STAT 457 - FINAL

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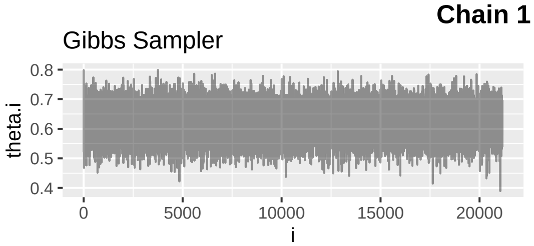
Problem 1

Problem 1a

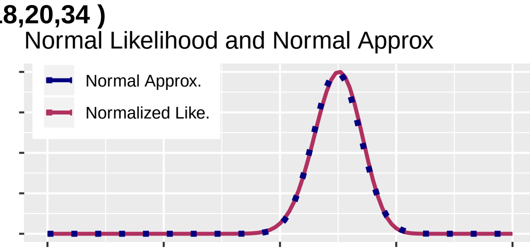
For the data $Y = (125, 18, 20, 34)$, implement the Gibbs sampler algorithm. Use a flat prior on θ.
Plot θ^i versus iteration i. $Y = (y_1, y_2, y_3, y_4) \propto (2 + \theta, 1 - \theta, 1 - \theta, \theta)$

1. Draw a starting value, $t \sim$Uniform(0,1)
2. Draw a latent value, $Z \sim \text{Binomial}\left(y_1,\ \frac{\theta}{2+\theta}\right)$
3. Draw a parameter, $\theta \sim \text{Beta}(Z + y_4 + 1, y_2 + y_3 + 1)$

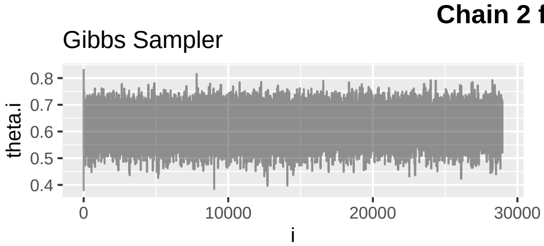
Chain 1 for data for Y=(125,18,20,34)



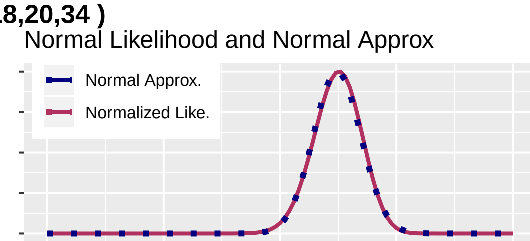
	Name	Value
1	Mean	0.62270
2	SD	0.05086
3	it	21156
4	Start	0.59298



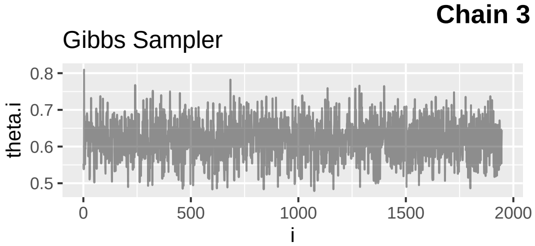
Chain 2 for data for Y=(125,18,20,34)



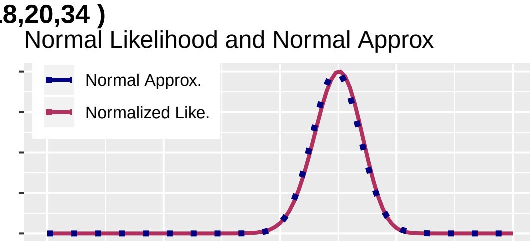
	Name	Value
1	Mean	0.62266
2	SD	0.05097
3	it	28962
4	Start	0.37667



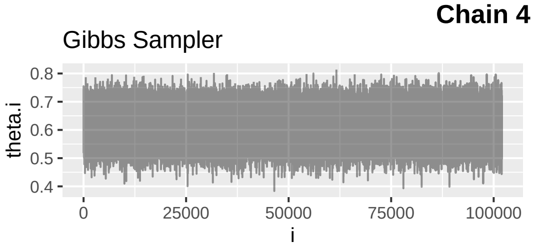
Chain 3 for data for Y=(125,18,20,34)



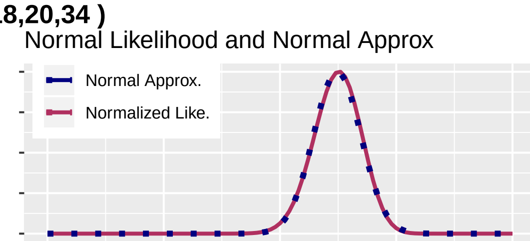
	Name	Value
1	Mean	0.62083
2	SD	0.05121
3	it	1947
4	Start	0.55304



