1. Prove that $\mathbb{E}\left[(X-\mathbb{E}[X])^2\right]=\mathbb{E}[X^2]-(\mathbb{E}[X])^2$.

$$\begin{aligned}\mathbb{E}\left[\left(X-\mathbb{E}[X]^2\right)\right] &= \mathbb{E}\left[X^2-2X\mathbb{E}[X]+\mathbb{E}[X]^2\right] \\ &= \mathbb{E}\left[X^2\right]-2\mathbb{E}[X]\,\mathbb{E}[X]+\mathbb{E}[X]^2 \\ &= \mathbb{E}\left[X^2\right]-2\mathbb{E}[X]^2+\mathbb{E}[X]^2 \\ &= \mathbb{E}\left[X^2\right]-\mathbb{E}[X]^2 \qquad \square\end{aligned}$$

2. Prove that if X_1 and X_2 are independent, then $\mathbb{E}[X_1X_2]=\mathbb{E}[X_1]\,\mathbb{E}[X_2]$.

Continuous

$$\begin{aligned}\mathbb{E}[X_1X_2] &= \int_{-\infty}^{\infty} x_1x_2p_{X_1X_2}(x_1,x_2)dx_1dx_2 \\ &= \int_{-\infty}^{\infty} x_1x_2p_{X_1}(x_1)p_{X_2}(x_2)dx_1dx_2 && X_1 \perp\!\!\!\perp X_2 \implies p_{X_1X_2}(x_1,x_2)=p_{X_1}(x_1)p_{X_2}(x_2) \\ &= \int_{-\infty}^{\infty} x_1p_{X_1}(x_1)dx_1 \int_{-\infty}^{\infty} x_2p_{X_2}(x_2)dx_2 \\ &= \mathbb{E}[X_1]\,\mathbb{E}[X_2]\end{aligned}$$

Discrete

$$\begin{aligned}\mathbb{E}[X_1X_2] &= \sum_{x_1}\sum_{x_2} x_1x_2p_{X_1,X_2}(x_1,x_2) \\ &= \sum_{x_1}\sum_{x_2} x_1x_2p_{X_1}(x_1)p_{X_2}x_2) && X_1 \perp\!\!\!\perp X_2 \implies p_{X_1X_2}(x_1,x_2)=p_{X_1}(x_1)p_{X_2}(x_2) \\ &= \left(\sum_{x_1} x_1p_{X_1}(x_1)\right)\left(\sum_{x_2} x_2p_{X_2}(x_2)\right) \qquad \square\end{aligned}$$

3. Prove that $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)$, where $\text{Cov}(X_1, X_2) = \mathbb{E}\left[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])\right] = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$.

$$\begin{aligned}
\text{Var}(X_1 + X_2) &= \mathbb{E}\left[(X_1+X_2)^2\right] - \mathbb{E}[X_1+X_2]^2 \\
&= \mathbb{E}\left[X_1^2 + 2X_1X_2 + X_2^2\right] - (\mathbb{E}[X_1] + \mathbb{E}[X_2])^2 \\
&= \mathbb{E}\left[X_1^2\right] + 2\mathbb{E}[X_1X_2] + \mathbb{E}\left[X_2^2\right] - \left(\mathbb{E}[X_1]^2 + 2\mathbb{E}[X_1]\mathbb{E}[X_2] + \mathbb{E}[X_2]^2\right) \\
&= \mathbb{E}\left[X_1^2\right] + 2\mathbb{E}[X_1X_2] + \mathbb{E}\left[X_2^2\right] - \mathbb{E}[X_1]^2 - 2\mathbb{E}[X_1]\mathbb{E}[X_2] - \mathbb{E}[X_2]^2 \\
&= \underbrace{\mathbb{E}\left[X_1^2\right] - \mathbb{E}[X_1]^2}_{\text{Var}(X_1)} + \underbrace{\mathbb{E}\left[X_2^2\right] - \mathbb{E}[X_2]^2}_{\text{Var}(X_2)} + 2\cdot\underbrace{(\mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2])}_{\text{Cov}(X_1,X_2)} \\
&= \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)
\end{aligned}$$

□

4. Prove or disprove via a counterexample: If $\text{Corr}(X_1, X_2) = 0$ then X_1 and X_2 are independent

$\text{Corr}(X_1, X_2) = 0 \implies \text{Cov}(X_1, X_2) = 0 \implies \mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$
Need to find an example where $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$ but $X_1 \not\perp X_2$.

