Operations Research: Homework 01

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CONVEX SET

Question 1

Exercise 1.3 extreme points ⇔ feasible base solutions.

Theorem: vector set V in field R,rank(V)=n:
S=$\{\vec{\alpha_1}, \vec{\alpha_2}, \ldots, \vec{\alpha_r}\}$ is linearly independent.
$\exists \alpha_{\vec{r+1}}, \ldots, \vec{\alpha_n} \in V$ subject to rank$\{\vec{\alpha_1}, \vec{\alpha_2}, \ldots, \vec{\alpha_n}\}$=rank(V)=n.
Solution space $V_A = \{\vec{x}|A\vec{x} = \vec{0}\}$ is a linear space in R.
And, W=$\{\vec{x}|A\vec{x} = \vec{b}\}$ is a congruence class mod V_A.

Statement: $\vec{x}$ is a feasible base solution ⇔ the positive components of $\vec{x}$:$\{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}$ such that rank$\{\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}}\}$=k.

Proof: sufficiency is obvious.
necessity:

(1). The row rank and column rank of a matrix is equal.
rank(A)=m ⇔ rank$\{\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}}\} = k \leq m$

(2). If k=m, rank$\{\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}}\} = m$.
Hence,B=$(\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}})$ and $\vec{x_B} = (x_{j_1}, x_{j_2}, \ldots, x_{j_m})$.
$\vec{x}$ is a feasible base solution.

(3). If k<m, rank(A)=m.
There are other vectors of A:$\{\alpha_{\vec{j_{k+1}}}, \alpha_{\vec{j_{k+2}}}, \ldots, \vec{\alpha_{j_m}}\}$ such that $x_{j_i} = 0,\ i = k+1, k+2, \ldots, m$ and rank$\{\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_m}}\} = m$.
Hence,B=$(\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}})$ and $\vec{x_B} = (x_{j_1}, x_{j_2}, \ldots, x_{j_m})$.
$\vec{x}$ is a feasible base solution.

Proof:
sufficiency: extreme points ⇒ feasible base solutions.

$\vec{x}$ is an extreme points ⇔ $\vec{x} : \forall \vec{x_1}, \vec{x_2} \in S, \lambda \in (0,1), \vec{x} = \lambda\, \vec{x_1} + (1-\lambda)\vec{x_2} \Rightarrow \vec{x} = \vec{x_1} = \vec{x_2}$.

the positive components of $\vec{x}$ are $\vec{x_B} = (x_{j_1}, x_{j_2}, \ldots, x_{j_k})$.

B=$(\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \ldots, \vec{\alpha_{j_k}})$ such that B$\vec{x_B} = \vec{b}$. **rank(B)=k:**

If rank(B)<k, B$\vec{x_B} = \vec{0}$ have a non-zero solution $\vec{y_B}$ expending to a n-dim vector $\vec{y}$.

$\vec{y}$ is a solution of A$\vec{x} = \vec{0}$.

$\exists \lambda_1 > 0$, subject to $\vec{x_1} = \vec{x} + \lambda_1 \vec{y} > \vec{0}$ and $\exists \lambda_2 > 0$, subject to $\vec{x_2} = \vec{x} - \lambda_2 \vec{y} > \vec{0}$.

Question 1 continued on next page...

Thus, $\vec{x_1}, \vec{x_2} \in S, \vec{x_1} \neq \vec{x_2}$ subject to $\vec{x} = \frac{\lambda_2}{\lambda_1+\lambda_2}\vec{x_1} + \frac{\lambda_1}{\lambda_1+\lambda_2}\vec{x_2}$ which is contradictory to the definition of extreme point.

k=rank(B)≤m, $rank\{\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \dots, \vec{\alpha_{j_k}}\} = k \leq m.$

According to the statement, $\vec{x}$ is a feasible base solution.

necessity: extreme points ⇐ feasible base solutions.