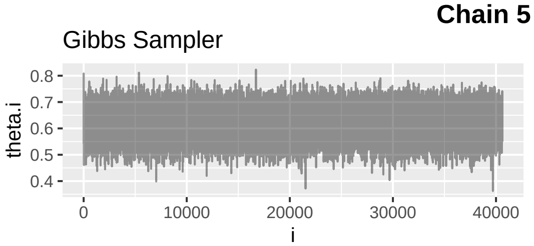
Chain 4 for data for Y=(125,18,20,34)



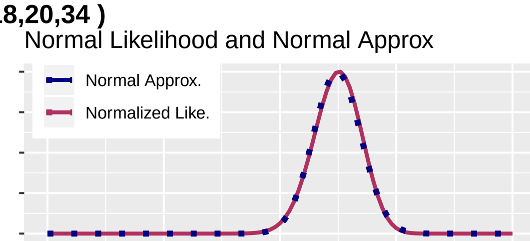
	Name	Value
1	Mean	0.62306
2	SD	0.05108
3	it	102018
4	Start	0.56521



Chain 5 for data for Y=(125,18,20,34)



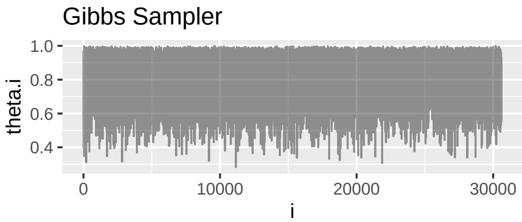
	Name	Value
1	Mean	0.62276
2	SD	0.05112
3	it	40623
4	Start	0.73643



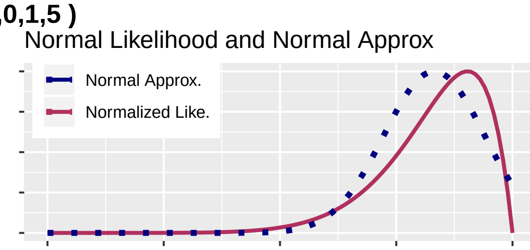
Problem 1b

Repeat 1a for $Y = (14, 0, 1, 5)$.

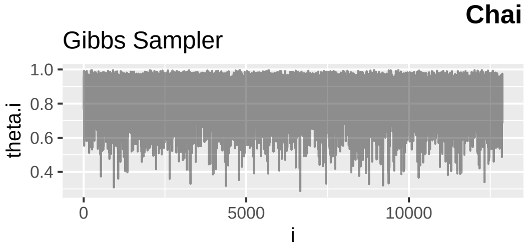
Chain 1 for data for Y=(14,0,1,5)



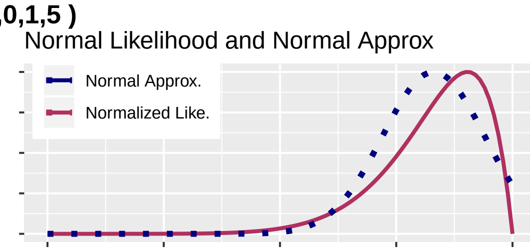
	Name	Value
1	Mean	0.83158
2	SD	0.10790
3	it	30607
4	Start	0.39922



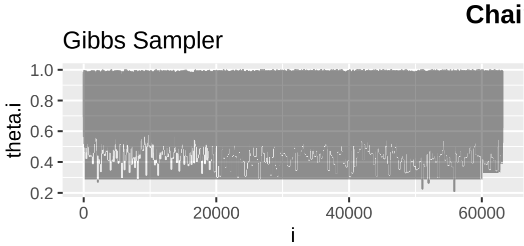
Chain 2 for data for Y=(14,0,1,5)



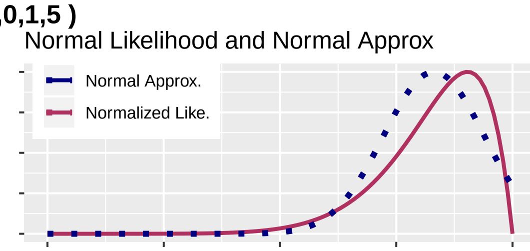
	Name	Value
1	Mean	0.83034
2	SD	0.10933
3	it	12875
4	Start	0.90502



