2. 由题意知: (1) $A_1\bar{A}_2\bar{A}_3+\bar{A}_1A_2\bar{A}_3+\bar{A}_1\bar{A}_2A_3$
(2) $A_1\cup A_2\cup A_3$
(3) $A_1\cap(A_2\cup A_3)$
(4) $A_1\bar{A}_2\bar{A}_3+\bar{A}_1A_2\bar{A}_3+\bar{A}_1\bar{A}_2A_3+\bar{A}_1\bar{A}_2\bar{A}_3$
(表示方法有多种, 答案不唯一)

8. 由$P(AC)=0$, 且$ABC\subset AC$, 有$P(ABC)=0$. 再由加法公式, 得A,B,C至少发生一个的概率为

$$\begin{aligned}P(A\cup B\cup C)&=P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)+P(ABC)\\&=1-\frac{2}{8}=\frac{3}{4}\end{aligned}$$

12. 三局两胜时, 甲获胜的概率为

$$P_1=p^2+C_2^1p^2(1-p)=p^2(3-2p)$$

五局三胜时, 甲获胜的概率为

$$P_2=p^3+C_3^1p^3(1-p)+C_4^2p^3(1-p)^2=p^3\left(6p^2-15p+10\right).$$

当$\frac{1}{2}<p\leq 1$ 时, 易得$P_1<P_2$, 所以, 五局三胜对甲更有利。

13. 记事件A_i 为甲掷硬币的次数为i, 则$P\left(A_i\right)=(1/2)^i$. 记事件$B$ 为甲获胜，则

$$P(B)=\sum_{i=1}^{n}P\left(B\mid A_i\right)P\left(A_i\right)=\sum_{i=1}^{n}\frac{1}{2^i}\cdot\frac{1}{i+1}$$

当$n\to\infty$ 时, $P(B)=\sum_{i=1}^{n}\frac{1}{2^i}\cdot\frac{1}{i+1}<\sum_{i=1}^{n}\frac{1}{2^i}\cdot\frac{1}{2}=\frac{1}{2}$, 所以该规则对乙更有利。

17.由题意可知，甲乙两人可乘$3:15,3:30,3:45$和$4:00$班次的公交车，且乘任一班次的车时等可能的。记$A:=$'甲乙两人同乘一辆车'，所以

$$P(A)=\frac{C_4^1}{4^2}=\frac{1}{4}.$$

20.掷一枚均匀的骰子，记事件A为投出的点数大于2，事件B为投出的点数

为4或6，事件C为投出的点数为偶数，则

$$P(A)=\frac{2}{3},\quad P(B)=\frac{1}{3},\quad P(C)=\frac{1}{2},\quad P(AB)=P(AC)=P(BC)=\frac{1}{3}.$$

所以

$$P(A|B)=\frac{P(AB)}{P(B)}=1>P(A),\quad P(B|C)=\frac{P(BC)}{P(C)}=\frac{2}{3}>P(B).$$

但是

$$P(A|C)=\frac{P(AC)}{P(C)}=\frac{2}{3}=P(A).$$

23.(1)记事件A 为谣言传播r次还没回到第一个造谣者所在群，则

$$P(A)=\frac{n(n-1)^{r-1}}{n^r}=\left(\frac{n-1}{n}\right)^{r-1}.$$

(2)记事件B_k为传播k次没有一个微信群两次收到谣言，$k=1,2,...,n$。则

$$P(B_k)=\frac{A_n^k}{n^k},\quad k=1,2,...,n.$$

(3)每次随机向m个群传播谣言，记事件A' 为谣言传播r次还没回到第一个造谣者所在群，记事件B'_k为传播k次没有一个微信群两次收到谣言，此时谣言经历了$m+m^2+...+m^r=(m^{r+1}-m)/(m-1)$个群，要令没有一个群两次收到谣言，则经历的群数小于等于n，有$k=1,2,...,\lfloor\log_m(nm+m-n)\rfloor-1$，则

$$P(A')=\frac{C_n^m(C_{n-1}^m)^m...(C_{n-1}^m)^{m^{r-1}}}{C_n^m(C_n^m)^m...(C_n^m)^{m^{r-1}}}=\frac{n}{n-m}\left(\frac{C_{n-1}^m}{C_n^m}\right)^{\frac{m^r-1}{m-1}}=\left(\frac{n-m}{n}\right)^{\frac{m^r-m}{m-1}}.$$

$$P(B'_k)=\frac{C_n^m\left(C_{n-m}^m...C_{n-m^2-m}^m\right)...\left(C_{n-m^{k-1}-...-m}^m...C_{n-m^k-...-m}^m\right)}{C_n^m(C_n^m)^m...(C_n^m)^{m^{r-1}}}=\frac{A_n^{(m^{k+1}-m)/(m-1)}}{(n!/(n-m)!)^{\frac{m^k-1}{m-1}}}.$$

24.记事件A_i为第i次取到的零件为一等品，事件B_i为从第i个箱子中取样，$i=1,2$。则

(1)

$$P(A_1)=P(A_1|B_1)P(B_1)+P(A_1|B_2)P(B_2)=\frac{1}{2}\times\frac{1}{5}+\frac{1}{2}\times\frac{3}{5}=\frac{2}{5}.$$

(2)

$$P(A_2|A_1)=\frac{P(A_1A_2)}{P(A_1)}=\frac{P(A_1A_2B_1)+P(A_1A_2B_2)}{P(A_1)}=\frac{\frac{1}{2}\times\frac{1}{5}\times\frac{9}{49}+\frac{1}{2}\times\frac{3}{5}\times\frac{17}{29}}{\frac{2}{5}}\approx 0.49.$$

30. 记$A :=$ “该人为带菌者”, $B_i :=$ “第i 次检测为阳性”, 则

(1)

$$P(A|B)=\frac{P(AB)}{P(B)}=\frac{0.95\times 0.1}{0.95\times 0.1+0.01\times 0.9}=\frac{0.095}{0.104}=0.91.$$

(2)

$$P\left(A|B_1B_2\right)=\frac{P\left(AB_1B_2\right)}{P\left(B_1B_2\right)}=\frac{0.95^2\times 0.1}{0.95^2\times 0.1+0.01^2\times 0.9}=0.99.$$

33. 记A := 从从乙袋中取出的球为白球", $B_i :=$ "第i 次从甲袋中取出的球为白球", $i=1,2$ 。则

(1)

$$\begin{aligned}P(A)&=P\left(AB_1B_2\right)+P\left(AB_1\bar{B}_2\right)+P\left(A\bar{B}_1B_2\right)+P\left(A\bar{B}_1\bar{B}_2\right)\\&=\frac{5}{7}\times\frac{4}{6}\times\frac{6}{11}+\frac{5}{7}\times\frac{2}{6}\times\frac{5}{11}+\frac{2}{7}\times\frac{5}{6}\times\frac{5}{11}+\frac{2}{7}\times\frac{1}{6}\times\frac{4}{11}\\&=\frac{38}{77}\end{aligned}$$

(2)

$$1-P\left(\bar{B}_1\bar{B}_2|A\right)=1-\frac{P\left(A\bar{B}_1\bar{B}_2\right)}{P(A)}=1-\frac{\frac{2}{7}\times\frac{1}{6}\times\frac{4}{11}}{\frac{38}{77}}=1-\frac{2}{57}=\frac{55}{57}.$$

第二周作业答案

March 29, 2023

37. $P(A)=P(AC)+P(A\bar{C})=P(A\mid C)P(C)+P(A\mid\bar{C})P(\bar{C})=0.55$, 同理, $P(B)=0.5$, 又

$$P(AB)=P(ABC)+P(AB\bar{C})=P(AB\mid C)P(C)+P(AB\mid\bar{C})P(\bar{C})$$
$$=P(A\mid C)P(B\mid C)P(C)+P(A\mid\bar{C})P(B\mid\bar{C})P(\bar{C})=0.415$$

所以$P(AB)\neq P(A)P(B)$

38 : (1) 恰有一次的射中概率$P_1=0.5\times(1-0.6)\times(1-0.8)+0.5\times0.6\times(1-0.8)$

$$+0.5\times(1-0.6)\times0.8=0.26$$

(2) 至少有一次射中的概率$P_2=1-0.5\times0.4\times0.2=0.96$

39 (4) 考虑$A{\cdot}B{\cdot}C$ 至少有一个正常工作 $\quad P=\left[1-\left(1-P_A\right)\left(1-P_B\right)\left(1-P_C\right)\right]P_D^2$

(5) 考虑C 是否能正常工作, 则

$P=\left(2P_AP_B-P_A^2P_B^2\right)\left(1-P_C\right)+\left(4P_AP_B+P_A^2P_B^2-2P_AP_B^2-2P_A^2P_B\right)P_C=$
$2P_A\left(1-P_A\right)\left[1-\left(1-P_B\right)\left(1-P_BP_C\right)\right]+P_A^2\left[1-\left(1-P_B\right)^2\right]$

40. 电路断开的概率 $\quad P=0.3+0.4*0.6-0.3*0.4*0.6=0.468$

补充题：取8个球，分别编号为1,2,2,3,3,12,13,123。$A_i=\{$ 随机取一个球，

球上有数字i $\}, i=1,2,3$.

$$
\begin{aligned}
& P\left(A_1\right)=P\left(A_2\right)=P\left(A_3\right)=\frac{1}{2} . P\left(A_1 A_2 A_3\right)=\frac{1}{8}, \\
& P\left(A_1 A_2 A_3\right)=P\left(A_1\right) P\left(A_2\right) P\left(A_3\right) . \\
& P\left(A_1 A_2\right)=\frac{1}{4}=P\left(A_1\right) P\left(A_2\right) \\
& P\left(A_1 A_3\right)=\frac{1}{4}=P\left(A_1\right) P\left(A_3\right) \\
& P\left(A_2 A_3\right)=\frac{1}{8} \neq P\left(A_2\right) P\left(A_3\right) .
\end{aligned}
$$

2. 考虑n次投篮投进次数为$X_n, X_n \in\{1,2, \cdots, n-1\}$. 数学归纳法：$n=2, P\left(X_2=i\right)=1, n=k, P\left(X_k=i\right)=1/(k-1), \forall i \in\{1,2, \cdots, k-1\}$, 当$n=k+1$时，则$P\left(X_{k+1}=i\right)=P\left(X_k=i-1 \mid A\right) P(A)+P\left(X_k=i-1 \mid A^C\right) P(A^C)$ $P\left(x_{k+1}=i\right)=\frac{1}{k-1} \times \frac{i-1}{k}+\frac{1}{k-1} \times \frac{k-i}{k}=\frac{k-1}{k(k-1)}=\frac{1}{k}, \forall i \in\{1,2, \cdots, k-1\}$. 所以$n=100, P\left(X_{100}=i\right)=1/99, \forall i \in\{1,2, \cdots, 99\}$

$$
\begin{aligned}
& 10 . P=P_4+P_5+P_6+P_7=0.6^4+\binom{4}{1} 0.6^4 0.4+\binom{5}{2} 0.6^4 0.4^2+\binom{6}{3} 0.6^4 0.4^3=0.71 \\
& P^{\prime}=P_2^{\prime}+P_3^{\prime}=0.6^2+\binom{2}{1} 0.6^2 0.4=0.65
\end{aligned}
$$

$P>P^{\prime}$, 所以三局两胜制对乙队更有利。

11. 以X 表示赌徒赌完一局后的收益, 则有

$$
\begin{aligned}
& P(X=-1)=\left(\frac{5}{6}\right)^3=\frac{125}{216} \\
& P(X=1)=\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2=\frac{75}{216} \\
& P(X=2)=\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)=\frac{15}{216} \\
& P(X=3)=\left(\frac{1}{6}\right)^3=\frac{1}{216}
\end{aligned}
$$

所以, X 的分布律为

$$
\left(\begin{array}{ccccc}
X & -1 & 1 & 2 & 3 \\
P & \frac{125}{216} & \frac{75}{216} & \frac{15}{216} & \frac{1}{216}
\end{array}\right) .
$$

14

$\lambda = np = 400 \times 0.02 = 8, X \sim P(8)$
$P(X \geq 2) = 1 - P(X = 1) - P(X = 0) = 1 - 8e^{-8} - e^{-8} = 1 - 9e^{-8} \approx 0.997$

16. 没来的乘客人数可近似为poisson分布, $\lambda = 52 \times 0.05 = 2.6$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = (1 + 2.6)e^{-2.6} \approx 0.27$$

20.

$$P(1<X<2)=\int_1^2 axdx=\frac{a}{2}x^2\ |_1^2=\frac{3}{2}a,$$

$$P(2<X<3)=\int_2^3 bdx=b=\frac{3}{2}a,$$

又因为$P(1<X<2)+P(2<X<3)=1$, 所以$a=\frac{1}{3}$, $b=\frac{1}{2}$.

21.(1)

$$\int_{-\infty}^{+\infty}\frac{a}{1+x^2}dx=a\arctan x\ |_{-\infty}^{+\infty}=a\pi=1.$$

所以，$a=\frac{1}{\pi}$

(2)

$$F(x)=\int_{-\infty}^{x}\frac{1}{\pi(1+x^2)}dx=\frac{1}{\pi}\arctan x+\frac{1}{2},\quad x\in R.$$

(3)

$$P(|x|<1)=\int_{-1}^{1}\frac{1}{\pi(1+x^2)}dx=\frac{1}{2}.$$

22.

$$S=\int_0^2(2x-x^2)dx=[x^2-\frac{1}{3}x^3]|_0^2=\frac{4}{3},$$

$$P(X\le x)=\int_0^x(2x-x^2)dx/S=\frac{3}{4}(x^2-\frac{1}{3}x^3)(x\in(0,2)),$$

所以

$$F(x)=\begin{cases}0, & x<0\\ \frac{3}{4}x^2-\frac{1}{4}x^3, & 0\le x<2\\ 1, & x\ge 2\end{cases}$$

$$f(x)=\begin{cases}\frac{3}{2}x-\frac{3}{4}x^2, & 0<x<2\\ 0, & other\end{cases}$$

27.证：令F(x)为X的分布函数，对于$\forall n\in(1,\infty)$，由题意得,

$$F(\frac{1}{n})-F(0)=F(\frac{2}{n})-F(\frac{1}{n})=...=F(1)-F(\frac{n-1}{n}),$$

又因为

$$\sum_{i=1}^{n}(F(\frac{i}{n})-F(\frac{i-1}{n}))=F(1)-F(0)=1,$$

所以

$$F(\frac{m}{n}) = \sum_{i=1}^{m}(F(\frac{i}{n}) - F(\frac{i-1}{n})) = \frac{m}{n},\ m \leq n,$$

所以有$F(x) = x$.

28.$X \sim \exp(\lambda)$.

(1)

$$P(X > 2) = \int_2^{\infty} e^{-x}dx = 1 - e^{-x}\Big|_2^{\infty} = e^{-2}.$$

(2)

$$P(X > 4) = \int_4^{\infty} e^{-x}dx = e^{-4},$$

$$P(X > 4 \mid X > 2) = \frac{P(X > 4)}{P(X > 2)} = e^{-2}.$$

31.(1)由题意可得：$\frac{X-1}{2} \sim N(0,1)$，所以

$$P(0 \leq X \leq 4) = P(\frac{0-1}{2} \leq \frac{X-1}{2} \leq \frac{4-1}{2}) = \Phi(1.5) - \Phi(-0.5) \approx 0.6247.$$

$$P(X > 2.4) = P(\frac{X-1}{2} > \frac{2.4}{2}) = 1 - \Phi(0.7) \approx 0.2420.$$

$$P(|X| > 2) = 1 - P(-2 \leq X \leq 2) = 1 - \Phi(0.5) + \Phi(-1.5) \approx 0.3753.$$

(2)由题意易知$1 - \Phi(\frac{c-1}{2}) = 2\Phi(\frac{c-1}{2})$，所以$\Phi(\frac{c-1}{2}) = \frac{1}{3}$。查表可知$\frac{c-1}{2} \approx -0.4307$，所以$c \approx 0.1386$.

