Numerical calculus

Evaluation: WR ex (just problems) 70% + Lab 30%

Chapter one

$\Delta x = x^* - \tilde{x}$ $\qquad$ δx = relative error

$|\Delta x|$ – abs. error $\qquad$ $\delta x = \frac{|\Delta x|}{|x^*|}$

$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$ (use this for proof)

1.2. Finite and divided diff.

$a_i = a + ih$, $i = \overline{0,m}$, $a, b \in \mathbb{R}^*$, $m \in \mathbb{N}^*$

$(\Delta_h f)(a_i) = f(a_{i+1}) - f(a_i)$ = finite diff of 1st order

$(\Delta_h^k f)(a_i) = (\Delta_h^{k-1} f)(a_{i+1}) - (\Delta_h^{k-1} f)(a_i)$ = k^{th} order finite diff

(ex) $h = 0.25$ $\quad a = 1$ $\quad a_i = a + ih$ $\quad i = \overline{0,4}$ $\quad f_0 = 0$, $f_1 = 2$, $f_2 = 6$, $f_3 = 14$, $f_4 = 17$

a	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
$a_0 = 1$	0	2	2	2	-11
$a_1 = 1.25$	2	4	4	-9	
$a_2 = 1.5$	6	8	-5		
$a_3 = 1.75$	14	3			
$a_4 = 2$	17				

$\Delta f(a_0) = f_1 - f_0$

$\Delta f(a_1) = f_2 - f_1$

Divided diff

x_i, $i = \overline{0,m}$

$(\mathcal{D} f)(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$: first order divided diff $:= [x_n, x_{n+1}; f]$

$(\mathcal{D}^k f)(x_n) = \frac{(\mathcal{D}^{k-1} f)(x_{n+1}) - (\mathcal{D}^{k-1} f)(x_n)}{x_{n+k} - x_n}$: k^{th} order divided diff

(ex)

x	f	$\mathcal{D} f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$x_0 = 0$	3	1	1	0
$x_1 = 1$	4	3	1	
$x_2 = 2$	7	6		
$x_3 = 4$	19			

$\mathcal{D} f(x_0) = \frac{f_1 - f_0}{x_1 - x_0} = \frac{4-3}{1-0} = 1$

$\mathcal{D} f(x_1) = \frac{f_2 - f_1}{x_2 - x_1} = \frac{7-4}{2-1} = 3$

$\mathcal{D}^2 f(x_0) = \frac{3-1}{2-0} = \frac{\mathcal{D} f(x_1) - \mathcal{D} f(x_0)}{x_{0+2} - x_0} = 1$

$\mathcal{D}^2 f(x_1) = \frac{6-3}{4-1} = 1$

$\mathcal{D}^3 f(x_0) = 0 = \frac{1-1}{4-0} = \frac{(\mathcal{D}^2 f)(x_1) - (\mathcal{D}^2 f)(x_0)}{x_3 - x_0}$

ex 17 x	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$x_0 = 2$	4	$\frac{8-4}{4-2} = 2$	$\frac{6-2}{6-2} = 1$	$\frac{2-1}{8-2} = \frac{1}{6}$
$x_1 = 4$	8	$\frac{20-8}{6-4} = 6$	$\frac{14-6}{8-4} = 2$	
$x_2 = 6$	20	$\frac{48-20}{8-6} = 14$		
$x_3 = 8$	48			

Chapter two = Polynomial interpolation =

2.1 Taylor interpolation

$x_0 \in [a,b]$, $f \in C^n[a,b]$

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$$

$f \approx T_n$ $\quad f(x) = T_n(x) + R_n(x)$

$$R_n(x) = \frac{(x_n - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \ \xi \text{ between } x \text{ and } x_0$$

ex 24 $\quad T_1(3) = ? \quad T_1(3) = \sum_{k=0}^{1} \frac{(3-1)^k}{k!} \cdot \left(\frac{1}{x}\right)^k$

$T_2(3) = ?$

$f(x) = 1/x$

$x_0 = 1$

2.2 Lagrange interpolation

x_i, $i = \overline{0,m}$ $\quad x_i \neq x_j$, $i \neq j$

$f : [a,b] \to \mathbb{R}$

$P(x_i) = f(x_i)$, $i = \overline{0,m}$

$L_m f$ - Lagrange interp. poly.

$(L_m f)(x_i) = f(x_i)$, $i = \overline{0,m}$

$$(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)$$

l_i - fundamental Lagrange poly

$u(x) = (x-x_0)(x-x_1)\dots(x-x_m) = \prod_{j=0}^{m}(x-x_j)$

$u_i(x) = \frac{u(x)}{x-x_i} = \prod_{\substack{j=0\\ j\neq i}}^{m}(x-x_j)$

$$l_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

Prop 29 $\quad l_i(x) = \frac{u(x)}{(x-x_i)u'(x_i)}$

$(fg)' = f'g + fg'$

$u(x) = (x-x_i)\,u_i(x)$

$u'(x) = u_i(x) + (x-x_j)\,u_i'(x)$

$u'(x_i) = u_i(x_i)$

$\left.\begin{aligned} l_i(x) &= \frac{u_i(x)}{u_i(x_i)} \\ u_i(x) &= \frac{u(x)}{x-x_i} \end{aligned}\right\} \Rightarrow \quad l_i(x) = \frac{u(x)}{(x-x_i)u_i(x_i)}$

$\Downarrow \quad l_i(x) = \frac{u(x)}{(x-x_i)u_i(x_i)}$

(Present applications ??)

↳ in last lectures

Lecture two

interpolation : in the interval $\qquad$ extrapolation : outside the interval

x_i, $i = \overline{0,m}$

$(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)$

$l_i(x) = \frac{u_i(x)}{u_i(x_i)}$

$u_i(x) = \frac{u(x)}{x-x_i}$

$u(x) = \prod_{i=0}^{m}(x-x_i)$

Lagrange poly

Example 2a) x_0, x_1

$f(x_0), f(x_1)$

$(m=1)$

$$(L_m f)(x) = (L_1 f)(x) = \sum_{i=0}^{1} \ell_i(x) f(x_i) = \ell_0(x) f(x_0) + \ell_1(x) f(x_1)$$

$$\ell_0(x) = \frac{u_0(x)}{u_0(x_0)} \qquad \ell_1(x) = \frac{u_1(x)}{u_1(x_1)} \Rightarrow \ell_0(x) = \frac{x-x_1}{x_0-x_1} \qquad \ell_1(x) = \frac{x-x_0}{x_1-x_0}$$

$u(x) = (x-x_0)(x-x_1)$

$u_0(x) = x - x_1 \; ; \quad u_1(x) = x - x_0$

$$\Downarrow$$

$$(L_1 f)(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

2b)

x	-1	0	3
$f(x)$	8	-2	4

$f(-0.5) = ?$

$\boxed{m=2}$

$x_0 = 1, \; x_1 = 0, \; x_2 = 3$

$f(x_0) = 8, \; f(x_1) = -2, \; f(x_2) = 4$

$$(L_2 f)(x) = \ell_0(x) f(x_0) + \ell_1(x) f(x_1) + \ell_2(x) f(x_2) = 8\ell_0(x) - 2\ell_1(x) + 4\ell_2(x)$$

$u(x) = (x-x_0)(x-x_1)(x-x_2)$

$u_0(x) = (x-x_1)(x-x_2) = x(x-3)$

$u_1(x) = (x-x_0)(x-x_2) = (x+1)(x-3)$

$u_2(x) = (x-x_0)(x-x_1) = (x+1)\,x$

$$\ell_0(x) = \frac{u_0(x)}{u_0(x_0)} = \frac{x(x-3)}{(-1)\cdot(-4)} = \frac{x(x-3)}{4}$$

$$\ell_1(x) = \frac{u_1(x)}{u_1(x_1)} = \frac{(x+1)(x-3)}{-3}$$

$$\ell_2(x) = \frac{u_2(x)}{u_2(x_2)} = \frac{(x+1)\,x}{12}$$

$$\Rightarrow (L_2 f)(x) = \frac{x(x-3)}{4}\cdot 8 + \frac{(x+1)(x-3)}{-3}\cdot(-2) + \frac{x(x+1)}{12}\cdot 4$$

