Problem 3a

$$
\begin{aligned}
p(\mu, \sigma_\epsilon^2, \sigma_\theta^2) &= p(\mu)p(\sigma_\theta^2)p(\sigma_\epsilon^2) \\
&= \mathcal{N}(\mu_0, \sigma_0^2)\text{IG}(a_1, b_1)\text{IG}(a_2, b_2) \\
&= \left[\frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left\{ -\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} \right\} \right] \cdot \left[(\sigma_\theta^2)^{a_1-1}\exp\left\{ -b_1\sigma_\theta^2 \right\} \right]^{-1} \cdot \left[(\sigma_\epsilon^2)^{a_2-1}\exp\left\{ -b_2\sigma_\epsilon^2 \right\} \right]^{-1} \\
&\propto \sigma_\theta^{-2(a_1-1)} \cdot \sigma_\epsilon^{-2(a_2-1)} \cdot \exp\left\{ -\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} + b_1\sigma_\theta^2 + b_2\sigma_\epsilon^2 \right\}
\end{aligned}
$$

$$
\begin{aligned}
p(\mu, \theta, \sigma_\theta^2, \sigma_\epsilon^2 \mid Y) &\propto p(Y \mid \mu, \theta, \sigma_\theta^2, \sigma_\epsilon^2) \cdot p(\theta \mid \mu, \sigma_\theta^2, \sigma_\epsilon^2) \cdot p(\mu, \sigma_\theta^2, \sigma_\epsilon^2) \\
&= \left[\prod_{i=1}^{K}\prod_{j=1}^{J} \left(p(y_{ij}) \sim \mathcal{N}(\theta_i, \sigma_\epsilon^2) \right) \right] \cdot \left[\prod_{i=1}^{K} \left(p(\theta_i) \sim \mathcal{N}(\mu, \sigma_\theta^2) \right) \right] \cdot \prod_{i=1}^{K} p(\mu, \sigma_\theta^2, \sigma_\epsilon^2) \\
&= \left[\sigma_\epsilon^{-(KJ)} \cdot \exp\left\{ -\frac{1}{2}\frac{\sum_{i=1}^{K}\sum_{j=1}^{J}(y_{ij}-\theta_i)^2}{\sigma_\epsilon^2} \right\} \right] \\
&\quad \cdot \left[\sigma_\theta^{-K}\exp\left\{ -\frac{1}{2}\frac{\sum_{i=1}^{K}(\theta_i-\mu)^2}{\sigma_\theta^2} \right\} \right] \\
&\quad \cdot \left[\sigma_\theta^{-2K(a_1-1)} \cdot \sigma_\epsilon^{-2K(a_2-1)} \cdot \exp\left\{ K\left(-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} + b_1\sigma_\theta^2 + b_2\sigma_\epsilon^2 \right) \right\} \right]
\end{aligned}
$$

Problem 3a(1)

$$
\begin{aligned}
p(\mu \mid \theta, \sigma_\epsilon^2, \sigma_\theta^2, Y) &\propto= \exp\left\{ -\frac{1}{2}\frac{\sum_{i=1}^{K}(\theta_i-\mu)^2}{\sigma_\theta^2} \right\} \cdot \exp\left\{ K\left(-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} \right) \right\} \\
&= \mathcal{N}\left(\frac{\sigma_\theta^2\mu_0 + \sigma_0^2\sum\theta_i}{\sigma_\theta^2 + K\sigma_0^2}, \; \frac{\sigma_\theta^2\sigma_0^2}{\sigma_\theta^2 + K\sigma_0^2} \right)
\end{aligned}
$$

Problem 3a(2)

$$
\begin{aligned}
p(\theta_i \mid \mu, \sigma_\epsilon^2, \sigma_\theta^2, Y) &\propto \cdot \exp\left\{ -\frac{1}{2}\frac{\sum_{i=1}^{K}\sum_{j=1}^{J}(y_{ij}-\theta_i)^2}{\sigma_\epsilon^2} \right\} \cdot \exp\left\{ -\frac{1}{2}\frac{\sum_{i=1}^{K}(\theta_i-\mu)^2}{\sigma_\theta^2} \right\} \\
&= \mathcal{N}\left(\frac{J\sigma_\theta^2}{J\sigma_\theta^2 + \sigma_\epsilon^2}\cdot\overline{Y}_i + \frac{\sigma_\epsilon^2}{J\sigma_\theta^2 + \sigma_\epsilon^2}\cdot\mu, \; \frac{\sigma_\theta^2\sigma_\epsilon^2}{J\sigma_\theta^2 + \sigma_\epsilon^2} \right)
\end{aligned}
$$

Problem 3a(3)

$$p(\sigma_\epsilon^2 \mid \mu, \theta, \sigma_\theta^2, Y) \varpropto= \sigma_\epsilon^{-(KJ)} \cdot \exp\left\{ -\frac{1}{2} \frac{\sum_{i=1}^K \sum_{j=1}^J (y_{ij} - \theta_i)^2}{\sigma_\epsilon^2} \right\} \cdot \sigma_\epsilon^{-2K(a_2-1)} \cdot \exp\left\{ K\left(b_2\sigma_\epsilon^2\right) \right\}$$

$$= \text{IG}\left(a_2 + \frac{KJ}{2}, \ b_2 + \frac{1}{2}\sum_{i=1}^K \sum_{j=1}^J (Y_{ij} - \theta_i)^2 \right)$$

Problem 3a(4)

$$p(\sigma_\theta^2 \mid \mu, \theta, \sigma_\epsilon^2, Y) \propto \sigma_\theta^{-K} \exp\left\{ -\frac{1}{2} \frac{\sum_{i=1}^K (\theta_i - \mu)^2}{\sigma_\theta^2} \right\} \cdot \sigma_\theta^{-2K(a_1-1)} \cdot \exp\left\{ K\left(b_1\sigma_\theta^2\right) \right\}$$

$$= \text{IG}\left(a_1 + \frac{K}{2}, \ b_1 + \frac{1}{2}\sum_{i=1}^K (\theta_i - \mu)^2 \right)$$