33.$X_1 \sim N(30, 100), \quad X_2 \sim N(40, 16)$,

(1)

$$P_1 = P\left(X_1 \leq 50\right) = P\left(\frac{X_1 - 30}{10} \leq \frac{50-30}{10}\right) = \Phi(2),$$

$$P_2 = P\left(X_2 \leq 50\right) = P\left(\frac{X_2 - 40}{4} \leq \frac{50-40}{4}\right) = \Phi(2.5).$$

所以$P_2 > P_1$.

(2)

$$P_1' = P\left(X_1 \leq 45\right) = \Phi(1.5), P_2' = P\left(X_2 \leq 45\right) = \Phi(1.25),$$

所以$P_1' > P_2'$.

34.记X为点数之和，则$X=2,3,...,12$, 所以

$$P(X=2)=P(X=12)=\frac{C_2^2}{6\times 6}=\frac{1}{36},$$

$$P(X=3)=P(X=11)=\frac{C_2^1}{6\times 6}=\frac{2}{36},$$

$$P(X=4)=P(X=10)=\frac{C_2^1+C_2^2}{6\times 6}=\frac{3}{36},$$

$$P(X=5)=P(X=9)=\frac{C_2^1+C_2^1}{6\times 6}=\frac{4}{36},$$

$$P(X=6)=P(X=8)=\frac{C_2^1+C_2^1+C_2^2}{6\times 6}=\frac{5}{36},$$

$$P(X=7)=\frac{C_2^1+C_2^1+C_2^1}{6\times 6}=\frac{6}{36}.$$

所以

$$X\sim\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{6} & \frac{5}{36} & \frac{1}{9} & \frac{1}{12} & \frac{1}{18} & \frac{1}{36} \end{pmatrix}.$$

36. 由X 的分布律, 易得Y_1,Y_2,Y_3 的分布律分别为:

$$Y_1\sim\begin{pmatrix} -3 & -1 & 1 & 3 \\ 0.4 & 0.1 & 0.3 & 0.2 \end{pmatrix},\quad Y_2\sim\begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix},\quad Y_3\sim\begin{pmatrix} 0 & 1 & 4 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}.$$

37. (1) 由分布函数的有界性:

$$\begin{cases} F(-\infty)=a-\frac{\pi}{2}b=0 \\ F(\infty)=a+\frac{\pi}{2}b=1 \end{cases} \Rightarrow \begin{cases} a=1/2 \\ b=1/\pi \end{cases}$$

(2) 由分布函数可得X 的密度函数为

$$f(x)=\frac{1}{\pi\left(1+x^2\right)},\quad x\in\mathcal{R}.$$

因为$y=3-\sqrt[3]{x}$ 为严格减函数, 其反函数为$x=(3-y)^3$. 所以$Y=3-\sqrt[3]{X}$ 的密度函数为

$$f_Y(y)=\frac{3(y-3)^2}{\pi\left[1+(3-y)^6\right]},\quad x\in\mathcal{R}.$$

(3) $Z = 1/X$ 的密度函数为

$$f_Z(z) = \frac{1}{\pi\,[1 + (1/z)^2]} \cdot \left| -\frac{1}{z^2} \right| = \frac{1}{\pi\,(1 + z^2)}, \quad z \in \mathcal{R}.$$

所以X 与$1/X$ 具有相同的分布。

第四周作业答案

April 12, 2023

40. $X \sim U(0,1)$, 则其密度函数为

$$f_X(x)=\left\{\begin{array}{ll}1, & 0<x<1, \\ 0 & \text{其他}.\end{array}\right.$$

(1) 因为$Y_1=e^X$ 的可能取值范围为$(1,e)$, 且$y_1=e^x$ 在(0,1) 上为严格增函数, 其反函数为$x=h\left(y_1\right)=\ln y_1$, 对应导数$h'\left(y_1\right)=\frac{1}{y_1}$. 所以$Y_1$ 的密度函数为

$$f_1\left(y_1\right)=\left\{\begin{array}{cc}f_X\left(\ln y_1\right)\left|\frac{1}{y_1}\right|, & 1<y_1<e \\ 0, & \text{其他}\end{array}=\left\{\begin{array}{cc}\frac{1}{y_1}, & 1<y_1<e, \\ 0, & \text{其他}.\end{array}\right.\right.$$

(2) $Y_2=X^{-1}$ 的可能取值范围为$(1,\infty)$, 且$y_2=x^{-1}$ 在(0,1) 上为严格减函数, 反函数及对应的导数为$x=h\left(y_2\right)=1/y_2, h'\left(y_2\right)=-1/y_2^2$. 所以Y_2 的密度函数为

$$f_2\left(y_2\right)=\left\{\begin{array}{cc}f_X\left(y_2^{-1}\right)\left|-1/y_2^2\right|, & y_2>1 \\ 0, & \text{其他}\end{array}=\left\{\begin{array}{cc}\frac{1}{y_2^2}, & y_2>1, \\ 0, & \text{其他}.\end{array}\right.\right.$$

(3) $Y_3=-\frac{1}{\lambda}\ln X$ 的可能取值范围为$(0,\infty)$, 且$y_3=-\frac{1}{\lambda}\ln x(\lambda>0)$ 在(0,1) 上为严格减函数, 反函数及对应的导数为$x=h\left(y_3\right)=e^{-\lambda y_3}, h'\left(y_3\right)=-\lambda e^{-\lambda y_3}$. 所以$Y_3$ 的密度函数为

$$f_3\left(y_3\right)=\left\{\begin{array}{cc}f_X\left(e^{-\lambda y_3}\right)\left|-\lambda e^{-\lambda y_3}\right|, & y_3>0 \\ 0, & \text{其他}\end{array}=\left\{\begin{array}{cc}\lambda e^{-\lambda y_3}, & y_3>0, \\ 0, & \text{其他}.\end{array}\right.\right.$$

42 $Y=F(x)$ 的取值范围为(0,1) ,且$F(x)$严格单调增，则$F^{-1}(x)$也是严格单调增。Y 的分布函数为

$$F_Y(y)=P(Y\leq y)=P(F(x)\leq y)=P\left(x\leq F^{-1}(y)\right)=\left\{\begin{array}{l}0, \quad y<0 \\ F\left(F'(y)\right)=y, \quad 0\leq y<1 \\ 1, \quad y\geq 1\end{array}\right.$$

即证$Y=F(x)$ 为均匀分布。

44. $Y=g(X)$. 则Y 的密度函数满足

$$f_Y(y)=f\left(g^{-1}(y)\right)\left(g^{-1}\right)'(y)=2\left(1-g^{-1}(y)\right)\left(g^{-1}\right)'(y)=e^{-y}.$$

由$x=g^{-1}(y)$则

$$2(1-x)\frac{dx}{dy}=e^{-y}$$

解得$2x-x^2=-e^{-y}+C$, 即$y=\ln\left(x^2-2x+C\right)\quad(0<x<1)$,由$Y\in(0,+\infty)$取$c=1$ 取$y=\ln\left(x^2-2x+1\right)=-2\ln(1-x)$ ，即$g(x)=-2\ln(1-x)$

48. 由密度函数的可知$\int_0^3\frac{1}{a}x^2dx=\frac{9}{a}=1$, 所以$a=9$. (1) Y 的可能取值范围为$[1,2]$, 且

$$P(Y=1)=P(X>2)=\int_2^3\frac{1}{9}x^2dx=\frac{19}{27},$$
$$P(Y=2)=P(X\le 1)=\int_0^1\frac{1}{9}x^2dx=\frac{1}{27},$$

$\forall 1<y<2, P(Y\le y)=P(Y=1)+P(1<Y\le y)=\dfrac{19}{27}+\displaystyle\int_1^y\frac{1}{9}x^2dx=\frac{y^3}{27}+\frac{2}{3}.$

所以Y 的分布函数为

$$F_Y(y)=\begin{cases}0, & y<1,\\ \frac{y^3}{27}+\frac{2}{3} & 1\le y<2,\\ 1, & y\ge 2.\end{cases}$$

(2) $P(X\le Y)=P(X\le 1)+P(1<X<2)=\int_0^2\frac{1}{9}x^2dx=\frac{8}{27}$.

49. (1)$Y=\frac{X}{1-X}$,在(0,1)上严格单调增且$Y\in(0,+\infty)$. 由$X=\frac{Y}{1+Y}$的导函数为$X'=\frac{1}{(y+1)^2}$且$X\sim U(0,1)$,则Y的密度函数

$$f_Y(y)=\begin{cases}\frac{1}{(y+1)^2}, & y\in(0,+\infty)\\ 0, & else\end{cases}$$

(2) $z=xI(a,1](x)=\begin{cases}0, & 0<x\le a\\ x, & a<x<1\end{cases}$

$$F_Z(z)=\begin{cases}0, z<0\\ \int_0^a f(x)dx=a, 0\le z<a\\ \int_0^a f(x)dx+\int_z^a f(x)dx=z, a\le z<1\\ 1, z\ge 1\end{cases}$$

(3) $W=X^2+XI_{[0,b]}(X),(0<b<1)$

$$X\in(0,b],\quad W=X^2+X,X=\frac{\sqrt{1+4W}-1}{2},\quad X'(W)=\frac{1}{\sqrt{1+4W}}$$

则密度函数为$f_{W1}(w)=\frac{1}{\sqrt{1+4w}}I_{(0,b^2+b]}(w)$。

$$X\in(b,1),\quad W=X^2,\quad X=\sqrt{W},\quad X'(W)=\frac{1}{2\sqrt{W}}$$

则密度函数为$f_{W2}(w)=\frac{1}{2\sqrt{W}}I_{(b^2,1)}(w)$.
当$b^2+b<1$时,W的密度函数为

$$f_W(w)=\begin{cases} f_{W1}(w)=\frac{1}{\sqrt{1+4w}}, & 0<w\le b^2\\ f_{W1}(w)+f_{W2}(w)=\frac{1}{\sqrt{1+4w}}+\frac{1}{2\sqrt{W}}, & b^2<w\le b^2+b\\ f_{W2}(w)=\frac{1}{2\sqrt{W}}, & b^2+b<w<1\\ 0, & else\end{cases}$$

当$b^2+b\ge 1$时,W的密度函数为

$$f_W(w)=\begin{cases} f_{W1}(w)=\frac{1}{\sqrt{1+4w}}, & 0<w\le b^2\\ f_{W1}(w)+f_{W2}(w)=\frac{1}{\sqrt{1+4w}}+\frac{1}{2\sqrt{W}}, & b^2<w<1\\ f_{W1}(w)=\frac{1}{\sqrt{1+4w}}, & 1\le w\le b^2+b\\ 0, & else\end{cases}$$

1. (X,Y)的分布律为

X \ Y	0	1	2
0	1/5	2/5	1/15
1	1/5	2/15	

$$F(x,y)=\begin{cases} 0, & x<0\text{ 或}y<0\\ \frac{1}{5}, & 0\le x<1\text{ 且}0\le y<1\\ \frac{3}{5}, & 0\le x<1\text{ 且}1\le y<2\\ \frac{2}{3}, & 0\le x<1\text{ 且}y\ge 2\\ \frac{2}{5}, & x\ge 1\text{ 且}0\le y<1\\ \frac{14}{15}, & x\ge 1\text{ 且}1\le y<2\\ 1, & x\ge 1\text{ 且}y\ge 2\end{cases}$$

3. (X,Y)的分布律为

Y \ X	0	1	2	3
1		3/8	3/8	
3	1/8			1/8

6. (1). (X,Y)联合分布律

$$P(X=x,Y=y)=\begin{cases}p^2(1-p)^{y-2},0<x<y,x,y\in N\\0,else\end{cases}$$

(2)

$$P(X=x)=\sum_{n=x+1}^{\infty}P(X=x,Y=n)=(1-p)^{x-1}\cdot P\cdot\sum_{n=1}^{\infty}(1-p)^{n-1}p=(1-p)^{x-1}\cdot p,\quad x\in N$$

$$P(Y=y)=\sum_{n=1}^{\infty}P(X=n,Y=y)=\sum_{n=1}^{y-1}P(X=n,Y=y)=(y-1)(1-p)^{y-2}p^2,\quad y>1\quad and\quad y\in N$$

9.(1)

$F(\infty,\infty)=1.$

$a\left(b+\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right)=1$

$\begin{cases}F(\infty,-\infty)=0\\F(-\infty,\infty)=0\end{cases}\quad\begin{cases}a\left(b+\frac{\pi}{2}\right)\left(c-\frac{\pi}{2}\right)=0\\a\left(b-\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right)=0.\end{cases}$

$\begin{cases}a=\frac{1}{\pi^2}\\b=\frac{\pi}{2}\\c=\frac{\pi}{2}\end{cases}$

(2).

$$P(X>0,Y>0)=1-F(0,+\infty)-F(+\infty,0)+F(0,0)=\frac{1}{4}$$

(3) $F(x,+\infty)=\frac{1}{2}+\frac{1}{\pi}\arctan x$ 可见边缘密度函数

$$f_X(x)=\frac{d}{dx}F(x,+\infty)=\frac{1}{\pi\left(1+x^2\right)},\quad x\in R$$

，同理

$$f_Y(x)=\frac{1}{\pi\left(1+y^2\right)}, \quad y \in R$$

10. (1) $F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}\cos t\sin u dt du$

$$=\begin{cases}\sin x\sin y & \left(0\leq x,y<\frac{\pi}{2}\right)\\ \sin x & \left(0\leq x<\frac{\pi}{2},y\geq\frac{\pi}{2}\right)\\ \sin y & \left(0\leq y<\frac{\pi}{2},\quad x\geq\frac{\pi}{2}\right)\\ 1 & \left(x\cdot y\geq\frac{\pi}{2}\right)\\ 0 & (x<0\text{ 或}y<0)\end{cases}$$

(2)

$$\begin{aligned}&P\left(0<x<\frac{\pi}{4},\frac{\pi}{4}<Y<\frac{\pi}{2}\right)\\ =&\int_0^{\frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos u\cos v du dv\\ =&\int_0^{\frac{\pi}{4}}\cos v dv\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos u du.\\ =&\sin v|_0^{\frac{\pi}{4}}\cdot\sin u\Big|_{\frac{\pi}{4}}^{\frac{\pi}{4}}\\ =&\frac{\sqrt{2}}{2}\left(1-\frac{\sqrt{2}}{2}\right)\\ =&\frac{\sqrt{2}-1}{2}\end{aligned}$$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\int_{A\cap B}f(x)dx}{\int_B f(x)dx}$$

11.(1)$f_X(x)=\int_0^x f(x,y)dy=xe^{-x},\quad x>0$

$$f_{Y|X}(y\mid x)=\frac{f(x,y)}{f_X(x)}=\begin{cases}\frac{1}{x},\ 0<y<x\\ 0,\text{其他}.\end{cases}$$

(2)因为$f_Y(y)=\int_y^\infty f(x,y)dx=e^{-y},\ y>0$，所以有

$$P(X\le 1,Y\le 1)=\int_0^1\int_0^x e^{-x}dydx=1-2e^{-1},$$

$$P(Y\le 1)=\int_0^1 e^{-y}dy=1-e^{-1},$$

$$P(X\le 1\mid Y\le 1)=\frac{P(x\le 1,Y\le 1)}{P(Y\le 1)}=\frac{e-2}{e-1}.$$

17.(1)因为$f(x,y)=f_{Y|X}(y|x)f_X(x)$，所以

$$f(x,y)=\begin{cases}\frac{9y^2}{x},\ 0<y<x<1\\ 0,\text{其他}.\end{cases}\qquad .$$