$f \simeq L_2 f$

$\Rightarrow f(-0.5) \simeq (L_2 f)(-0.5) = \dots$ with (-0.5) replacing x $= 2.25$

The barycentric formula of Lagrange interpolation poly

(should be used in lab instead of the classical one)

$$(L_m f)(x) = \frac{\displaystyle\sum_{i=0}^{m} \frac{A_i f(x_i)}{x-x_i}}{\displaystyle\sum_{i=0}^{m} \frac{A_i}{x-x_i}}$$

$f \simeq L_m f$

$f = L_m f + (R_m f)$ → the reminder

$\alpha = \min\{x, x_0 \dots x_m\}$, $\beta = \max\{x, x_0, \dots x_m\}$

$f \in C^m[a,b]$, $f^{(m)}$ derivable, $\exists\, \xi \in (\alpha, \beta)$ s.t

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$$

Proof $\quad F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}$

$F(x) = 0$ (det. cu x in loc de z)

$F(x_i) = ?$ $i = \overline{0,m}$ $\qquad f = L_m f + R_m f$

$u(x_i) = 0$; $i = \overline{0,m}$ $\qquad R_m f = f - L_m f$

$(R_m f)(x_i) = f(x_i) - \underbrace{(L_m f)(x_i)}_{= f(x_i)} = 0$

$= f(x_i)$ (ceva legat de condition for Lagrange poly ?!)

$F(x) = 0$ and $F(x_i) = 0$ for $i = \overline{0,m}$ $\Rightarrow$ f has $m+2$ zeros

$\overset{\text{Rolle}}{\underset{\text{Th}}{\Longrightarrow}}$ F' has $m+1$ zeros $\Rightarrow \dots \Rightarrow F^{(m+1)}$ has one zero

$\Longrightarrow \exists\, \xi \in (\alpha, \beta)$ s.t $F^{(m+1)}(\xi) = 0$

$u^{(m+1)}(x) = \left((x-x_0)(x-x_1)\dots(x-x_m)\right)^{(m+1)} = (m+1)!$

$(R_m f)^{(m+1)}(x) = (f - L_m f)^{m+1}(x) = f^{(m+1)}(x) - \underbrace{(L_m f)}_{\text{poly of } m \text{ degree}}{}^{(m+1)}(x) = f^{(m+1)}(x)$

$= 0$

$\exists\, \xi \in (\alpha, \beta)$ s.t. $F^{(m+1)}(\xi) = 0$

$\begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & R_m f(x) \end{vmatrix} = 0 \Rightarrow (m+1)!\, R_m f(x) = u(x)\, f^{(m+1)}(\xi)$

$\Rightarrow R_m f(x) = \dfrac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$

$$|(R_m f)(x)| \le \frac{|u(x)|}{(m+1)!} \cdot \underbrace{\|f^{(m+1)}\|_\infty}_{\max |f^{(m+1)}(x)|\ x \in (a,b)}$$

Ex. 8 $\quad \sqrt{115}$ to be aprox. $\qquad f(x) = \sqrt{x}$

~~$x_0 = 4$, $x_1 = 9$, $x_2 = 25$~~

$x_0 = 100$
$x_1 = 121$
$x_2 = 144$

(look for values close to the given one ex 115) $\qquad \boxed{m = 2}$

$|(R_2 f)(x)| \le \dfrac{|u(x)|}{3!} \cdot \underbrace{\max_{x \in [100,144]} |f^{(3)}(x)|}_{= M_3 f}$

$M_n f = \max |f^{(n)}(x)|$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$

$M_3 f = \frac{3}{8} \cdot (10^2)^{-\frac{5}{2}}$

$$|(R_2 f)(x)| \le \frac{|u(x)|}{6} \cdot \frac{3}{8} 10^{-5}$$

$$|(R_2 f)(115)| \le \frac{|(115-100)(115-121)(115-144)|}{6} \cdot \frac{3}{8} \cdot 10^{-5}$$

Ex. 7 $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$

$\lg 76 = ?$

$x_0 = 20$

$x_1 = 15$

$x_2 = 18$

$x_3 = 16$

$76 = 4 \cdot 19 \to \lg 19 = ?$

↳ take nrs. around 19, for which we know lg

Lecture three

$P(x_i) = f(x_i)$, $i = \overline{0,m}$

The Aitken's algorithm

x_i, $i = \overline{0,m}$, $f : [a,b] \to \mathbb{R}$

$$|(R_m f)(x)| < \frac{|u(x)|}{(m+1)!} M_{m+1} f \le \varepsilon$$

$$M_{m+1} f = \max_{x \in [a,b]} |f^{(m+1)}(x)|$$

$$f_{i0} = f(x_i)$$

$$f_{i,j+1} = \frac{1}{x_i - x_j} \cdot \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}$$

x_0	f_{00}				
x_1	f_{10}	f_{11}			
x_2	f_{20}	f_{21}	f_{22}	.....	
⋮	⋮	⋮	⋮		
x_m	f_{m0}	f_{m1}	f_{m2}		f_{mm}

$$\underset{i=1\ j=0}{f_{11}} = \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = \frac{x_1 - x}{x_1 - x_0} \underbrace{f_{00}}_{f(x_0)} - \frac{(x_0 - x)}{x_1 - x_0} \underbrace{f_{10}}_{f(x_1)}$$

$f_{ii} = (L_i f)(x)$

$|f_{ii} - f_{i-1,i-1}| < \varepsilon$ | f_{ii} - aprox with precision ε

$|x_i - x| \le |x_j - x|$ if $i < j$ $i,j = \overline{1,m}$

ex 1 $\sqrt{115}$ $\varepsilon = 10^{-3}$ $f = \sqrt{x}$

$x_0 = 100$ $x_1 = 121$ $x_2 = 144$

x	$f(x)$		
121	11		
100	10	$f_{11} = 10.71$	
144	12	f_{21}	f_{22}

$$f_{11} = \frac{1}{x - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = \frac{1}{100 - 121} \cdot \begin{vmatrix} 11 & 121 - 115 \\ 10 & 100 - 115 \end{vmatrix}$$

$= 10.71$

Check $|f_{22} - f_{11}| < \varepsilon$

Newton interpolation polynomial

$x_i,\ i=\overline{0,m},\ f(x_i)$

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x-x_0)\dots(x-x_{i-1})(\mathcal{D}^i f)(x_0)$$

$f \simeq N_m f$

$f = N_m f + R_m f$

$$(N_i f)(x) = (N_{i-1} f)(x) + (x-x_0)\dots(x-x_{i-1})(\mathcal{D}^i f)(x_0)$$

ex 4) $L_2 f,\ f(x) = \sin \pi x \qquad x_0 = 0,\ x_1 = 1/6,\ x_2 = 1/2$

a) $(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)$

$l_i(x) = \frac{u_i(x)}{u_i(x_i)}$; $u_i(x) = \frac{u(x)}{x-x_i}$, $u(x) = \prod_{i=0}^{m} (x-x_i)$

$$(L_2 f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) = l_0(x)\cdot 0 + l_1(x)\cdot \tfrac{1}{2} + l_2(x)\cdot 1$$

$$l_0(x) = \frac{u_0(x)}{u_0(x_0)} = \frac{(x-1/6)(x-1/2)}{(-1/6)\cdot(-1/2)}$$

$u(x) = (x-x_0)(x-x_1)(x-x_2)$

$\quad = x\,(x-1/6)(x-\tfrac{1}{2})$

$$l_1(x) = \frac{u_1(x)}{u_1(x_1)} = \frac{x(x-1/2)}{1/6\cdot(1/6-\frac{1}{2})} \qquad l_2 = \frac{u_2(x)}{u_2(x_2)} = \frac{x(x-\frac{1}{6})}{1/2(\frac{1}{2}-\frac{1}{6})}$$

