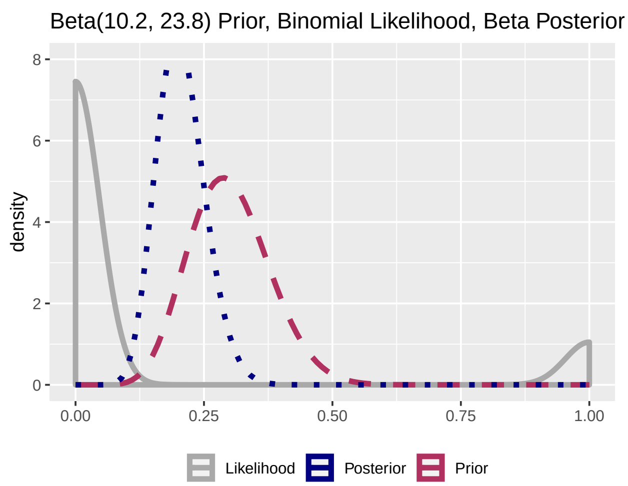
The Uniform has a mean of 0.27 and both beta distributions have a mean of 0.30. The uniform distribution has the lowest variance and a block shape. The Beta(10.2,23.8) has the highest variance and a lower peak than the Beta(20.4,47.6).

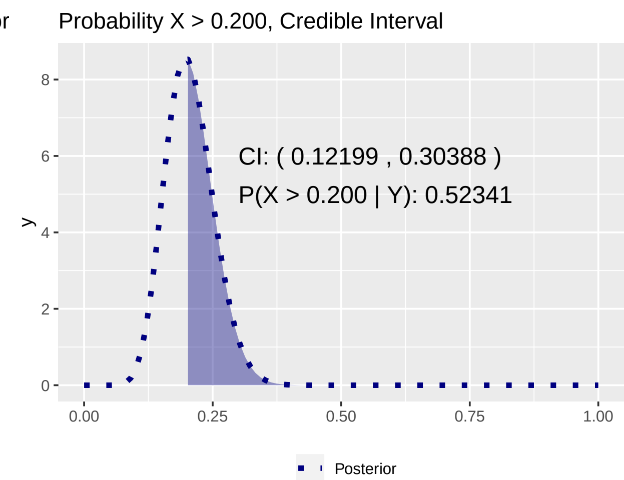
Problem 3b

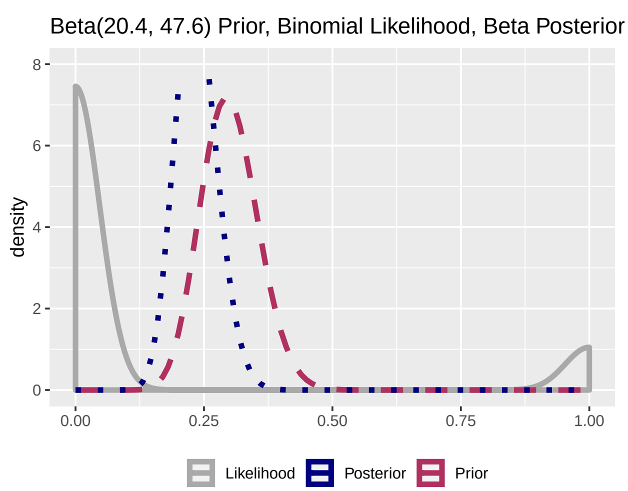
Suppose a player gets 5 hits in 40 at-bats. For each of the above priors: plot the likelihood, posterior and prior; compute the probability that he player is better than a .200 hitter; compute your best guess as to the batting average of the player; compute a 95% credible interval for p.

