五、15、$X_1-2X_2\sim N(0,20)$，$3X_3-4X_4\sim N(0,100)$

则 $\frac{X_1-2X_2}{\sqrt{20}}\sim N(0,1)$，$\frac{1}{20}(X_1-2X_2)^2\sim \chi_1^2$

$\frac{3X_3-4X_4}{\sqrt{100}}\sim N(0,1)$，$\frac{1}{100}(3X_3-4X_4)^2\sim \chi_1^2$

于是当 $a=\frac{1}{20}$，$b=\frac{1}{100}$ 时，$T\sim \chi_2^2$

（当 a 或 b 其之一为 0 时，$T\sim \chi_1^2$）

六、29.（1）此时 $X\sim U(\theta,0)$

$EX=\frac{\theta}{2}$，则 $\hat{\theta}_M=2\bar{X}$

$L(\theta)=\prod_{i=1}^{n}f(x;\theta)=\theta^{-n}$，$\theta\le X_1,\cdots,X_n\le 0$

当 $\theta\le X_1,\cdots,X_n\le 0$ 时，$L(\theta)$ 为增函数，$\therefore \hat{\theta}_L=X_{(1)}$

（2）此时 $X\sim U(\theta,2\theta)$

$EX=\frac{3}{2}\theta$，则 $\hat{\theta}_M=\frac{2}{3}\bar{X}$

$L(\theta)=\prod_{i=1}^{n}f(x;\theta)=\theta^{-n}$，$\theta\le X_1,\cdots,X_n\le 2\theta$

即 $\frac{X_{(n)}}{2}\le\theta\le X_{(1)}$，$L(\theta)$ 为减函数，$\therefore \hat{\theta}_L=\frac{X_{(n)}}{2}$

30. $\int_{-\infty}^{+\infty}ce^{-(a+bx)^2}dx=1\ \Rightarrow c=\frac{b}{\sqrt{\pi}}$

$L(\theta)=L(a,b)=\left(\frac{b}{\sqrt{\pi}}\right)^n e^{-\sum_{i=1}^{n}(a+bx_i)^2}$

$\ell(a,b)=\ln(L(a,b))=n\ln\left(\frac{b}{\sqrt{\pi}}\right)-\sum_{i=1}^{n}(a+bx_i)^2$

$$\begin{cases}\frac{\partial \ell(a,b)}{\partial a}=-2na-2b\sum_{i=1}^{n}x_i=0\\ \frac{\partial \ell(a,b)}{\partial b}=\frac{n}{b}-2\sum_{i=1}^{n}x_i(a+bx_i)=0\end{cases}$$

解得 $b=\sqrt{\frac{n}{2\sum_{i=1}^{n}x_i^2-\frac{2}{n}\left(\sum_{i=1}^{n}x_i\right)^2}}=\sqrt{\frac{n}{2\sum_{i=1}^{n}(x_i-\bar{x})^2}}=\frac{1}{\sqrt{2m_2}}$

$a=-b\bar{x}=-\frac{\bar{x}}{\sqrt{2m_2}}$

36. $L(\theta)=L(\theta_1,\theta_2)=(\theta_2-\theta_1)^{-n}$，$\theta_1\le X_1,\cdots X_n\le\theta_2$

$L(\theta_1,\theta_2)$ 关于 $(\theta_2-\theta_1)$ 为减函数，要使 $L(\theta_1,\theta_2)$ 最大，应使 $(\theta_2-\theta_1)$ 最小。

即 $\hat{\theta}_{1L}=X_{(1)}$，$\hat{\theta}_{2L}=X_{(n)}$

44.（1）$EX=\int_{\theta}^{+\infty}\frac{x}{\sigma}e^{-\frac{x-\theta}{\sigma}}dx=\sigma+\theta$　$\therefore \hat{\theta}_1=\bar{X}-\sigma$

$L(\theta)=\prod_{i=1}^{n}f(x_i;\theta)=\sigma^{-n}e^{-\frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\theta)}$，$\theta<X_1,\cdots,X_n$