b) using Newton's form

$$(L_2 f)(x) = f(x_0) + \sum_{i=1}^{2} (x-x_0)\dots(x-x_{i-1})(\mathcal{D}^i f)(x_0) = f(x_0) + (x-x_0)(\mathcal{D} f)(x_0) + (x-x_0)(x-x_1)(\mathcal{D}^2 f)(x_0)$$

x	f	$\mathcal{D}f$	$\mathcal{D}^2 f$
$x_0=0$	0	3	-3
$x_1=1/6$	1/2	3/	
$x_2=1/2$	1		

$$\mathcal{D}f(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{1/2}{1/6} = 3$$

$$\mathcal{D}f(x_1) = \frac{1-1/2}{1/2-1/6} = \frac{3}{2}$$

$$\mathcal{D}^2 f(x_0) = \frac{3/2-3}{1/2} = -3$$

$$(L_2 f)(x) = 0 + x\cdot 3 + x(x-1/6)\cdot(-3)$$

Hermite interpolation

$x_i,\ i=\overline{0,m}$

$f^{(j)}(x_k),\ j=0,\dots,r_k \quad r_k\in N$

$n=m+r_0+r_1+\dots+r_m$

$p^{(j)}(x_k)=f^{(j)}(x_k),\ k=\overline{0,m};\ j=\overline{0,r_k}$

H_mf - poly of n-th degree

$$\boxed{\begin{aligned}&(H_mf)^{(j)}(x_k)=f^{(j)}(x_k),\ k=\overline{0,m}\quad j=\overline{0,r_k}\\&(H_mf)(x)=\sum_{k=0}^{m}\sum_{j=0}^{r_k}h_{kj}(x)\cdot f^{(j)}(x_k)\in P_n\end{aligned}}$$

$h_{kj}^{(p)}(x_\nu)=0,\ \nu\neq k$

$h_{kj}^{(p)}(x_k)=\delta_{jp}=\begin{cases}1 & j=p\\0 & j\neq p\end{cases}$

$$\boxed{\begin{aligned}&u(x)=\prod_{k=0}^{m}(x-x_k)^{r_k+1}\\&u_k(x)=\frac{u(x)}{(x-x_k)^{r_k+1}}\end{aligned}}$$

ex 7) $f(0)=1 \quad f'(0)=2 \quad f(1)=-3 \qquad H_2f\simeq f$

$x_0=0;\ r_0=1$ (max order of derivative for current x)

$x_1=1;\ r_1=0$

$m=1 \qquad n=m+r_0+r_1=2$

$(H_2f)(x)=\sum_{k=0}^{1}\sum_{j=0}^{r_k}h_{kj}(x)f^{(j)}(x_k)=h_{00}(x)f(x_0)+h_{01}(x)f'(x_0)+h_{10}(x)f(x_1)$

$=h_{00}(x)+2h_{01}(x)-3h_{10}(x)$

$h_{00}(x)=ax^2+bx+c$

$\begin{cases}h_{00}(x_0)=1\\h_{00}'(x_0)=0\\h_{10}(x_1)=0\end{cases} \Rightarrow \begin{aligned}&h_{00}(0)=c=1\\&h_{00}'(0)=b=0\\&h_{00}(1)=a+b+c=0\Rightarrow a=-1\end{aligned} \Rightarrow h_{00}(x)=-x^2+1$

$h_{01}(x)=ax^2+bx+c$

$\begin{cases}h_{01}(x_0)=0\Rightarrow c=0\\h_{01}'(x_0)=1\Rightarrow b=1\\h_{01}(x_1)=0\Rightarrow a=-1\end{cases}$

$\Rightarrow h_{01}(x)=-x^2+x$

$h_{10}(x)=ax^2+bx+c$

$\begin{cases}h_{10}(x_0)=0\Rightarrow c=0\\h_{10}'(x_0)=0\Rightarrow b=0\\h_{10}(x_1)=1\Rightarrow a+b+c=1\Rightarrow a=1\end{cases}$

$\Rightarrow h_{10}(x)=x^2$

$$\Rightarrow (H_2 f)(x) = (-x^2+1)\cdot 1 + (-x^2+x)\,2 + x^2 - 3$$
$$= -6x^2 + 2x + 1 \in \mathcal{P}_2$$

Lecture 4

$$(H_m f) = \sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \qquad n = m + r_0 + \dots + r_m$$

$f \simeq H_m f$, $f = H_m f + R_m f$

$f \in C^m[\alpha,\beta]$, $\alpha = \min\{x, x_0 \dots x_m\}$, $\beta = \max\{x_1, x_0 \dots x_m\}$ $\Rightarrow \exists\, \xi \in (\alpha,\beta)$ s.t

$$\boxed{(R_m f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi)}$$

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{(r_k+1)}$$

proof

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix} \qquad F(x) = 0,\ F^{(j)}(x_k) = ?$$

$u^{(j)}(x_k) = 0$, $j = \overline{0, r_k}$ $\quad u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1}$

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_m f)^{(j)}(x_k) = 0$$

$f = H_m f + R_m f \Rightarrow R_m f = f - H_m f$

$F^{(j)}(x_k) = 0$ for $k = \overline{0,m}$, $j = \overline{0, r_k}$ $\rightarrow$

$x_0 \rightarrow 0 \dots r_0 \Rightarrow r_0 + 1$

$x_1 \rightarrow 0 \dots r_1 \Rightarrow r_1 + 1$

- - - - - - -

$x_m \rightarrow 0 \dots r_m \Rightarrow r_m + 1$

$\Rightarrow \underbrace{r_0 + \dots + r_m + m}_{n} + 1$

$\Rightarrow n + 1 + 1 = n + 2$ zeros

Rolle Th $\Rightarrow$ F' has $n+1$ zeros $\rightarrow F^{(n+1)}$ has one zero $\rightarrow \exists\, \xi \in (\alpha,\beta)$ st $F^{(n+1)}(\xi) = 0$

$F^{(m+1)}(\xi) = ?$ $\qquad u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1}$

$u^{(r+1)}(z)$ $\qquad r_0 + 1 + r_1 + 1 + \dots + r_m + 1 = \underbrace{r_0 + \dots + r_m + m}_{n} + 1 = n + 1$

$$(R_m f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_m f)^{(n+1)}(z) = f^{(n+1)}(z) - 0 = f^{(n+1)}(z)$$

$\exists\, \xi$ s.t. $F^{(m+1)}(\xi) = 0$

$$\begin{vmatrix} (n+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0 \iff (n+1)!\,(R_m f)(x) = u(x) f^{(m+1)}(\xi)$$

$$\Rightarrow (R_m f)(x) = \frac{u(x) f^{(n+1)}(\xi)}{(n+1)!}$$

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \cdot M_{n+1} f \quad , \quad M_{n+1} f = \max_{x \in (\alpha, \beta)} |f^{n+1}(x)|$$

Hermite interpolation with double nodes

$x_0 \dots x_m$

$f(x_0) \dots f(x_m)$ $\quad (Hf)(x_k) = f(x_k)$

$f'(x_0) \dots f'(x_m)$ $\quad (Hf)'(x_k) = f'(x_k)$, $k = \overline{0, m}$

$r_0 = 1$

$r_1 = 1$ $\qquad n = m + r_0 + \dots + r_m = m + m + 1 = 2m+1$

$\dots\dots$ $\qquad$ degree of the poly. that meets the conditions

$r_m = 1$

$z_0 = x_0, \ z_1 = x_0, \ z_2 = x_1, \ z_3 = x_1, \ \dots \ z_{2m} = x_m, \ z_{2m+1} = x_m$

z	f				
z_0	$f(z_0)$	$f'(z_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D} f)(z_1)$	$(\mathcal{D}^2 f)(z_1)$		
z_2	$f(z_2)$	$f'(z_1)$			
z_3	$f(z_3)$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$			
z_m	$f(z_m)$	$f'(z_{2m})$			
z_{m+1}	$f(z_{m+1})$				