$\vec{x}$ is a feasible base solution ⇔ the positive components $\vec{x_B}$ $\{x_{j_1}, x_{j_2}, \dots, x_{j_k}\}$

such that rank($B = (\vec{\alpha_{j_1}}, \vec{\alpha_{j_2}}, \dots, \vec{\alpha_{j_k}})$)=k.

B$\vec{x_B} = \vec{0}$ have only zero solution.

$\forall \vec{x_1}, \vec{x_2} \in S, \lambda \in (0,1)$, subject to $\vec{x} = \lambda\vec{x_1} + (1-\lambda)\vec{x_2} \Rightarrow \vec{x_B} = \lambda\vec{x_{1,B}} + (1-\lambda)\vec{x_{2,B}}$

$\vec{x}, \vec{x_1}, \vec{x_2} \in S \Rightarrow A(\vec{x} - \vec{x_1}) = A(\vec{x} - \vec{x_2}) = \vec{0} \Rightarrow B(\vec{x_B} - \vec{x_{1,B}}) = B(\vec{x_B} - \vec{x_{2,B}}) = \vec{0}$

Thus, $\vec{x_B} = \vec{x_{1,B}} = \vec{x_{1,B}}$

$\vec{x_N}$ is $\vec{x} \setminus \vec{x_B}$: $\vec{x_N} = \lambda\vec{x_{1,N}} + (1-\lambda)\vec{x_{2,N}} = \vec{0} \Rightarrow \vec{x_N} = \vec{x_{1,N}} = \vec{x_{2,N}} = \vec{0}$.

Hence, $\vec{x} = \vec{x_1} = \vec{x_2} \Rightarrow \vec{x}$ is a extreme point.

Question 2

Exercise 1.2 Display the extreme points and the feasible base solutions of the Linear Programming as follow.

$$\begin{aligned} &\text{min } -x_1 + 3x_2 \\ &\text{s.t. } x_1 + 3x_2 \leq 8 \\ &\quad x_2 \leq 2 \\ &\quad x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

The extreme points

extreme point ⇔ vertex .

The vertexs of inequations as follows:
$x_1 + 3x_2 \leq 8$
$x_2 \leq 2$
$x_1 \geq 0,\ x_2 \geq 0$

are (0,0), (8,0), (0,2), (2,2).

Question 2 continued on next page...

The feasible base solutions.

The normalized form of the LP:

$$
\begin{aligned}
min \quad -x_1 + 3x_2 \\
s.t. \;\; x_1 + 3x_2 + x_3 = 8 \\
x_2 + x_4 = 2 \\
x_1,\ x_2,\ x_3,\ x_4 \geq 0
\end{aligned}
$$

$$A\vec{x} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{b} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

(1).

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Thus,

$$\vec{x}^T = \begin{pmatrix} 2 & 2 & 0 & 0 \end{pmatrix}$$

corresponding to the vertex (2,2).

(2).

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Thus,

$$\vec{x}^T = \begin{pmatrix} 8 & 0 & 0 & 2 \end{pmatrix}$$

corresponding to the vertex (8,0).

(3).

$$\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Thus,

$$\vec{x}^T = \begin{pmatrix} 0 & 2 & 2 & 0 \end{pmatrix}$$

corresponding to the vertex (0,2).

(4).

$$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Thus,

$$\vec{x}^T = \begin{pmatrix} 0 & \frac{8}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

is not a feasible solution.

(6).

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

Thus,

$$\vec{x}^T = \begin{pmatrix} 0 & 0 & 8 & 2 \end{pmatrix}$$

corresponding to the vertex (0,0).