(2)

$$f_Y(y)=\int_y^1 f(x,y)dx=\begin{cases}-9y^2lny,\ 0<y<1,\\ 0,\text{其他}.\end{cases}$$

18.(1)由题意可得$Y\mid X\sim U(0,X)$，所以$f_{Y|X}(y\mid x)=\frac{1}{x},\quad 0<y<x$。则

$$f(x,y)=f_{Y|X}(y\mid x)\cdot f_X(x)=\frac{1}{x}\cdot xe^{-x}=e^{-x},\quad 0<y<x$$

$$f(x,y)=\begin{cases}e^{-x},\ 0<y<x,\\ 0,\text{其他}.\end{cases}$$

(2)因为

$$f_Y(y)=\int_y^\infty f(x,y)dx=\int_y^\infty e^{-x}dx=e^{-y},\ y>0,$$

所以

$$f_Y(y)=\begin{cases}e^{-y},\ y>0,\\ 0,\text{其他}.\end{cases}$$

22.(1)由题意可知$S=1/2$，所以联合密度为

$$f(x,y)=\begin{cases}2,\ 0<x<y<1,\\ 0,\text{其他}.\end{cases}$$

(2)当$0\le x\le 1$时，$f_X(x)=\int_x^1 2dy=2(1-x)$，所以

$$f_X(x)=\begin{cases}2(1-x),\ 0<x<1,\\ 0,\text{其他}.\end{cases}$$

当$0\le y\le 1$时，$f_Y(y)=\int_0^y 2dx=2y$，所以

$$f_Y(y)=\begin{cases}2y,\ 0<y<1,\\ 0,\text{其他}.\end{cases}$$

(3)当$0\le x\le y$时，有$f_{X|Y}(x|y)=f(x,y)/f_Y(y)=1/y$，所以

$$f_{X|Y}(x|y)=\begin{cases}\frac{1}{y},\ 0<x<y,\\ 0,\text{其他}.\end{cases}$$

(4)

$$P(X\le 0.5|Y=y)=\int_0^{0.5}\frac{1}{y}dx=\frac{1}{2y},\quad 0<y<1.$$

25.(1)由题意可知，$(X_1,X_2,...,X_n)$是多项分布，则当$m_1,m_2,...,m_n=1,2,...,m$且$m_1+m_2+...+m_n=m$时，

$$P(X_1=m_1,X_2=m_2,...,X_n=m_n)=\frac{m!}{m_1!m_2!...m_n!}p_1^{m_1}p_2^{m_2}...p_n^{m_n}.$$

(2)显然$X_k\sim b(m,p_k),\quad k=1,2,...,n.$

(3)由题意可知，$(X_1,X_2,X_3+...+X_n)\sim b(m;p_1,p_2,1-p_1-p_2)$，所以

$$\begin{aligned}P(X_1=m_1,X_2=m_2)&=P(X_1=m_1,X_2=m_2,X_3+...+X_n=m-m_1-m_2)\\&=\frac{m!}{m_1!m_2!(m-m_1-m_2)!}p_1^{m_1}p_2^{m_2}(1-p_1-p_2)^{m-m_1-m_2},\end{aligned}$$

其中$m_1+m_2\le m, m_1, m_2\in\mathbf{N}$.

(4)

$$\begin{aligned}P(X_2=m_2,...,X_n=m_n|X_1=m_1)&=\frac{P(X_1=m_1,X_2=m_2,...,X_n=m_n)}{P(X_1=m_1)}\\&=\frac{(m-m_1)!}{m_2!...m_n!}\left(\frac{p_2}{1-p_1}\right)^{m_2}...\left(\frac{p_n}{1-p_1}\right)^{m_n},\sum_i m_i=m, m_i\in\mathbf{N}.\end{aligned}$$

30.(1)因为X,Y相互独立，所以

$$f(x,y)=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}},\ 0<x<1, y>0\\0,\text{其他}.\end{cases}$$

(2)令$\Delta=4X^2-4Y\ge 0$，则有$Y\le X^2$，所以

$$\begin{aligned}P(Y\le X^2)&=\int_0^1\int_0^{x^2}\frac{1}{2}e^{-\frac{y}{2}}dxdy=\int_0^1(1-e^{-\frac{x^2}{2}})dx\\&=1-\sqrt{2\pi}\int_0^1\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=1+\sqrt{2\pi}/2-\sqrt{2\pi}\Phi(1)\approx 0.1446.\end{aligned}$$

32.(1)由题意可知，$P(X=1,Y=-1)=P(X=1,Y=1)=0$，所以可得

X \ Y	-1	0	1	
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

(2)不独立。例如

$$\frac{1}{4}=P(X=0,Y=-1)\ne P(X=0)P(Y=-1)=\frac{1}{8}.$$

34.由题意可知

$$P(X=x)=P(Y=y)=\frac{1}{K},\quad x,y=0,1,...,K-1,$$

则

$$P(Z_i=z_i)=P(X+iY=z_i)=\sum_{y=0}^{\lfloor z_i/i\rfloor}P(X=Z_i-iy|Y=y)P(Y=y)=(\lfloor z_i/i\rfloor+1)\frac{1}{K^2},\quad z_i=1,2,...,K.$$

对$\forall i<k,\ 0\le z_i\le z_k\le (k+1)(K-1)$，有

$$P(Z_i=z_i,Z_k=z_k)=P(X+iY=z_i,X+kY=z_k)=P\left(X=\frac{kz_i-iz_k}{k-i},Y=\frac{z_k-z_i}{k-i}\right)=\frac{1}{K^2}$$

显然$P(Z_i=z_i,Z_k=z_k)\ne P(Z_i=z_i)P(Z_k=z_k)$。所以$Z_n$之间不独立也不两两独立。

例：

$$P(Z_1=2,Z_2=3)=P(X+Y=2,X+2Y=3)=P(X=1,Y=1)=\frac{1}{K^2},$$

$$P(Z_1=2)=P(X+Y=2)=P(X=0,Y=2)+P(X=1,Y=1)+P(X=2,Y=0)=\frac{3}{K^2},$$

$$P(Z_2=3)=P(X+2Y=3)=P(X=1,Y=1)+P(X=3,Y=0)=\frac{2}{K^2}.$$

所以$P(Z_1=2,Z_2=3)\ne P(Z_1=2)P(Z_1=3)$。所以两两不独立。

37.由对称性可知, X,Y 的分布相同, X^2,Y^2 的分布相同。(1)

$$f_X(x)=\int_{-1}^{1}f(x,y)dy=\begin{cases}\frac12, & |x|<1,\\ 0, & \text{其他}.\end{cases}\qquad f_Y(y)=\begin{cases}\frac12, & |y|<1,\\ 0, & \text{其他}.\end{cases}$$

当$|x|<1,|y|<1$时，

$$f\left(y|x\right)=\frac{f(x,y)}{f(x)}=\frac{1+xy}{2},$$

所以

$$f\left(y\middle|X=\frac12\right)=\begin{cases}\frac{2+y}{4}, & |y|<1,\\ 0, & \text{其他}\end{cases}$$

(2) 令$U=X^2,V=Y^2$, 取值范围为$0\le U,V\le 1$. 则(U,V) 的联合分布为

$$\begin{aligned}P(U\le u,V\le v)&=\iint_{x^2\le u,y^2\le v}f(x,y)dxdy\\&=\begin{cases}0, & u,v<0\\ \int_{-\sqrt{u}}^{\sqrt{u}}\int_{-\sqrt{v}}^{\sqrt{v}}\frac{1+xy}{4}dxdy, & 0\le u,v<1,\\ \int_{-\sqrt{u}}^{\sqrt{u}}\int_{-1}^{1}\frac{1+xy}{4}dxdy, & 0\le u<1,v\ge 1;\\ \int_{-1}^{1}\int_{-\sqrt{v}}^{\sqrt{v}}\frac{1+xy}{4}dxdy, & u\ge 1,0\le v<1;\\ 1, & u,v\ge 1.\end{cases}\quad\begin{cases}0, & u,v<0,\\ \sqrt{uv}, & 0\le u,v<1,\\ \sqrt{u}, & 0\le u<1,v\ge 1,\\ \sqrt{v}, & u\ge 1,0\le v<1,\\ 1, & u,v\ge 1.\end{cases}\end{aligned}$$

再由(1) 中X, Y 的边际密度函数可求

$$F_U(u) = \int_{x^2 \le u} f_X(x)dx = \begin{cases} 0, & u < 0, \\ \sqrt{u}, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases} \quad F_V(v) \ = \begin{cases} 0, & v < 0, \\ \sqrt{v}, & 0 \le v < 1, \\ 1, & v \ge 1. \end{cases}$$

$F_{UV}(u, v) = F_U(u)F_V(v)$, 所以$U := X^2, V := Y^2$ 相互独立.

42.

$$f_X(x) = \int_0^{2\pi} \int_0^{2\pi} f(x, y, z)dydz = \begin{cases} \frac{1}{2\pi}, \ 0 < x < 2\pi, \\ 0, \text{其他}. \end{cases}$$

$$f(x, y) = \int_0^{2\pi} f(x, y, z)dz = \begin{cases} \frac{1}{4\pi^2}, \ 0 < x, y < 2\pi, \\ 0, \text{其他}. \end{cases}$$

由对称性可知，$f_Y(y), f_Z(z)$与$f_X(x)$，$f(x, z), f(y, z)$与$f(x, y)$有相同的形式，因为$f(x, y) = f_X(x)f_Y(y)$,
所以x, y相互独立，同理可得X, Y, Z两两独立。
因为$f(x, y, z) \neq f_X(x)f_Y(y)f_Z(z)$，所以$X, Y, Z$不相互独立。

第六周作业答案

April 21, 2023

44. (1) 设Y 取n个值$a_1, a_2, ..., a_n$，对应概率分别为$p_1, p_2, ..., p_n$，先求$U := X+Y$ 的分布函数$F_U(\cdot)$. 由题意知X, Y相互独立,即

$$
\begin{aligned}
F_U(u) &= P(X+Y \leq u) \\
&= \sum_{i=1}^{n} P\left(Y=a_i\right) P\left(X+Y \leq u \mid Y=a_i\right) \\
&= \sum_{i=1}^{n} P\left(Y=a_i\right) P\left(X \leq u-a_i\right) \\
&= \sum_{i=1}^{n} p_i F_X\left(u-a_i\right)
\end{aligned}
$$

上式对u 求导得U 的密度函数$h(u)=\sum_{i=1}^{n} p_i f\left(u-a_i\right)$. 由$X$的密度函数$f$是连续的知$h$也是连续的，所以$Z=X+Y$是连续型随机变量.

(2)先求$W := XY$ 的分布函数$F_W(\cdot)$,其中$X \sim N(\mu, \sigma^2), Y \sim B(1, p)$.由题意知$XY$相互独立,当$w<0$时

$$
\begin{aligned}
F_W(w) &= P(XY \leq w) \\
&= P\left(Y=1\right) P\left(X \leq w \mid Y=1\right) \\
&= pP\left(X \leq w\right) \\
&= pF_X\left(w\right)
\end{aligned}
$$

当$w \geq 0$时

$$
\begin{aligned}
F_W(w) &= P(XY \leq w) \\
&= P\left(Y=1\right) P\left(X \leq w \mid Y=1\right) + P\left(Y=0\right) \\
&= pP\left(X \leq w\right) + 1 - p \\
&= pF_X\left(w\right) + 1 - p
\end{aligned}
$$

可知$F_W(\cdot)$在$w=0$处不连续，所以不是连续型随机变量。

47. $(X,Y)\sim N\left(a,b,\sigma_1^2,\sigma_2^2,\rho\right)$, 其联合密度为

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{(x-a)^2}{\sigma_1^2}\right.\right.$$
$$\left.\left.-2\rho\frac{(x-a)(y-b)}{\sigma_1\sigma_2}+\frac{(y-b)^2}{\sigma_2^2}\right]\right\},-\infty<x,y<\infty,$$

$\left\{\begin{array}{l}U=X+cY,\\V=X-cY,\end{array}\right.$ 的反函数为$\left\{\begin{array}{l}X=\frac{1}{2}(U+V),\\Y=\frac{1}{2c}(U-V).\end{array}\right.$ 对应的雅可比行列式为

$$J=\left|\begin{array}{cc}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2c}&-\frac{1}{2c}\end{array}\right|=-\frac{1}{2c}$$

则(U,V) 的联合密度为

$$\begin{aligned}f_{UV}(u,v)=&f\left(\left(\frac{1}{2}(u+v),\frac{1}{2c}(u-v)\right)\cdot|J|\right.\\=&\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\left(\frac{1}{2}(u+v)-a\right)^2}{\sigma_1^2}\right.\right.\\&\left.\left.-2\rho\frac{\left(\frac{1}{2}(u+v)-a\right)\left(\frac{1}{2c}(u-v)-b\right)}{\sigma_1\sigma_2}+\frac{\left(\frac{1}{2c}(u-v)-b\right)^2}{\sigma_2^2}\right]\right\}\cdot\left|\frac{1}{2c}\right|\\=&\frac{1}{4|c|\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[A_2u^2+A_1v+B_2v^2+B_1v+C_2uv+D\right]\right\},\\&-\infty<u,v<\infty.\text{ 其中}C_2=\left(\frac{1}{2\sigma_1^2}-\frac{1}{2c^2\sigma_2^2}\right).\end{aligned}$$

由定理3.2 , 要使U,V 独立, 即要求uv 项系数C_2 为0 即可:

$$C_2=\left(\frac{1}{2\sigma_1^2}-\frac{1}{2c^2\sigma_2^2}\right)=0\Rightarrow c=\frac{\sigma_1}{\sigma_2}\left(\text{ 或}-\frac{\sigma_1}{\sigma_2}\right)$$

48.(1) $(X,Y)\sim N(0,0;1,1;\rho),Z=(Y-\rho X)/\sqrt{1-\rho^2}$, 则$Y=\sqrt{1-\rho^2}Z+\rho X$ 且

$$J^{-1}=\frac{\partial(x,z)}{\partial(x,y)}=\left|\begin{array}{cc}1&0\\-\frac{\rho}{\sqrt{1-\rho^2}}&\frac{1}{\sqrt{1-\rho^2}}\end{array}\right|=\frac{1}{\sqrt{1-\rho^2}}.$$

所以(X,Z) 的密度函数为

$$\begin{aligned} f(x,z) &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2-2\rho xy+y^2}{2\left(1-\rho^2\right)}\right\}\cdot\sqrt{1-\rho^2}\left(\text{ 其中}y=\sqrt{1-\rho^2}z+\rho x\right) \\ &= \frac{1}{2\pi}\exp\left\{-\frac{(y-\rho x)^2+x^2-\rho^2x^2}{2\left(1-\rho^2\right)}\right\} \\ &= \frac{1}{2\pi}\exp\left\{-\frac{\left(\sqrt{1-\rho^2}z\right)^2+\left(1-\rho^2\right)x^2}{2\left(1-\rho^2\right)}\right\} \\ &= \frac{1}{2\pi}e^{-\frac{1}{2}\left(x^2+z^2\right)},\quad x,z\in R. \end{aligned}$$

由(X,Z) 的密度函数可知: $(X,Z)\sim N(0,0;1,1;0)$, 两者相关系数为0 ，所以X,Z 相互独立. 或求得密度函数:

$$f_X(x)=\int_{-\infty}^{\infty}f(x,z)dz=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{x^2+z^2}{2}}dz=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},x\in R.$$

同理

$$f_Z(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}},z\in R$$

所以$f(x,z)=f_X(x)f_X(z),X,Z$ 相互独立. 对于Y,Z, 与(1) 过程类似可求得(Y,Z) 密度函数为

$$g(y,z)=\frac{1}{2\pi|\rho|}\exp\left\{-\frac{y^2-2\sqrt{1-\rho^2}yz+z^2}{2\rho^2}\right\},y,z\in R.$$

