# 第 10 章综合习题

1. 计算二重积分 $I=\iint_D \operatorname{sgn}(x^2-y^2+2)\,\mathrm{d}x\mathrm{d}y$, 其中

$$D=\{(x,y):\ x^2+y^2\leqslant 4\}.$$

**解** 设 $D$ 在第一象限部分为 $D_1$, 则由对称性

$$I=4\iint_{D_1}\operatorname{sgn}(x^2-y^2+1)\mathrm{d}x\mathrm{d}y.$$

设 $D_2$ 是 $D_1$ 中使得 $x^2-y^2+2<0$ 的部分, $D_3$ 是 $D_1$ 中使得 $x^2-y^2+2\geqslant 0$ 的部分, 则 $D_1=D_2\cup D_3$. 因此

$$\begin{aligned} I&=4\left[\iint_{D_3}\mathrm{d}x\mathrm{d}y-\iint_{D_2}\mathrm{d}x\mathrm{d}y\right]=4\left[\sigma(D_3)-\sigma(D_2)\right]\\ &=4\left[\frac{1}{4}\cdot\pi\cdot 2^2-2\sigma(D_2)\right]=4\pi-8\sigma(D_2)\end{aligned}$$

其中 $\sigma(D_2),\sigma(D_3)$ 分别表示 $D_2$ 和 $D_3$ 的面积. 在极坐标 $x=r\cos\varphi, y=r\sin\varphi$ 之下, $D_2$ 为 $\{(r,\varphi): \frac{\pi}{3}\leqslant\varphi\leqslant\frac{\pi}{2}, \sqrt{-\frac{2}{\cos 2\varphi}}\leqslant r\leqslant 2\}$. 因而

$$
\begin{aligned}
\sigma(D_2) &= \iint_{D_2}\mathrm{d}x\mathrm{d}y = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi\int_{\sqrt{-\frac{2}{\cos 2\varphi}}}^{2} r\mathrm{d}r \\
&= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(4+\frac{2}{\cos 2\varphi}\right)\mathrm{d}\varphi = \frac{\pi}{3}+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}\frac{1}{\cos\varphi}\mathrm{d}\varphi \\
&= \frac{\pi}{3}-\frac{1}{2}\ln(2+\sqrt{3}).
\end{aligned}
$$

故,

$$
I=\frac{4\pi}{3}+4\ln(2+\sqrt{3}).
$$

2. 计算三重积分

$$
I=\iiint\limits_{[0,1]^3}\frac{\mathrm{d}u\mathrm{d}v\mathrm{d}w}{(1+u^2+v^2+w^2)^2}.
$$

**解**: 作变量代换

$$u = r\cos\theta,\ v = r\sin\theta,\ w = \tan\varphi,$$

则其 Jacobian 行列式为

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & \sec^2\varphi \end{vmatrix} = r\sec^2\varphi,$$

所以

$$\begin{aligned} I &= 2\int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sec\theta} \frac{r\sec^2\varphi}{(1+r^2+\tan^2\varphi)^2} dr \\ &= \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{\pi}{4}} \left(\frac{\sec^2\varphi}{r^2+\sec^2\varphi}\right)\Bigg|_{r=\sec\theta}^{r=0} d\varphi \\ &= \left(\frac{\pi}{4}\right)^2 - \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta} d\theta d\varphi. \end{aligned}$$

由于

$$A=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}d\theta d\varphi=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\varphi+\sec^2\theta}d\varphi d\theta,$$

所以

$$2A=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\sec^2\varphi+\sec^2\theta}{\sec^2\varphi+\sec^2\theta}d\theta d\varphi=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}d\theta d\varphi=\left(\frac{\pi}{4}\right)^2,$$

因而 $A=\frac{1}{2}(\frac{\pi}{4})^2$, 于是

$$I=\left(\frac{\pi}{4}\right)^2-\frac{1}{2}\left(\frac{\pi}{4}\right)^2=\frac{\pi^2}{32}.$$

3. 设 $a>0, b>0$. 试求下面的积分:

(1) $I_1=\displaystyle\int_0^1\sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x;$

(2) $I_2=\displaystyle\int_0^1\cos\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x.$

