Bekenstein-Hawking entropy

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The **Bekenstein-Hawking entropy** or **black hole entropy** is the amount of entropy that must be assigned to a black hole in order for it to comply with the laws of thermodynamics as they are interpreted by observers external to that black hole. This is particularly true for the first and second laws. Black hole entropy is a concept with geometric root but with many physical consequences. It ties together notions from gravitation, thermodynamics and quantum theory, and is thus regarded as a window into the as yet mostly hidden world of quantum gravity.

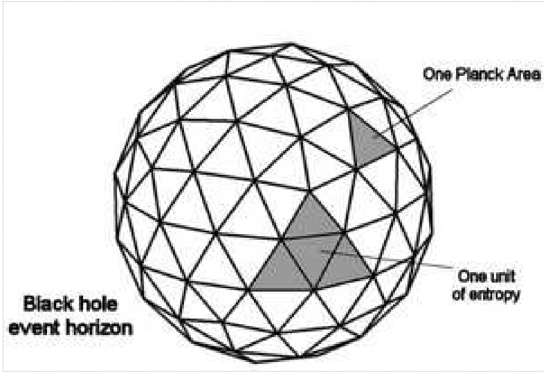

Figure 1: The Bekenstein-Hawking entropy is the entropy to be ascribed to any black hole: one quarter of its horizon area expressed in units of the Planck area [see equation (1)].

Contents

Why black hole entropy?

A black hole may be described as a blemish in spacetime, or a locale of very high curvature. Is it meaningful or desirable to associate entropy with it ? Is this possible at all ?

There are several ways to justify the concept of black hole entropy (Bekenstein 1972, 1973).

- A black hole is usually formed from the collapse of a quantity of matter or radiation, both of which carry entropy. However, the hole's interior and contents are veiled to an exterior observer. Thus a thermodynamic description of the collapse from that observer's viewpoint cannot be based on the entropy of that matter or radiation because these are unobservable. Associating entropy with the black hole provides a handle on the thermodynamics.
- A stationary black hole is parametrized by just a few numbers (Ruffini and Wheeler 1971): its mass, electric charge and angular momentum (and magnetic monopole charge, except its actual existence in nature has not been demonstrated yet). For any specific choice of these parameters one can imagine many scenarios for the black hole's formation. Thus there are many possible internal states corresponding to that black hole. In thermodynamics one meets a similar situation: many internal microstates of a system are all compatible with the one observed (macro)state. Thermodynamic entropy quantifies the said multiplicity. Thus by analogy one needs to associate entropy with a

Figure 2: Due to the disposition of the local light cone, the event horizon stops any signals bearing interior information from exiting the black hole. Only the hole's mass M , angular momentum J and electric charge Q are sensed by an exterior observer.

black hole .

- By blocking all signal travel through it, the event horizon prevents an external observer from receiving information about the black hole (save for the mentioned few parameters; see Figure 2). Thus a black hole can be said to hide information. In ordinary physics entropy is a measure of missing information. Hence it makes sense to attribute entropy to a black hole.

Formula for black hole entropy

How to express the black hole entropy in a concrete formula? It is clear at the outset that black hole entropy should only depend on the observable properties of the black hole: mass, electric charge and angular momentum. It turns out that these three parameters enter only in the same combination as that which represents the surface area of the black hole. One way to understand why is to recall the "area theorem" (Hawking 1971, Misner, Thorne and Wheeler 1973): *the event horizon area of a black hole cannot decrease; it increases in most transformations of the black hole.* This increasing behavior is reminiscent of thermodynamic entropy of closed systems. Thus it is reasonable that the black hole entropy should be a monotonic function of area, and it turns out to be simplest such function.