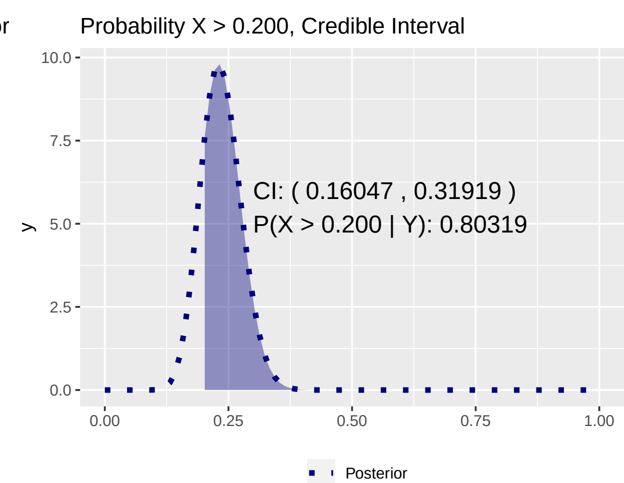






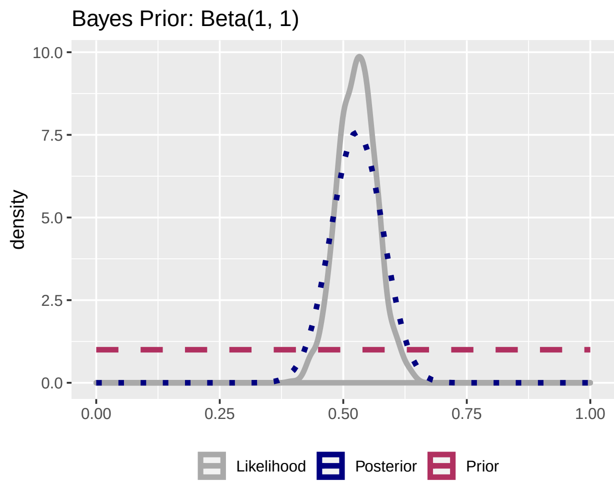




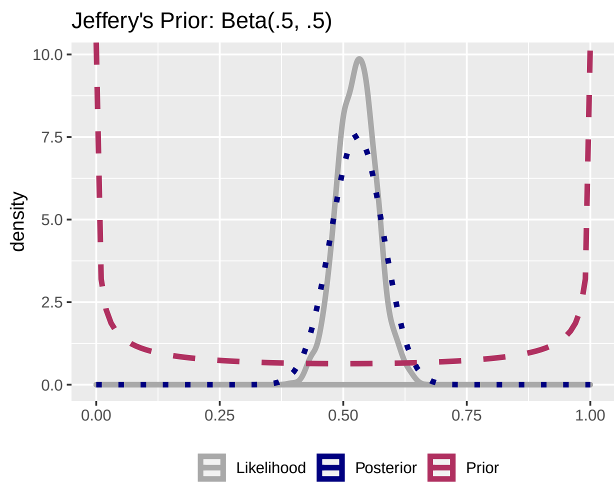


Problem 3c

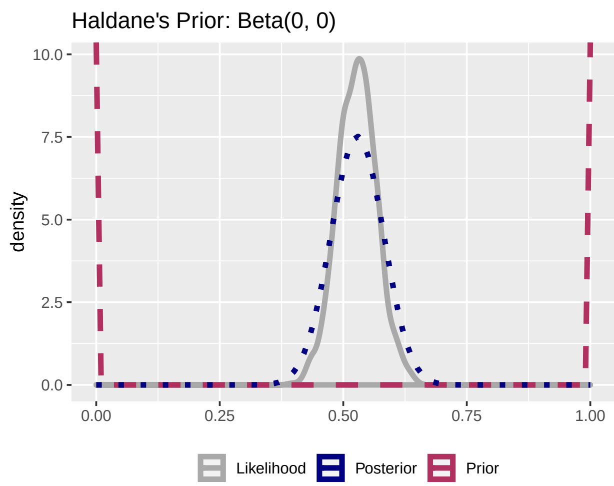
Look at the Cubs during the 2019 MLB season. As of the allstar break on 7/7/2019 the Cubs had win/loss record of 47/42 (played 89 games). In the remaining 73 games, i.e. the rest of the regular season, predict how many games the cubs will win. Note: the actual final win/loss record for the 2019 Cubs was 84/78



	Mean	Mode	Est..Median
Posterior Value	0.52747	0.52809	0.52768
Est Games Won	85.451	85.551	85.483
Actual Games Won	84	84	84
Difference	−1.451	−1.551	−1.483



	Mean	Mode	Est..Median
Posterior Value	0.52778	0.52841	0.52799
Est Games Won	85.500	85.602	85.534
Actual Games Won	84	84	84
Difference	−1.500	−1.602	−1.534