**解**: 应用公式 $\int_a^b x^y \mathrm{d}y = (x^b - x^a)/\ln x$, 可知

$$I_1 = \int_0^1 dx \int_a^b x^y \sin\left(\ln\frac{1}{x}\right)\mathrm{d}y,\ I_2 = \int_0^1 dx \int_a^b x^y \cos\left(\ln\frac{1}{x}\right)\mathrm{d}y.$$

交换积分顺序, 可得

$$I_1 = \int_a^b \mathrm{d}y \int_0^1 x^y \sin\left(\ln\frac{1}{x}\right)\mathrm{d}x,\ I_2 = \int_a^b \mathrm{d}y \int_0^1 x^y \cos\left(\ln\frac{1}{x}\right)\mathrm{d}x.$$

再作变换 $x = e^{-t}$, 得

$$\int_0^1 x^y \sin\left(\ln\frac{1}{x}\right)\mathrm{d}x = \int_0^{+\infty} e^{-(1+y)t}\sin t\mathrm{d}t = \frac{1}{1+(1+y)^2},$$

$$\int_0^1 x^y \cos\left(\ln\frac{1}{x}\right)\mathrm{d}x = \int_0^{+\infty} e^{-(1+y)t}\cos t\mathrm{d}t = \frac{1+y}{1+(1+y)^2}.$$

因而

$$I_1 = \int_a^b \frac{1}{1+(1+y)^2}\mathrm{d}y = \arctan\left(\frac{b-a}{1+(a+1)(b+1)}\right),$$

$$I_2 = \int_a^b \frac{1+y}{1+(1+y)^2}\mathrm{d}y = \frac{1}{2}\ln\left(\frac{b^2+2b+2}{a^2+2a+2}\right).$$

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4. 设 $D=\{(x,y)\mid x^2+y^2\leqslant 1\}$. 求 $I=\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\mathrm{d}y$.

**解** 由极坐标变换 $x=r\cos\varphi,\ y=r\sin\varphi,\ 0\leqslant r\leqslant 1, 0\leqslant\varphi\leqslant 2\pi$, 有

$$I=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\left|\frac{\cos\varphi+\sin\varphi}{\sqrt{2}}-r\right|r^2\mathrm{d}r\mathrm{d}\varphi$$

$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\left|\sin(\varphi+\frac{\pi}{4})-r\right|r^2\mathrm{d}r\mathrm{d}\varphi=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi$$

$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi+\iint\limits_{\substack{0\leqslant r\leqslant 1\\\pi\leqslant\varphi\leqslant 2\pi}}(\sin\varphi-r)\,r^2\mathrm{d}r\mathrm{d}\varphi$$

$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi+\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}(\sin\varphi+r)\,r^2\mathrm{d}r\mathrm{d}\varphi$$

因此, 有

$$
\begin{aligned}
I &= \int_0^\pi \mathrm{d}\varphi \int_0^{\sin\varphi} (\sin\varphi - r) r^2 \mathrm{d}r + \int_0^\pi \mathrm{d}\varphi \int_{\sin\varphi}^1 (\sin\varphi - r) r^2 \mathrm{d}r \\
&\quad + \int_0^\pi \mathrm{d}\varphi \int_0^{\sin\varphi} (\sin\varphi + r) r^2 \mathrm{d}r + \int_0^\pi \mathrm{d}\varphi \int_{\sin\varphi}^1 (\sin\varphi + r) r^2 \mathrm{d}r \\
&= \int_0^\pi \mathrm{d}\varphi \int_0^{\sin\varphi} 2\sin\varphi \cdot r^2 \mathrm{d}r + \int_0^\pi \mathrm{d}\varphi \int_{\sin\varphi}^1 2r \cdot r^2 \mathrm{d}r \\
&= \int_0^\pi \frac{2}{3} \sin^4\varphi \mathrm{d}\varphi + \int_0^\pi \frac{1}{2}(1 - \sin^4\varphi) \mathrm{d}\varphi \\
&= \frac{1}{6} \int_0^\pi \sin^4\varphi \mathrm{d}\varphi + \frac{\pi}{2} \\
&= \frac{1}{6} \cdot \frac{3\pi}{8} + \frac{\pi}{2} \\
&= \frac{9}{16}\pi.
\end{aligned}
$$