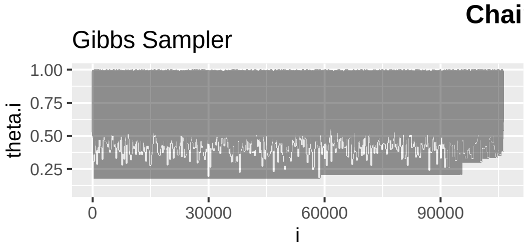
Chain 3 for data for Y=(14,0,1,5)



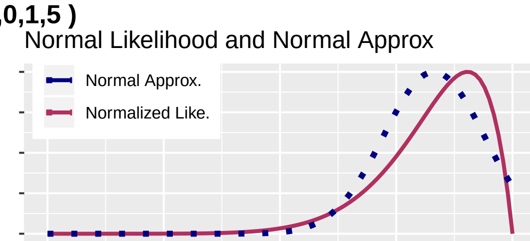
	Name	Value
1	Mean	0.83093
2	SD	0.10811
3	it	63107
4	Start	0.28758



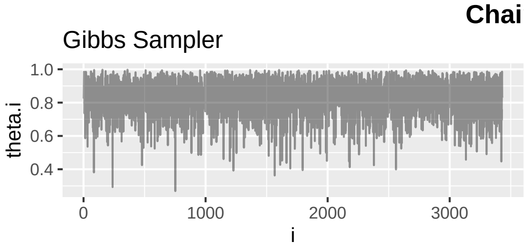
Chain 4 for data for Y=(14,0,1,5)



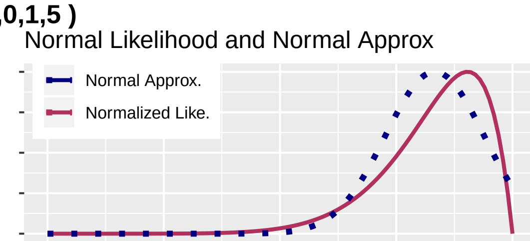
	Name	Value
1	Mean	0.83093
2	SD	0.10790
3	it	106106
4	Start	0.08302



Chain 5 for data for Y=(14,0,1,5)



	Name	Value
1	Mean	0.83272
2	SD	0.10589
3	it	3432
4	Start	0.82467



There is a lack of fit for the data in 1b, where the fit appears to be better for data in 1a. Convergence was assessed when values were had a difference less than 10^{-7}.

Problem 1c

20 Chains for Y=(125,18,20,34)

Mean	Standard Deviation	Standard Error	Iterations
0.62265	0.05147	0.00042	15243
0.62279	0.05103	0.00013	161378
0.62191	0.05203	0.00066	6229
0.62261	0.05126	0.00024	44139
0.62286	0.05113	0.00020	64679
0.62250	0.05089	0.00026	36893
0.62237	0.05087	0.00039	16865
0.62240	0.05085	0.00054	8893
0.62274	0.05114	0.00015	111670
0.62294	0.05100	0.00018	80166
0.62302	0.05108	0.00025	42245
0.62287	0.05109	0.00030	28917
0.62262	0.05099	0.00016	95966
0.62376	0.05110	0.00030	29344
0.62261	0.05111	0.00021	57875
0.62272	0.05060	0.00054	8760
0.62249	0.05108	0.00029	31756
0.62281	0.05068	0.00026	37926
0.62325	0.05090	0.00021	57245
0.62295	0.05102	0.00022	54595

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## [1] "0.000371 is the standard deviation of the 20 averages of theta"
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## [1] "0.000296 is the average of the standard errors of the 20 chains of theta"
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You would expected these values to be similar but it appears that our standard error average is under-estimating the variation in the chain means of θ in this case.

Problem 2

Problem 2a

For the genetic linkage model applied to $Y = (125, 18, 20, 34)$, implement the Metropolis algorithm.
(use a flat prior on θ). Use one long chain and plot θ^i versus i.
Try several driver functions:

Problem 2a(1) - Uniform on (0,1)

Problem 2a(2) - Normal Centered at the Current Point of the Chain and sd = 0.01

Problem 2a(3) - Normal Centered at the Current Point of the Chain and sd = 0.1

Problem 2a(4) - Normal Centered at the Current Point of the Chain and sd = 0.5

Problem 2a(5) - Normal Centered at 0.4 and sd = 0.1

Problem 2b

Repeat 2a for $Y = (14, 0, 1, 5)$

Problem 2c

Compute both the posterior mean and standard deviation for both data sets.
Compare to results from the previous problem.

Problem 2d

Problem 2d(1)

For each of the drives in part 2a, run 20 chains with independent starting values.
Compute the averages of the θ's in each chain.

Problem 2d(2)

Calculate the standard deviation of the 20 averages. Interpret this value.

Problem 2d(3)

Compute the standard deviation of the θ's in each chain. Divide each SD by the square root of the number of iterations.
Average these "standard errors".