可见$(Y,Z)\sim N\left(0,0;1,1;\sqrt{1-\rho^2}\right)(\rho\neq 0,\pm 1)$, 相关系数不为0 , 所以$Y,Z$ 不独立. (2)

$$\begin{aligned} P(X>0,Y>0) &= P\left(X>0,\sqrt{1-\rho^2}Z+\rho X>0\right) \\ &= P(X>0,Z>0)+P\left(Z>0,-\frac{\rho_X}{\sqrt{1-p^2}}<Z<0\right) \\ &= \frac{1}{4}+P\left(X>0,-\frac{\rho X}{\sqrt{1-\rho^2}}<Z<0\right). \end{aligned}$$

下面求$P\left(X>0,-\frac{\rho X}{\sqrt{1-\rho^2}}<Z<0\right)$, 令$X=r\cos\theta,Z=r\sin\theta$. 代入$X>$

$0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0$ 得θ 的取值范围为$\left(-\arctan\frac{\rho}{\sqrt{1-\rho^2}}, 0\right)$. 所以

$$P\left(X > 0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0\right) = \int_0^{\infty}\int_{-\arctan\frac{\rho}{\sqrt{1-\rho^2}}}^{0} \frac{1}{2\pi}e^{-\frac{r^2}{2}} \cdot r d\theta dr$$
$$= \arctan\frac{\rho}{\sqrt{1-\rho^2}}\int_0^{\infty}\frac{r}{2\pi}e^{-\frac{r^2}{2}}dr$$
$$= \frac{1}{2\pi}\arcsin\rho$$

即得

$$P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho$$

所以由XY对称性可知

$$P(XY < 0) = 1 - P(X > 0, Y > 0) - P(X < 0, Y < 0) = 1 - 2P(X > 0, Y > 0)$$
$$= 1 - \frac{1}{2} - \frac{1}{\pi}\arcsin\rho = \frac{1}{\pi}\arccos\rho$$

例。$X, Y \sim iid \quad Exp(\lambda)$,求证$min\{X, Y\}$与$max\{X, Y\} - min\{X, Y\}$相互独立，并求它们的分布。

proof: 由$X, Y \sim iid \quad Exp(\lambda)$知$X, Y$的联合密度函数为

$$f(x, y) = \lambda^2 e^{-\lambda(x+y)}$$

联合分布函数为$F(x, y)$。记$U = min\{X, Y\}, V = max\{X, Y\}$，所以$U, V$的联合分布函数为

$$G(u, v) = P(X \le u, Y \le v) + P(Y \le u, X \le v) = 2F(u, v)$$

联合密度函数为

$$g(u, v) = 2f(u, v) = 2\lambda^2 e^{-\lambda(u+v)}$$

令$Z = U, W = V - U$,有$U = Z, V = Z + W$,所以雅克比行列式为

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

所以Z, W的联合密度函数为

$$f_{Z,W}(z, w) = g(z, z + w) = 2\lambda^2 e^{-\lambda(2z+w)}$$

$$= 2\lambda e^{-2\lambda z} \times \lambda e^{-\lambda w} = f_Z(z) f_W(w)$$

所以$Z = min\{X, Y\}, W = max\{X, Y\} - min\{X, Y\}$相互独立，且

$$f_Z(z) = 2\lambda e^{-2\lambda z}, f_W(w) = \lambda e^{-\lambda w}$$

2. 证明: (1) X 为非负整值随机变量,

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} P(X = k) \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(X = k) = \sum_{n=1}^{\infty} P(X \geq n) (\text{ 第一个“ }=\text{” 成立}) \\ &= \sum_{n=1}^{\infty} P(X > n - 1) = \sum_{n=0}^{\infty} P(X > n) (\text{ (第二个“=” 成立}) \end{aligned}$$

(2) X 为非负连续型随机变量且$X \sim F$, 设对应概率密度为$f(\cdot)$.

$$\begin{aligned} E(X) &= \int_0^{\infty} x f(x) dx = \int_0^{\infty} \left(\int_0^{x} f(x) dt \right) dx \\ &= \int_0^{\infty} \left(\int_t^{\infty} f(x) dx \right) dt \\ &= \int_0^{\infty} F(x) \Big|_t^{\infty} dt = \int_0^{\infty} 1 - F(t) dt \\ &= \int_0^{\infty} 1 - F(x) dx \end{aligned}$$

(3) X 为非负随机变量,

$$\begin{aligned} E(X) &= E\left[\int_0^{X} 1 dx \right] = E\left[\int_0^{\infty} I_{(X > x)} dx \right] \\ &= \int_0^{\infty} E\left(I_{(X > x)} \right) dx = \int_0^{\infty} P(X > x) dx \\ &= \int_0^{\infty} 1 - F(x) dx \end{aligned}$$

3. 记$\phi(x)$ 为标准正态分布的密度函数, 则X 的密度函数为$f(x) = 0.5\phi(x) +$

$0.25 \cdot \phi\left(\frac{x-4}{2}\right)$, 则

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx = 0.5 \int_{-\infty}^{\infty} x\phi(x) dx + 0.25 \int_{-\infty}^{\infty} x\phi\left(\frac{x-4}{2}\right) dx \\
&= 0.25 \int_{-\infty}^{\infty} (2y+4)\phi(y) d(2y+4) \\
&= 0.5 \int_{-\infty}^{\infty} 2y\phi(y) dy + 0.5 \int_{-\infty}^{\infty} 4\phi(y) dy \\
&= 2 \int_{-\infty}^{\infty} \phi(y) dy = 2
\end{aligned}
$$

7. (1) 记$X_i = \begin{cases} 1, \text{ 第i 个盒子为空} \\ 0, \text{ 第i 个盒非空.} \end{cases}$ $, i = 1, \ldots, n$ 则空盒子总数为$Y = \sum_{i=1}^{n} X_i$, 因为$P\left(X_i = 1\right) = \left(1 - \frac{1}{n}\right)^n$ 所以$Y \sim b\left(n, \left(1 - \frac{1}{n}\right)^n\right), EY = n\left(1 - \frac{1}{n}\right)^n$
(2) $n \to \infty$ 时,

$$
\lim_{n\to\infty} \frac{n\left(1-\frac{1}{n}\right)^n}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-n\cdot(-1)} = e^{-1}
$$

8. $X = n, n+1, \ldots,$ 记Y_j 为抽到i $-$ 1 种卡后, 抽到新卡所需的次数, 则$X_n = \sum_{j=1}^{n} Y_j$

$$
P\left(Y_j = k\right) = \frac{n-j+1}{n} \cdot \left(\frac{j-1}{n}\right)^{k-1},
$$

$$
EY_j = \frac{n-j+1}{n} \sum_{k=1}^{\infty} k \left(\frac{j-1}{n}\right)^{k-1} = \frac{n-j+1}{n} \cdot \frac{n^2}{(n-j+1)^2} = \frac{n}{n-j+1},
$$

所以

$$
EX_n = E\sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} EY_j = \sum_{j=1}^{n} \frac{n}{n-j+1} = \sum_{k=1}^{n} \frac{n}{k}
$$

(1) $n = 12$ 时,

$$
EX_n = \sum_{k=1}^{n} \frac{n}{k} = 12 \sum_{k=1}^{12} \frac{1}{k} \approx 37.24
$$

(2)

$$
\lim_{n\to\infty} \frac{EX_n}{n \ln n} = \lim_{n\to\infty} \frac{n \sum_{k=1}^{n} \frac{1}{k}}{n \ln n} = \lim_{n\to\infty} \frac{\ln n + \gamma + \frac{1}{2n}}{\ln n} = 1.
$$

17.已知X 的密度函数, 有

$$E[\min\{|X|,1\}]=\int_{|x|>1}f(x)dx+\int_{|x|\leq 1}|x|\cdot f(x)dx$$

$$=2\int_{1}^{\infty}\frac{1}{\pi\left(1+x^{2}\right)}dx+2\int_{0}^{1}\frac{x}{\pi\left(1+x^{2}\right)}dx(\text{ 对称性})$$

$$=\left.\frac{2\arctan x}{\pi}\right|_{1}^{\infty}+\int_{0}^{1}\frac{d\left(1+x^{2}\right)}{\pi\left(1+x^{2}\right)}=\frac{1}{2}+\left.\frac{1}{\pi}\ln\left(1+x^{2}\right)\right|_{0}^{1}$$

$$=\frac{1}{2}+\frac{\ln 2}{\pi}$$

18.(1)当$X=1$时，$Y\in(0,1)$，当$X=2$时，$Y\in(0,2)$，所以

$$P(Y<y)=P(x=1)\cdot P(Y<y\mid x=1)+P(x=2)P(Y<y\mid x=2)$$

$$=\frac{1}{2}\int_{0}^{y}du+\frac{1}{2}\int_{0}^{y}\frac{1}{2}du=\frac{3}{4}y\quad(0<y<1)$$

$$P(Y<y)=P(Y<1)+P(X=2)P(1\leq Y<2)$$

$$=\frac{3}{4}+\frac{1}{2}\int_{1}^{y}\frac{1}{2}du=\frac{1}{2}+\frac{1}{4}y\quad(1\leq y<2)$$

$$\text{所以}F_Y(y)=\begin{cases}0, & y<0\\ \frac{3}{4}y, & 0\leq y<1\\ \frac{1}{2}+\frac{1}{4}y & ,1\leq y<2\\ 1, & y\geq 2\end{cases}$$

(2)

$$EY=\int ydF(y)=\int_{0}^{1}\frac{3}{4}ydy+\int_{1}^{2}\frac{y}{4}dy=\frac{3}{4}.$$

19.$(X,Y)\sim N(1,1,0.5,0.5,0.5)$, 则$(X,Y)$ 的联合密度为

$$f(x,y)=\frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{2}{3}\left[2(x-1)^{2}-2(x-1)(y-1)+2(y-1)^{2}\right]\right\}.$$

(1) $Z=|X-Y|$, 可以先求$X-Y$ 的密度。令$\left\{\begin{array}{l}S=X-Y\\ T=Y\end{array}\right.$ 则$\left\{\begin{array}{l}X=S+T\\ Y=T\end{array}\right.$,
对应的雅可比行列式为

$$J=\left|\begin{array}{ll}1 & 1\\ 0 & 1\end{array}\right|=1$$

所以(S,T) 的联合密度为

$$
\begin{aligned}
f_{ST}(s,t) &= f(s+t,t)\cdot|J| \\
&= \frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{2}{3}\left[2(s+t-1)^2-2(s+t-1)(t-1)+2(t-1)^2\right]\right\} \\
&= \frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{4}{3}s^2-\frac{4}{3}s(t-1)-\frac{4}{3}(t-1)^2\right\}, -\infty<s,t<\infty.
\end{aligned}
$$

则U 的边际密度为

$$
\begin{aligned}
f_S(s) &= \int_{-\infty}^{\infty}\frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{4}{3}s^2-\frac{4}{3}s(t-1)-\frac{4}{3}(t-1)^2\right\}dt \\
&= \int_{-\infty}^{\infty}\frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{4}{3}\left[\left(t+\frac{s-2}{2}\right)^2++\frac{3}{4}s^2\right]\right\}dt \\
&= \frac{1}{\sqrt{\pi}}e^{-s^2}\int_{-\infty}^{\infty}\frac{2}{\sqrt{3}\pi}\exp\left\{-\frac{\left(t+\frac{s-2}{2}\right)^2}{3/4}\right\}dt \\
&= \frac{1}{\sqrt{\pi}}e^{-s^2}, \quad -\infty<s<\infty
\end{aligned}
$$

对于$Z=|S|, P(Z\le z)=P(-z\le S\le z)=2\int_0^z \frac{1}{\sqrt{\pi}}e^{-s^2}ds, (z>0)$, 可得$Z$ 的密度函数为

$$
f_Z(z)=\frac{2}{\sqrt{\pi}}e^{-z^2}, z>0.
$$

期望为

$$
E(Z)=\int_0^{\infty} z\cdot\frac{2}{\sqrt{\pi}}e^{-z^2}dz=-\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-z^2}d\left(-z^2\right)=\frac{1}{\sqrt{\pi}}.
$$

(2) $U=\max(X,Y), V=\min(X,Y)$, 则有$\begin{cases} U+V=X+Y \\ U-V=|X-Y| \end{cases}$ 由期望的性质有:

$$
\begin{cases} E(U+V)=E(X+Y) \\ E(U-V)=E|X-Y| \end{cases} \Rightarrow \begin{cases} E(U)+E(V)=E(X)+E(Y)=2 \\ E(U)-E(V)=EZ=\frac{1}{\sqrt{\pi}} \end{cases}
$$

有$\begin{cases} E(U)=1+\frac{1}{2\sqrt{\pi}} \\ E(V)=1-\frac{1}{2\sqrt{\pi}} \end{cases}$

21.(1)由泊松分布的可加性可知，$X+Y\sim P(\lambda+\mu)$，所以

$$P(X=k|X+Y=m)=\frac{P(X=k,X+Y=m)}{P(X+Y=m)}=\frac{P(X=k)P(Y=m-k)}{P(X+Y=m)}=\frac{\frac{\lambda^k}{k!}e^{-\lambda}\frac{\mu^{m-k}}{(m-k)!}e^{-\mu}}{\frac{(\lambda+\mu)^m}{m!}e^{-(\lambda+\mu)}}$$
$$=C_m^k\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(1-\frac{\lambda}{\lambda+\mu}\right)^{m-k},\quad k=0,1,..,m.$$

所以$X|X+Y=m\sim b(m,\lambda/(\lambda+\mu))$，$E(X|X+Y=m)=m\lambda/(\lambda+\mu)$。
(2)由二项分布的可加性可知，$X+Y\sim b(2n,p)$，所以

$$P(X=k|X+Y=m)=\frac{P(X=k)P(Y=m-k)}{P(X+Y=m)}=\frac{C_n^kp^k(1-p)^{n-k}C_n^{m-k}p^{m-k}(1-p)^{n-m+k}}{C_{2n}^mp^m(1-p)^{2n-m}}$$
$$\frac{C_n^kC_n^{m-k}}{C_{2n}^m},\quad k=0,1,...,m.$$

由题意可知，X,Y独立同分布。所以$E(X|X+Y=m)=E(Y|X+Y=m)$。
又因为$E(X+Y|X+Y=m)=m$，所以$E(X|X+Y=m)=m/2$。

22.

$$X\sim\begin{pmatrix}0&1&2\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix},Y\mid X=k\sim N(k,1)$$

(1) Y 的概率密度函数为

$$f_Y(y)=\sum_{k=0}^{2}f_Y(y\mid X=k)P(X=k)=\frac{1}{3}\sum_{k=0}^{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-k)^2}{2}},-\infty<y<\infty.$$

期望为

$$E(Y)=\frac{1}{3}\sum_{k=0}^{2}\left[\int_{-\infty}^{\infty}y\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-k)^2}{2}}dy\right]=\frac{1}{3}\sum_{k=0}^{2}k=1$$

(2) $Z:=X+Y$, 则其分布函数为

$$\begin{aligned}F_Z(z)&=P(Z\le z)=P(X+Y\le z)\\&=\sum_{k=0}^{2}P(Y\le z-X\mid X=k)P(X=k)\\&=\frac{1}{3}\sum_{k=0}^{2}P(Y\le z-X\mid X=k)\\&=\frac{1}{3}\sum_{k=0}^{2}\Phi(z-2k),-\infty<z<\infty.\ (\text{其中}\Phi(\cdot)\ \text{为标准正态的分布函数.})\end{aligned}$$