	Mean	Mode	Est..Median
Posterior Value	0.52809	0.52874	0.52830
Est Games Won	85.551	85.655	85.585
Actual Games Won	84	84	84
Difference	−1.551	−1.655	−1.585

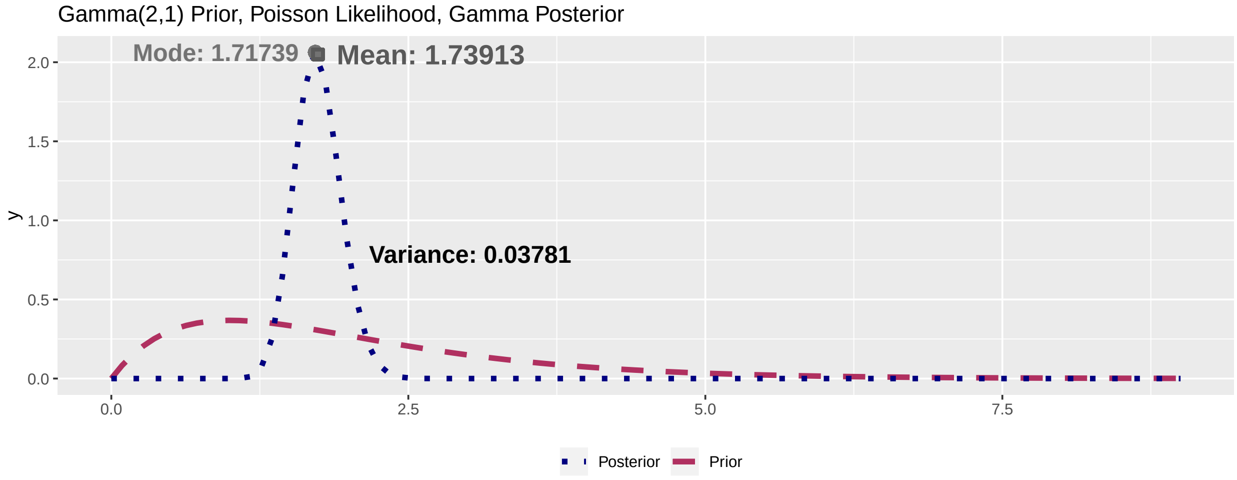
Problem 4

The following data represents the number of arrivals for 45 time intervals of length 2 minutes at a cashier's desk at a supermarket and are taken from Andersen (1980):

```
Arrival <- c(rep(0,6), rep(1,18), rep(2,9), rep(3,7), rep(4, 4), rep(5, 1))
```

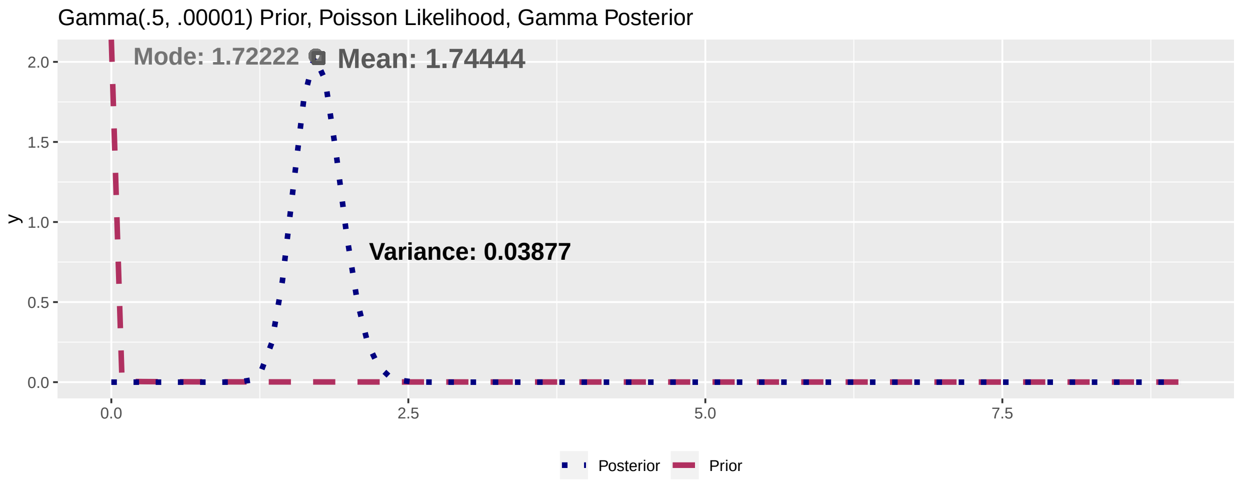
Problem 4a

For a Gamma(2,1) prior, obtain the posterior distribution under a Poisson (λ) model for the data. Draw the prior and the posterior. Note on your plot the mean, variance and mode of the posterior.