Problem 2d(4)

Compare the 2d(2) values to 2d(3). Would you expect these number to be similar or different?
Compare to the results of Exercise 1c.

Problem 3

Problem 3a

Consider the 1-way variance components model

$$Y_{ij} = \theta_i + \epsilon_{ij}$$

where Y_{ij} is the jth observation from the ith group, θ_i is the effect, ϵ_{ij}=error, $i = 1, ..., K$ and $j = 1, ..., J$. It is assume that $\epsilon_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ and $\theta_i \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma_\theta^2)$. Under the prior specification $p(\sigma_\epsilon^2, \sigma_\theta^2, \mu) = p(\sigma_\epsilon^2)p(\sigma_\theta^2)p(\mu)$, with $p(\sigma_\theta^2) = \text{InverseGamma}(a_1, b_1)$, $p(\sigma_\epsilon^2) = \text{InverseGamma}(a_2, b_2)$, and $p(\mu) = \mathcal{N}(\mu_0, \sigma_0^2)$.
Let $\overline{Y}_i = \frac{1}{J}\sum_{j=1}^{J} Y_{ij}$ and $\theta = (theta_1, \cdots, \theta_k)$.
Show the following:

Problem 3a(1)

$$p(\mu \mid \theta, \sigma_\epsilon^2, \sigma_\theta^2, Y) = \mathcal{N}\left(\frac{\sigma_\theta^2\mu_0 + \sigma_0^2\sum\theta_i}{\sigma_\theta^2 + K\sigma_0^2},\ \frac{\sigma_\theta^2\sigma_0^2}{\sigma_\theta^2 + K\sigma_0^2}\right)$$

Problem 3a(2)

$$p(\theta_i \mid \mu, \sigma_\epsilon^2, \sigma_\theta^2, Y) = \mathcal{N}\left(\frac{J\sigma_\theta^2}{J\sigma_\theta^2 + \sigma_\epsilon^2} \cdot \overline{Y}_i + \frac{\sigma_\epsilon^2}{J\sigma_\theta^2 + \sigma_\epsilon^2} \cdot \mu, \quad \frac{\sigma_\theta^2\sigma_\epsilon^2}{J\sigma_\theta^2 + \sigma_\epsilon^2}\right)$$

Problem 3a(3)

$$p(\sigma_\epsilon^2 \mid \mu, \theta, \sigma_\theta^2, Y) = \text{InverseGamma}\left(a_2 + \frac{KJ}{2},\ b_2 + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{J}(Y_{ij} - \theta_i)^2\right)$$

Problem 3a(4)

$$p(\sigma_\theta^2 \mid \mu, \theta, \sigma_\epsilon^2, Y) = \text{InverseGamma}\left(a_1 + \frac{K}{2},\ b_1 + \frac{1}{2}\sum_{i=1}^{K}(\theta_i - \mu)^2\right)$$

Problem 3b

Run the Gibbs sampler for the data below. Use one chain of length 75,000. Take $p(\mu) = \mathcal{N}(0, 10^{12})$, $p(\sigma_\epsilon^2) = IG(0,0)$, and $p(\sigma_\theta^2) = IG(1,1)$. For each θ_i, for σ_ϵ, and for θ_θ, plot the simulated value at iteration j versus j. Summarize each posterior marginal.

Problem 3c

Repeat 3b using the prior specification $p(\mu) = \mathcal{N}(0, 10^{12})$, $p(\sigma_\epsilon^2) = IG(0,0)$, and $p(\sigma_\theta^2) = IG(0,0)$. Does this specification violate the Hobart_Casella conditions? Describe what happens to the Gibbs sampler chain in this case.

Problem 5

Suppose that X and Y have exponential conditional distributions restricted over the interval $(0, B)$, i.e. $p(x \mid y) \propto y \exp\{-yx\}$ for $0 < x < B < \infty$ and $p(y \mid x) \propto x \exp\{-xy\}$ for $0 < y < B < \infty$, where B is known constant.

Problem 5a

Take $m = 1$ and $B = 3$. Run the data augmentation algorithm using these conditionals. (Hist: Reject the exponential deviates that lie outside $(0, B)$). How did you assess convergence of this chain? Obtain the marginal for x using the mixture of conditionals $p(x \mid y)$, mixed over the simulated y deviates in your chain.

Problem 5b

Show that the marginal for x is proportional to $(1 - \exp\{-Bx\})/x$. Compare your results in 5a to this curve.

Problem 5c

Repeat 5a and 5b using $B = \infty$. Describe what happens. Is the marginal for x a proper density in this case?