4.(1)

$$
\begin{aligned}
EX &= \int_0^\infty x \cdot \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2\pi}\sigma} \int x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \sigma^2 = \sigma \sqrt{\frac{\pi}{2}}
\end{aligned}
$$

$$
\begin{aligned}
EX^2 &= \int_0^\infty x^2 \cdot \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = -\int_0^\infty x^2 d\left(e^{-\frac{x^2}{2\sigma^2}}\right) \\
&= - x^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^\infty + 2 \int_0^\infty x e^{-\frac{x^2}{2\sigma^2}} dx = 0 - 2\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^\infty = 2\sigma^2
\end{aligned}
$$

$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{4-\pi}{2}\sigma^2$

(2)

$$
\begin{aligned}
EX &= \int_0^1 x \cdot \frac{T(\alpha+\beta)}{T(\alpha)T(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{T(\alpha+\beta)}{T(\alpha)T(\beta)} \int_0^1 x^{\alpha-1+1}(1-x)^{\beta-1} dx \\
&= \frac{T(\alpha+\beta)}{\Gamma(\alpha)T(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta} \\
EX^2 &= \int_0^1 x^2 \frac{T(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha+\beta)}{T(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+2-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)T(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\
Var(X) &= EX^2 - (EX)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}
$$

(3)

$$
\begin{aligned}
EX &= \int_0^\infty x \cdot \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k} dx = \lambda \int_0^\infty \left[\left(\frac{x}{\lambda}\right)^k\right]^{\frac{1}{k}+1-1} e^{-(\frac{x}{\lambda})^k} d\left(\frac{x}{\lambda}\right)^k \\
&= \lambda \Gamma\left(1 + \frac{1}{k}\right) \\
EX^2 &= \int_0^\infty x^2 \cdot \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k} dx = \lambda^2 \int_0^\infty \left[\left(\frac{x}{\lambda}\right)^k\right]^{\frac{2}{k}+1-1} e^{-(\frac{x}{\lambda})^k} d\left(\frac{x}{\lambda}\right)^k \\
&= \lambda^2 \Gamma\left(1 + \frac{2}{k}\right) \\
Var(X) &= EX^2 - (EX)^2 = \lambda^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]
\end{aligned}
$$

24.(1)令$Z=X+Y$，由题意易知$Z\sim N(3\mu,4\sigma^2)$。所以

$$E((X+Y-3\mu)_+)=E(Z-3\mu|z>3\mu)=\int_{3\mu}^{\infty}\frac{z-3\mu}{\sqrt{2\pi}2\sigma}e^{-\frac{(z-3\mu)^2}{8\sigma^2}}dz$$

$$\int_0^{\infty}\frac{t}{\sqrt{2\pi}2\sigma}e^{-\frac{t^2}{8\sigma^2}}dt=-\frac{8\sigma^2}{2}\frac{1}{\sqrt{2\pi}2\sigma}e^{-\frac{t^2}{8\sigma^2}}|_0^{\infty}=\frac{2\sigma}{\sqrt{2\pi}}.$$

(2)

$$E((Z-3\mu)^2|Z>3\mu)=\frac{1}{2}E(Z-3\mu)^2=\frac{1}{2}\mathrm{Var}(Z)=2\sigma^2.$$

所以$\mathrm{Var}((X+Y-3\mu)_+)=2\sigma^2-(\sigma/\sqrt{2\pi})^2=2\sigma^2-2\sigma^2/\pi$。

28.(1) 设投掷的点数大于等于n 时投烘次数的期望为E_n, 则题中所要求的为E_{10}. 则

$$E_1=1$$
$$E_2=1\times\frac{5}{6}+2\times\frac{1}{6}=\frac{7}{6}$$

对于$3\leq n\leq 6$, 对第一次投出的点数取条件，则有

$$\begin{aligned}E_n&=\sum_{i=1}^{6}E_{n|X_{1-1}}P\left(X_1=i\right)\\&=P\left(X_1=1\right)\left(1+E_{n-1}\right)+\cdots P\left(X_1=n-1\right)\left(1+E_1\right)\\&\quad+P\left(X_1=n\right)(1+0)+\cdots P\left(X_1=6\right)(1+0)\\&=\sum_{i=1}^{6}P\left(X_1=i\right)+P\left(X_1=1\right)E_{n-1}+P\left(X_1=2\right)E_{n-2}+\cdots P\left(X_1=n-1\right)E_1\\&=1+\frac{1}{6}\sum_{i=1}^{n-1}E_i\end{aligned}$$

$$E_n=\frac{7}{6}E_{n-1}=\left(\frac{7}{6}\right)^{n-1}(1\leq n\leq 6)$$

$n>6$ 时, $E_n=\sum_{i=1}^{6}E_{n|X_{1-i}}P\left(X_1=i\right)=1+\sum_{i=1}^{6}P\left(X_1=i\right)E_{n-1}=1+\frac{1}{6}\sum_{i=1}^{6}E_{\mathrm{n}-i}$

$$\Rightarrow E_{\mathrm{n}+1}-E_n=\frac{1}{6}\left(E_n-E_{n-6}\right)$$

$$E_{n+1}=\frac{7}{6}E_n-\frac{1}{6}E_{n-6}(n\geq 7)$$

两式结合, 有$E_7=\left(\frac{7}{6}\right)^6, E_9=\left(\frac{7}{6}\right)^7-\frac{1}{6}, E_9=\left(\frac{7}{6}\right)^8-\frac{7}{18}$,

$$E_{10}=\left(\frac{7}{6}\right)^9-\frac{49}{72}\approx 3.3237$$

(2) 记直到点数大于等于10 所需的投推次数为Y_{10}, 由(1) 知$E\left(Y_{10}\right)=E_{10}$.
再由Wald 等式有

$$E\left(\sum_{i=1}^{Y_{10}}X_i\right)=E\left(X_1\right)E_{10}=\frac{7}{2}E_{10}\approx 11.6329$$

29.(1)由题意易知

$P(Y<y)=1-P(X_1>y,X_2>Y)=1-P(X_1>y)P(X_2>y)=1-e^{-\frac{3}{2}y},\quad y>0,$

$P(Z<z)=P(X_1<z,X_2<z)=P(X_1<z)P(X_2<z)=1-e^{-z}-e^{-\frac{1}{2}z}+e^{-\frac{3}{2}z},\quad z>0.$

所以有

$$EY=\int_0^\infty yF_Y'(y)dy=\int_0^\infty y\frac{3}{2}e^{-\frac{3}{2}y}dy=\frac{2}{3},$$

$$EZ=\int_0^\infty zF_Z'(z)dz=\int_0^\infty z(e^{-z}+\frac{1}{2}e^{-\frac{z}{2}}-\frac{3}{2}e^{-\frac{3}{2}z})dz=\frac{7}{3}.$$

(2)因为$Y\sim EXP(\frac{3}{2})$，所以$\mathrm{Var}(Y)=4/9$。

$$EZ^2=\int_0^\infty z^2(e^{-z}+\frac{1}{2}e^{-\frac{z}{2}}-\frac{3}{2}e^{-\frac{3}{2}z})dz=\frac{82}{9},$$

所以$\mathrm{Var}(Z)=82/9-(7/3)^2=33/9=11/3$。

第八周作业答案

May 17, 2023

31. (1)

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$E(X) = \sum_{x=1}^{6} P(X = x) \cdot x = \sum_{x=1}^{6} \frac{1}{6} x = \frac{7}{2}$$

$$\mathrm{Var}(X) = \sum_{x=1}^{66} P(X = x) \cdot x^2 - (EX)^2 = \frac{35}{12}$$

$$Y \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{36} & \frac{1}{12} & \frac{5}{36} & \frac{7}{36} & \frac{7}{36} & \frac{11}{36} \end{pmatrix}$$

$$E(Y) = \sum_{y=1}^{6} P(Y = y) \cdot y = \frac{161}{36}$$

$$\mathrm{Var}(Y) = \sum_{y=1}^{6} P(Y = y) y^2 - (EY)^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 \approx 1.97$$

(2)

$Y \backslash X$	1	2	3	4	5	6
1	$\frac{1}{36}$					
2	$\frac{1}{36}$	$\frac{2}{36}$				
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{3}{36}$			
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{4}{36}$		
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}$

$$EXY = \sum xyP(X = x, Y = y) = \frac{616}{36} = \frac{154}{9}$$

$$\mathrm{Cov}(X, Y) = EXY - EXEY = \frac{35}{24}$$

33.

$$
\begin{aligned}
&\operatorname{Cov}(\alpha X+\beta Y, \alpha X-\beta Y)=\operatorname{Cov}(\alpha X, \alpha X-\beta Y)+\operatorname{Cov}(\beta Y, \alpha X-\beta Y) \\
=&\operatorname{Cov}(\alpha X, \alpha X)-\operatorname{Cov}(\alpha X, \beta Y)+\operatorname{Cov}(\beta Y, \alpha x)-\operatorname{Cov}(\beta Y, \beta Y) \\
=&\alpha^2 \operatorname{Cov}(X, X)-\beta^2 \operatorname{Cov}(Y, Y)
\end{aligned}
$$

$(X, Y) \sim N\left(\mu, \mu, \sigma^2, \sigma^2, \rho\right)$, 则$X, Y \sim N\left(\mu, \sigma^2\right)$ 有

$$
\begin{gathered}
\operatorname{Cov}(\alpha X+\beta Y, \alpha X-\beta Y)=\left(\alpha^2-\beta^2\right) \sigma^2=0 \\
\Rightarrow \alpha=\pm \beta .
\end{gathered}
$$

37 (1)

$$
\begin{gathered}
Cov(X_1, X_2)=E(X_1 X_2)-E(X_1) E(X_2) \\
Cov(X_1, E(X_2 \mid X_1))=E(X_1 E(X_2 \mid X_1))-E(X_1) E(E(X_2 \mid X_1)) \\
=E(X_1 E(X_2 \mid X_1))-E(X_1) E(X_2)=E(X_1 X_2)-E(X_1) E(X_2)
\end{gathered}
$$

所以$Cov(X_1, X_2)=Cov(X_1, E(X_2 \mid X_1))$.

(2)

$$
\begin{gathered}
Cov(X_1, X_2)=Cov(X_1, E(X_2 \mid X_1)) \\
=Cov(X_1, 1+c X_1)=c Var(X_1)
\end{gathered}
$$

所以$c=Cov(X_1, X_2) / Var(X_1)$.

39 (1) 由$X_i \stackrel{iid}{\sim} Ge(p), i=1,2,...,N \sim B(n, q)$,且$X_i, N$相互独立,

$$
\begin{gathered}
E(S \mid N=n)=E(\sum_{i=1}^N X_i \mid N=n)=\sum_{i=1}^n E(X_i)=n / p \\
Var(S \mid N=n)=\sum_{i=1}^n Var(X_i \mid N=n)=\sum_{i=1}^n Var(X_i)=n(1-p) / p^2 \\
Var(S)=E(Var(S \mid N))+Var(E(S \mid N)) \\
=E(N(1-p) / p^2)+Var(N / p)=n q(1-p) / p^2+n q(1-q) / p^2
\end{gathered}
$$

40

(1) 由

$$E(T) = a, Var(T) = b > 0$$

$$N(t) \sim poisson(\lambda t), E(N(t)) = \lambda t, Var(N(t)) = \lambda t$$

有

$$E(N(T)) = E(E(N(T) \mid T)) = E(\lambda T) = \lambda a$$

$$Cov(T, N(T)) = E(TN(T)) - E(T)E(N(T))$$

$$= E(E(TN(T) \mid T)) - \lambda a^2 = E(\lambda T^2) - \lambda a^2$$

$$= \lambda(b + a^2) - \lambda a^2 = \lambda b$$

(2)

$$Var(N(T)) = E(N^2(T)) - E^2(N(T))$$

$$= E(E(N^2(T) \mid T)) - (\lambda a)^2$$

$$= E(\lambda T + (\lambda T)^2) - (\lambda a)^2$$

$$= \lambda a + \lambda^2(b + a^2) - (\lambda a)^2 = \lambda a + \lambda^2 b$$

42

$$X_n + Y_n \xrightarrow{P} X + Y$$

$$\Longleftrightarrow \forall \epsilon > 0, lim_{n\to\infty} P(| X_n + Y_n - X - Y| \geq \epsilon) = 0$$

$$\because | X_n + Y_n - X - Y| \leq |X_n - X| + |Y_n - Y|$$

$$\therefore P(| X_n + Y_n - X - Y| \geq \epsilon) \leq P((|X_n - X| + |Y_n - Y|) \geq \epsilon)$$

$$\leq P(|X_n - X| \geq \epsilon/2) + P(|Y_n - Y| \geq \epsilon/2)$$

$$\because X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y \quad \therefore \forall \epsilon/2 > 0,$$

$$lim_{n\to\infty} P(|X_n - X| \geq \epsilon/2) = 0, \quad lim_{n\to\infty} P(|Y_n - Y| \geq \epsilon/2) = 0$$

$$lim_{n\to\infty} P(| X_n + Y_n - X - Y| \geq \epsilon) \leq lim_{n\to\infty} P(|X_n - X| \geq \epsilon/2) + lim_{n\to\infty} P(|Y_n - Y| \geq \epsilon/2) = 0$$

45 记X为500次实验中A发生的次数，$X \sim B(500, 0.2)$,所以

$$E(X) = 500 * 0.2 = 100, Var(X) = 500 * 0.2 * 0.8 = 80$$

由切比雪夫不等式知$P(|X - 100| \geq 20) \leq Var(X)/20^2 = 80/400 = 0.2$. 所以$X$落在80到120之间的概率为

$$P(|X - 100| < 20) = 1 - P(|X - 100| \geq 20) \geq 0.8$$

由中心极限定理知n比较大时可近似认为

$$\frac{X-E(X)}{\sqrt{Var(X)}} \sim N(0,1)$$

所以

$$P(|X-100|<20)=P(|X-100|/\sqrt{80}<20/\sqrt{80})=\Phi(20/\sqrt{80})-\Phi(-20/\sqrt{80})\approx 0.975$$

46. $X_1,\cdots,X_2$ 为独立同分布随机变量,

$$E\left(\sum_{i=1}^{n} X_i^2\right)=nE\left(X_1^2\right)=n\alpha_2,$$

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i^2\right)=n\operatorname{Var}\left(X_1^2\right)=n\left(E\left(X_1^4\right)-\left(E\left(X_1^2\right)\right)^2\right)=n\left(\alpha_4-\alpha_2^2\right).$$

则由中心极限定理有

$$\frac{\sum_{i=1}^{n} X_i^2-n\alpha_2}{\sqrt{n\left(\alpha_4-\alpha_2^2\right)}} \xrightarrow{d} N(0,1)$$

48

$X_1,\cdots,X_n, i.i.d., \sim b(1,0.9)$

$$E\left(\sum_{i=1}^{n} X_i\right)=nE\left(X_1\right)=0.9n,\quad \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)=n\operatorname{Var}\left(X_1\right)=0.09n$$

(1) 当$n=100, E\left(\sum_{i=1}^{n} X_i\right)=90, \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)=9$. 由中心极限定理,

$$\begin{aligned}
P\left(\sum_{i=1}^{n} X_i \geq 85\right) &= 1-P\left(\sum_{i=1}^{n} X_i \leq 85\right) \\
&= P\left(\frac{\sum_{i=1}^{n} X_i-90}{3} \leq \frac{85-90}{3}\right) \\
&\approx 1-\Phi\left(-\frac{5}{3}\right)=\Phi\left(\frac{5}{3}\right) \\
&\approx 0.9522
\end{aligned}$$

(2) 由中心极限定理, 至少有80% 的部件正常工作的概率为

$$\begin{aligned}
P\left(\sum_{i=1}^{n} X_i > 0.8n\right) &= 1 - P\left(\sum_{i=1}^{n} X_i \le 0.8n\right) \\
&= 1 - P\left(\frac{\sum_{i=1}^{n} X_i - 0.9n}{0.3\sqrt{n}} \le \frac{0.8n - 0.9n}{0.3\sqrt{n}}\right) \\
&\approx 1 - \Phi\left(-\frac{n}{3\sqrt{n}}\right) = \Phi\left(\frac{\sqrt{n}}{3}\right)
\end{aligned}$$

要使所求概率不小于0.95, 即

$$\Phi\left(\frac{\sqrt{n}}{3}\right) \ge 0.95$$
$$n \ge 24.35.$$

所以n 至少取25 .