Problem 4b

For the noninformative prior, i.e. a Gamma(?, ?), repeat part a.



Problem 5

197 animals are distributed into four categories: $Y = (y_1, y_2, y_3, y_4)$ according ot the genetic linkage model $\left(\frac{2+\theta}{4}, \frac{1-\theta}{4}, \frac{1-\theta}{4}, \frac{\theta}{4}\right)$

$$
\begin{aligned}
L(\theta \mid \boldsymbol{Y}) &= \frac{(y_1+y_2+y_3+y_4)!}{y_1!y_2!y_2!y_4!} \left(\frac{2+\theta}{4}\right)^{y_1} \left(\frac{1-\theta}{4}\right)^{y_2} \left(\frac{1-\theta}{4}\right)^{y_3} \left(\frac{\theta}{4}\right)^{y_4} \\
&\propto (2+\theta)^{y_1} \cdot (1-\theta)^{y_2+y_3} \cdot (\theta)^{y_4} \\
\ell(\theta \mid \boldsymbol{Y}) &\propto y_1 \log(2+\theta) + (y_2+y_3)\log(1-\theta) + y_4 \log(\theta) \\
\frac{\partial \ell}{\partial \theta} &= \frac{y_1}{2+\theta} - \frac{y_2+y_3}{1-\theta} + \frac{y_4}{\theta} \\
\frac{\partial^2 \ell}{\partial \theta^2} &= -\frac{y_1}{(2+\theta)^2} - \frac{y_2+y_3}{(1-\theta)^2} - \frac{y_4}{\theta^2} \\
\theta^{(i+1)} &= \theta^{(i)} - \frac{\frac{y_1}{2+\theta^{(i)}} - \frac{y_2+y_3}{1-\theta^{(i)}} + \frac{y_4}{\theta^{(i)}}}{-\frac{y_1}{(2+\theta^{(i)})^2} - \frac{y_2+y_3}{(1-\theta^{(i)})^2} - \frac{y_4}{(\theta^{(i)})^2}}
\end{aligned}
$$

Problem 5a

$$L(\theta \mid \boldsymbol{Y} = (125, 18, 20, 34)) \propto (2+\theta)^{125} \cdot (1-\theta)^{38} \cdot (\theta)^{34}$$

Problem 5b

$$L(\theta \mid \boldsymbol{Y} = (14, 0, 1, 5)) \propto (2+\theta)^{14} \cdot (1-\theta)^{1} \cdot (\theta)^{5}$$

Problem 5c

Use th Newton-Raphson to obtain the MLE $(\hat{\theta})$ for $Y = (125, 18, 20, 34)$. Start the algorithm at $\theta = .1, .2, .3, .4, .6, .8$.

```
## [1] "Start at 0.1 : Root approximation is 0.626821497870988 with 6 iterations"

## [1] "Start at 0.2 : Root approximation is 0.626821497871005 with 5 iterations"

## [1] "Start at 0.3 : Root approximation is 0.626821497870983 with 5 iterations"

## [1] "Start at 0.4 : Root approximation is 0.626821497870984 with 4 iterations"

## [1] "Start at 0.6 : Root approximation is 0.626821497874505 with 3 iterations"

## [1] "Start at 0.8 : Root approximation is 0.626821497870986 with 5 iterations"
```

Problem 5d

Repeat 5c for $Y = (14, 0, 1, 15)$.

```
## [1] "Start at 0.1 : Root approximation is 0.903440114240028 with 8 iterations"

## [1] "Start at 0.2 : Root approximation is 0.903440114216673 with 6 iterations"

## [1] "Start at 0.3 : diverges"

## [1] "Start at 0.4 : Root approximation is 0.903440114216814 with 6 iterations"

## [1] "Start at 0.6 : diverges"

## [1] "Start at 0.8 : Root approximation is 0.903440114216679 with 8 iterations"
```