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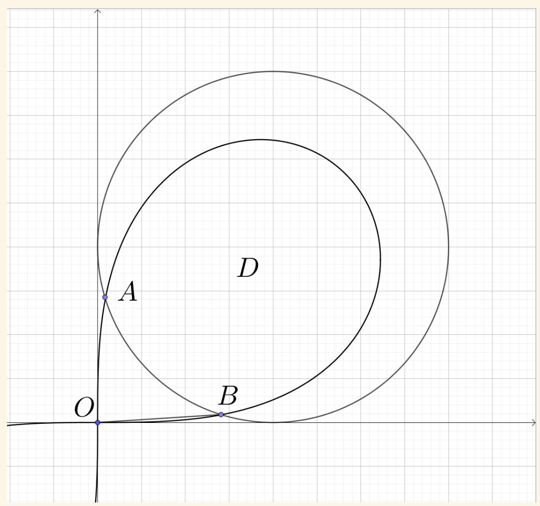
5. 试求圆盘 $(x-a)^2+(y-a)^2 \leqslant a^2$ 与曲线 $(x^2+y^2)^2=8a^2xy$ 所围部分相交的区域 $D$ 的面积 $S$.

**解** 如图, 圆 $(x-a)^2+(y-a)^2=a^2$ 与曲线 $(x^2+y^2)^2=8a^2xy$ 的交点为 $A, B$. 不妨设 $a>0$. 解方程可得这两点的坐标 $A\left(\frac{3-\sqrt{7}}{8}a, \frac{3+\sqrt{7}}{8}a\right)$, $B\left(\frac{3+\sqrt{7}}{8}a, \frac{3-\sqrt{7}}{8}a\right)$.




设线段 $OB$ 与 $x$ 轴正向的夹角为 $\theta$. 因为 $OB$ 的长为 $\frac{\sqrt{2}}{2}a$, 所以

$$\sin\theta=\frac{3-\sqrt{7}}{8}a\Big/\frac{\sqrt{2}}{2}a=\frac{3\sqrt{2}-\sqrt{14}}{8}.$$

计算可得 $\sin\left(\frac{1}{2}\arcsin\frac{1}{8}\right)=\frac{3\sqrt{2}-\sqrt{14}}{8}$. 故, $\theta=\frac{1}{2}\arcsin\frac{1}{8}$.

在极坐标 $x=r\cos\varphi$, $y=r\sin\varphi$ 之下, $D$ 为

$$D=\left\{(r,\varphi):\begin{array}{l}a[(\sin\varphi+\cos\varphi)-\sqrt{\sin 2\varphi}]\leqslant r\leqslant 2a\sqrt{\sin 2\varphi};\\ \dfrac{1}{2}\arcsin\dfrac{1}{8}\leqslant\varphi\leqslant\dfrac{\pi}{2}-\dfrac{1}{2}\arcsin\dfrac{1}{8}\end{array}\right\}.$$

注意到 $D$ 关于 $\varphi=\frac{\pi}{4}$ 对称, 有

$$\begin{aligned}S&=\iint_D \mathrm{d}x\mathrm{d}y=2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4}\mathrm{d}\varphi\int_{a[(\sin\varphi+\cos\varphi)-\sqrt{\sin 2\varphi}]}^{2a\sqrt{\sin 2\varphi}}r\mathrm{d}r\\
&=a^2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4}\left[2\sin 2\varphi+2(\sin\varphi+\cos\varphi)\sqrt{\sin 2\varphi}-1\right]\mathrm{d}\varphi\\
&=a^2\left[\cos\left(\arcsin\frac{1}{8}\right)-\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}\right]+2a^2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4}(\sin\varphi+\cos\varphi)\sqrt{\sin 2\varphi}\mathrm{d}\varphi\end{aligned}$$

因为 $\cos\left(\arcsin\frac{1}{8}\right)=\sqrt{1-\frac{1}{64}}=\frac{3\sqrt{7}}{8}$, 以及

$$-\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}=-\frac{1}{2}\left(\frac{\pi}{2}-\arcsin\frac{1}{8}\right)=-\frac{1}{2}\arccos\frac{1}{8},$$