Question 3

Exercise 1.1(1) The extreme point set P is non-empty and finite $\{\vec{p^{(1)}}, \vec{p^{(2)}}, \ldots, \vec{p^{(k)}}\}$**.**

Theorem: the optimal solution of LP is a basic feasible solution.
proof:

$$\vec{c}^T\vec{x} = \vec{c_B}^T\vec{x_B} + (\vec{c_N}^T - \vec{c_B}^T B^{-1}N)\vec{x_N}$$

The optimality:

$$\vec{c_N}^T - \vec{c_B}^T B^{-1}N \geq \vec{0}$$

Thus, $\vec{x_N} \geq \vec{0} \Rightarrow \vec{c}^T\vec{x} = \vec{c_B}^T\vec{x_B} + (\vec{c_N}^T - \vec{c_B}^T B^{-1}N)\vec{x_N} \geq \vec{c_B}^T\vec{x_B}$
if $\vec{x}$ is not a basic feasible solution, $\vec{x}$ is not a optimal solution.
⇔ if $\vec{x}$ is a optimal solution, $\vec{x}$ is a basic feasible solution(converse-negative proposition).

The extreme point set P is non-empty.

extreme point ⇔ feasible base solution.

S is not empty.

(1). S = $\{\vec{0}\}$ ⇔ $\vec{0}$ is a extreme point.

(2). S have non-zero solutions ⇒ $\exists\ \vec{0} \leq \vec{x} \in S \Rightarrow \exists k > 0, x_k > 0.$

A auxiliary LP, $\vec{y} \in R^{m\times 1}$:

$$\begin{aligned} min \quad & \vec{1}^T\vec{y} \\ s.t. \quad & A\vec{x} + \vec{y} = \vec{b} \\ & \vec{x}, \vec{y} \geq \vec{0} \end{aligned}$$

$\begin{pmatrix} \vec{0} \\ \vec{b} \end{pmatrix}$ is a feasible base solution of the auxiliary LP.

$\begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}$ is also an optimal solution of the auxiliary LP⇒ the auxiliary LP is bounded.

Use simplex method to get an optimal solution $\begin{pmatrix} \vec{x^*} \\ \vec{y^*} \end{pmatrix}$ of the auxiliary LP.

⇒ $\begin{pmatrix} \vec{x^*} \\ \vec{y^*} \end{pmatrix}$ is a basic feasible solution and $\vec{y^*} = \vec{0}$.

⇒ $\vec{x^*}$ is a basic feasible solution of S.

Thus, the extreme point set P is not empty.

The extreme point set P is finite.

There are at most $\binom{n}{m}$ distinct maximal linearly independent vectors in $\{\vec{\alpha_1}, \vec{\alpha_2}, \ldots, \vec{\alpha_n}\}$.

$\Rightarrow$ There are at most $\binom{n}{m}$ distinct basic solutions for $A\vec{x} = \vec{b}$.

$\Rightarrow$ There are at most $\binom{n}{m}$ distinct feasible base solutions in S.

$\Rightarrow$ There are at most $\binom{n}{m}$ distinct extreme points in S.

Thus, P is finite.

Question 4

Exercise 1.1(2) The direction set D is empty $\Leftrightarrow$ S is bounded(*the point in S is bounded*).

sufficency: S is unbounded $\Rightarrow$ The direction set D is not empty.

S is unbouned $\Rightarrow \forall \vec{x} \in S, \exists k, x_k$ is unbouned $\Rightarrow 0 \leq x_k < +\infty$.

$\vec{d} = \begin{pmatrix} 0 & \cdots & 0 & \underset{k}{1} & 0 & \cdots & 0 \end{pmatrix}$ subject to $\forall \vec{x} \in S, \lambda \geq 0, \vec{x} + \lambda\vec{d} \in S$.

Thus ,$\vec{d}$ is a direction of S.D is not empty.The extreme direction set E is also not empty.

necessity: S is bounded $\Rightarrow$ The direction set D is empty.

S is bouned $\Rightarrow \forall \vec{x} \in S, \forall k, x_k$ is bouned $\Rightarrow 0 \leq x_k \leq N_k$.

$\forall \vec{0} \neq \vec{d} \in R^m$:

(1). If there is a negative component d_k in $\vec{d}$, $\exists N \in N^+$ subject to $\forall \lambda > N$ satisfying $x_k + \lambda d_k < 0$

Hence, $\vec{d}$ is not a direction.