51 (1)记X为生产线上组装每件产品的时间，$X \sim exp(1/10), E(X) = 10, Var(X) = 100$. 由

$$E(\sum_{i=1}^{100} X_i) = 100 * 10 = 1000, Var(\sum_{i=1}^{100} X_i) = 100^2$$

由中心极限定理知

$$P(15 * 60 \le \sum_{i=1}^{100} X_i \le 20 * 60) = P(-1 \le (\sum_{i=1}^{100} X_i - 1000)/100 \le 2)$$

$$= \Phi(2) - \Phi(-1) \approx 0.819$$

(2)

$$P(\sum_{i=1}^{n} X_i \le 16 * 60) = P((\sum_{i=1}^{100} X_i - 10n)/(10\sqrt{n}) \le (960 - 10n)/(10\sqrt{n}))$$

$$\Phi((960 - 10n)/(10\sqrt{n})) \ge 0.95$$
$$(960 - 10n)/(10\sqrt{n}) \ge 1.645$$

解得$n \le 81$.

第五章

11.(1)由$(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$,所以由卡方分布的密度函数为

$$g_{n-1}(x)=\frac{x^{(n-1)/2-1}e^{-x/2}}{2^{(n-1)/2}\Gamma((n-1)/2)}.$$

由χ^2_{n-1}分布的方差为$2(n-1)$,即

$$var((n-1)S^2/\sigma^2)=2(n-1)$$

$$(n-1)^2/\sigma^4 Var(S^2)=2(n-1), Var(S^2)=2\sigma^4/(n-1)$$

(2)设$Y=(n-1)S^2/\sigma^2$, 有$S=\sigma\sqrt{Y}/\sqrt{n-1}$.所以

$$E(S)=E(\sigma\sqrt{Y}/\sqrt{n-1})=\sigma/\sqrt{n-1}\int_0^{+\infty}\sqrt{y}g_{n-1}(y)dy$$

$$=\sigma/\sqrt{n-1}\int_0^{+\infty}\sqrt{y}\frac{y^{(n-1)/2-1}e^{-y/2}}{2^{(n-1)/2}\Gamma((n-1)/2)}dy$$

$$=\sigma/\sqrt{n-1}\frac{2^{n/2}\Gamma(n/2)}{2^{(n-1)/2}\Gamma((n-1)/2)}\int_0^{+\infty}\frac{y^{n/2-1}e^{-y/2}}{2^{n/2}\Gamma(n/2)}dy$$

后一个积分即为χ^2_n分布密度函数的积分为1，所以有

$$E(S)=\frac{\sigma\sqrt{2}\Gamma(n/2)}{\sqrt{n-1}\Gamma((n-1)/2)}$$

14. $X_1,X_2\overset{iid}{\sim}N(0,1)$,所以$X_1-X_2,X_1+X_2\sim N(0,2)$.
下证X_1-X_2,X_1+X_2相互独立。因为正态分布的独立性与协方差为0等价，即证

$$Cov(X_1-X_2,X_1+X_2)=Var(X_1)-Var(X_2)=0$$

所以X_1-X_2,X_1+X_2相互独立。所以

$$\frac{(X_1-X_2)^2}{(X_1+X_2)^2}\sim F_{1,1}.$$

15. X_1,X_2,X_3,X_4 i.i.d. $\sim N\left(0,2^2\right)$, 则有

$$(X_1-2X_2)\sim N(0,20),\quad (3X_3-4X_4)\sim N\left(0,10^2\right)$$

要使$T=a\left(X_1-2X_2\right)^2+b\left(3X_3-4X_4\right)^2$ 服从χ^2 分布,

$$
\begin{aligned}
&\sqrt{a}\left(X_1-2X_2\right)\sim N(0,1), \quad \sqrt{b}\left(3X_3-4X_4\right)\sim N(0,1)\\
\Rightarrow & a=\frac{1}{20}, \quad b=\frac{1}{100}
\end{aligned}
$$

此时$T\sim\chi^2(2)$. 或$(a=1/20, b=0)$ 及$(a=0, b=1/100)$ 也可, 此时$T\sim\chi^2(1)$

16. $X_1, X_2, \cdots, X_9 i.i.d.\sim N\left(\mu, \sigma^2\right)$, 则有

$$
\begin{aligned}
&Y_1\sim N\left(\mu, \frac{1}{6}\sigma^2\right), \quad Y_2\sim N\left(\mu, \frac{1}{3}\sigma^2\right), \quad \sqrt{2}\left(Y_1-Y_2\right)\sim N\left(0, \sigma^2\right),\\
&\frac{2S^2}{\sigma^2}\sim\chi^2(2). \quad \left(Y_2 \text{ 是样本} X_7, X_8, X_9 \text{ 的样本均值, } S^2 \text{ 为样本方差. }\right)\\
&Z=\frac{\frac{\sqrt{2}}{\sigma}\left(Y_1-Y_2\right)}{S}=\frac{\sqrt{2}\left(Y_1-Y_2\right)}{\sqrt{\frac{2S^2}{2\sigma^2}}}\sim t(2).
\end{aligned}
$$

17. $X_1, X_2, \cdots, X_{15}$ i.i.d. $\sim N\left(0, 2^2\right)$. 则有

$$
\begin{aligned}
&\frac{1}{4}\left(X_1^2+\cdots+X_{10}^2\right)\sim\chi^2(10),\\
&\frac{1}{4}\left(X_{11}^2+\cdots+X_{15}^2\right)\sim\chi^2(5),\\
&Y=\frac{X_1^2+\cdots+X_{10}^2}{2\left(X_{11}^2+\cdots+X_{15}^2\right)}=\frac{\left(\frac{1}{4}\left(X_1^2+\cdots+X_{10}^2\right)\right)/10}{\left(\frac{1}{4}\left(X_{11}^2+\cdots+X_{15}^2\right)\right)/5}\sim F(10,5).
\end{aligned}
$$

2.总体为射击的环数，即$\{0,1,2,...,10\}$。样本为五次射击的结果，为8，9，7，10，6环。

4.由题意可知，总体分布为$B(1,p)$。样本分布为

$$P(X_1=x_1,...,X_{10}=x_{10})=p^{\sum\limits_{i=1}^{10}x_i}(1-p)^{10-\sum\limits_{i=1}^{10}x_i},\quad x_i=0,1.$$

8.(1)样本空间为$\Omega=\{(X_1,X_2,X_3,X_4,X_5):X_1,...,X_5\in\{0,1\}\}$，抽样分布为

$$P((X_1,X_2,X_3,X_4,X_5)=(x_1,x_2,x_3,x_4,x_5))=p^{\sum\limits_{i=1}^{5}x_i}(1-p)^{5-\sum\limits_{i=1}^{5}x_i},x_i\in\{0,1\},i=1,...,5$$

(2)X_5+p不是统计量，因为含有未知参数。

9.

$$\overline{X}=\frac{1}{7}(74.001+74.005+...+74.002)=73.9893,$$

$$S=\sqrt{\frac{1}{n-1}((74.001-73.9893)^2+...+(74.002-73.9893)^2)}=0.0359.$$

1.

$$\overline{X}=\frac{53\times1+16\times2+21\times3}{100}=1.48,$$

$$EX=2\theta(1-\theta)+2\theta^2+3(1-2\theta)=3-4\theta,$$

所以$\theta=\frac{3-EX}{4}$,
所以$\widehat{\theta}=\frac{3-\overline{X}}{4}=0.38$

4. (1)

$$EX=\int_0^\theta\frac{x}{\theta^2}2(\theta-x)dx=\frac{1}{\theta^2}\left[\theta x^2-\frac{2}{3}x^3\right]_0^\theta=\frac{\theta}{3},$$

所以$\theta=3EX$，所以$\widehat{\theta}=3\overline{X}$。

(2)

$$EX=\int_0^1x(\theta+1)x^\theta dx=\int_0^1(\theta+1)x^{\theta+1}=\frac{\theta+1}{\theta+2}x^{\theta+2}|_0^1=1-\frac{1}{\theta+1},$$

所以$\theta = \frac{1}{1-EX} - 2$，所以$\widehat{\theta} = \frac{1}{1-\overline{X}} = \frac{2\overline{X}-1}{1-\overline{X}}$

(3)

$$\int_0^1 x\sqrt{\theta}x^{\sqrt{\theta}-1}dx = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}x^{\sqrt{\theta}+1}|_0^1 = \frac{\sqrt{\theta}}{\sqrt{\theta}+1},$$

所以$\theta = \left(\frac{EX}{1-EX}\right)^2$，所以$\widehat{\theta} = \left(\frac{\overline{X}}{1-\overline{X}}\right)^2$。

(4)

$$EX = \int_c^\infty x\theta c^\theta x^{-(\theta+1)}dx = \int_c^\infty \theta c^\theta x^{-\theta}dx = \frac{\theta c^\theta}{-\theta+1}x^{-\theta+1}|_c^\infty = \frac{\theta c}{\theta-1},$$

所以$\theta = \frac{EX}{EX-c}$，所以$\widehat{\theta} = \frac{\overline{X}}{\overline{X}-c}$

(5)

$$\int_0^\theta \frac{x}{\theta^3}6x(\theta-x)dx = \frac{1}{\theta^3}\left[2\theta x^3 - \frac{3}{2}x^4\right]_0^\theta = \frac{\theta}{2},$$

所以$\theta = 2EX$，所以$\widehat{\theta} = 2\overline{X}$。

(6)

$$EX = \int_0^\infty x\frac{\theta^2}{x^3}e^{-\frac{\theta}{x}}dx = \int_0^\infty \frac{\theta^2}{x^2}e^{-\frac{\theta}{x}} = \theta e^{-\frac{\theta}{x}}|_0^\infty = \theta,$$

所以$\widehat{\theta} = \overline{X}$

5. (1) X 的期望为

$$\begin{aligned} EX &= \int_0^\infty x\cdot\frac{4x^2}{\theta^3\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}}dx = \int_0^\infty \frac{4}{\sqrt{\pi}}y^{\frac{3}{2}}e^{-y}\cdot\frac{\theta}{2\sqrt{y}}dy \\ &= \frac{2\theta}{\sqrt{\pi}}\int_0^\infty ye^{-y}dy = \frac{2\theta}{\sqrt{\pi}}\Gamma(2) = \frac{2\theta}{\sqrt{\pi}} \\ \Rightarrow \theta &= \frac{\sqrt{\pi}}{2}EX \end{aligned}$$

所以, θ 的矩估计为$\hat{\theta} = \frac{\sqrt{\pi}}{2}\bar{X}$. (2) 先求$X$ 的方差:

$$\begin{aligned} EX^2 &= \int_0^\infty x^2\cdot\frac{4x^2}{\theta^3\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}}dx = \int_0^\infty \frac{4\theta}{\sqrt{\pi}}y^2e^{-y}\cdot\frac{\theta}{2\sqrt{y}}dy \\ &= \frac{2\theta^2}{\sqrt{\pi}}\int_0^\infty y^{\frac{3}{2}}e^{-y}dy = \frac{2\theta^2}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right) = \frac{2\theta^2}{\sqrt{\pi}}\cdot\frac{3\sqrt{\pi}}{4} = \frac{3\theta^2}{2} \\ \operatorname{Var}(X) &= E\left(X^2\right) - (EX)^2 = \frac{3\pi-8}{2\pi}\theta^2 \end{aligned}$$

样本均值$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, 所以$\operatorname{Var}(\bar{X}) = \frac{1}{n}\operatorname{Var}(X)$, 由此

$$\operatorname{Var}(\hat{\theta}) = \operatorname{Var}\left(\frac{\sqrt{\pi}}{2}\bar{X}\right) = \frac{\pi}{4}\operatorname{Var}(\bar{X}) = \frac{3\pi-8}{8n}\theta^2.$$

7.(1)

$$EX_1 = \int_0^\infty \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} dx = -\frac{2}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} |_0^\infty = \sqrt{\frac{2}{\pi}} E|X_1|,$$

所以$\sigma = \sqrt{\frac{\pi}{2}} E|X_1|$，所以$\widehat{\sigma} = \sqrt{\frac{\pi}{2}} \overline{|X|}$。
(2)因为$\sigma^2 = \mathrm{Var}(X_1) = EX_1^2$，所以$\widehat{\sigma} = \sqrt{\overline{X^2}}$。

17.(1)

$$E(a_n \bar{X}) = a_n E\bar{X} = a_n \frac{\theta}{2},$$

所以$a_n = 2$时，$\hat{\theta}_1$无偏。由46题可知

$$E(b_n \min\{X_i\}) = b_n E(\min\{X_i\}) = b_n \frac{\theta}{n+1},$$

$$E(c_n \max\{X_i\}) = c_n E(\{X_i\}) = c_n \frac{n}{n+1} \theta,$$

所以$b_n = n+1$，$c_n = \frac{n+1}{n}$。
(2)

$$\mathrm{Var}(2\bar{X}) = \frac{4}{n} \mathrm{X} = \frac{\theta^2}{3n},$$

$$\mathrm{Var}((n+1)\max\{X_i\}) = (n+1)^2 \mathrm{Var}(\max\{X_i\}) = \frac{n}{n+2} \theta^2,$$

$$\mathrm{Var}\left(\frac{n+1}{n} \min\{X_i\}\right) = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}(\min\{X_i\}) = \frac{1}{n^2+2n} \theta^2.$$

因为$\mathrm{Var}(\hat{\theta}_3) \leq \mathrm{Var}(\hat{\theta}_1) \leq \mathrm{Var}(\hat{\theta}_2)$，当且仅当$n=1$时等号成立，所以$\hat{\theta}_3$更有效。

25. (1)$p(x;\theta) = 1/\theta, \quad x = 0, 1, 2, ..., \theta - 1$。则似然函数为

$$L(\theta) = \frac{1}{\theta^n},$$

显然$L(\theta)$随着θ的增大而减小，所以θ的MLE为$\hat{\theta} = X_{(n)} + 1$。
(2)$p(x;\theta) = \binom{m}{x} \theta^x (1-\theta)^{m-1}, \quad x = 0, 1, ..., m$。则似然函数为

$$L(\theta) = \prod_{i=1}^n \binom{m}{x_i} \theta^{\sum\limits_{i=1}^n x_i} (1-\theta)^{\sum\limits_{i=1}^n (m - x_i)},$$

忽略常数项，对数似然函数为

$$\ln L(\theta) = \sum_{i=1}^{n} x_i \ln\theta + \sum_{i=1}^{n} (m - x_i) \ln(1-\theta).$$

对其关于θ 求导并令为0，

$$\frac{\sum\limits_{i=1}^{n} x_i}{\theta} - \frac{\sum\limits_{i=1}^{n} (m - x_i)}{1-\theta} = 0.$$

所以，θ的MLE为$\hat{\theta} = \overline{X}/m$。

(3) $p(x;\theta) = (x-1)\theta^2(1-\theta)^{x-2}, x = 2, 3, \cdots; 0 < \theta < 1$. 则似然函数

$$L(\theta) = \theta^{2n}(1-\theta)^{\sum_{n-1}^{n} x_i - 2n} \prod_{i=1}^{n} (x_i - 1)$$

对数似然函数为

$$\ln L(\theta) = 2n\ln\theta + \left(\sum_{i=1}^{n} x_i - 2n\right) \ln(1-\theta) + C$$

将其关于θ 求导并令为0，

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{2n}{\theta} - \frac{\sum_{i=1}^{n} x_i - 2n}{1-\theta} = 0$$

所以，θ 的MLE为$\hat{\theta} = \frac{2\mathrm{n}}{\sum_{i-1}^{*} X_i} = \frac{2}{\overline{X}}$.