$L(\theta)$ 为 θ 的增函数，$\therefore \hat{\theta}_2=X_{(1)}$

（2）$E\hat{\theta}_1=E\bar{X}-\sigma=\theta$，$\therefore \hat{\theta}_1$ 为无偏的

$P(X_{(1)}\le x)=1-(1-F(x))^n=1-e^{-\frac{n}{\sigma}(x-\theta)}$，$x>\theta$

$\therefore EX_{(1)} = \int_{\theta}^{+\infty} \frac{nx}{\sigma} e^{-\frac{n}{\sigma}(x-\theta)} dx = \theta + \frac{\sigma}{n} = E\hat{\theta}_2$

$\therefore \hat{\theta}_2$ 是有偏的，修正为 $\hat{\theta}_2 = \hat{\theta}_2 - \frac{\sigma}{n} = X_{(1)} - \frac{\sigma}{n}$

(3). $Var X = EX^2 - (EX)^2 = \int_{\theta}^{+\infty} \frac{x^2}{\sigma} e^{-\frac{x-\theta}{\sigma}} dx - (\theta+\sigma)^2 = \sigma^2$

$Var X_{(1)} = EX_{(1)}^2 - (EX_{(1)})^2 = \int_{\theta}^{+\infty} \frac{nx}{\sigma} e^{-\frac{n}{\sigma}(x-\theta)} dx - (\theta + \frac{\sigma}{n})^2 = \frac{\sigma^2}{n^2}$

$\therefore Var\,\hat{\theta}_1 = \frac{1}{n} Var X = \frac{\sigma^2}{n}$

$Var\,\tilde{\theta}_2 = Var X_{(1)} = \frac{\sigma^2}{n^2} < Var\,\hat{\theta}_1$

$\therefore \hat{\theta}_2$ 更优

47. $EX = \int_0^{+\infty} \lambda\alpha x^{\alpha} e^{-\lambda x^{\alpha}} dx = \lambda^{-\frac{1}{\alpha}} \int_0^{+\infty} (\lambda x^{\alpha})^{\frac{1}{\alpha}} e^{-(\lambda x^{\alpha})} d(\lambda x^{\alpha}) = \lambda^{-\frac{1}{\alpha}} \Gamma(\frac{1}{\alpha}+1)$

$\therefore \hat{\lambda}_M = \left(\frac{\Gamma(\frac{1}{\alpha}+1)}{\bar{X}}\right)^{\alpha}$

$L(\theta) = L(\lambda) = \lambda^n \alpha^n (\prod_{i=1}^{n} x_i)^{\alpha-1} e^{-\lambda \sum_{i=1}^{n} x_i^{\alpha}}$

$\ell(\lambda) = \ln(L(\lambda)) = n\ln\lambda + n\ln\alpha + (\alpha-1)\sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^{\alpha}$

$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\alpha} = 0 \Rightarrow \hat{\lambda}_L = \frac{n}{\sum_{i=1}^{n} x_i^{\alpha}} = \frac{1}{a_{\alpha}}$

49. 由 29.(2) 知 $\hat{\theta}_L = \frac{X_{(n)}}{2}$

$P(X_{(n)} \le x) = F^n(x) = (\frac{x-\theta}{\theta})^n$ ， $\theta < x < 2\theta$

$EX_{(n)} = \int_{\theta}^{2\theta} x \cdot \frac{n(x-\theta)^{n-1}}{\theta^n} dx = \frac{2n+1}{n+1}\theta$

$\therefore E\hat{\theta}_L = \frac{2n+1}{2n+2}\theta$ ，不是无偏估计，修正 $\hat{\theta}_L = \frac{2n+2}{2n+1}\hat{\theta}_L = \frac{n+1}{2n+1} X_{(n)}$

50. $L(\theta) = L(\mu_1, \mu_2, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{m+n} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right)}$

$\ell(\theta) = -(m+n)\ln(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right)$

$\frac{\partial \ell}{\partial \mu_1} = -\frac{1}{2\sigma^2}\sum_{i=1}^{m} 2(x_i-\mu_1) = 0 \Rightarrow \hat{\mu}_{1L} = \bar{X}$

