~~作~~ 15、17、18、21、25、28

15.

$$E\sin X = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx = 0$$

$$E\cos x = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx = \frac{2}{\pi}$$

$$E x\cos X = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\, dx = 0$$

17.

$$p(x>1) = \int_1^{\infty} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\arctan x\Big|_1^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{1}{4}$$

$\because$ 对称 $\therefore p(|X|\le 1) = 1-\frac{1}{4}\times 2 = \frac{1}{2}$

$$E\min\{|X|,1\} = \frac{1}{2}\times 1 + 2\int_0^1 \frac{x}{\pi(1+x^2)}dx$$

$$= \frac{1}{2} + 2\int_0^1 \frac{d(1+x^2)}{2\pi(1+x^2)}$$

$$= \frac{1}{2} + \frac{1}{\pi}\ln(1+x^2)\Big|_0^1 = \frac{1}{2} + \frac{1}{\pi}\ln 2$$

18. $f(y) = \frac{1}{2}I_{[0,1]} + \frac{1}{4}I_{[0,2]} = \frac{3}{4}I_{[0,1)} + \frac{1}{4}I_{[1,2]}$

(1)

$$F(y) = \begin{cases} 0 & y<0 \\ \frac{3}{4}y & 0\le y<1 \\ \frac{1}{4}y+\frac{1}{2} & 1\le y\le 2 \\ 1 & y>2 \end{cases}$$

(2) $EY = E(E(Y|X)) = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1 = \frac{3}{4}$

21.

(1) $$p(X=k|X+Y=m) = \frac{P(X=k, X+Y=m)}{P(X+Y=m)} = \frac{P(X=k, Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\frac{\lambda^k}{k!}\frac{\mu^{m-k}}{(m-k)!}e^{-\lambda-\mu}}{\frac{(\lambda+\mu)^m}{m!}e^{-\lambda-\mu}} = \binom{m}{k}\frac{\lambda^k\mu^{m-k}}{(\lambda+\mu)^m}$$

$$E(X|X+Y=m) = \sum_{k=1}^{m} k\binom{m}{k}\frac{\lambda^k\mu^{m-k}}{(\lambda+\mu)^m}$$

$$= \frac{1}{(\lambda+\mu)^m}\sum_{k=1}^{m} k\binom{m}{k}\lambda^k\mu^{m-k}$$

$$= \frac{m\lambda}{(\lambda+\mu)^m}\sum_{k=1}^{m}\binom{m-1}{k-1}\lambda^{k-1}\mu^{m-k} = \frac{m\lambda}{(\lambda+\mu)^m}(\lambda+\mu)^{m-1} = \frac{m\lambda}{\lambda+\mu}$$

(2) $$p(X=k \mid X+Y=m) = \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n+k-m}}{\binom{2n}{m}p^m(1-p)^{2n-m}}$$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

$$E(X \mid X+Y=m) = \sum_{k=1}^{m} k\cdot\frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} = n\cdot\sum_{k=1}^{m}\frac{\binom{n-1}{k-1}\binom{n}{m-k}}{\binom{2n}{m}}$$

$$= \frac{n}{\binom{2n}{m}}\sum_{k=1}^{m}\binom{n-1}{k-1}\binom{n}{m-k} = \frac{n}{\binom{2n}{m}}\binom{2n-1}{m-1} = \frac{m}{2}$$

25. $$p(\min\{X_1,X_2\}>x) = p(X_1>x)p(X_2>x) = (e^{-2x})^2$$

$$p(\max\{X_1,X_2\}\le x) = p(X_1\le x)p(X_2\le x) = (1-e^{-2x})^2$$

$\because EX = \int_0^{\infty}(1-F(x))dx$ （习题结论）

$$E\min\{X_1,X_2\} = \int_0^{\infty} e^{-4x}dx = \frac{1}{4}$$

$$E\max\{X_1,X_2\} = \int_0^{\infty}[1-(1-e^{-2x})^2]dx = \frac{3}{4}.$$

28. 令 $E(n)$ 为点数和为 $n$ 时，还需要掷的次数的期望.

(1) 则 $$E(n) = \begin{cases} 0 & 10\le n\le 15 \\ 1+\frac{1}{6}[E(n+1)+\cdots+E(n+6)] & 0\le n\le 9 \end{cases}$$ $\approx 3.32$

$E(9) = 1$

$E(8) = 1+\frac{1}{6} = \frac{7}{6}$

$E(7) = 1+\frac{1}{6}(\frac{7}{6}+1) = (\frac{7}{6})^2$

$\cdots$

$E(3) = 1+\frac{1}{6}((\frac{7}{6})^5+(\frac{7}{6})^4+\cdots+\frac{7}{6}+1) = (\frac{7}{6})^6$

$E(2) = 1+\frac{1}{6}[(\frac{7}{6})^6+(\frac{7}{6})^5+\cdots+\frac{7}{6}] = (\frac{7}{6})^7-\frac{1}{6}$

$E(1) = 1+\frac{1}{6}[(\frac{7}{6})^7-\frac{1}{6}+(\frac{7}{6})^6+\cdots+(\frac{7}{6})^2]$

$= (\frac{7}{6})^8-\frac{7}{18}$

$E(0) = 1+\frac{1}{6}[(\frac{7}{6})^8-\frac{7}{18}+(\frac{7}{6})^7-\frac{1}{6}+\cdots+(\frac{7}{6})^3]$

$= (\frac{7}{6})^9-(\frac{7}{6})^3+\frac{49}{54}$

(2). 设共掷 $T$ 次，第 $i$ 次掷的点数为 $X_i$.

$$E\sum_{i=1}^{T} X_i = E\left(E\left(\sum_{i=1}^{T} X_i \,\middle|\, T\right)\right)$$

$$= E(T \cdot EX_i) = ET \cdot EX_i$$

$$EX_i = \frac{1}{6}(1+2+\cdots+6) = \frac{7}{2} \Rightarrow E\sum_{i=1}^{T} X_i \approx 11.63$$