(4) $p(x;\theta) = -\frac{1}{\ln(1-\theta)}\frac{\theta^x}{2}, x = 1, 2, \cdots$. 则似然函数为

$$L(\theta) = [-\ln(1-\theta)]^{-\mathrm{n}} \theta^{\sum_{n-1}^{n} x_i} \prod_{i=1}^{n} x_i^{-1}$$

对数似然函数为

$$\ln L(\theta) = -n\ln(-\ln(1-\theta)) + \ln\theta \sum_{i=1}^{n} x_i + C$$

将其关于θ 求导并令为0，

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{\sum_{i=1}^{n} I_i}{\theta} + \frac{n}{(1-\theta)\ln(1-\theta)} = 0$$

则

$$\frac{(1-\theta)}{\theta} \ln(1-\theta) = -\frac{n}{\sum_{i=1}^{n} x_i}.$$

此时MLE无法得出显示解，有

$$\widehat{\theta}=\arg\max_{0<0<1}L(\theta)=\arg\max_{0<\theta<1}\ln L(\theta).$$

(5)$p(x;\theta)=\theta^x e^{-\theta}/x!,\quad x=0,1,2...$，则似然函数为

$$L(\theta)=\frac{\theta^{\sum\limits_{i=1}^{n}x_i}e^{-n\theta}}{\prod\limits_{i=1}^{n}x!},$$

忽略常数项，对数似然函数为

$$\ln L(\theta)=\sum_{i=1}^{n}x_i\ln\theta-n\theta.$$

对其关于θ求导并令其为0，则$\sum_{i=1}^{n}x_i/\theta-n=0$。所以$\theta$的MLE为$\hat{\theta}=\overline{X}$。

30.因为

$$\int f(x;a,b)dx=\frac{\sqrt{2\pi}}{\sqrt{2}b}c\int\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+a/b)^2}{1/b^2}}dx=\frac{\sqrt{\pi}}{b}=1$$

所以$c=\frac{b}{\sqrt{\pi}}$。所以$X\sim N(-a/b,1/2b^2)$。又因为

$$L(\mu,\sigma^2)=\prod_{i=1}^{n}f(x_i)=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{\sum\limits_{i=1}^{n}(x_i-\mu)^2}{\sigma^2}},$$

令$\frac{\partial\ln L(\mu,\sigma^2)}{\partial\mu}=\frac{\partial\ln L(\mu,\sigma^2)}{\partial\sigma^2}=0$，有

$$\hat{\mu}=\bar{x},\quad\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

所以$\hat{b}=\sqrt{\frac{n}{2\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}}$。令$\frac{\partial ln(a,b)}{\partial a}=-2\sum\limits_{i=1}^{n}a+bx_i=0$，有$\hat{a}=-\hat{b}\bar{x}=-\bar{x}\sqrt{\frac{n}{2\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}}$。

42.由题意可知$X\sim Ge(p)$，所以$EX=p^{-1}$，$\mathrm{Var}(X)=p^{-2}(1-p)$。所以$p^{-1}=EX$，$p^{-2}=\mathrm{Var}(X)+p^{-1}=\mathrm{Var}(X)+EX$。由矩估计可得$\widehat{p}^{-1}=\overline{X}$，$\widehat{p}^{-2}=S^2+\overline{X}$。又$EX=E\overline{X}$，$ES^2=\mathrm{Var}(X)$，所以$\widehat{p}^{-1}$和$\widehat{p}^{-2}$是无偏估计。

44.(1)

$$EX=\int_{\theta}^{\infty}\frac{x}{\sigma}e^{-\frac{x-\theta}{\sigma}}dx=\int_{0}^{\infty}\frac{t+\theta}{\sigma}e^{-\frac{t}{\sigma}}dt=\sigma+\theta,$$

所以$\hat{\theta}_1=\bar{x}-\sigma$。因为

$$L(\theta)=\prod_{i=1}^{n}f(x_i)=\frac{1}{\sigma^n}e^{-\frac{\sum_{i=1}^{n}(x_i-\theta)}{\sigma}},\quad \theta<x_{(1)}\le...\le x_{(n)}.$$

显然，当θ增加时，$L(\theta)$也随之增加。所以$\hat{\theta}_2=X_{(1)}$。

(2)$E\hat{\theta}_1=E\bar{X}-\sigma=EX-\sigma=\theta$，所以$\hat{\theta}_1$是一个无偏估计。令$Y=X_{(1)}$，所以

$$P(Y\le y)=1-P(X_1>y,...,X_n>y)=1-\left(\int_{y}^{\infty}f(x)dx\right)^n=1-e^{-\frac{n(y-\theta)}{\sigma}},\quad y>\theta,$$

所以

$$EY=\int_{\theta}^{\infty}ydF(y)=\int_{\theta}^{\infty}\frac{ny}{\sigma}e^{-\frac{n(y-\theta)}{\sigma}}dy=\int_{0}^{\infty}(\theta+\frac{t}{n})\frac{1}{\sigma}e^{-\frac{t}{\sigma}}dt=\theta+\frac{\sigma}{n},$$

所以$\hat{\theta}_2$不是无偏估计，可修正为$\widetilde{\theta}_2=X_{(1)}-\frac{\sigma}{n}$。

(3)

$$EX^2=\int_{\theta}^{\infty}\frac{x^2}{\sigma}e^{-\frac{x-\theta}{\sigma}}dx=\int_{0}^{\theta}\frac{(t+\theta)^2}{\sigma}e^{-\frac{t}{\sigma}}dt=2\sigma^2+2\theta\sigma+\theta^2,$$

所以

$$\mathrm{Var}(\hat{\theta}_1)=\frac{1}{n}\mathrm{Var}(X)=\frac{1}{n}\left(EX^2-(EX)^2\right)=\frac{\sigma^2}{n}.$$

因为

$$EY^2=\int_{\theta}^{\infty}\frac{ny^2}{\sigma}e^{-\frac{n(y-\theta)}{\sigma}}dy=\int_{0}^{\infty}(\theta+\frac{t}{n})^2\frac{1}{\sigma}e^{-\frac{t}{\sigma}}dt=\theta^2+\frac{2\theta\sigma}{n}+\frac{2\sigma^2}{n^2},$$

所以

$$\mathrm{Var}(\widetilde{\theta}_2)=\mathrm{Var}(Y)=EY^2-(EY)^2=\frac{\sigma^2}{n^2}<\frac{\sigma^2}{n},$$

所以$\widetilde{\theta}_2$更优。

Week11

May 28, 2023

49. $X_1, X_2, \ldots, X_n$的联合密度函数为

$$f(X_1, X_2, \ldots, X_n) = (1/\theta)^n I(\theta \leq X_{(1)} \leq X_{(n)} \leq 2\theta)$$

,所以$\theta \geq min\{X_{(1)}, \frac{X_{(n)}}{2}\}$.
当$\hat{\theta}_M = \frac{X_{(n)}}{2}$时上式取到极大值，所以$\hat{\theta}_M$即为极大似然估计。因为$X_{(n)}$的密度函数为$f_n(x) = \frac{n(x-\theta)^{n-1}}{\theta^n}$所以

$$E(X_{(n)}) = \int_\theta^{2\theta} x \frac{n(x-\theta)^{n-1}}{\theta^n} dx = \frac{2n+1}{n+1}\theta$$

所以$E(\hat{\theta}_M) = \frac{2n+1}{2n+2}\theta$, 不是无偏估计。可修正为$\tilde{\theta}_M = \frac{2n+2}{2n+1}\hat{\theta}_M = \frac{n+1}{2n+1}X_{(n)}$。

50. 由题意可知, 对数似然函数为

$$\begin{aligned}
&\ln L\left(\mu_1, \mu_2, \sigma^2; X_1, \ldots, X_m, Y_1, \ldots, Y_n\right) \\
=&-\frac{1}{2\sigma^2} \sum_{i=1}^m \left(x_i - \mu_1\right)^2 - \frac{m}{2} \ln \sigma^2 - \frac{m}{2} \ln 2\pi - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(y_i - \mu_2\right)^2 - \frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln 2\pi \\
=&-\frac{1}{2\sigma^2} \sum_{i=1}^m \left(x_i - \mu_1\right)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(y_i - \mu_2\right)^2 - \frac{m+n}{2} \ln \sigma^2 + C.
\end{aligned}$$

$$\text{令} \frac{\partial L}{\partial \mu_1} = -\frac{1}{2\sigma^2} \left(\sum_{i=1}^m 2\left(\mu_1 - x_i\right)\right) = 0$$

$$\frac{\partial L}{\partial \mu_2} = -\frac{1}{2\sigma^2} \left(2\sum_{j=1}^n \left(\mu_2 - y_j\right)\right) = 0$$

所以

$$\widehat{\mu}_{1M} = \bar{x}, \quad \widehat{\mu}_{2M} = \bar{y}$$

将σ^2 看成参数，令

$$\frac{\partial L}{\partial \sigma^2} = -\frac{m+n}{2\sigma^2} + \frac{1}{2\sigma^4}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{j=1}^{n}(y_j-\mu_2)^2\right) = 0$$

则$\widehat{\sigma}^2_{MLE} = \frac{1}{m+n}\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{j=1}^{n}(y_j-\mu_2)^2\right)$. 又因为$\mu_1,\mu_2$ 末知, 所以将$\widehat{\mu}_{1M},\widehat{\mu}_{2M}$ 代入上式, 得到σ^2 的极大似然估计

$$\widehat{\sigma}^2_{MLE} = \frac{1}{m+n}\left(\sum_{i=1}^{m}(x_i-\bar{x})^2 + \sum_{j=1}^{n}(y_j-\bar{y})^2\right).$$

52. (1) 由题意知, 似然函数为

$$L(\mu) = \prod_{i=1}^{n} e^{-(x_i-\mu)} I(x_i \geq \mu) = e^{-\sum_{i=1}^{n}(x_i-\mu)} I\left(x_{(1)} \geq \mu\right),$$

要使$L(\mu)$ 达到最大, 首先示性函数取值应为1 , 其次$e^{-\sum_{i=1}^{n}(x_i-\mu)}$ 尽可能大, 所以mu 取值应尽可能大, 但示性函数为1 确定了$\mu \leq x_{(1)}$, 由此μ 的极大似然估计$\hat{\mu}^* = X_{(1)}$. 由最小值的分布结论可知, $X_{(1)}$ 的密度函数为

$$f_1(x) = \begin{cases} n(1-F(x))^{n-1}f(x) = ne^{-n(x-\mu)}, & x \geq \mu, \\ 0, & \text{其他}. \end{cases}$$

$$E\left(X_{(1)}\right) = \int_{\mu}^{\infty} x \cdot ne^{-n(x-\mu)}dx = \int_{0}^{\infty}(y+\mu)\cdot ne^{-ny}dy = \mu + \frac{1}{n}$$

所以$\hat{\mu}^* = X_{(1)}$ 不是μ 的无偏估计. 修正之后的无偏估计$\hat{\mu}^{**} = X_{(1)} - \frac{1}{n}$.
(2)

$$E(X) = \int_{\mu}^{\infty} x \cdot e^{-(x-\mu)}dx = \int_{0}^{\infty}(y+\mu)\cdot e^{-y}dy = \mu + 1.$$

记$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, 所以$\mu$ 的矩估计$\hat{\mu} = \bar{X} - 1$, , 且

$$E(\hat{\mu}) = E(\bar{X}) - 1 = E(X) - 1 = \mu,$$

$\hat{\mu}$ 是μ 的无偏估计.

(3) $\hat{\mu}^{**}$ 及$\hat{\mu}$ 都是μ 的无偏估计, 比较两者方差

$$\begin{aligned}
\operatorname{Var}\left(\hat{\mu}^{**}\right) &= \operatorname{Var}\left(X_{(1)}-\frac{1}{n}\right)=\operatorname{Var}\left(X_{(1)}\right) \\
&= \int_{\mu}^{\infty} x^{2} \cdot n e^{-n(x-\mu)} d x-\left(\mu+\frac{1}{n}\right)^{2}=\frac{1}{n^{2}} \\
\operatorname{Var}(\hat{\mu}) &= \operatorname{Var}(\bar{X}-1)=\operatorname{Var}(\bar{X})=\frac{1}{n} \operatorname{Var}(X) \\
&= \frac{1}{n}\left[\int_{\mu}^{\infty} x^{2} \cdot e^{-(x-\mu)} d x-(\mu+1)^{2}\right]=\frac{1}{n}
\end{aligned}$$

所以$\hat{\mu}^{**}$ 更有效.

54.
法一: 由Jensen 不等式有,

$$E\left(\frac{1}{\bar{X}}\right) \geq \frac{1}{E(\bar{X})}=\lambda, \text{ 等号当且仅当} P(\bar{X}=E(\bar{X}))=1 \text{ 时成立.}$$

所以$1/\bar{X}$ 不是λ 的无偏估计. Jensen 不等式: 如果$\varphi(\cdot)$ 是凸函数, X 是随机变量, 则:

$$E[\varphi(X)] \geq \varphi[E(X)]$$

若$\varphi(\cdot)$ 严格凸, 则等号当且仅当$P(X=E(X))=1$ 时成立.
法二: 由指数分布与伽玛分布的关系知: $X_1, X_2, \cdots, X_n i.i.d. \sim \operatorname{Exp}(\lambda)=\operatorname{Gamma}(1, \lambda)$, 再由伽玛分布的可加性, 知

$$\begin{aligned}
& Y:=X_1+X_2+\cdots+X_n \sim \operatorname{Gamma}(n, \lambda), \text{ i.e.,} \\
& f_Y(y)= \begin{cases}\frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}, & y \geq 0, \\ 0, & y<0.\end{cases} \\
& E\left(\frac{1}{\bar{X}}\right)=n E\left(\frac{1}{Y}\right)=n \int_0^{\infty} \frac{1}{y} \cdot \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} d y=\frac{n}{n-1} \lambda
\end{aligned}$$

所以$1/\bar{X}$ 不是λ 的无偏估计. 伽玛分布: $X \sim \operatorname{Gamma}(\alpha, \lambda)$, 其中$\alpha>0$ 为形状参数, $\lambda>0$ 为尺度参数. 密度函数为

$$f_X(x)= \begin{cases}\frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, & x \geq 0, \\ 0, & x<0.\end{cases}$$

利用卷积公式易证其可加性: $U \sim \operatorname{Gamma}\left(\alpha_1, \lambda\right), V \sim \operatorname{Gamma}\left(\alpha_2, \lambda\right)$, 且$U, V$ 相互独立, 则$U+V \sim \operatorname{Gamma}\left(\alpha_1+\alpha_2, \lambda\right)$.