作变换 $\varphi+\frac{\pi}{4}=t$, 我们有

$$S=a^2\left(\frac{3\sqrt{7}}{8}-\frac{1}{2}\arccos\frac{1}{8}+2\sqrt{2}\int_{\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/2}\sqrt{-\cos 2t}\sin t\,\mathrm{d}t\right).$$

记上式括号中的积分为 $I$, 我们有

$$I=2\int_{\pi/2}^{\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}}\sqrt{1-(\sqrt{2}\cos t)^2}\,\mathrm{d}(\sqrt{2}\cos t).$$

作变换 $u=\sqrt{2}\cos t$, 得

$$I=2\int_0^{\arcsin\frac{\sqrt{7}}{2\sqrt{2}}}\cos^2 u\,\mathrm{d}u=\arcsin\frac{\sqrt{7}}{2\sqrt{2}}+\frac{\sqrt{7}}{8}.$$

于是

$$S=a^2\left(\frac{\sqrt{7}}{2}+\arcsin\frac{\sqrt{7}}{2\sqrt{2}}-\frac{1}{2}\arccos\frac{1}{8}\right)=a^2\left(\frac{\sqrt{7}}{2}+\arcsin\frac{\sqrt{14}}{8}\right).$$

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6. 计算曲面 $(x^2+y^2)^2+z^4=y$ 所围的区域 $V$ 的体积 $\sigma(V)$.

**解** 设 $V$ 在第一挂限中的部分为 $V_1$, 则根据对称性, $V$ 的体积是 $V_1$ 的体积的4倍. $V_1$ 在 $xy$ 平面的投影趋于是 $D:(x^2+y^2)^2+z^4\leqslant y, x\geqslant 0, y\geqslant 0$. 因此,

$$\sigma(V)=4\iint_D \left(y-(x^2+y^2)^2\right)^{\frac{1}{4}}\mathrm{d}x\mathrm{d}y.$$

用极坐标变换 $x=r\cos\varphi,\ y=r\sin\varphi$. 有

$$\begin{aligned}\sigma(V)&=4\iint\limits_{\substack{0\leqslant\varphi\leqslant\frac{\pi}{2}\\0\leqslant r\leqslant\sin^{1/3}\varphi}}\left(r\sin\varphi-r^4\right)^{\frac{1}{4}}\cdot r\mathrm{d}r\\&=4\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\sin^{1/3}\varphi}\left(\sin\varphi-r^3\right)^{1/4}\cdot r^{5/4}\mathrm{d}r\end{aligned}$$

对上式最右边的积分作变换 $r=(x\sin\varphi)^{1/3}$, 得

$$\sigma(V)=\frac{4}{3}\int_0^{\frac{\pi}{2}}\sin\varphi\mathrm{d}\varphi\int_0^1 x^{-1/4}(1-x)^{1/4}\mathrm{d}x.$$

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故,

$$\sigma(V)=\frac{4}{3}\mathrm{B}(\frac{3}{4},\frac{5}{4})=\frac{1}{3}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\sqrt{2}}{3}\pi.$$

注 解法中用到 $\Gamma$ 函数, 此题应放在第13章的后面.

7. 证明: $\displaystyle\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1 t^t\,\mathrm{d}t.$

**解** 首先化为累次积分

$$\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^1(xy)^{xy}\,\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^x\frac{t^t}{x}\,\mathrm{d}t$$

$$=\int_0^1\frac{f(x)}{x}\mathrm{d}x,$$

其中 $f(x)=\int_0^x t^t\mathrm{d}t$. 由分部积分,

$$\int_0^1\frac{f(x)}{x}\mathrm{d}x=f(x)\ln x\Big|_0^1-\int_0^1 x^x\ln x\mathrm{d}x=-\int_0^1 x^x\ln x\mathrm{d}x$$

因为 $(x^x)' = x^x \ln x + x^x$, 所以

$$\int_0^1 x^x \ln x \mathrm{d}x = \int_0^1 \left((x^x)' - x^x\right) \mathrm{d}x = -\int_0^1 x^x \mathrm{d}x.$$

于是

$$\iint\limits_{[0,1]^2} (xy)^{xy} \,\mathrm{d}x\mathrm{d}y = \int_0^1 t^t \,\mathrm{d}t.$$

