Common inequalities[1]

Prepared by HL

Chebychev's inequality Let X be a random variable and let $g(x)$ be a nonnegative function. For any $r > 0$,

$$P(g(X) \geq r) \leq \frac{E[g(X)]}{r}$$

Different forms of Chebychev's inequality

- If g is nondecreasing, then another form of Chebychev's inequality is, for $\varepsilon > 0$,

$$P(X \geq \varepsilon) \leq \frac{E[g(X)]}{g(\varepsilon)}$$

- Suppose that X has expectation μ and variance σ^2. For $g(x) = (x-\mu)^2/\sigma^2$, we have

$$P(|X-\mu| \geq t\sigma) = P\left(\frac{(X-\mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2}$$

- If X has a finite kth moment with an integer k, then, for $t > 0$,

$$P(|X-\mu| \geq t) \leq \frac{E|X-\mu|^k}{t^k}$$

- **Chernoff inequality** If X has a finite mgf $M_X(t)$ for $t \in (-h,h)$, then, for $r > 0$ and $t > 0$,

$$P(X \geq r) \leq \frac{E(e^{tX})}{e^{tr}} = \frac{M_X(t)}{e^{tr}}, \quad P(X \leq -r) \leq \frac{E(e^{-tX})}{e^{tr}} = \frac{M_X(-t)}{e^{tr}}$$

$$P(|X| \geq r) \leq \frac{M_X(t) + M_X(-t)}{e^{tr}}$$

Cauchy-Schwartz's inequality If X and Y are random variables with $E(X^2) < \infty$ and $E(Y^2) < \infty$, then the following Cauchy-Schwartz's inequality holds:

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

with equality holds iff $P(X = cY) = 1$ for a constant c.

Hölder's inequality If p and q are positive constants satisfying $p > 1$ and $p^{-1} + q^{-1} = 1$ and X and Y are random variables, then

$$E|XY| \leq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}$$

Liapounov's inequality If r and s are constants satisfying $1 \leq r \leq s$ and X is a random variable, then

$$(E|X|^r)^{1/r} \leq (E|X|^s)^{1/s}$$

Minkowski's inequality If $p \geq 1$ is a constant and X and Y are random variables, then

$$(E|X+Y|^p)^{1/p} \leq (E|X|^p)^{1/p} + (E|Y|^p)^{1/p}$$

Jensen's inequality If g is a convex function on a convex $A \subset \mathcal{R}$ and X is a random variable with $P(X \in A) = 1$, then

$$g(E(X)) \leq E[g(X)]$$

provided that the expectations exist. If g is strictly convex, then $\leq$ in the previous inequality can be replaced by $<$ unless $P(g(X) = c) = 1$ for a constant c.

- The function $g(x) = x^{-1}$ is strictly convex. Hence,

$$(EX)^{-1} < E(X^{-1})$$

unless $P(X = c) = 1$ for a constant c.

- The function $g(x) = -\log x$ is strictly convex ($\log x$ is strictly concave). Then

$$-\log(EX) < -E(\log X) \quad \text{i.e.,} \qquad E(\log X) < \log(EX)$$

unless $P(X = c) = 1$ for a constant c.

- Let f and g be positive functions satisfying $0 < \int_{-\infty}^{\infty} g(x)dx \leq \int_{-\infty}^{\infty} f(x)dx = 1$. We want to show that

$$\int_{-\infty}^{\infty} f(x) \log \frac{g(x)}{f(x)} dx \leq 0$$

Cantelli's inequality

$$Pr(X-\mu \leq \lambda) = \begin{cases} \leq \frac{\sigma^2}{\sigma^2+\lambda^2} & \text{if } \lambda < 0, \\ \geq 1 - \frac{\sigma^2}{\sigma^2+\lambda^2} & \text{if } \lambda > 0, \end{cases}$$

where $\mu = E(X)$ and $\sigma^2 = \text{var}(X)$

Gnedenko $g(x)$ is positive and nondecreasing.

$$P(|X-\mu| \geq c) \leq \frac{Eg(|X-\mu|)}{g(c)}$$

$EX = 0$, then

$$P(|X| \geq c\sigma) \geq \frac{\mu_4 - \sigma^4}{\mu_4 + c^4\sigma^4 - 2c^2\sigma^4}$$

for $c > 1$.