56. 由题意知, 对于总体X,

$$E(X)=\theta+3 \theta+3(1-3 \theta)=3-5 \theta,$$

θ 的矩估计$\hat{\theta}_M=\frac{3-\bar{X}}{5}$. 记$n_i$ 为样本取到i 的次数, 且$n=\sum_{i=0}^{3} n_i$. 则似然函数, 对数似然为

$$
\begin{aligned}
& L(\theta)=\left(\frac{\theta}{2}\right)^{n_0} \theta^{n_1}\left(\frac{3 \theta}{2}\right)^{n_2}(1-3 \theta)^{n_3} \\
& \ln L(\theta)=n_0(\ln \theta-\ln 2)+n_1 \ln \theta+n_2(\ln \theta+\ln (3 / 2))+n_3 \ln (1-3 \theta) \\
& \frac{\partial \ln L(\theta)}{\partial \theta}=\frac{n_0+n_1+n_2}{\theta}-\frac{3 n_3}{1-3 \theta}=0 \Rightarrow \theta_0=\frac{n_0+n_1+n_2}{3 n} \\
& \text { 且可验证} \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \mid \theta=\theta_0<0 .
\end{aligned}
$$

θ 的极大似然估计$\hat{\theta}_L=\frac{n_0+n_1+n_2}{3 n}=\frac{n-n_3}{3 n}$. 由观测值可计算具体的估计值: $\hat{\theta}_M=\frac{2}{5}, \hat{\theta}_L=\frac{4}{15}$.
(2) $E\left(\hat{\theta}_M\right)=\frac{3-E(\bar{X})}{5}=\theta, \hat{\theta}_M$ 显然无偏. 对于$\hat{\theta}_L=\frac{n-n_3}{3 n}$, 这里$n=10$, 且$n_3 \sim b(n, 1-3 \theta), E\left(n_3\right)=n(1-3 \theta)$.

$$
E\left(\hat{\theta}_L\right)=\frac{n-E\left(n_3\right)}{3 n}=\frac{n-n(1-3 \theta)}{3 n}=\theta .
$$

所以$\hat{\theta}_L$ 也是无偏估计.
(3) 对于$\hat{\theta}_M$,

$$
\begin{aligned}
\operatorname{Var}(X) & =E\left(X^2\right)-(E X)^2=9-20 \theta-(3-5 \theta)^2=10 \theta-25 \theta^2 \\
\operatorname{Var}\left(\hat{\theta}_M\right) & =\frac{1}{25} \operatorname{Var}(\bar{X})=\frac{1}{25 n} \operatorname{Var}(X)=\frac{\theta(2-5 \theta)}{5 n}
\end{aligned}
$$

对于$\hat{\theta}_L$, 由$n_3 \sim b(n, 1-3 \theta)$

$$
\begin{aligned}
& \operatorname{Var}\left(n_3\right)=3 n \theta(1-3 \theta) \\
& \operatorname{Var}\left(\hat{\theta}_L\right)=\frac{1}{(3 n)^2} \operatorname{Var}\left(n_3\right)=\frac{\theta(1-3 \theta)}{3 n}
\end{aligned}
$$

比较两者方差, 可以发现$\hat{\theta}_L$ 更有效.

Week13

June 21, 2023

3. (1) 犯第一类错误的概率为

$$\alpha_1 = P\left(4|\bar{X}| \geq u_{0.05} \mid H_0\right) = 2\left(1-\Phi\left(u_{0.05}\right)\right) = 0.1$$
$$\alpha_2 = P\left(4|\bar{X}| \geq u_{0.45} \mid H_0\right) = 2\Phi\left(u_{0.45}\right) - 1 = 0.1$$
$$\alpha_3 = P\left(4\bar{X} \geq u_{0.10} \mid H_0\right) = 1-\Phi\left(u_{0.10}\right) = 0.1$$
$$\alpha_4 = P\left(4\bar{X} \leq -u_{0.10} \mid H_0\right) = 1-\Phi\left(u_{0.10}\right) = 0.1$$

(2) 犯第二类错误的概率为

$$\beta_1 = P\left(4|\bar{X}| < u_{0.05} \mid H_1\right) = P\left(-u_{0.05} - 4 < 4\bar{X} - 4 < u_{0.05} - 4\right) = 0.009258$$
$$\beta_2 = P\left(4|\bar{X}| > u_{0.45} \mid H_1\right) = P\left(\bar{X} - 4 > u_{0.45} - 4\right) + P\left(\bar{X} - 4 < -u_{0.45} - 4\right) = 0.9999$$
$$\beta_3 = P\left(4\bar{X} < u_{0.10} \mid H_1\right) = \Phi\left(u_{0.10} - 4\right) = 0.003279$$
$$\beta_4 = P\left(4\bar{X} > -u_{0.10} \mid H_1\right) = 1-\Phi\left(-u_{0.10} - 4\right) = 0.99999$$

所以V_3 的二类错误最小。

4. 由题意知, 要求的值为

$$\alpha = P\left(X > 1/2 \mid H_0\right) = \int_{1/2}^{1} 6x^5 dx = \frac{63}{64}$$
$$\beta = P\left(X \leq 1/2 \mid H_1\right) = \int_{0}^{1/2} 4x^3 dx = \frac{1}{16}$$
$$g(\theta) = P(X > 1/2 \mid \theta = 2) \int_{1/2}^{1} 3x^2 dx = \frac{7}{8}$$

7. (1) 检验$H_0 : \mu = 7 \leftrightarrow H_1 : \mu \neq 7$. 方差已知, 检验统计量为$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim$

$N(0,1)$, 所以拒绝域为$\left\{\left|\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| \geq u_{\frac{\alpha}{2}}\right\}$. 计算得

$$\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| = 1.739 < u_{0.025} = 1.96$$

由此可见样本末落入拒绝域内, 所以接受原假设。

(2) 检验$H_0 : \mu \geq 7 \leftrightarrow H_1 : \mu < 7$, 检验统计量同上, 拒绝域为$\left\{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq u_{1-\alpha}\right\}$. 计算得

$$\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = -1.739 < u_{0.95} = -1.645$$

由此可见样本落在拒绝域内, 所以拒绝原假设。

(3) 检验$H_0 : \mu \leq 7 \leftrightarrow H_1 : \mu > 7$, 拒绝域为$\left\{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \geq u_\alpha\right\}$. 计算得

$$\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = 2.2136 > u_{0.05} = 1.645$$

样本落在拒绝域内, 所以拒绝原假设。

8. $H_0 : \mu > 1550 \leftrightarrow H_1 : \mu \leq 1550$, 拒绝域为$\{\frac{\bar{X}-\mu}{S/\sqrt{n}} \leq -t_\alpha(n-1)\}$, 查表得$-t_\alpha(15) = -1.753$, 因为$\frac{\bar{X}-\mu_0}{S/\sqrt{n}} = 13.33 > -1.753$ 末落入拒绝域内, 所以接受原假设。

9. $H_0 : \mu = 105.2 \leftrightarrow H_1 : \mu \neq 105.2$, 拒绝域为$\left\{\left|\frac{\bar{X}-\mu}{S/\sqrt{n}}\right| \geq t_{\frac{\alpha}{2}}(n-1)\right\}$, 查表得$t_{0.025}(7) = 2.365$, 因为$\left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| = 0.01750 < 2.365$ 末落入拒绝域内, 所以接受原假设。

13 均值已知$\mu = 2.5, n = 4, \sigma_0 = 1, \alpha = 0.05$, $H_0 : \sigma^2 = 1 \leftrightarrow H_1 : \sigma^2 \neq 1$,检验统计量$nS_\mu^2/\sigma_0^2 = 3.42 \sim \chi_4^2$,拒绝域为$\{nS_\mu^2/\sigma_0^2 < \chi_4^2(1-\alpha/2)\}$或$\{nS_\mu^2/\sigma_0^2 > \chi_4^2(\alpha/2)\}$,$\chi_4^2(1-\alpha/2) = 0.484, \chi_4^2(\alpha/2) = 8.496, 0.484 \leq 3.42 \leq 8.496$,接受原假设。

15均值末知$\bar{x} = 3.2, n = 25, S = 0.031, \sigma_0 = 0.02, \alpha = 0.05$, $H_0 : \sigma = 0.02 \leftrightarrow H_1 : \sigma \neq 0.02$,检验统计量$(n-1)S^2/\sigma_0^2 = 57.66 \sim \chi_{24}^2$,拒绝域为$\{(n-$

$1)S^2/\sigma_0^2 < \chi^2_{24}(1-\alpha/2)\}$或$\{(n-1)S^2/\sigma_0^2 > \chi^2_{24}(\alpha/2)\}$,$\chi^2_{24}(1-\alpha/2) = 12.40, \chi^2_{24}(\alpha/2) = 34.57, 57.66 > 34.57$,拒绝原假设。

22设甲方法和乙方法的均值为μ_1, μ_2,$H_0 : \mu_2 - \mu_1 = 0 \leftrightarrow H_1 : \mu_2 - \mu_1 \neq 0$,方差未知但方差相同，$\bar{x} = 31.75, \bar{y} = 21.92, m = n = 12, \alpha = 0.05, S_w^2 = [(m-1)/S_1^2 + (n-1)S_2^2]/(n+m-2) = 12.326$,检验统计量$\sqrt{mn/(m+n)}(\bar{y}-\bar{x})/S_w = -2.572 \sim t_{m+n-2}, t_{22}(\alpha/2) = 2.074$,因为$-2.572 < -2.074$,拒绝原假设。

27(1)均值未知的两样本方差比检验：$H_0 : \sigma_2^2/\sigma_1^2 = 1 \leftrightarrow H_1 : \sigma_2^2/\sigma_1^2 \neq 1$,$m = 13, n = 10, \alpha = 0.1$,$S_2^2/S_1^2 = 0.776 \sim F_{n-1,m-1}, F_{9,12}(1-\alpha/2) = 0.0.325, F_{9,12}(\alpha/2) = 2.796$,接受原假设。

(2)$H_0 : \mu_2 - \mu_1 = 0 \leftrightarrow H_1 : \mu_2 - \mu_1 \neq 0$,方差未知但方差相同，$\bar{x} = 1.752, \bar{y} = 2.507, m = 13, n = 10, \alpha = 0.1, S_w^2 = [(m-1)/S_1^2 + (n-1)S_2^2]/(n+m-2) = 0.031$,检验统计量$\sqrt{mn/(m+n)}(\bar{y}-\bar{x})/S_w = 10.242 \sim t_{m+n-2}, t_{21}(\alpha/2) = 1.721$,因为$10.242 > 1.721$,拒绝原假设。

28. 成对数据检验: 令$Y_i = X_{1i} - X_{2i}$, 其中X_{1i} 为训练前体重, X_{2i} 为训练后体. 检验$H_0 : \mu \leq 8 \leftrightarrow H_1 : \mu > 8$. 方差末知, 在$\mu = \mu_0 = 8$ 时, 检验统计量$\frac{\bar{Y}-\mu}{S/\sqrt{n}} \sim t(n-1)$, 拒绝域为$\left\{\frac{\bar{Y}-\mu}{S/\sqrt{n}} \geq t_\alpha(n-1)\right\}$. 经计算,

$$\frac{\bar{Y}-\mu}{S/\sqrt{n}} = 0.1457 < t_{0.05}(8) = 1.860.$$

由此可见, 样本末落入拒绝域, 接受原假设H_0, 即认为该乐部的宣传不可信. 注: 根据原假设的提法原则, 此题应站在保护消费者的角度考虑原假设的取法. 若取原假设与备择假设为$H_0 : \mu \geq 8 \leftrightarrow H_1 : \mu < 8$, 则得到该乐部的宣传可信的结论. 所以上面的检验原假设的提法更可取.

8.(1)由题意可知，X服从对数似然分布，所以

$$EX=\int_0^\infty \frac{x}{\sqrt{2\pi}x}e^{-\frac{(\ln x-\mu)^2}{2}}dx=\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^te^{-\frac{(t-\mu)^2}{2}}dt=\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(t-\mu-1)^2-2\mu-1}{2}}dt=e^{\mu+\frac{1}{2}}.$$

所以$a=e^{\mu+\frac{1}{2}}$。

(2)因为$(\overline{Y}-\mu)/2\sim N(0,1)$，所以$\mu$的置信区间为$[\overline{Y}-u_{\alpha/2}/2,\overline{Y}+u_{\alpha/2}/2]$，代入样本数据，则:

μ的95%置信区间为$[-0.9222,1.0378]$，90%置信区间为$[-0.7646,0.8802]$。

(3)由(1)(2)可知，a的置信区间为$[\exp(\overline{Y}-u_{\alpha/2}/2+1/2),\exp(\overline{Y}+u_{\alpha/2}/2+1/2)]$，所以:

a的95%置信区间为$[0.6556,4.6542]$，90%置信区间为$[0.7675,3.9757]$。

9.(1)设包月用户的使用时间为X，由题意可知$n=900,\overline{X}=220,S_X=90$，所以$(\overline{X}-\mu_X)/(S_X/\sqrt{n})\sim t(n-1)$，所以平均使用时间的置信区间为

$$\left[\overline{X}-\frac{S_X}{\sqrt{n}}t_{\alpha/2}(n-1),\overline{X}+\frac{S_X}{\sqrt{n}}t_{\alpha/2}(n-1)\right],$$

所以95%置信区间为$[214.1122,225.8878]$。(2)同理可求按流量收费用户的平均使用时间为$[154.448,165.552]$。

13.由题意可知$(X-\mu)/\sigma\sim N(0,1)$，所以$(X-\mu)^2/\sigma^2\sim\chi^2(1)$,所以$\sum_{i=1}^4(X_i-2.5)^2/\sigma^2\sim\chi^2(4)$。所以置信区间为

$$\left[\frac{\sum_{i=1}^4(X_i-2.5)^2}{\chi^2_{\alpha/2}(4)},\frac{\sum_{i=1}^4(X_i-2.5)^2}{\chi^2_{1-\alpha/2}(4)}\right].$$

代入样本数据并查表可得，σ^2的95%置信区间为$[0.3069,7.0600]$。

16.(1)μ已知时，有$\frac{X_i-\mu}{\sigma}\sim N(0,1)$，

所以$\sum\limits_{i=1}^n\frac{(X_i-\mu)^2}{\sigma^2}\sim\chi^2(n)$，

置信区间为

$$\left[\frac{\sum\limits_{i=1}^n(X_i-\mu)^2}{\chi^2_{0.025}(n)},\quad \frac{\sum\limits_{i=1}^n(X_i-\mu)^2}{\chi^2_{0.975}(n)}\right],$$

置信区间为[0.14, 0.89]

(2)μ未知时，有$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$，

置信区间为

$$\left[\frac{(n-1)S^2}{\chi^2_{0.025}(n-1)},\quad \frac{(n-1)S^2}{\chi^2_{0.975}(n-1)}\right],$$

置信区间为[0.15, 1.05]。

22.(1)由题意可知，一件产品是否为次品服从参数为p的$0-1$分布，所以$\hat{p}=\overline{X}$。由中心极限定理可知，$(\overline{X}-p)/(p(1-p)/\sqrt{n})\sim N(0,1)$。因为n足够大，所以在这里我们忽略$u^2_{\alpha/2}/n$，置信区间为

$$\left[\hat{p}-u_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\hat{p}+u_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right],$$

所以95%的置信区间为$[0.0279,0.1221]$。
(2)置信上限为$\hat{p}+u_\alpha\sqrt{\hat{p}(1-\hat{p})/n}$，代入可得95%置信上限为0.1145。

29.总体分布为$N\left(\mu,\sigma^2\right)$, 对于样本均值$\overline{X}$, $\sqrt{n}(\overline{X}-\mu)/\sigma\sim N(0,1)$ 且$\mu=80$.

$$P(\overline{X}>75)=P\left(\frac{\overline{X}-\mu}{\sqrt{\sigma/n}}\geq\frac{75-\mu}{\sqrt{\sigma/n}}\right)=0.99,$$

得$\sqrt{n}(75-80)/5=-2.3263, n=5.4119$, 所以至少6 块试验田。

33.因为$\sqrt{n}(\overline{X}-\mu)/S\sim t(n-1)$，所以

$$P(\mu\geq\overline{X}-\frac{S}{\sqrt{n}}t_\alpha(n-1))=1-\alpha,$$

置信下限为41147.53。

34.(1) 由题意可得

$$P\left(\mu\geq\overline{X}-\frac{S}{\sqrt{n}}t_\alpha(n-1)\right)=1-\alpha,$$

所以置信下限为1593.4262 。
(2)同理

$$P\left(\sigma^2\leq\frac{(n-1)S^2}{\chi^2_{1-\alpha}(n-1)}\right)=1-\alpha,$$

置信上限为464.8120 。