If A stands for the surface area of a black hole (area of the event horizon), then the black hole entropy, in dimensionless form, is given by

$$S_{BH} = \frac{A}{4L_P^2} = \frac{c^3 A}{4G\hbar}, \tag{1}$$

where L_P stands for the Planck length $G\hbar/c^3$ while $G, \hbar$ and c denote, respectively, Newton's gravity constant, the Planck-Dirac constant ($h/(2\pi)$) and the speed of light. Of course, if the entropy in the usual (chemist's) form is required, the above should be multiplied by Boltzmann's constant k .

For the spherically symmetric and stationary, or Schwarzschild, black hole (see Schwarzschild metric (http://en.wikipedia.org/wiki/Schwarzschild_Black_Hole)), the only parameter is the black hole's mass$M$, the horizon's radius is $r_h = 2GM/c^2$, and its area is naturally given by $4\pi r_h^2$, or

$$A = 16\pi(GM/c^2)^2\ . \tag{2}$$

Note that a one-solar mass Schwarzschild black hole has an horizon area of the same order as the municipal area of Atlanta or Chicago. Its entropy is about 4×10^{77} , which is about twenty orders of magnitude larger than the thermodynamic entropy of the sun. This observation underscores the fact that one should not think of black hole entropy as the entropy that fell into the black hole when it was formed.

For the most general type of stationary black hole, the Kerr-Newman black hole (rotating black hole (http://en.wikipedia.org/wiki/Kerr_Black_Hole)), the hole's parameters are mass M , electric charge Q and angular momentum J , and the horizon is no longer spherical. Nevertheless, in the popular Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$ (see Misner, Thorne and Wheeler 1973 or (http://en.wikipedia.org/wiki/Kerr-Newman_metric) Kerr-Newman metric) it lies at the fixed radial coordinate

$$r = r_h \equiv GM/c^2 + \sqrt{(GM/c^2)^2 - (G^{1/2}Q/c^2)^2 - (J/Mc)^2}. \tag{3}$$

Consequently the horizon area is given by

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{g_{\theta\theta}\, g_{\varphi\varphi}} = 4\pi(r_h^2 + (J/Mc)^2). \tag{4}$$

The first law of black hole thermodynamics

When near to equilibrium a thermodynamic system at temperature T changes its state, the consequent increments of its energy E and entropy S are related by the first law of thermodynamics:

$$TdS = dE - dW \tag{5}$$

Here dW is the work done on the system by exterior agents. When the system is one rotating with angular frequency Ω and charged up to electric potential Φ , the changes in its angular momentum J and charge Q contribute the work

$$dW = \Omega dJ + \Phi dQ \tag{6}$$

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to the above formula.
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A stationary black hole admits a similar relation (Bekenstein 1973). The differential dA from equation (4), when multiplied by a suitable factor, takes the form

$$\Theta\, dA = d(Mc^2) - \Omega_{BH}\, dJ - \Phi_{BH}\, dQ \tag{7}$$

where

$$\Theta \equiv c^4(2GA)^{-1}(r_h - GM/c^2); \tag{8}$$

$$\Omega_{BH} \equiv (J/Mc)\left(r_h^2 + (J/Mc)^2\right)^{-1}; \tag{9}$$

$$\Phi_{BH} \equiv Qr_h\left(r_h^2 + (J/Mc)^2\right)^{-1}. \tag{10}$$

Just now this is just a relation between increments in mechanical and geometrical properties. It turns out that Ω_{BH} is precisely the angular rotation frequency of the black hole in the sense that any test body dropped into it, as it approaches the horizon no matter where, ends up circumnavigating it at just this frequency. And Φ_{BH} turns out to be black hole's electric potential in the sense that it equals the line integral of the hole's electric field from infinity to any location on the horizon.