8. 设 $a, b$ 是不全为 0 的常数. 求证:

$$\iint\limits_{x^2+y^2\leqslant 1} f(ax+by+c)\,\mathrm{d}x\mathrm{d}y = 2\int_{-1}^1 \sqrt{1-t^2}\, f\left(t\sqrt{a^2+b^2}+c\right)\,\mathrm{d}t.$$

**证明** 作变换

$$x = \frac{a}{\sqrt{a^2+b^2}}t - \frac{b}{\sqrt{a^2+b^2}}s, y = \frac{b}{\sqrt{a^2+b^2}}t + \frac{a}{\sqrt{a^2+b^2}}s.$$

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则有 $x^2+y^2=s^2+t^2$, 且 $\frac{\partial(x,y)}{\partial(t,s)}=1$. 因此

$$
\begin{aligned}
&\iint\limits_{x^2+y^2\leqslant 1} f(ax+by+c)\,\mathrm{d}x\mathrm{d}y \\
=&\iint\limits_{t^2+s^2\leqslant 1} f\left(t\sqrt{a^2+b^2}+c\right)\mathrm{d}t\mathrm{d}s \\
=&\int_{-1}^{1} f\left(t\sqrt{a^2+b^2}+c\right)\mathrm{d}t\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}}\mathrm{d}s \\
=&\,2\int_{-1}^{1}\sqrt{1-t^2}f\left(t\sqrt{a^2+b^2}+c\right)\mathrm{d}t.
\end{aligned}
$$

9. 设 $f$ 是连续可导的单变量函数. 令 $F(t)=\iint\limits_{[0,t]^2} f(xy)\,\mathrm{d}x\mathrm{d}y$. 求证:

(1) $F'(t)=\dfrac{2}{t}\left(F(t)+\iint\limits_{[0,t]^2} xyf'(xy)\,\mathrm{d}x\mathrm{d}y\right)$;

(2) $F'(t) = \dfrac{2}{t}\displaystyle\int_0^{t^2} f(s)\,\mathrm{d}s$.

**证明** (1) 作变换 $x = tu,\ y = tv$. 有

$$F(t) = \iint\limits_{[0,1]^2} f(t^2uv)t^2\mathrm{d}u\mathrm{d}v.$$

因为上式中被积函数关于 $t$ 连续可导, 所以 $F(t)$ 可导, 且

$$\begin{aligned}
F'(t) &= \iint\limits_{[0,1]^2} \left(2tf(t^2uv) + 2t^3uvf'(t^2uv)\right)\mathrm{d}u\mathrm{d}v \\
&= \frac{2}{t}\iint\limits_{[0,1]^2} f(t^2uv)t^2\mathrm{d}u\mathrm{d}v + \frac{2}{t}\iint\limits_{[0,1]^2} t^4uvf'(t^2uv)\mathrm{d}u\mathrm{d}v \\
&= \frac{2}{t}F(t) + \frac{2}{t}\iint\limits_{[0,t]^2} xyf'(xy)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

(2) 用累次积分

$$F(t)=\int_0^t \mathrm{d}x\int_0^t f(xy)\mathrm{d}y=\int_0^t \frac{1}{x}\mathrm{d}x\int_0^{tx} f(s)\mathrm{d}s=\int_0^t \frac{1}{x}g(tx)\mathrm{d}x,$$

其中 $g(u)=\int_0^u f(s)\mathrm{d}s$. 于是

$$\begin{aligned} F'(t) &= \frac{1}{t}g(t^2)+\int_0^t \frac{1}{x}g'(tx)x\mathrm{d}x \\ &= \frac{1}{t}\int^{t^2} f(s)\mathrm{d}s+\int_0^t f(tx)\mathrm{d}x \\ &= \frac{2}{t}\int^{t^2} f(s)\mathrm{d}s. \end{aligned}$$

注 此题的证明用到了积分号里面求导.