(2). If there is a positive component d_k in $\vec{d}$, $\exists N \in N^+$ s.t. $\forall \lambda > N$ satisfying $x_k + \lambda d_k > N_k$

Hence, $\vec{d}$ is not a direction.

Thus, D is empty.The extreme direction set E is also empty.

Question 5

Exercise 1.1(3) if S is unbound $\Rightarrow$ the extreme direction set E is finite $\{\vec{d^{(1)}}, \vec{d^{(2)}}, \ldots, \vec{d^{(l)}}\}$.

Question 5 continued on next page...

theorem:$\forall$ m $\times$ n matrix A$\in R^{m\times n}$, rank(A)=r, A is equivalent to the canonical form:

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, there are m-order invertible matrix P and n-order invertible matrix Q satisfying:

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$

Proof: rank(A)=m $\Rightarrow A = P \begin{pmatrix} I_m & 0 \end{pmatrix} Q$

$\forall \vec{x_0} \in S$, and A$\vec{x} = \vec{b} \Rightarrow$ A$(\vec{x} - \vec{x_0}) = \vec{0} \Rightarrow P \begin{pmatrix} I_m & 0 \end{pmatrix} Q(\vec{x} - \vec{x_0}) = \vec{0}$.

P is invertible $\Rightarrow \begin{pmatrix} I_m & 0 \end{pmatrix} Q(\vec{x} - \vec{x_0}) = \vec{0} \Rightarrow$ the first-m components of $Q(\vec{x} - \vec{x_0})$ is zero.

$\vec{\varepsilon_k} = \begin{pmatrix} 0 & \dots & 0 & \underset{k}{1} & 0 & \dots & 0 \end{pmatrix}$

$Q(\vec{x} - \vec{x_0}) = \sum_{k=m+1}^{n} \lambda_k \vec{\varepsilon_k}, \lambda_k \in R \Rightarrow \vec{x} = \vec{x_0} + \sum_{k=m+1}^{n} \lambda_k Q^{-1} \vec{\varepsilon_k}, \lambda_k \in R.$

Let $V_A = \{\vec{x} \in R^m | \vec{x} = \vec{x_0} + \sum_{k=m+1}^{n} \lambda_k Q^{-1} \vec{\varepsilon_k}, \lambda_k \in R, \forall \vec{x_0} \in S\}$

rank(S=$V_A \cap \{\vec{x} \geq \vec{0}\}$) $\leq$ rank(A) = n-m.

$\{\lambda_{m+1} Q^{-1} \vec{\varepsilon_{m+1}}, \lambda_{m+2} Q^{-1} \vec{\varepsilon_{m+2}}, \dots, \lambda_n Q^{-1} \vec{\varepsilon_n}\}$ is a set of bases(extreme directions) of V_A.

There are at most (n-m) extreme directions in V_A.
$\Rightarrow$ There are at most (n-m) extreme directions in S.

Thus, E is finite.

S is unbound $\Rightarrow$ D is not empty $\Rightarrow$ E is not empty.

Thus, E =$\{\vec{d^{(1)}}, \vec{d^{(2)}}, \dots, \vec{d^{(l)}}\}$

Question 6

Exercise 1.1(4) $\vec{x} \in S \ \Leftrightarrow \vec{x} = \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} + \sum_{j=1}^{l} \mu_j \vec{d^{(j)}},\ \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0$**.**

Proof:
sufficiency:$\vec{x} \in S \ \Rightarrow \vec{x} = \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} + \sum_{j=1}^{l} \mu_j \vec{d^{(j)}},\ \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0.$

$A \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} = \sum_{i=1}^{k} \lambda_i A \vec{p^{(i)}} = \sum_{i=1}^{k} \lambda_i \vec{b} = \vec{b} \Rightarrow \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} \in S.$

$\vec{x} - \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}}$ satisfying $A(\vec{x} - \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}}) = \vec{0}$.