Hajek-Renyi $X_1, \ldots, X_n$ are independent random variables with zero mean and finite variances σ_i^2. Let $c_1, \cdots$ be positive and non-increasing, and $S_k = \sum_{i=1}^k X_i$. Then we have, for any integer $m (< n)$ and $\varepsilon > 0$,

$$P\left(\max_{m \leq k \leq n} c_k|S_k| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\left(c_m^2 \sum_{i=1}^m \sigma_i^2 + \sum_{i=m+1}^n c_i^2\sigma_i^2\right)$$

A useful lower bound Let $Y \geq 0$ with $E(Y^2) < \infty$ and let $a < E(Y)$. Then we have $P(Y > a) \geq (EY - a)^2/EY^2$. This is often used with $a = 0$.

Feller-Chung Theorem For each integral number j, $A_j A_{j-1}^C \cdots A_0^C$ and B_j are independent, where $A_0 = \emptyset$. Then $P(\cup_j A_j B_j) \geq \alpha P(\cup_j A_j) \quad$ for $\alpha = \inf_j P(B_j)$.

$X_1, \cdots, X_n$ are independent and **symmetric** random variables. Write $S_k = \sum_{i=1}^k X_i$ for $k = 1, \cdots, n$. Then

$$P(\max_{1 \leq k \leq n} S_k > a) \leq 2P(S_n < a).$$

X and Y are independent with means zero. Then $E|X+Y|^r \geq \max(E|X|^r, E|Y|^r)$ for any $r \geq 1$.

Generalized Kolmogorov inequality $X_1, \cdots, X_n$ are independent random variables with mean zeros. Write $S_k = \sum_{i=1}^k X_i$ for $k = 1, \cdots, n$ and $A = \{\sup_{k \leq n} |S_k| \geq C\}$ for some positive constant C. Then

$$C^r P(A) \leq E(|S_n|^r I_A) \leq E|S_n|^r \quad \text{for } r \geq 1.$$

Doob Inequality For independent sequence $\{X_n\}$ with mean zero and $p > 1$,

$$E\left(\max_{1 \leq k \leq n}\left|\sum_{j=1}^k X_j\right|^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left(\left|\sum_{j=1}^n X_j\right|^p\right)$$

Tail Normal $X \sim \mathcal{N}(0,1)$, then to show that for $x > 0$,

$$P(X > x) \leq \frac{\exp(-x^2/2)}{x\sqrt{2\pi}}.$$

Bernstein inequalities Let $X_1, \cdots, X_n$ be independent Bernoulli random variables taking values $+1$ and -1 with probability $1/2$, then for every positive ε,

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geq \varepsilon\right\} \leq 2\exp\left\{-\frac{n\varepsilon^2}{2(1+\varepsilon/3)}\right\}.$$

Let $X_1, \cdots, X_n$ be independent zero-mean random variables. Suppose that $|X_i| \leq M$ almost surely, for all i. Then, for all positive t,

$$P\left\{\sum_{i=1}^n X_i > t\right\} \leq 2\exp\left\{-\frac{t^2}{\sum EX_j^2 + Mt/3}\right\}.$$

Hoeffding's inequality If $X_1, \cdots, X_n$ are independent. Assume that the X_i are almost surely bounded; that is, assume for $1 \leq i \leq n$ that $P(X_i \in [a,b]) = 1$. Then, for the empirical mean of these variables, $\overline{\mathbf{X}}$, we have the inequalities:

$$P\{\overline{\mathbf{X}} - E(\overline{\mathbf{X}}) \geq t\} \leq \exp\left\{-\frac{2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\},$$

$$P\{|\overline{\mathbf{X}} - E(\overline{\mathbf{X}})| \geq t\} \leq 2\exp\left\{-\frac{2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}.$$

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