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0)(x - z_1) \dots (x - z_{i-1})(\mathcal{D}^i f)(z_0)$$

Example 6 $\quad x_0=-1\ ,\ x_1=1\ ,\ f(-1)=-3\ ,\ f'(-1)=10\ ,\ f(1)=1\ ,\ f'(1)=2$

$m=1\ ,\ z_0=x_0\ ,\ z_1=x_0\ ,\ z_2=x_1\ ,\ z_3=x_1$

$n=2m+1=3$

z	f	$\mathcal{D}f$	$\mathcal{D}^2f$	$\mathcal{D}^3f$
$z_0=-1$	-3	10	$\frac{2-10}{2}=-4$	$\frac{0-(-4)}{2}=2$
$z_1=-1$	-3	$\frac{1+3}{2}=2$	$\frac{2-2}{2}=0$	
$z_2=1$	1	2		
$z_3=1$	1			

$$(H_3f)(x)=f(z_0)+\sum_{i=1}^{3}(x-z_0)\dots(x-z_{i-1})(\mathcal{D}^if)(z_0)=$$

$$=f(z_0)+(x-z_0)\underbrace{(\mathcal{D}f)(z_0)}_{10}+(x-z_0)(x-z_1)\underbrace{(\mathcal{D}^2f)(z_0)}_{-4}+(x-z_0)(x-z_1)(x-z_2)\underbrace{(\mathcal{D}^3f)(z_0)}_{2}$$

$$=-3+(x+1)\cdot 10+(x+1)^2\cdot(-4)+(x+1)^2(x-1)\cdot 2=2x^3-2x^2+1$$

b)

classical method: $\quad m=1\ ,\ r_0=1\ ,\ r_1=1\ ,\ n=3 \qquad (H_3f)(x)=\sum_{k=0}^{m}\sum_{j=0}^{r_k}h_{kj}(x)f^{(j)}(x_k)$

$$\Rightarrow\ h_{00}(x)f(x_0)+h_{01}(x)f'(x_0)+h_{10}(x)f(x_1)+h_{11}(x)f'(x_1)$$

$h_{00}(x)=ax^3+bx^2+cx+d \quad\Rightarrow\quad h_{00}'(x)=3ax^2+2bx+c$

$$\begin{cases} h_{00}(x_0)=1 & \Rightarrow -a+b-c+d=1\\ h_{00}'(x_0)=0 & \Rightarrow 3a-2b+c=0\\ h_{00}(x_1)=0 & \Rightarrow a+b+c+d=0\\ h_{00}'(x_1)=0 & \Rightarrow 3a+2b+c=0 \end{cases}$$

. . . . (tbc as homework)

Birkhoff interpolation

$x_K \in [a,b]$, $K=\overline{0,m}$ $\quad I_K \in \{0,1,\dots r_K\}$; $r_K \in \mathbb{N}$

$j \in I_K$, $n = |I_0| + |I_1| + \dots + |I_m| - 1$

$$p^{(j)}(x_K) = f^{(j)}(x_K),\ K=\overline{0,m},\ j \in I_K$$

$f(x_0)$	$f(x_0)$	$f(x_0)$
$\cancel{f(x_1)}, f'(x_1)$	$f'(x_1)$	$f(x_1)$
$f(x_2)$ Hermite	$f(x_2)$ Birkhoff	$f(x_2)$ Lagrange

$P(x) = a_m x^n + \dots + a_0$

$p^{(j)}(x_K) = f^{(j)}(x_K)$, $K=\overline{0,m}$; $j \in I_K$ $\qquad b_{Kj}(x)$ - fundamental Birkhoff polynomial

$$\boxed{(B_m f)(x) = \sum_{K=0}^{m} \sum_{j \in I_K} b_{Kj}(x) f^{(j)}(x_K)}$$

$b_{Kj}^{(p)}(x_\upsilon) = 0 \quad K \neq \upsilon$

$b_{Kj}^{(p)}(x_K) = \delta_{jp} = \begin{cases} 1 & j=p \\ 0 & j=0 \end{cases}$

Example 11 $\quad x_0 = 0$, $x_1 = 1$, $f(0) = 1$, $f'(1) = \frac{1}{2}$ $\quad$ ($f(1)$ is missing, so we can't use Hermite)

$m = 1 \qquad n = 1+1-1 = 1 \Rightarrow$ poly of 1st degree

$I_0 = \{0\}$, $I_1 = \{1\}$

$P(x) = ax + b$

$\begin{cases} P(x_0) = f(x_0) \\ P'(x_1) = f'(x_1) \end{cases} \quad \begin{cases} b = f(x_0) \\ a = f'(x_1) \end{cases} \qquad \Delta = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow \exists$ sol for this problem

$$(B_1 f)(x) = \sum_{K=0}^{1} \sum_{j \in I_K} b_{Kj}(x) f^{(j)}(x_K) = b_{00} f(x_0) + b_{11}(x) f'(x_1)$$

$b_{00}(x) = ax + b = 1 \qquad\qquad b_{11}(x) = ax + b = x$

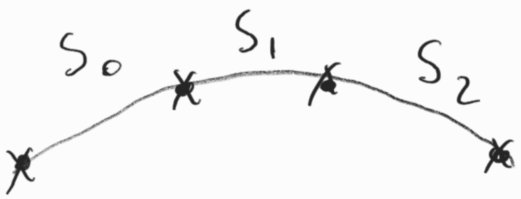
$\begin{cases} b_{00}(x_0) = 1 \\ b_{00}'(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases} \qquad \begin{cases} b_{11}(x_0) = 0 \Rightarrow b = 0 \\ b_{11}'(x_0) = 1 \Rightarrow a = 1 \end{cases}$

$$\Rightarrow (B_n f)(x) = f(x_0) + x f'(x_1) = 1 + \frac{1}{2} x$$

Lecture 5 Cubic spline interpolation

$a = x_0 < x_1 < \dots < x_n = b$

a) $S(x)$ is a cubic poly., $S_j(x), [x_j, x_{j+1}] \Rightarrow S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \\ \dots & \\ S_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$



b) $S_j(x_j) = f(x_j)$
$S_j(x_{j+1}) = f(x_{j+1})$, $(\forall)\ j = \overline{0, n-2}$

c) $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$; $(\forall)\ j = \overline{0, n-2}$

d) $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1})$

e) $S_j''(x_{j+1}) = S_{j+1}''(x_{j+1})$

i) $S''(x_0) = S''(x_n) = 0 \Rightarrow$ natural spline function

ii) $\left.\begin{matrix} S'(x_0) = f'(x_0) \\ S'(x_n) = f'(x_n) \end{matrix}\right\} \Rightarrow$ clamped spline function

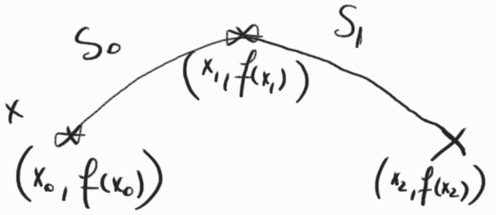
iii) $S_1(x) = S_2(x)$ and $S_{n-2} = S_{n-1} \Rightarrow$ de Boor spline function

Form of the cubic spline:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

example 9

Construct a cubic spline with the points:
$(1,2), (2,3), (3,5)$



(condition a))

$$S(x) = \begin{cases} S_0(x), & x \in [1,2] \\ S_1(x_1), & x \in (2,3] \end{cases} \qquad n = 2$$

$$S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3$$
$$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3$$

(condition b))
$$S_0(x_0) = f(x_0)$$
$$S_0(x_1) = f(x_1)$$
$$S_1(x_1) = f(x_1)$$
$$S_1(x_2) = f(x_2)$$