Thus, $\vec{x} - \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} \in V_A = \{\vec{x} \in R^m | A\vec{x} = \vec{0}, \vec{x} + \vec{x_0} \geq \vec{0}, \forall \vec{x_0} \in S\}$.

$\{\vec{d^{(1)}}, \vec{d^{(2)}}, \ldots, \vec{d^{(l)}}\}$ is a set of 'bases' of V_A.

Thus, $\vec{x} - \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} = \sum_{j=1}^{l} \mu_j \vec{d^{(j)}}, \mu_j \geq 0$.

necessity: $\vec{x} = \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} + \sum_{j=1}^{l} \mu_j \vec{d^{(j)}},\ \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0\ \Rightarrow \vec{x} \in S.$

$\sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} \in S$ and $\sum_{j=1}^{l} \mu_j \vec{d^{(j)}} \in V_A$.

$\vec{x} = \sum_{i=1}^{k} \lambda_i \vec{p^{(i)}} + \sum_{j=1}^{l} \mu_j \vec{d^{(j)}}$ satisfying A$\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$.

Thus, $\vec{x} \in S$.

SIMPLEX MODEL

Question 7

Exercise 1.4 The pivoting ensures that the solution is a feasible base solution.

proof: $\{\vec{\alpha_{j_1}}, \dots, \vec{\alpha_{j_{l-1}}}, \vec{\alpha_{j_{l+1}}}, \dots, \vec{\alpha_{j_m}}, \vec{\alpha_{j_{m+k}}}\}$ is linearly independent.

$(\vec{x_B}, \vec{x_N})^T = (x_{j_1}, x_{j_2}, \dots, x_{j_m}, 0, \dots, 0)$ is a feasible base solution.
$\Leftrightarrow B = \{\vec{\alpha}_{j_1}, \vec{\alpha}_{j_2}, \dots, \vec{\alpha}_{j_m}\}$ satisfies $rank(B) = m$.

for $c_{j_{m+k}} - \vec{c_B} B^{-1} \vec{\alpha_{j_{m+k}}} < 0$,
$\{\vec{\alpha}_{j_1}, \vec{\alpha}_{j_2}, \dots, \vec{\alpha}_{j_m}, \vec{\alpha}_{j_{m+k}}\}$ is linearly dependent. $\Rightarrow \vec{\alpha}_{j_{m+k}} = \sum_{i=1}^m \lambda_i \vec{\alpha}_{j_i}, \sum_{i=1}^m \lambda_i^2 \neq 0.$

$$\begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} \vec{x_B} \\ \vec{x_N} \end{pmatrix} = \vec{b}$$

$$\vec{\Delta} \in R^{(n-m)\times 1}, \vec{\Delta} = \begin{pmatrix} 0 \\ \vdots \\ \Delta \\ \vdots \\ 0 \end{pmatrix}, \Delta > 0.$$

$$\begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} \vec{x_B} + \vec{\delta} \\ \vec{x_N} + \vec{\Delta} \end{pmatrix} = \vec{b}$$

$$\Downarrow$$

$$\vec{\delta} = -B^{-1} N \vec{\Delta} = -\Delta B^{-1} \vec{\alpha_{j_{m+k}}} = -\Delta B^{-1} \sum_{i=1}^m \lambda_i \vec{\alpha}_{j_i} = -\Delta \vec{\lambda}.$$

for $\vec{x_B} - \vec{\delta} = \vec{x_B} - \Delta B^{-1} \sum_{i=1}^m \lambda_i \vec{\alpha}_{j_i} = \vec{x_B} - \Delta B^{-1} B \vec{\lambda} = \vec{x_B} - \Delta \vec{\lambda}$:
B^{-1} is invertible and $\vec{\alpha_{j_{m+k}}} \neq \vec{0} \Rightarrow \vec{\lambda} = B^{-1} \vec{\alpha_{j_{m+k}}} \neq \vec{0}$.

