四. 3. 设 $Y \sim N(0,1)$，$\frac{Z-4}{2}=Y$，即 $Z=2Y+4 \sim N(4,4)$

则 $EX = 0.5EY + 0.5EZ = 2$

4.(2). $EX = \int_0^1 x f(x)dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot B(\alpha+1,\beta) = \frac{\alpha}{\alpha+\beta}$

$\mathrm{Var}X = EX^2-(EX)^2 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

7: 记 $I_i = \begin{cases} 1, & \text{第}i\text{个盒子为空} \\ 0, & \text{其他} \end{cases}$，$i=1,\cdots,n$，空盒子数 $X=\sum_{i=1}^{n} I_i$

而 $EI_i = P(I_i=1) = (1-\frac{1}{n})^n$，$\therefore EX = \sum_{i=1}^{n} E(I_i) = n(1-\frac{1}{n})^n$

$n\to\infty$ 时平均比例即 $\lim_{n\to\infty}\frac{EX}{n} = \lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$

8.(1) 记 Y_i 为得到第 i 张新卡片所需次数，$X_n = \sum_{i=1}^{n} Y_i$，$Y_i \sim Ge(\frac{n-i+1}{n})$

则 $EY_i = \frac{n}{n-i+1}$，$EX_n = \sum_{i=1}^{n} E(\frac{1}{n-i+1}) = n\sum_{i=1}^{n}\frac{1}{i} = 12\sum_{i=1}^{12}\frac{1}{i}$

(2). $\lim_{n\to\infty} E(\frac{X_n}{n\ln n}) = \lim_{n\to\infty}\frac{EX_n}{n\ln n} = \lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{1}{i}}{\ln n} = 1$

12: (1) $f_X(x) = \lambda e^{-\lambda x}$，$x>0$

记 $Y=cX$，即 $X=\frac{Y}{c}$

$\therefore f_Y(y) = f_X(\frac{y}{c})\cdot|(\frac{y}{c})'| = \frac{\lambda}{c}e^{-\lambda\frac{y}{c}}$，$y>0$

即 $Y \sim Exp(\frac{\lambda}{c})$

(2): $EX^n = \int_0^{+\infty} x^n f_X(x)dx = \int_0^{+\infty} x^n \lambda e^{-\lambda x}dx \xlongequal{\text{令}t=\lambda x} \int_0^{+\infty}(\frac{t}{\lambda})^n\cdot\lambda e^{-t}d(\frac{t}{\lambda}) = \frac{1}{\lambda^n}\int_0^{+\infty}t^n e^{-t}dt = \frac{n!}{\lambda^n}$

17: $E(\min\{|X|,1\}) = \int_{-\infty}^{+\infty}\min\{|X|,1\}f(x)dx = \int_{-1}^{1}\frac{|x|}{\pi(1+x^2)}dx + 2\int_1^{+\infty}\frac{1}{\pi(1+x^2)}dx = \frac{\ln 2}{\pi}+\frac{1}{2}$

18: (1). $F(y) = P(Y\le y) = P(Y\le y|X=1)P(X=1) + P(Y\le y|X=2)P(X=2)$

当 $y<0$ 时，$P(Y\le y)=0$

当 $0\le y<1$ 时，$P(Y\le y) = \frac{1}{2}\cdot y + \frac{1}{2}\cdot\frac{y}{2} = \frac{3}{4}y$

当 $1\le y<2$ 时，$P(Y\le y) = \frac{1}{2}\times 1 + \frac{1}{2}\cdot\frac{y}{2} = \frac{y}{4}+\frac{1}{2}$

当 $y\ge 2$ 时，$P(Y\le y)=1$

综上，$F(y) = \begin{cases} 0, & y<0 \\ \frac{3}{4}y, & 0\le y<1 \\ \frac{y}{4}+\frac{1}{2}, & 1\le y<2 \\ 1, & y\ge 2 \end{cases}$

(2): $EY = \int_{-\infty}^{+\infty} y\,dF(y) = \int_0^1 \frac{3}{4}y\,dy + \int_1^2 \frac{y}{4}dy = \frac{3}{4}$

19：(1)：令 $W=X-Y\sim N\left((1,-1)\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix},(1,-1)\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\right)=N(0,0.5)$

即 $f_W(w)=\frac{1}{\sqrt{\pi}}e^{-w^2}$，而 $Z=|W|$，有 $f_Z(z)=\frac{2}{\sqrt{\pi}}e^{-z^2}$

$EZ=\int_{-\infty}^{+\infty}z\frac{2}{\sqrt{\pi}}e^{-z^2}dz=\frac{1}{\sqrt{\pi}}$

(2)：$EU=E(\max\{X,Y\})=E\left(\frac{X+Y+|X-Y|}{2}\right)=\frac{1}{2}EX+\frac{1}{2}EY+\frac{1}{2}EZ=1+\frac{1}{2\sqrt{\pi}}$

$EV=E(\min\{X,Y\})=E\left(\frac{X+Y-|X-Y|}{2}\right)=\frac{1}{2}EX+\frac{1}{2}EY-\frac{1}{2}EZ=1-\frac{1}{2\sqrt{\pi}}$

25：$E(\min\{X_1,X_2\})=\int_0^{+\infty}\int_0^{+\infty}\min\{X_1,X_2\}f(x_1)f(x_2)dx_1dx_2=\int_0^{+\infty}dx_1\int_0^{x_1}x_2f(x_1,x_2)dx_2+\int_0^{+\infty}dx_1\int_{x_1}^{+\infty}x_1f(x_1,x_2)dx_2$

$=4\int_0^{+\infty}dx_1\int_0^{x_1}x_2e^{-2x_1-2x_2}dx_2+4\int_0^{+\infty}dx_1\int_{x_1}^{+\infty}x_1e^{-2x_1-2x_2}dx_2=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$

$E(\max\{X_1,X_2\})=E(X_1+X_2-\min\{X_1,X_2\})=\frac{1}{2}+\frac{1}{2}-\frac{1}{4}=\frac{3}{4}$

28：(1)：设点数大于等于n时停止投掷次数为 X_n

$n\le 6$ 时，$EX_n=1+\frac{1}{6}\sum_{i=1}^{n-1}EX_i$

$n>6$ 时，$EX_n=1+\sum_{i=n-6}^{n-1}EX_i$

递推得 $EX_{10}=\left(\frac{7}{6}\right)^9-\frac{1}{2}\cdot\left(\frac{7}{6}\right)^2\approx 3.3237$