10. (Poincaré 不等式) 设 $\varphi(x), \psi(x)$ 是 $[a, b]$ 上的连续函数, $f(x, y)$ 在区域 $D = \{(x, y) : a \leqslant x \leqslant b, \varphi(x) \leqslant y \leqslant \psi(x)\}$ 上连续可微, 且有 $f(x, \varphi(x)) = 0$, 则存在 $M > 0$, 使得

$$\iint_D f^2(x, y)\,\mathrm{d}x\mathrm{d}y \leqslant M \iint_D (f_y'(x, y))^2\,\mathrm{d}x\mathrm{d}y.$$

**证明** 由 Newton-Leibeniz 公式和 Cauchy 不等式,

$$\begin{aligned} f^2(x, y) &= \left[f(x, y) - f(x, \varphi(x)\right]^2 = \left(\int_{\varphi(x)}^{y} \frac{\partial f}{\partial t}(x, t)\,\mathrm{d}t\right)^2 \\ &\leqslant (y - \varphi(x)) \int_{\varphi(x)}^{y} \left(\frac{\partial f}{\partial t}(x, t)\right)^2 \mathrm{d}t \end{aligned}$$

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因此

$$
\begin{aligned}
&\iint_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} f^2(x,y)\mathrm{d}y \\
\leqslant & \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} (y-\varphi(x))\mathrm{d}y \int_{\varphi(x)}^{y} \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t \\
= & \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t \int_t^{\psi(x)} (y-\varphi(x))\mathrm{d}y \\
\leqslant & \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t \int_{\varphi(x)}^{\psi(x)} (y-\varphi(x))\mathrm{d}y \\
= & \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \frac{1}{2}(\psi(x)-\varphi(x))^2 \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t \\
\leqslant & M \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t = M \iint_D \left(\frac{\partial f}{\partial y}(x,y)\right)^2 \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

这里 $M$ 是满足 $M > \max\limits_{a\leqslant x\leqslant b} \frac{1}{2}(\psi(x)-\varphi(x))^2$ 的常数.

11. 设 $a>0$, $\Omega_n(a)$ : $x_1+\cdots+x_n\leqslant a$, $x_i\geqslant 0$ $(i=1,2,\cdots,n)$. 求积分

$$I_n(a)=\int\cdots\int\limits_{\Omega_n(a)} x_1x_2\cdots x_n\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

**解** 作变换 $x_i=at_i$, $i=1,2,\cdots,n$, 则

$$I_n(a)=a^{2n}\int\cdots\int\limits_{\Omega_n(1)} t_1t_2\cdots t_n\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n=a^{2n}I_n(1). \tag{1}$$

用累次积分, 可得

$$\begin{aligned}I_n(1)&=\int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n\\&=\int_0^1 t_n\mathrm{d}t_n\int\cdots\int\limits_{t_1+\cdots+t_{n-1}\leqslant 1-t_n} t_1\cdots t_{n-1}\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\\&=\int_0^1 t_nI_{n-1}(1-t_n)\mathrm{d}t_n=\int_0^1 t_n(1-t_n)^{2(n-1)}I_{n-1}(1)\mathrm{d}t_n.\end{aligned}$$

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因此

$$I_n(1) = \frac{1}{2n(2n-1)} I_{n-1}(1).$$

注意到 $I_1(1) = \int_0^1 t\mathrm{d}t = \frac{1}{2}$. 由上面的递推公式, 可得

$$I_n(1) = \frac{1}{(2n)!}.$$

故,

$$I_n(a) = \frac{a^{2n}}{(2n)!}.$$

12. 设 $f(x_1, x_2, \cdots, x_n)$ 为 $n$ 元连续函数. 证明:

$$\int_a^b \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_{n-1}} f(x_1, x_2, \cdots, x_n)\mathrm{d}x_n$$
$$= \int_a^b \mathrm{d}x_n \int_{x_n}^b \mathrm{d}x_{n-1} \cdots \int_{x_2}^b f(x_1, x_2, \cdots, x_n)\mathrm{d}x_1.$$

**证明** $n = 1$ 时, 无需证明. $n = 2$ 时, 就是要证

$$\int_a^b \mathrm{d}x_1 \int_a^{x_1} f(x_1, x_2)\mathrm{d}x_2 = \int_a^b \mathrm{d}x_2 \int_{x_2}^b f(x_1, x_2)\mathrm{d}x_1.$$

上式左右两边都是 $f(x_1, x_2)$ 在区域 $D: a \leqslant x_1 \leqslant b,\ 0 \leqslant x_2 \leqslant x_1$ 上的累次积分, 因而它们相等. 假设 $n-1$ 时结论成立.