if $\vec{\lambda} < \vec{0}$, and $c_{j_{m+k}} - \vec{c_B} B^{-1} \vec{\alpha_{j_{m+k}}} < 0 \Rightarrow c_{m+k} < 0$

let Δ as large as possible $\Rightarrow \begin{pmatrix} \vec{x_B} + \vec{\delta} \\ \vec{x_N} + \vec{\Delta} \end{pmatrix}$ is a feasible base and $\vec{c}^T \vec{x} \to -\infty$.
The LP is unbounded.

if $\exists \lambda_l > 0$,$\Rightarrow \vec{\alpha_{j_{m+k}}} - \sum_{i \neq j, i=1}^m \lambda_i \vec{\alpha_{j_i}} = \lambda_l \vec{\alpha_{j_l}} \neq \vec{0}$

$\{\vec{\alpha_{j_1}}, \dots, \vec{\alpha_{j_{l-1}}}, \vec{\alpha_{j_{l+1}}}, \dots, \vec{\alpha_{j_m}}, \vec{\alpha_{j_{m+k}}}\}$ is linearly independent.

Then,let $\Delta = min\{\frac{x_{j_l}}{\lambda_l}|\lambda_l > 0\}$
$\begin{pmatrix} \vec{x_B} + \vec{\delta} \\ \vec{x_N} + \vec{\Delta} \end{pmatrix}$ is a feasible base solution.(As the statement in Question 1)

Question 8

Exercise 1.6 pivoting$\Leftrightarrow$ elementary base tansformation.

proof: as the description above

$\vec{\alpha}_{j_{m+k}} = \sum_{i=1}^{m} \lambda_i \vec{\alpha}_{j_i} = B\vec{\lambda}, \vec{\lambda} \neq 0, \lambda_l > 0 \Rightarrow \vec{\alpha_{j_l}} = \frac{1}{\lambda_l}(\vec{\alpha_{m+k}} - \sum_{l\neq i,i=1}^{m} \lambda_i \vec{\alpha}_{j_i})$

$$B = \begin{pmatrix} \vec{\alpha_{j_1}} & \dots & \vec{\alpha_{j_{l-1}}} & \vec{\alpha_{j_l}} & \vec{\alpha_{j_{l+1}}} & \dots & \vec{\alpha_{j_1}} \end{pmatrix}$$
$$B^{new} = \begin{pmatrix} \vec{\alpha_{j_1}} & \dots & \vec{\alpha_{j_{l-1}}} & \vec{\alpha_{j_{m+k}}} & \vec{\alpha_{j_{l+1}}} & \dots & \vec{\alpha_{j_1}} \end{pmatrix}$$

$\forall 1 \leq i \leq n, \vec{\alpha_i} = \sum_{c=1}^{m} \mu_{i_c} \vec{\alpha_{j_c}} = B\vec{\mu_i} = B^{new}\vec{\mu_i}^{new}.$

$$\Rightarrow \vec{\mu_i}^{new} = \begin{pmatrix} \mu_{i_1} - \frac{\lambda_1}{\lambda_l}\mu_{i_l} & \dots & \mu_{i_{l-1}} - \frac{\lambda_{l-1}}{\lambda_l}\mu_{i_l} & \frac{\mu_{i_l}}{\lambda_l} & \mu_{i_{l+1}} - \frac{\lambda_{l+1}}{\lambda_l}\mu_{i_l} & \dots & \mu_{i_m} - \frac{\lambda_m}{\lambda_l}\mu_{i_l} \end{pmatrix}^T$$
$$\Rightarrow \vec{\mu_i}^{new} = \vec{\mu_i} - \frac{\mu_{i_l}}{\lambda_l}(\vec{\lambda} - \begin{pmatrix} 0 & \dots & \underset{l}{1} & \dots & 0 \end{pmatrix})$$