$\frac{\partial \ell}{\partial \mu_2} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(y_i-\mu_2) = 0 \Rightarrow \hat{\mu}_{2L} = \bar{Y}$

$\frac{\partial \ell}{\partial \sigma^2} = -\frac{m+n}{2\sigma^2} + \frac{1}{2\sigma^4}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right) = 0 \Rightarrow \hat{\sigma}_L^2 = \frac{1}{m+n}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right)$

52. (与44题完全相同，取 $\sigma=1$ 即可) $\hat{\mu}^* = X_{(1)}$ ，$\hat{\mu}^{**} = X_{(1)} - \frac{1}{n}$ ，$\hat{\mu} = \bar{X} - 1$ ，$Var(\hat{\mu}) = \frac{1}{n}$ ，$Var(\hat{\mu}^{**}) = \frac{1}{n^2}$ ，$\hat{\mu}^{**}$ 更有效

56. (新教材23年印刷版)：

(1). $EX = 0$ ，$EX^2 = 1-\theta$ ，$\therefore \hat{\theta}_M = 1 - a_2 = \frac{1}{2}$

$L(\theta) = (\frac{1-\theta}{2})^{n_{-1}+n_1} \cdot \theta^{n_0}$

$\ell(\theta) = (n_{-1}+n_1)\ln(\frac{1-\theta}{2}) + n_0 \ln\theta$

$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n_{-1}+n_1}{\theta-1} + \frac{n_0}{\theta} = 0 \Rightarrow \hat{\theta}_L = \frac{n_0}{n} = \frac{1}{2}$

(2). $E\hat{\theta}_M = 1 - \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \theta$

$E\hat{\theta}_L = \frac{1}{n} E n_0 = \frac{1}{n} \cdot n\theta = \theta$ ，均为无偏的

(3). $Var X^2 = EX^4 - (EX^2)^2 = \theta(1-\theta)$

$\therefore Var\hat{\theta}_M = \frac{1}{n} Var X^2 = \frac{1}{n}\theta(1-\theta)$

$Var\hat{\theta}_L = \frac{1}{n^2} Var n_0 = \frac{1}{n^2}\cdot n\theta(1-\theta) = \frac{1}{n}\theta(1-\theta) = Var\hat{\theta}_M$

$\therefore$ 有效性相同、

(充教材22年印刷版)：

(1). $EX = 3-5\theta$ ，即 $\hat{\theta}_M = \frac{3-\bar{X}}{5} = \frac{2}{5}$

$L(\theta) = (\frac{\theta}{2})^{n_0} \cdot \theta^{n_1} (\frac{3}{2}\theta)^{n_2} (1-3\theta)^{n_3}$

$l(\theta) = n_0 \ln(\frac{\theta}{2}) + n_1 \ln\theta + n_2 \ln(\frac{3}{2}\theta) + n_3 \ln(1-3\theta)$

$\frac{\partial l(\theta)}{\partial\theta} = \frac{n_0+n_1+n_2}{\theta} + \frac{-3n_3}{1-3\theta} = 0 \Rightarrow \hat{\theta}_L = \frac{n-n_3}{3n} = \frac{4}{15}$

(2). $E\hat{\theta}_M = \frac{3-E\bar{X}}{5} = \theta$

$E\hat{\theta}_L = \frac{n-En_3}{3n} = \frac{n-n(1-3\theta)}{3n} = \theta$ ，均为无偏的.

(3). $Var X = EX^2 - (EX)^2 = 9-20\theta-(3-5\theta)^2 = 10\theta-25\theta^2$

$\therefore Var\hat{\theta}_M = \frac{1}{25n} Var X = \frac{\theta(2-5\theta)}{50}$

$Var\hat{\theta}_L = \frac{1}{9n^2} Var n_3 = \frac{1}{900}\cdot n(1-3\theta)\cdot 3\theta = \frac{\theta(1-3\theta)}{30} < Var\hat{\theta}_M$

$\therefore \hat{\theta}_L$ 更有效.