Let X_1 be a random variable with possible values of $-1,\ 1$ where
$\mathbb{P}(X_1 = -1) = \mathbb{P}(X_2 = 1) = 0.5$

Let X_2 be a random variable with possible values of $-1,\ 0,\ 1$

$$X_2 = \begin{cases} 0 & \text{if } X_1 = -1 \\ -1 \text{ or } 1 & \text{if } X_1 = 1 \text{ (both with probability 0.5)} \end{cases}$$

Note that X_2 is dependent on X_1

$$\begin{aligned}
\mathbb{E}[X_1] &= \textstyle\sum_{x_1} x_1 p_{X_1}(x_1) \\
&= -1 \cdot p_{X_1}(-1) + 1 \cdot p_{X_1}(1) \\
&= -1 \cdot 0.5 + 1 \cdot 0.5 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X_2] &= \textstyle\sum_{x_2} x_2 p_{X_2}(x_2) \\
&= 0 \cdot \underbrace{p_{X_2}(0)}_{p_{X_1}(-1)} + -1 \cdot \underbrace{p_{X_2}(-1)}_{0.5p_{X_1}(1)} + 1 \cdot \underbrace{p_{X_2}(1)}_{0.5p_{X_1}(1)} \\
&= 0 \cdot 0.5 - 1 \cdot 0.25 + 1 \cdot 0.25 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X_1X_2] &= \textstyle\sum_{x_1}\sum_{x_1} x_1x_2 p_{X_1X_2}(x_1, x_2) \\
&= -1 \cdot 0 \cdot \mathbb{P}(X_1 = -1) + 1 \cdot -1 \cdot \mathbb{P}(X_1 = 1, X_2 = -1) + 1 \cdot 1 \cdot \mathbb{P}(X_1 = 1, X_2 = 1) \\
&= 0 + \mathbb{P}(X_1 = 1, X_2 = -1) - \mathbb{P}(X_1 = 1, X_2 = 1) \\
&\quad \text{Note: } \mathbb{P}(X_1 = 1, X_2 = -1) = \mathbb{P}(X_1 = 1, X_2 = 1) \\
&= 0
\end{aligned}$$

$\mathbb{E}[X_1]\mathbb{E}[X_2] = \mathbb{E}[X_1X_2] \implies \text{Corr}(X_1, X_2) = 0$ but $X_1 \not\perp X_2$. □

5. Let f and g be non-decreasing on $\mathbb{R}$. Assume that $\mathbb{E}\left[g^2(X)\right] < \infty$ and $\mathbb{E}\left[f^2(X)\right] < \infty$. Show $\text{Cov}\left(f(X), g(X)\right) \geq 0$. Hint: Let Y be independent of X, with the same distribution as X. Consider $\mathbb{E}\left[\{g(X) - g(Y)\}\{f(X) - f(Y)\}\right]$.

$$X \perp\!\!\!\perp Y \quad X, Y \text{ are from the same distribution}$$

Look at $\{g(X) - g(Y)\}\{f(X) - f(Y)\}$

Option 1: $X < Y$

$$\left.\begin{array}{r} g(X) - g(Y) \leq 0 \\ f(X) - f(Y) \leq 0 \end{array}\right\} \implies \{g(X) - g(Y)\}\{f(X) - f(Y)\} \geq 0$$

Option 2: $X = Y$

$$\left.\begin{array}{r} g(X) - g(Y) = 0 \\ f(X) - f(Y) = 0 \end{array}\right\} \implies \{g(X) - g(Y)\}\{f(X) - f(Y)\} = 0$$

Option 3: $X > Y$

$$\left.\begin{array}{r} g(X) - g(Y) \geq 0 \\ f(X) - f(Y) \geq 0 \end{array}\right\} \implies \{g(X) - g(Y)\}\{f(X) - f(Y)\} \geq 0$$

$$\implies \{g(X) - g(Y)\}\{f(X) - f(Y)\} \geq 0$$
$$\implies \mathbb{E}\left[\{g(X) - g(Y)\}\{f(X) - f(Y)\}\right] \geq 0$$

Let $g(Y) = \mathbb{E}\left[g(X)\right]$ and $f(Y) = \mathbb{E}\left[f(X)\right]$

$$\mathbb{E}\left[\{g(X) - \mathbb{E}\left[g(X)\right]\}\{f(X) - \mathbb{E}\left[f(X)\right]\}\right] \geq 0$$
$$\text{Cov}\left(f(X), g(X)\right) \geq 0 \qquad \square$$