Problem 3a

$$\begin{aligned}
p(\mu, \sigma_\epsilon^2, \sigma_\theta^2) &= p(\mu)p(\sigma_\theta^2)p(\sigma_\epsilon^2) \\
&= \mathcal{N}(\mu_0, \sigma_0^2)\mathrm{IG}(a_1, b_1)\mathrm{IG}(a_2, b_2) \\
&= \left[\frac{1}{\sqrt{2\pi}\ \sigma_0} \exp\left\{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}\right\}\right] \cdot \left[(\sigma_\theta^2)^{a_1-1}\exp\left\{-b_1\sigma_\theta^2\right\}\right]^{-1} \cdot \left[(\sigma_\epsilon^2)^{a_2-1}\exp\left\{-b_2\sigma_\epsilon^2\right\}\right]^{-1} \\
&\propto {\sigma_\theta}^{-2(a_1-1)} \cdot {\sigma_\epsilon}^{-2(a_2-1)} \cdot \exp\left\{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} + b_1\sigma_\theta^2 + b_2\sigma_\epsilon^2\right\}
\end{aligned}$$

$$\begin{aligned}
p(\mu, \theta, \sigma_\theta^2, \sigma_\epsilon^2 \mid Y) &\propto p(Y \mid \mu, \theta, \sigma_\theta^2, \sigma_\epsilon^2) \cdot p(\theta \mid \mu, \sigma_\theta^2, \sigma_\epsilon^2) \cdot p(\mu, \sigma_\theta^2, \sigma_\epsilon^2) \\
&= \left[\prod_{i=1}^K \prod_{j=1}^J \left(p(y_{ij}) \sim \mathcal{N}(\theta_i, \sigma_\epsilon^2)\right)\right] \cdot \left[\prod_{i=1}^K \left(p(\theta_i) \sim \mathcal{N}(\mu, \sigma_\theta^2)\right)\right] \cdot \prod_{i=1}^K p(\mu, \sigma_\theta^2, \sigma_\epsilon^2) \\
&= \left[{\sigma_\epsilon}^{-(KJ)} \cdot \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K\sum_{j=1}^J (y_{ij}-\theta_i)^2}{\sigma_\epsilon^2}\right\}\right] \\
&\quad \cdot \left[\sigma_\theta^{-K} \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K (\theta_i-\mu)^2}{\sigma_\theta^2}\right\}\right] \\
&\quad \cdot \left[{\sigma_\theta}^{-2K(a_1-1)} \cdot {\sigma_\epsilon}^{-2K(a_2-1)} \cdot \exp\left\{K\left(-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2} + b_1\sigma_\theta^2 + b_2\sigma_\epsilon^2\right)\right\}\right]
\end{aligned}$$

Problem 3a(1)

$$\begin{aligned}
p(\mu \mid \theta, \sigma_\epsilon^2, \sigma_\theta^2, Y) \propto &= \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K(\theta_i-\mu)^2}{\sigma_\theta^2}\right\} \cdot \exp\left\{K\left(-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}\right)\right\} \\
&= \mathcal{N}\left(\frac{\sigma_\theta^2\mu_0 + \sigma_0^2\sum\theta_i}{\sigma_\theta^2 + K\sigma_0^2},\ \frac{\sigma_\theta^2\sigma_0^2}{\sigma_\theta^2 + K\sigma_0^2}\right)
\end{aligned}$$

Problem 3a(2)

$$\begin{aligned}
p(\theta_i \mid \mu, \sigma_\epsilon^2, \sigma_\theta^2, Y) &\propto \cdot \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K\sum_{j=1}^J (y_{ij}-\theta_i)^2}{\sigma_\epsilon^2}\right\} \cdot \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K(\theta_i-\mu)^2}{\sigma_\theta^2}\right\} \\
&= \mathcal{N}\left(\frac{J\sigma_\theta^2}{J\sigma_\theta^2+\sigma_\epsilon^2}\cdot \overline{Y}_i + \frac{\sigma_\epsilon^2}{J\sigma_\theta^2+\sigma_\epsilon^2}\cdot\mu, \quad \frac{\sigma_\theta^2\sigma_\epsilon^2}{J\sigma_\theta^2+\sigma_\epsilon^2}\right)
\end{aligned}$$

Problem 3a(3)

$$p(\sigma_\epsilon^2 \mid \mu, \theta, \sigma_\theta^2, Y) \propto= \sigma_\epsilon^{-(KJ)} \cdot \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K \sum_{j=1}^J (y_{ij}-\theta_i)^2}{\sigma_\epsilon^2}\right\} \cdot \sigma_\epsilon^{-2K(a_2-1)} \cdot \exp\left\{K\left(b_2\sigma_\epsilon^2\right)\right\}$$

$$= \mathrm{IG}\left(a_2 + \frac{KJ}{2},\ b_2 + \frac{1}{2}\sum_{i=1}^K\sum_{j=1}^J (Y_{ij}-\theta_i)^2\right)$$

Problem 3a(4)

$$p(\sigma_\theta^2 \mid \mu, \theta, \sigma_\epsilon^2, Y) \propto \sigma_\theta^{-K} \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^K (\theta_i-\mu)^2}{\sigma_\theta^2}\right\} \cdot \sigma_\theta^{-2K(a_1-1)} \cdot \exp\left\{K\left(b_1\sigma_\theta^2\right)\right\}$$

$$= \mathrm{IG}\left(a_1 + \frac{K}{2},\ b_1 + \frac{1}{2}\sum_{i=1}^K (\theta_i-\mu)^2\right)$$

Problem 5b

$$
\begin{aligned}
p_{X|Y}(x \mid y) &= ye^{-yx} \\
p_X(x) &\propto \int_0^B e^{-yx} dy \\
&= \frac{1}{x} \int_0^B xe^{-xy} dy \\
&= \frac{1}{x} \Big[-e^{-xy} \Big]_{y=0}^{y=B} \\
&= \frac{1}{x} \Big(\big[-e^{-Bx}\big] - \big[-e^{-0 \cdot x}\big] \Big) \\
&= \frac{1}{x} \Big(-e^{-Bx} + 1 \Big) \\
&= \frac{1 - e^{-Bx}}{x} \\
p_X(x) &\propto \frac{1 - e^{-Bx}}{x}
\end{aligned}
$$

Problem 5c

$$
\lim_{B \to \infty} \frac{1 - e^{-Bx}}{x} = \frac{1 - \lim_{B \to \infty} e^{-Bx}}{x} = \frac{1}{x}
$$