(cond. c)) $S_0(x_1) = S_1(x_1)$

(cond. d)) $S_0'(x_1) = S_1'(x_1)$

(cond. e)) $S_0''(x_1) = S_1''(x_1)$

(cond. i))
$$S_0''(x_0) = 0$$
$$S_1''(x_2) = 0$$

$$\begin{cases} a_0 = 2 \\ a_0 + b_0 + c_0 + d_0 = 3 \\ a_1 = 3 \\ a_1 + b_1 + c_1 + d_1 = 5 \\ a_0 + b_0 + c_0 + d_0 = a_1 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_0 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases}$$

(solve the system, find the 8 coeff. to obtain the 2 splines)

$$S'(0) = b_0 + 2c_0(x-x_0) + 3d_0(x-x_0)^2$$
$$S''(0) = 2c_0 + 6d_0(x-x_0)$$

Homework: example 10

Least squares approximation

$f(x_i),\ i = \overline{0,m}$

$$\varphi \approx f \quad \left(\sum_{i=0}^{m}[f(x_i) - \varphi(x_i)]^2\right)^{1/2} \longrightarrow \text{minimal}$$

$$\left(\int_a^b [f(x) - \varphi(x)]^2\,dx\right)^{1/2} \rightarrow \text{minimal}$$

$\varphi(x_i) = f(x_i),\ i = \overline{0,m}$

(the example has 5 points)

Liniar case: $\varphi(x) = ax + b$

$$E(a,b) = \sum_{i=0}^{4}[f(x_i) - \varphi(x_i)]^2$$

x	x_0	x_1	...	x_m
$f(x)$	$f(x_0)$	$f(x_1)$	...	$f(x_m)$

$f_i = f(x_i)$

$$E(a,b) = \sum_{i=0}^{m} \left[f(x_i) - (a x_i + b) \right]^2$$

$$\left\{ \begin{array}{l} \dfrac{\partial E}{\partial a} = 2 \sum_{i=0}^{m} \left[f(x_i) - (a x_i + b) \right] \cdot (-x_i) = 0 \\ \dfrac{\partial E}{\partial a} = -2 \sum_{i=0}^{m} \left[f(x_i) - (a x_i + b) \right] = 0 \end{array} \right\} \text{normal equations}$$

General case:

$(x_i, f(x_i))$ $i = \overline{0, m}$

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$f(x_i) = f_i$

$$E(a_0, a_1 \dots a_n) = \sum_{i=0}^{m} \left[f(x_i) - f(x_i) \right]^2 = \sum_{i=0}^{m} \left[f(x_i) - \sum_{i=0}^{m} a_k x_j^k \right]^2$$

$$\frac{\partial E}{\partial a_0} = 0$$

$$\dots$$

$$\frac{\partial E}{\partial a_n} = 0$$

$$\frac{\partial E}{\partial a_j} = -2 \sum_{i=0}^{m} x_i^j y_i + 2 \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{m} x_i^{j+k} \right) = 0, \quad j = \overline{0, n}$$

example 13

$m = 2$

x	-3	-1	2
$f(x)$	-4	-2	3

a) $f(x) = ax + b$

b) $P(x) = a_2 x^2 + a_1 x + a_0$

a) $$E(a,b) = \sum_{i=0}^{2} \left[f(x_i) - (a x_i + b) \right]^2$$

$$\frac{\partial E}{\partial a} = 2 \sum_{i=0}^{2} \left[f(x_i) - (a x_i + b) \right] (-x_i) = 0 \quad \Big| : (-2)$$

$$\frac{\partial E}{\partial b} = 2 \sum_{i=0}^{2} \left[f(x_i) - (a x_i + b) \right] \cdot (-1) = 0 \quad \Big| : (-2)$$

$$\left\{ \begin{array}{l} \left[-4 - (-3a + b) \right] (-3) + \left[-2 - (a(-1) + b) \right] \cdot (-1) + \left[3 - (2a + b) \right] \cdot 2 = 0 \\ (-4 + 3a - b) + (-2 + a - b) + (3 - 2a - b) = 0 \end{array} \right.$$

$\rightarrow$ find a & b from the syst.

b) $$\begin{cases} a_0 \sum_{i=0}^{2} x_i^0 + a_1 \sum_{i=0}^{m} x_j + a_2 \sum_{i=0}^{m} x_i^2 = \sum_{i=0}^{2} x_i^0 y_i \\ a_0 \cdot 3 + a_1(-3-1+2) + a_2(9+1+4) = (-4-2+3) \\ a_0 \cdot (-2) + a_1 \cdot 14 + a_2(-3^3 - 1 + 2^3) = \sum_{i=0}^{2} x_i y_i = (12+2+6) \\ a_0 \cdot 14 + a_1(-20) + a_2 \sum_{i=0}^{2} x_i^4 = \sum_{i=0}^{2} x_i^2 y_i = 9\cdot(-4)+(-2)+4\cdot 3 \end{cases}$$

solve system to find coeff. for the polynomial

Lecture 6

Numerical integration of functions

$f : [a,b] \to \mathbb{R}$ $\qquad \int_a^b f(x)\,dx = ?$

x_k, $k = \overline{0,m}$

$$\boxed{\int_a^b f(x)\,dx = \sum_{k=0}^{m} A_k f(x_k) + R(f)}$$

quadrature formula

A_k = coeff of the formula $\qquad x_k$ - nodes $\qquad R(f)$ - reminder

Degree of exactness is $r \Leftrightarrow R(f) = 0$, $f \in \mathbb{P}_r$ and
and $\exists$ g of $r+1$ degree st. $R(g) \neq 0$

$\Leftrightarrow$

$e_i(x) = x^i \qquad R(e_i) = 0$, $i = \overline{0,r}$ and $R(e_{r+1}) \neq 0$

Interpolatory quadrature formulas

$$f(x) = \sum_{k=0}^{m} l_k(x) f(x_k) + R_m f \quad \text{(Lagrange)}$$

$$\int_a^b f(x)\,dx = \sum_{k=0}^{m} \underbrace{\left(\int_a^b l_k(x)\,dx\right)}_{A_k} f(x_k) + \underbrace{\int_a^b R_m f(x)\,dx}_{R_m(f)}$$

(Usually:)

x_k, $k = \overline{0,m}$ equidistant $\Rightarrow$ Newton - Cotes formulas

$$\left.\begin{aligned} &m = 1 \qquad h = \frac{b-a}{m} \\ &x_k = a + kh, \quad h = b-a \\ &x_a = a, \quad x_1 = b \end{aligned}\right\} \Longrightarrow \begin{aligned} &f(x) = (L_1 f)(x) + (R_1 f)(x) \\ &x_0 = a \\ &x_1 = b \end{aligned}$$

$$(L_1 f)(x) = \frac{u_0(x)}{u_0(x_0)} f(x_0) + \frac{u_1(x)}{u_1(x_1)} f(x_1) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

$$u(x) = (x-a)(x-b)$$

$$A_0 = \int_a^b \frac{x-b}{a-b}\,dx = \frac{(x-b)^2}{(a-b)}\Big|_a^b = -\frac{(a-b)^2}{2(a-b)} = \frac{b-a}{2}$$

$$A_1 = \int_a^b \frac{x-a}{b-a}\,dx = \frac{(x-a)^2}{(b-a)}\Big|_a^b = \frac{(b-a)^2}{2(b-a)} = \frac{b-a}{2}$$

$$\Rightarrow \int_a^b f(x)\,dx = A_0 f(x_0) + A_1 f(x_1) + R_1(f) = \frac{b-a}{2}\Big[f(a) + f(b)\Big] + R_1(f)$$

$$(R_1 f)(x) = \frac{u(x)}{2} f''(\xi) = \frac{(x-a)(x-b)}{2} f''(\xi)$$

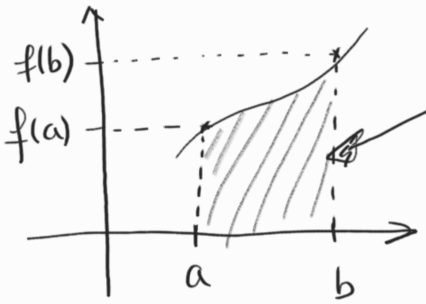
$$R_1(f) = f''(\xi) \int_a^b \frac{(x-a)(x-b)}{2}\,dx \quad ; \quad \xi \in (a,b)$$