$c_i - \vec{c_B}^T B^{-1}\vec{\alpha_i} = c_i - \vec{c_B}^T \vec{\mu_i}.$
$c_i - \vec{c_B^{new}}^T \vec{\mu_i}^{new} - (c_i - \vec{c_B}^T \vec{\mu_i}) = \vec{c_B}^T \vec{\mu_i} - \vec{c_B^{new}}^T \vec{\mu_i}^{new}$
$= (\vec{c_B}^T - \vec{c_B^{new}}^T)\vec{\mu_i}^{new} + \frac{\mu_{i_l}}{\lambda_l}\vec{c_B}^T\vec{\lambda} - \frac{\mu_{i_l}}{\lambda_l}c_l$
$= \frac{\mu_{i_l}}{\lambda_l}(c_l - c_{m+k} - c_l) + \frac{\mu_{i_l}}{\lambda_l}\vec{c_B}^T\vec{\lambda}$
$= \frac{\mu_{i_l}}{\lambda_l}(\vec{c_B}^T B^{-1}\vec{\alpha_{m+k}} - c_{m+k}).$

Thus,
$$c_i - \vec{c_B^{new}}^T B^{new-1}\vec{\alpha_i} = c_i - \vec{c_B}^T B^{-1}\vec{\alpha_i} - \frac{\mu_{i_l}}{\lambda_l}(c_{m+k} - \vec{c_B}^T B^{-1}\vec{\alpha_{m+k}}).$$

$$x_B^{\vec{new}} = \begin{pmatrix} x_{j_1} - \Delta\lambda_1 & \dots & x_{j_{l-1}} - \Delta\lambda_{l-1} & \Delta & x_{j_{l+1}} - \Delta\lambda_{l+1} & \dots & x_{j_m} - \Delta\lambda_m \end{pmatrix}$$
$$= \vec{x_B} - \Delta\vec{\lambda} + \begin{pmatrix} 0 & \dots & \underset{l}{\Delta} & \dots & 0 \end{pmatrix}$$
$$\vec{c_B^{new}}^T x_B^{\vec{new}} - \vec{c_B}^T \vec{x_B} = (c_{m+k} - c_l)\Delta - \Delta\vec{c_B}^T\vec{\lambda} + c_l\Delta = \Delta(c_{m+k} - \vec{c_B}^T\vec{\lambda})$$

Thus,
$$\vec{c_B^{new}}^T x_B^{\vec{new}} = \vec{c_B}^T \vec{x_B} + \frac{x_{j_l}}{\lambda_l}(c_{m+k} - \vec{c_B}^T B^{-1}\vec{\alpha_{j_{m+k}}})$$

Question 9

Exercise 1.5

The algorithm of Simplex Method

step 1 : normalization

step 2 : Looking for a feasible base solution as initial solution.(There are two basic ways.)

step 3: optimality :

$$B = \begin{pmatrix} \vec{\alpha_{j_1}} & \dots & \vec{\alpha_{j_{l-1}}} & \vec{\alpha_{j_l}} & \vec{\alpha_{j_{l+1}}} & \dots & \vec{\alpha_{j_1}} \end{pmatrix}$$

$\forall \vec{\alpha_i}$, $\vec{\alpha_i} = B\vec{\lambda_i} \rightarrow$ examine $c_j - \vec{c_B}^T \vec{\lambda_i}$

step 4 : pivoting:

$$B^{new} = \begin{pmatrix} \vec{\alpha_{j_1}} & \dots & \vec{\alpha_{j_{l-1}}} & \vec{\alpha_{j_{m+k}}} & \vec{\alpha_{j_{l+1}}} & \dots & \vec{\alpha_{j_1}} \end{pmatrix}$$

return to step 2.

an example of Simplex Method

$$\begin{aligned} min \quad & -4x_1 - x_2 \\ s.t. \quad & -x_1 + 2x_2 \leq 4 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

step 1 : normalization

$$\begin{aligned} min \quad & -4x_1 - x_2 \\ s.t. \quad & -x_1 + 2x_2 + x_3 = 4 \\ & 2x_1 + 3x_2 + x_4 = 12 \\ & x_1 - x_2 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

$$A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

step 2 : Simplex tableau

B \ A		$\vec{c}^T$					$\vec{x_B}$
		-4	-1	0	0	0	
	x_3	-1	2	1	0	0	4
	x_4	2	3	0	1	0	12
I	x_5	1	-1	0	0	1	3
	$\vec{c}_B^T N - \vec{c}_N^T$	4	1	0	0	0	0