Because Mc^2 is the hole's energy, equation (7) obviously looks like the first law for an ordinary thermodynamic system. It will be the first law if black hole entropy is required to be a function of A and of nothing else, so that $dS_{BH} \propto dA$ (Gour and Mayo 2001). With the choice in equation (1) the black hole temperature T_{BH} must be

$$T_{BH} = 4L_P^2\Theta = \frac{\hbar c}{2\pi}\frac{\sqrt{(GM/c^2)^2 - (G^{1/2}Q^2/c^2)^2 - (J/Mc)^2}}{r_h^2 + (J/Mc)^2} \tag{11}$$

The reality of black hole temperature was brought home when Hawking showed (Hawking 1974, 1975) that a non-eternal black hole must spontaneously emit thermal radiation (Hawking radiation) with precisely this temperature (the original calculation was for $J = 0, Q = 0$, but it is now clear that equation (11) is valid for all J and Q). This discovery provided the calibration of the numerical factor in equation (1).

The generalized second law of thermodynamics

In ordinary thermodynamics the second law requires that the entropy of a closed system shall never decrease, and shall typically increase as a consequence of generic transformations. While this law may hold good for a system including a black hole, it is not informative in its original form. For example, if an ordinary system falls into a black hole, the ordinary entropy becomes invisible to an exterior observer, so from her viewpoint, saying that ordinary entropy increases does not provide any insight: the ordinary second law is transcended.

Including the black hole entropy in the entropy ledger gives a more useful law, the generalized second law of thermodynamics (GSL) (Bekenstein 1972, 1973, 1974): *the sum of ordinary entropy S_o outside black holes and the total black hole entropy never decreases and typically increases as a consequence of generic transformations of the black* *hole*. In equations

$$\Delta S_o + \Delta S_{BH} \geq 0. \tag{12}$$

The GSL extends the reach of the area theorem in both classical and quantum physics as follows:

When matter entropy flows into a black hole, the GSL demands that the increase in black hole entropy shall more than compensate for the disappearance of ordinary entropy from sight. This has been verified by examples (Bekenstein 1973).

During the process of Hawking radiation the black hole's area decreases (basically because of the decrement of black hole mass), in violation of the area theorem. This is known to reflect a failure of the energy condition (assumed by the theorem) as a result of the very quantum fluctuations which engender the radiation. The GSL predicts that the emergent Hawking radiation entropy shall more than compensate for the drop in black hole entropy. This has been verified amply (Bekenstein 1975, Hawking 1976), and stands as testament to the predictive power of the GSL which was formulated two years before the Hawking radiation was put in evidence.

Varied theoretical arguments have been given in support of the GSL (Frolov and Page 1993, Bombelli et al 1986)

Status of the third law of black hole thermodynamics

In ordinary thermodynamics the third law may be stated in two ways:

- Nernst-Simon statement: The entropy of a system at absolute zero temperature either vanishes or becomes independent of the intensive thermodynamic parameters, e.g. pressure, magnetic field, electric potential, etc.
- Unattainability statement: To bring a system to absolute zero temperature involves an infinite number of processes or steps.

From formula (11) it is clear that the black hole temperature vanishes when

$$(GM/c^2)^2 - (G^{1/2}Q^2/c^2)^2 - (J/Mc)^2 = 0. \tag{13}$$

Kerr-Newman black holes satisfying this condition are called *extreme*. From equations (1) and (3)-(4) it is clear that for $T_{BH} = 0$ the black hole entropy is not only nonvanishing, but depends on (J/Mc) . Now this last quantity is an analog of a thermodynamic intensive parameter. For example, it is directly related to the hole's angular velocity in equation (9), and angular velocity of a thermodynamic system is an intensive parameter. Thus the Nernst-Simon statement of the third law fails for black holes.

There is some evidence that the unattainability statement of the third law is satisfied by black holes. For example, in an astrophysical setting the process of spinning up a $Q = 0$ black hole gets "hung up" at $(J/Mc) \approx 0.998GM/c^2$, before the extremality condition can be satisfied (Thorne 1973).

What is behind the black hole entropy

In ordinary statistical mechanics, the entropy S is a measure of the multiplicity of microstates that hide behind one particular macrostate. A special case of this is Boltzmann's famous formula $S = \ln W$ where W stands for the number of equally probable microstates of a particular macrostate. Since black hole entropy plays a role quite analogous to that of ordinary entropy, e.g. it participates in the second law, many have wondered what the microstates that are counted by black hole entropy are. A partial list of interpretations of black hole entropy is the following:

- Black hole entropy counts the number of internal states of matter and gravity.