$$= \frac{1}{2} f''(\xi) \int_a^b (x^2 - (a+b)x + ab)\,dx = \frac{1}{2} f''(\xi) \left[\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx\right]\Big|_a^b$$

$$= \frac{1}{2\cdot 6} f''(\xi)\left[2b^3 - 2a^3 - 3(a+b)(b^2-a^2) + 6ab(b-a)\right]$$

$$= \frac{1}{12} f''(\xi)\left(-(b-a)^3\right)$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right] - \frac{(b-a)^3}{12} f''(\xi) \longleftarrow \text{trapezian formula}$$



area of the trapezium

Example 7: $\int_1^3 (2x+1)\,dx = x^2 + x\Big|_1^3 = 9 + 3 - 1 - 1 = 10$

trapezium: $\int_1^3 2x+1\,dx = \frac{3-1}{2}\left(f(3) + f(1)\right) = 7 + 3 = 10$

$$m = 2 \qquad x_k = a + kh, \quad h = \frac{b-a}{2} \qquad x_0 = a, \quad x_1 = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$$x_2 = a + 2\cdot\frac{b-a}{2} = b$$

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + R_2(f)$$

$$R_2(f) = \frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad \xi \in (a,b)$$

degree of exactness $\begin{cases} m \\ m+1 \end{cases}$

$A_i = A_{m-i}$, $i = \overline{0,m}$ (coeff. are symetric)

example 11 $\int_0^2 x^2\,dx = \frac{8}{3} = 2.66$

trapezian : $\int_0^2 x^2\,dx = \frac{2-0}{2}(0+4) = 4$

Simpson : $\int_0^2 x^2\,dx \simeq \frac{2-0}{6}(0+4\cdot 1+4) = \frac{8}{3}$

Lecture seven

Repeated quadrature formulas

example 1 $I = \int_0^4 e^x\,dx \xrightarrow[\text{Simpson}]{\text{apply}} I \simeq \frac{4}{6}\left[e^0+4e^2+e^4\right] = 56.76$

$\xrightarrow[\text{Simpson}]{\text{apply}} I = \int_0^2 e^x\,dx + \int_2^4 e^x\,dx = \frac{2}{6}\left[e^0+4e+e^2\right] + \frac{2}{6}\left[e^2+4e^3+e^4\right]$

Applying the formula more times $\Rightarrow$ smaller error

$$|R_1(f)| = \frac{(b-a)^3}{12}|f''(\xi)| \geq \frac{(b-a)^3}{12} m_2 f$$

$$m_2 f = \min_{x\in[a,b]} |f''(x)|$$

$x_k = a + kh$ $k = \overline{0,n}$ $h = \frac{b-a}{n}$ (h = the step)

$\int_a^b f(x)\,dx = \sum_{k=1}^{n} I_k$, $I_k = \int_{x_{k-1}}^{x_k} f(x)\,dx$

$\int_a^b f(x)\,dx = \sum_{k=1}^{n}\left\{\frac{x_k - x_{k-1}}{2}\left[f(x_{k-1})+f(x_k)\right] - \frac{(x_k-x_{k-1})^3}{12} f''(\xi_k)\right\}$ $\xi_k \in [x_{k-1}, x_k]$

$= \frac{h}{2}\left[f(a) + f(x_1) +\right.$

$f(x_1) + f(x_2) +$

$f(x_2) + f(x_3) +$

$\dots\dots$

$\left. f(x_{k-1}) + f(b)\right] + \frac{h^3}{12} f''(\xi_k)$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{2n}\left[f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)\right] + R_n(f)$$

$R_n(f) = -\frac{(b-a)^3}{12n^3}\sum_{k=1}^{n} f''(\xi_k)$

$\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k) = f''(\xi)$

$\Rightarrow$ $R_n(f) = -\frac{(b-a)^3}{12n^2} f''(\xi)$, $\xi \in [a,b]$

$$|R_n(f)| \le \frac{(b-a)^3}{12} M_2 f \quad , \quad M_2 f = \max_{x\in[a,b]} |f''(x)|$$

$\frac{(b-a)^3}{12n^2} M_2 f < \varepsilon$ (solve this to get the integral with precision ε)

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \frac{x_k - x_{k-1}}{6}\left[f(x_{k-1}) + 4f\left(\frac{x_{k-1}+x_k}{2}\right) + f(x_k)\right] + R_n(f)$$

$$= \frac{b-a}{6n}\left\{f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k) + 4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\right\} + R_n(f)$$

$$R_n(f) = \frac{(b-a)^5}{2880n^4} f^{(4)}(\xi) \quad , \quad \xi\in[a,b]$$

$$|R_n(f)| \le \frac{b-a}{2880n^4} M_4 f \quad , \quad M_4 f = \max_{x\in[a,b]} |f^{4}(x)|$$

ex2 $\int_1^3 (2x+1)\,dx =$

$n=2$

$x_k = a+kh$ | $x_0 = a$ $\quad x_2 = b$

$k=\overline{0,n}$, $h = \frac{b-a}{n} = \frac{b-a}{2}$ | $x_1 = a + \frac{b-a}{2} = \frac{a+b}{2} = 2$

$$\int_1^3 (2x+1)\,dx \approx \frac{3-1}{4}\Big[\underbrace{(2\cdot1+1)}_{f(a)} + \underbrace{(2\cdot3+1)}_{f(b)} + 2f(x_1)\Big] = \frac{1}{2}\Big(3+7+2(2\cdot2+1)\Big) = \frac{20}{2} = 10$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

ex3 $\frac{\pi}{4}$, $\varepsilon = 10^{-2}$

$|R_n(f)| \le \frac{(b-a)^3}{12n^2} M_2 f < \varepsilon$ ← solve this to find n

$\frac{\pi}{4} = \text{arctg}(1) = \int_0^1 \frac{1}{1+x^2}\,dx \qquad f(x) = \frac{1}{1+x^2}$

$M_2 f = \max_{x\in[0,1]} |f''(x)| \qquad f'(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad M_2 f = 2$

$\frac{(b-a)^3}{12n^2} M_2 f < \varepsilon \Leftrightarrow \frac{1}{12n^2}\cdot 2 < \frac{1}{100} \Rightarrow \frac{100}{6} < n^2 \Rightarrow n=5$ (the first value that fulfills the inecuation)

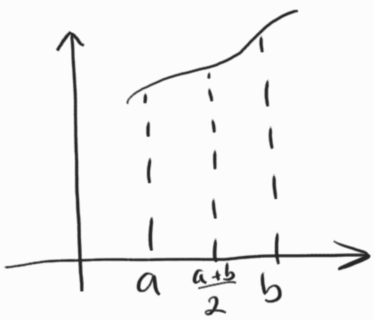
$x_k = a+kh$, $k=\overline{0,n}$, $h = \frac{b-a}{n} = \frac{1}{5}$

$x_0 = 0 \quad x_1 = \frac{1}{5} \quad x_2 = \frac{2}{5} \; , \; x_3 = \frac{3}{5} \; , \; x_4 = \frac{4}{5} \; , \; x_5 = 1$

$$\int_0^1 f(x)\,dx \simeq \frac{1}{10}\left[\frac{1}{1} + \frac{1}{2} + 2\left[\frac{1}{1+\frac{1}{25}} + \frac{1}{1+\frac{4}{25}} + \frac{1}{1+\frac{9}{25}} + \frac{1}{1+\frac{16}{25}}\right]\right] = \frac{1}{10}\left[1+\frac{1}{2}+2\left(\frac{25}{26}+\frac{25}{29}+\frac{25}{34}+\frac{25}{41}\right)\right]$$

$$\approx \frac{\pi}{4}$$

Romberg's iterative generation



$$Q_{T_0} = \frac{h}{2}\left[f(a)+f(b)\right]$$

$$Q_{T_1} = \frac{\frac{a+b}{2} - a}{2}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] + \frac{b - \frac{a+b}{2}}{2}\left[f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$Q_{T_K}(f) = \frac{1}{2} Q_{T_{K-1}} + \frac{h}{2^K} \sum_{j=1}^{2^{K-1}} f\left(a + \frac{2j-1}{2^K} h\right)$$