B \ A		$\vec{c}^T$					$\vec{x_B}$
		-4	-1	0	0	0	
I	x_3	0	1	1	0	1	7
	x_4	0	5	0	1	-2	6
	x_1	1	-1	0	0	1	3
	$\vec{c_B^T}N - \vec{c_N^T}$	0	5	0	0	-4	-12

B \ A		$\vec{c}^T$					$\vec{x_B}$
		-4	-1	0	0	0	
I	x_3	0	0	1	$-\frac{1}{5}$	$\frac{7}{5}$	$\frac{29}{5}$
	x_2	0	1	0	$\frac{1}{5}$	$-\frac{2}{5}$	$\frac{6}{5}$
	x_1	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{21}{5}$
	$\vec{c_B^T}N - \vec{c_N^T}$	0	0	0	-1	-2	-14

Thus, the optimal solution is ($x_1 = \frac{21}{5}$, $x_2 = \frac{6}{5}$)

DUAL THEORY

Question 10

Exercise 1.7

LP:
$min \quad \vec{c}^T\vec{x}$
$s.t. \quad A\vec{x} = \vec{b}$
$\vec{x} \geq \vec{0}$

DP:
$max \quad \vec{b}^T\vec{w}$
$s.t. \quad A^T\vec{w} \leq \vec{c}$

Weak duality theorem:

$$\vec{b}^T\vec{w} \leq max(\vec{b}^T\vec{w}) \leq min(\vec{c}^T\vec{x}) \leq \vec{c}^T\vec{x}$$

proof: LP has optimal solution $\vec{x^*} \Rightarrow$ DP has optimal solution $\vec{w^*}$ and $\vec{c}^T\vec{x^*} = \vec{b}^T\vec{w^*}$.

LP has optimal solution $\vec{x^*} \Rightarrow$ LP has feasible solutions $\Rightarrow$ DP is bounded.

$$A\vec{x^*} = (B\ N)(\vec{x_B}^T\ \vec{x_N}^T)^T = \vec{b},\ \vec{x_N} = \vec{0}$$
$$\Rightarrow \vec{b}^T = (\vec{x_B}^T\ \vec{x_N}^T)(B\ N)^T$$

Let $\vec{p} \in R^{m\times 1}$, satisfy $B^T\vec{p} = \vec{c_B}$.

$$\Rightarrow \vec{b}^T\vec{p} = (\vec{x_B}^T\ \vec{x_N}^T)(B\ N)^T\vec{p} = \vec{x_B}^T B^T\vec{p} = \vec{x_B}^T\vec{c_B} = \vec{c_B}^T\vec{x_B} = \vec{c}^T\vec{x^*}$$

The optimality:

$$\vec{c_N}^T - \vec{c_B}^T B^{-1}N \geq \vec{0}$$

Thus,

$$\vec{c_N}^T N^{-1} \geq \vec{c_B}^T B^{-1} \Leftrightarrow {N^{-1}}^T\vec{c_N} \geq {B^{-1}}^T\vec{c_B} \Leftrightarrow {N^T}^{-1}\vec{c_N} \geq {B^T}^{-1}\vec{c_B} = \vec{p}$$
$$N^T\vec{p} \leq \vec{c_N},\ B^T\vec{p} \leq \vec{c_B} \Rightarrow A^T\vec{p} \leq \vec{c}$$

Thus, $\vec{p}$ is a feasible solution of DP.

$\vec{b}^T\vec{p} = \vec{c}^T\vec{x^*} \Rightarrow \vec{p}$ is the optimal solution of DP.