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Exercise 01

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Recommended Reading Material

- Section 16.3.1 of the Handbook of Model Checking (definition and example of *semi-lattices*)

1 From Partial Orders to Lattices

PO Candidate	is a PO?	finite height?	height	$\top$	$\bot$
$(\mathbb{Z}, \leq)$					
$([-10{,}10], \leq)$					
$([-10{,}10], <)$					
$(\mathbb{Z}, \geq)$					
$(\mathbb{Z}, =)$					
$(2^Q, \subseteq)$, Q finite					
$(\Sigma^*$, lexicographic order$)$					
$([a-z]^3$, lexicographic order$)$					
$(\bigcup_{0\leq i\leq 3}[a-z]^i$, lexicographic order$)$					
$(\Sigma^*$, suffix$)$					
$([a-z]^3$, suffix$)$					
$(\bigcup_{0\leq i\leq 3}[a-z]^i$, suffix$)$					

(a) Which three properties need to be fulfilled by a set $\mathbb{X}$ and subsumption relation $\sqsubseteq$ to make a partial order?

(b) Decide for each of the listed partial order candidates whether it actually is a partial order. Note: $[-10{,}10]$ should be treated as an interval of integers.

(c) The height n of a partial order $(\mathbb{X}, \sqsubseteq)$ is defined as the number of subsumption operations in the longest ascending chain $x_1 \sqsubseteq x_2 \cdots \sqsubseteq x_{n+1}$ that can be constructed from pairwise different elements x_i of the underlying set $\mathbb{X}$. (As such, the height is always one smaller than the length of the longest ascending chain).

Decide for each of the tuples $(\mathbb{X}, \sqsubseteq)$ identified in (b) as partial orders whether it has a finite height. If so, also write down a formula that describes the value of that finite height.

(d) Which properties need to be fulfilled by a lattice?

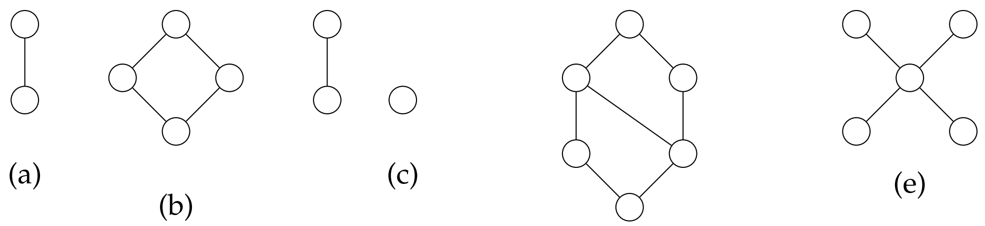
(e) Which of the identified partial orders can be extended into (semi-)lattices? For those who can, write down the $\top$ and $\bot$ element.

2 Upper and Lower Bounds

1. Consider the partial order $(\mathbb{N}, \sqsubseteq)$ with $x \sqsubseteq y$ if $x \leq y \wedge x \mod 2 = y \mod 2$.
   Compute for each of the following subsets the least upper bound and the greatest lower bound if they exist.
   a) $\{1,2\}$
   b) $\{2,4,8\}$
   c) $\{x \mid x \mod 2 = 0\}$
   d) $\emptyset$
   e) $\mathbb{N}$
2. Consider the following partial orders. What are the least upper bound and greatest lower bound operators for finite, non-empty subsets?
   a) $(2^{\mathbb{Q}}, \supseteq)$
   b) $(\mathbb{N}, \sqsubseteq_{\text{div}})$ with $x \sqsubseteq_{\text{div}} y$ if $y \mod x = 0$.

3 Lattices

Consider the following partial orders. Which of them are lattices?



(a) (b) (c) (d) (e)

4 Concretization

For describing the states of the system we are interested in, we model it with so-called concrete states from the set C of all concrete states.

A concrete state $c \in C$ is a pair of program counter and data state ($C = L \times \Sigma$). The second part of a concrete state is also sometimes referred to as the data state.

- Program counter $pc \in L$
  - Where am I?
  - Location in CFA
  - $c(pc)$ refers to program counter of concrete state
- Data state $\sigma \in \Sigma, \Sigma = V \to \mathbb{Z}$ (σ is a function that maps variables $v \in V$ to concrete values)
  - Fixes variable values

– $c(v)$ refers to the value that is assigned to variable v in the concrete state c

1. Consider the flat lattice $\mathcal{L}$ derived from the set L of program locations. Define a concretization function for the lattice $\mathcal{L}$ that maps each element to all concrete states with a program counter considered by this element. Consider also the $\top$ and $\bot$ element of the lattice.

5 Programming Exercise on Lattices: Lattice of Prime Factors

Go to the lecture's notebooks repository[1] and complete the programming tasks in the Jupyter Notebook named `01_Lattices.ipynb`.

You can either click on the badge labeled "launch binder" that will open the notebook in a clean environment on mybinder.org, or you can download the notebook and run it from your personal computer using Jupyter according to the installation instructions in the repository. Binder will sometimes take a while to create the environment, especially when a notebook recently changed or is not used that frequently.

All the necessary text is given in the notebook, here is a rough sketch of what awaits you:

We have a look at a lattice where the elements consist of numbers that are created by multiplying the numbers in each subset of the set $\{1,2,3,5\}$. For example, $\{3,5\}$ is a subset of $\{1,2,3,5\}$, so $3 \cdot 5 = 15$ is an element of the lattice.

The exercise contains three sub-exercises:

(a) The subsumption relation:
In order to get a nice lattice, we have to define a better subsumption relation. Your task is to implement in the next cell the subsumption relation that where $x \sqsubseteq y$ iff the orime factors of x are a subset of the prime factors of y. Execute the next three cells and observe the output. If you are not satisfied with your implementation of 'subsumption', refine it until you are happy.

(b) remove redundant edges to increase readability:
Some of the edges in the graph generated in a) are redundant, they could be left out because of the transitivity of the subsumption relation. For example, $1 \sqsubseteq 6$ could be left out because it follows from $1 \sqsubseteq 2$ and $2 \sqsubseteq 6$. To reflect this, let's add another function that returns *True* whenever y is a direct "successor" of x, i.e.: $successor(x, y) \Leftrightarrow x \sqsubset y \wedge \nexists x'.x \sqsubset x' \wedge x' \sqsubset y$

(c) $\top$ and $\bot$:
From the graph generated in b) it should be quite clear which element is $\top$ and which element is $\bot$. Can you determine this programmatically? Your solution should work even if we modified the set s by adding other primes like 7,11,13.

[1] https://gitlab2.cip.ifi.lmu.de/sosy-lab/sv-notebooks-ss25