As mentioned earlier, the perception that a particular black hole (specific M, J and Q) can be formed in many ways originally suggested the notion of black hole entropy; in this approach the internal states of matter and/or gravity are the sought for microstates. Examples of this viewpoint are provided by Frolov and Novikov (1993) and by Mukhanov (2003).

- Black hole entropy is the entropy of entanglement between degrees of freedom inside and outside the horizon.

For a black hole formed by collapse, the quantum field degrees of freedom external to the horizon should be entangled with those inside it. To an external observer the last are not accessible, so the meaningful state would be the one from which internal degrees of freedom have been traced out (removed). Even if the global state was pure and thus entropy free, this so reduced state will be a mixed one and have entropy associated with it. Calculation (Bombelli et al 1986, Srednicki 1993) shows that this entanglement entropy is proportional to the horizon area, just as required to explain black hole entropy, but the coefficient is ultraviolet divergent and thus requires renormalization by physical arguments.

- Black hole entropy counts the number of horizon gravitational states.

It has been suggested that the sought for states are the states of the gravitational degrees of freedom residing on the black hole's horizon. An example of this approach is a calculation by Carlip (1999) based on the group of symmetries at the horizon. It reproduces formula (1).

- Black hole entropy is a conserved quantity connected with coordinate invariance of the gravitational action.

This abstract approach has been championed by Wald (1993) as a road to black hole entropy in more general theories of gravity than general relativity. Technically, black hole entropy is the Noether charge of the diffeomorphism symmetry. This notion reproduces formula (1) when the gravitational action is of first order in the curvature, but gives a modified formula for higher order gravity theories.

- Black hole entropy is thermal entropy of the gas of quanta constituting the thermal atmosphere of the black hole.

The atmosphere concept comes from Thorne and Zurek (1985) and 't Hooft (1985,1996); the last introduced the concept of a "brick wall" to keep the said atmosphere from contacting the horizon, and thus making the entropy infinite. This approach recovers the proportionality of entropy to horizon area, but the coefficient has to be chosen by hand.

- Black hole entropy counts the number of states or excitations of a fundamental string.

Strings in string theory have a variety of excitations, so there is a multitude of string states. Therefore, a string has entropy, which turns out to be proportional to its *mass*. This is quite in contrast with black hole entropy. However, an argument by Bowick, Smolin and Wijewardhana (1987) suggests that by *adiabatically* (i.e. sufficiently slowly) reducing the string coupling constant g , it is possible to shrink a black hole's size as well as to reduce its mass (while keeping its entropy constant) until eventually it gets to be the size of the string length scale l_s when the black hole should not be distinguishable from a string. At the corresponding value of g , string and black hole entropy are quite similar (see e.g. Zwiebach 2004). This has been taken to mean that there is a one-to-one correspondence between black hole and string states, where both entities have the same entropy (Susskind 1993). This picture has been corroborated in the context of five-dimensional extreme black holes (Strominger and Vafa 1996). Hence black hole entropy can be understood in terms of string entropy.

- Black hole entropy is equivalent to the thermal entropy of the radiation residing on the boundary of the spacetime containing the black hole.

The AdS/CFT correspondence is a mapping between gravitational degrees of freedom of a certain spacetime and the matter (or field) degrees of freedom residing on its boundary. In particular, certain string theories in five dimensional Anti-deSitter (AdS) spacetime are so mapped to conformal field theories on the corresponding spacetime's four-dimensional boundary which bears some resemblance to Minkowski spacetime (Maldacena et al. 1997). Witten (1998) has shown that the entropy of a black hole residing in the bulk Anti-deSitter spacetime equals that of thermal radiation of the fields residing on its boundary.

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See also

Black hole, Bekenstein bound, Hawking radiation

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