$Q_{T_0}, Q_{T_1} \dots Q_{T_K}$

$|Q_{T_K} - Q_{T_{K-1}}| < \varepsilon$

$|Q_{S_K} - Q_{S_{K-1}}| < \varepsilon$

$Q_{S_0} = \frac{h}{6}\left[f(a) + 4f(a+\frac{h}{2}) + f(b)\right]$

$Q_{S_K} = \frac{1}{3}\left[4 Q_{T_{K+1}} - Q_{T_K}\right]$

T_{00}

$T_{10} \quad T_{11}$

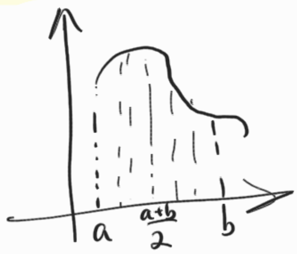
$- - - - -$

$T_{i0} \quad T_{i1} \dots T_{ii}$

$$T_{ij} = \frac{4^{-j} T_{i-1,j-1} - T_{i,j-1}}{4^{-j} - 1}$$

$|T_{ii} - T_{i-1,i-1}| \leq \varepsilon$

Adaptive quadrature methods



Lecture 8

General quadrature formulas

$$\int_a^b f(x)\,dx = \sum_{K=0}^{m} \sum_{j \in I_K} A_{Kj} f^{(j)}(x_K) + R(f)$$

$f = H_n f + R_n f$; $x_K, K = \overline{0,m}$

$r_0, r_1 \dots r_m \in \mathbb{N}$

$n = m + r_0 + r_1 + \dots + r_m$

$$(H_n f)(x) = \sum_{K=0}^{m} \sum_{j=0}^{r_K} h_{Kj} f^{(j)}(x_K)$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^b (H_n f)(x)\,dx + \underbrace{\int_a^b R(f)\,dx}_{R(f)}$$

$$\int_a^b f(x) \simeq \sum_{K=0}^{m} \sum_{j=0}^{r_K} \underbrace{\left(\int_a^b h_{Kj}(x)\,dx\right)}_{A_{Kj}} f^{(j)}(x_K)$$

example $f'(a), f(b) \Rightarrow$ Birkof interp. (because $f(a)$ is missing)

$x_0 = a \quad x_1 = b \qquad I_0 = \{1\} \to$ contains rank of derivative of x_0

$n = |I_0| + |I_1| - 1 \qquad I_1 = \{0\} \to$ — " — x_1

$n = 1 + 1 - 1 = 1 \Rightarrow$ poly of 1^{st} degree

$$(B_1 f)(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k) = \underline{b_{01}(x)}\, f'(x_0) + \underline{b_{10}(x)}\, f(x_1)$$

$\underline{b_{01}(x) = px + r = x - b}$

$$\begin{cases} b_{01}'(x_0) = 1 & p = 1 \\ b_{01}(x_1) = 0 & pb + r = 0 \Rightarrow r = -b \end{cases}$$

$\underline{b_{10}(x) = px + r = 1}$

$$\begin{cases} b_{10}'(x_0) = 0 & \Rightarrow p = 0 \\ b_{10}(x_1) = 1 & \Rightarrow px + r = 1 \Rightarrow r = 1 \end{cases}$$

$$\Rightarrow (B_1 f)(x) = (x-b) f'(a) + 1 \cdot f(b)$$

$f \simeq B_1 f$

$$\int_a^b f(x)\,dx \simeq \int_a^b (B_1 f)(x) = \int_a^b (x-b) f'(a)\,dx + \int_a^b f(b)\,dx$$

$$= \frac{(x-b)^2}{2}\Big|_a^b f'(a) + x\Big|_a^b f(b) = -\frac{(a-b)^2}{2} f'(a) + (b-a) f(b)$$

example 1 $\displaystyle\int_0^1 f(x)\,dx = Af(0) + Bf'(0) + Cf(1) + R(f)$ → Hermite interpolation

(HOMEWORK)

Quadrature formulas of Gauss type

$$\int_a^b f(x)\,dx = \sum_{k=1}^{m} A_k f(x_k) + R_m(f)$$

x_k and A_k are found s.t the formula will have max. degree of exactness

$x_k, A_k, k=\overline{1,m} \Rightarrow 2m$ unknowns

$e_k(x) = x^k$, $R(e_k)=0$ for $k=\overline{0,2m-1}$

$$\begin{cases} \sum_{k=1}^{m} A_k e_0(x_k) = A_0 e_0 = A_0 + A_1 + \dots + A_m = \int_a^b 1\,dx \\ \sum_{k=1}^{m} A_k e_1(x_k) = A_0 x_0 + A_1 x_1 + \dots + A_m x_m = \int_a^b x\,dx \\ \dots \\ \sum_{k=1}^{m} A_k e_{2m-1} = A_0 x_0^{2m-1} + A_1 x_1^{2m-1} + \dots + A_m x_m^{2m-1} = \int_a^b x^{2m-1}\,dx \end{cases}$$

$x_k, k=\overline{1,m}$ (are roots of Legendre poly.) $\quad u(x) = \frac{m!}{(2m)!}\cdot\left[(x-a)^m(x-b)^m\right]^{(m)}$

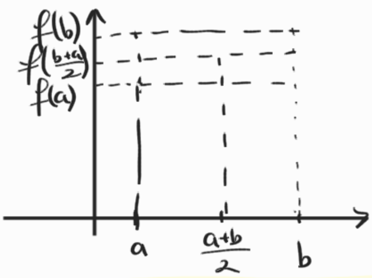
case $m=1$

$$\left.\begin{aligned} A_1 &= \int_a^b 1\,dx \Rightarrow A_1 = b-a \\ A_1 x_1 &= \int_a^b x\,dx \Rightarrow A_1 x_1 = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2} \end{aligned}\right\} \Rightarrow x_1 = \frac{a+b}{2}$$

$$\left.\begin{aligned} u(x) &= \frac{1}{2}\left((x-a)(x-b)\right)' = \frac{1}{2}\left(2x-(a+b)\right) = x - \frac{a+b}{2} \\ u(x) &= 0 \Rightarrow x_1 = \frac{a+b}{2} \end{aligned}\right\} \text{second way to find } x_1$$

$$\boxed{\int_a^b f(x)\,dx = A_1 f(x_1) + R_1(f) = (b-a) f\left(\frac{a+b}{2}\right) + R_1(f)} \leftarrow$$ rectangle formula (midpoint formula)

$$R_1(f) = \frac{(b-a)^2}{2} f'(\xi),\ \xi \in [a,b]$$



$$\left.\begin{aligned} \int_a^b f(x)\,dx &\simeq (b-a) f(a) \\ \int_a^b f(x)\,dx &\simeq (b-a) f(b) \end{aligned}\right\} \text{rectangle formulas}$$

$$\int_a^b f(x)\,dx = \frac{b-a}{n} \sum_{k=1}^{m} f(x_i) + R_n(f)$$ ← repeated rectangle formula

$x_1 = a + \frac{b-a}{2n}$ $\qquad x_i = x_1 + (i-1)\frac{b-a}{n}$

$$R_n(f) = \frac{(b-a)^3}{24 n^2} f''(\xi),\ \xi \in [a,b]$$

$$|R_n(f)| \leq \frac{(b-a)^3}{24n^2} M_2 f \quad ; \quad M_2 f = \max_{x\in[a,b]} |f''(x)|$$

$$\int_a^b f(x)\,dx \,,\ \mathcal{E} \Rightarrow \text{find } n \text{ s.t } \frac{(b-a)^3}{24n^2} M_2 f < \mathcal{E}$$