记 $g(x_1,\cdots,x_{n-1})=\int_a^{x_{n-1}} f(x_1,x_2,\cdots,x_n)\mathrm{d}x_n$. 则

$$\begin{aligned}
&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}} f(x_1,x_2,\cdots,x_n)\mathrm{d}x_n\\
=&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-2}} g(x_1,x_2,\cdots,x_{n-1})\mathrm{d}x_{n-1}\\
=&\int_a^b \mathrm{d}x_{n-1}\int_{x_{n-1}}^{b}\mathrm{d}x_{n-2}\cdots\int_{x_2}^{b} g(x_1,\cdots,x_{n-1})\mathrm{d}x_1\\
=&\int_a^b \mathrm{d}x_{n-1}\int_{x_{n-1}}^{b}\mathrm{d}x_{n-2}\cdots\int_{x_2}^{b} \left(\int_a^{x_{n-1}} f(x_1,x_2,\cdots,x_n)\mathrm{d}x_n\right)\mathrm{d}x_1\\
=&\int_a^b \mathrm{d}x_{n-1}\int_a^{x_{n-1}}\mathrm{d}x_n\int_{x_{n-1}}^{b}\mathrm{d}x_{n-2}\cdots\int_{x_2}^{b} f(x_1,x_2,\cdots,x_n)\mathrm{d}x_1\\
=&\int_a^b \mathrm{d}x_{n-1}\int_a^{x_{n-1}} h(x_{n-1},x_n)\mathrm{d}x_n,
\end{aligned}$$

这里 $h(x_{n-1},x_n)=\int_{x_{n-1}}^b \mathrm{d}x_{n-2}\cdots\int_{x_2}^b f(x_1,x_2,\cdots,x_n)\mathrm{d}x_1$. 再利用 $n=2$

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的结论, 得

$$\int_a^b \mathrm{d}x_{n-1} \int_a^{x_{n-1}} h(x_{n-1}, x_n)\mathrm{d}x_n = \int_a^b \mathrm{d}x_n \int_{x_{n-1}}^b h(x_{n-1}, x_n)\mathrm{d}x_{n-1}.$$

故,

$$\begin{aligned}&\int_a^b \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_{n-1}} f(x_1, x_2, \cdots, x_n)\mathrm{d}x_n \\ = &\int_a^b \mathrm{d}x_n \int_{x_n}^b \mathrm{d}x_{n-1} \cdots \int_{x_2}^b f(x_1, x_2, \cdots, x_n)\mathrm{d}x_1.\end{aligned}$$

习题10.4

4. 设 $f(x)$ 连续, 证明:

$$\int_0^a \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-1}} f(x_1)f(x_2)\cdots f(x_n)\mathrm{d}x_n = \frac{1}{n!}\left(\int_0^a f(t)\mathrm{d}t\right)^n$$

**证明** 记

$$g_n(t) = \int_0^t \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-1}} f(x_1)f(x_2)\cdots f(x_n)\mathrm{d}x_n.$$

则

$$g_1(t) = \int_0^t f(u)\mathrm{d}u.$$

假设

$$g_{n-1}(t) = \frac{1}{(n-1)!}\left(\int_0^t f(u)\mathrm{d}u\right)^{n-1}.$$

对 $g_n(t)$ 求导, 得

$$g_n'(t) = \int_0^t \mathrm{d}x_2 \cdots \int_0^{x_{n-1}} f(t)f(x_2)\cdots f(x_n)\mathrm{d}x_n,$$

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即,

$$g_n'(t) = f(t)g_{n-1}(t) = \frac{1}{(n-1)!}f(t)\left(\int_0^t f(u)\mathrm{d}u\right)^{n-1}$$
$$= \frac{1}{n!}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t f(u)\mathrm{d}u\right)^n.$$

于是

$$g_n(t) = \frac{1}{n!}\left(\int_0^t f(u)\mathrm{d}u\right)^n.$$

根据归纳原理, 结论得证.

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