Romberg's alg. for the rectangle quadrature formula

$Q_{D_0}(f) = (b-a) f\left(\frac{a+b}{2}\right)$

$Q_{D_1}(f) = \dots\dots\dots$

$|Q_{D_m}(f) - Q_{D_{m-1}}(f)| < \mathcal{E}$

example 6 $\ln 2 = \int_1^2 \frac{1}{x}\,dx$, $\mathcal{E} = 10^{-2}$

$|R_n(f)| < \mathcal{E} \Rightarrow \frac{(b-a)^3}{24n^2} M_2 f < \mathcal{E}$, $a=1$ $b=2$

$\Rightarrow \frac{1}{24n^2} M_2 f < \frac{1}{100}$

$f'(x) = \frac{-1}{x^2}$ $\qquad M_2 f = \max_{x\in[1,2]} |f''(x)| = 2$

$f''(x) = \frac{2}{x^3}$

$\Rightarrow \frac{1}{\cancel{24}n^2} \cdot \cancel{2} < \frac{1}{100} \Rightarrow 100 < 12n^2 \Rightarrow \frac{100}{12} < n^2 \Rightarrow n = 3$

$x_1 = a + \frac{b-a}{2n} = 1 + \frac{1}{6} = \frac{7}{6}$

$x_2 = x_1 + \frac{(i-1)}{3} = \frac{7}{6} + \frac{1}{3} = \frac{9}{6} = \frac{3}{2}$

$x_3 = x_1 + \frac{3-1}{3} = \frac{7}{6} + \frac{2}{3} = \frac{11}{6}$

$\ln 2 = \int_1^2 \frac{1}{x}\,dx \simeq \frac{1}{3}(f(x_1) + f(x_2) + f(x_3)) = \frac{1}{3}\left(\frac{6}{7} + \frac{2}{3} + \frac{6}{11}\right) = 0.6897$

case m=2 (Gauss formula)

$$\int_a^b f(x)\,dx = A_1 f(x_1) + A_2 f(x_2) + R_2(f)$$

$$\begin{cases} A_1 + A_2 = \int_a^b 1\,dx \\ A_1 x_1 + A_2 x_2 = \int_a^b x\,dx \\ A_1 x_1^2 + A_2 x_2^2 = \int_a^b x^2\,dx \\ A_1 x_1^3 + A_2 x_2^3 = \int_a^b x^3\,dx \end{cases}$$

$u(x) = \frac{2}{4!}\left[(x-a)^2(x-b)^2\right]'' =$

$= x^2 -$

$u(x) = 0 \Rightarrow x_1 = \frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}$

$x_2 = \frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}$

$$\begin{cases} A_1 + A_2 = b-a \\ A_1 x_1 + A_2 x_2 = \frac{b^2-a^2}{2} \end{cases} \Rightarrow A_1 = A_2 = \frac{b-a}{2}$$

$t = \frac{2x - a - b}{b - a} \Rightarrow x = \frac{1}{2}\left((b-a)t + a + b\right)$

$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t + (b+a)}{2}\right) \cdot \frac{(b-a)}{2}\,dt$

Example 11

a) $m = 1 \Rightarrow$ trapezium form. $\frac{2}{2}[f(1) + f(3)]$

Lecture 9 Numerical Methods for Solving Linear Systems

- direct methods : provide the exact sol.
- iterative methods : provide aprox. of the sol
- semiiterative methods : — " ——

Perturbation of linear systems

$Ax = b$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$ → conditioning number

$\text{cond}(A) < 1000 \Rightarrow$ syst is good conditioned

$\text{cond}(A) > 1000 \Rightarrow$ syst. is ill / bad conditioned

$\text{cond}(A) \geq 1$ (always)

* $Ax = b$ $\qquad A(x + \delta x) = b + \delta b$

$Ax = b \Rightarrow Ax + A\delta x = b + \delta b \Rightarrow A\delta x = \delta b \Rightarrow \delta x = A^{-1}\delta b$

$\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta b\|$

$b = Ax \Rightarrow \|b\| \leq \|A\| \cdot \|x\| \Rightarrow \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|} \Bigg\} \Rightarrow$

$\Rightarrow \frac{\|x + \delta x - x\|}{\|x\|} \leq \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\delta b\|}{\|b\|} = \text{cond}(A) \cdot \frac{\|\delta b\|}{\|b\|}$

$(A + \delta A)(x + \delta x) = b \Leftrightarrow Ax + A\delta x + \delta A x + \delta A \cdot \delta x = b$

$\Rightarrow A\delta x = -\delta A(x + \delta x) \Rightarrow \delta x = -A^{-1}\delta A(x + \delta x)$

$\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta A\| \, \|x + \delta x\|$

$\Rightarrow \frac{\|\delta x\|}{\|x + \delta x\|} \leq \|A^{-1}\| \cdot \|\delta A\| = \underbrace{\|A\| \cdot \|A^{-1}\|}_{\text{cond}(A)} \cdot \frac{\|\delta A\|}{\|A\|}$

Direct methods for solving linear systems

Gauss method

$$Ax = b$$

example 6

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 3 & 1 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 31 \\ -2 \\ 18 \end{pmatrix}$$

$L_2 - ?\,L_1 \to L_2$ $\qquad L_2 - \frac{2}{3} L_1 \to L_2$

$2 - ? \cdot 3 = 0 \Rightarrow ? = \frac{2}{3}$

$L_3 - \frac{-1}{3} L_1 \to L_3$

$L_4 - \frac{1}{3} L_1 \to L_4$

$$\begin{cases} 3x_1 + x_2 + 7x_3 - 2x_4 = 18 \\ 2.33\, x_2 - 3.66\, x_3 + 6.33\, x_4 = 19 \\ -0.57\, x_3 - 1.28\, x_4 = -6.85 \\ 0.5\, x_4 = 2 \Rightarrow x_4 = 4 \end{cases}$$

$$x_3 = \frac{-6.85 + 1.28 \cdot 4}{-0.57} = 3$$

$x_2 = 2$

$x_1 = 1$

example 7

$$\begin{cases} 2x + y = 3 \\ 3x - 2y = 1 \end{cases} \qquad \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 1 \end{array}\right]$$

pivot
- largest $|\text{number}|$ from first line

$$\to \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 2 & 1 & 3 \end{array}\right]$$

$L_2 - ?\,L_1 \to L_2$

$\uparrow 2/3$

$$\to \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & \frac{7}{3} & \frac{7}{3} \end{array}\right] \Rightarrow \begin{cases} 3x_1 - 2x_2 = 1 \\ \frac{7}{3} x_2 = \frac{7}{3} \end{cases}$$

$2 - \frac{2}{3} \cdot 3 = 0$

$1 - \frac{2}{3}(-2) = \frac{7}{3}$

$3 - \frac{2}{3}(1) = \frac{7}{3}$

$\Rightarrow x_2 = 1$

$x_1 = \frac{1+2}{3} = 1$

A matrix A is strictly diagonally dominant if $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$.

Lecture 10 a) Factorization methods

= LU method =

$Ax = b \iff LUx = b \Rightarrow$ matrix A will be decomposed s.t $A = LU$ where

L - lower triangular matrix

U - upper triangular matrix

$$\begin{cases} Lz = b \\ Ux = z \end{cases}$$

I. Doolittle method

$$l_{i,k} = \frac{a_{i,k}^{(k-1)}}{a_{k,k}^{(k-1)}}, \quad k = \overline{1, n-1} \qquad t^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{n,k} \end{bmatrix}$$

k zero-s

$M_k = I_n - t^{(k)} e_k$ - Gauss matrix

$e_k = (0 \dots \underset{\uparrow}{1} \dots 0)$

k-th position

$$U = M_{n-1} \cdot M_{n-2} \cdot \dots \cdot M_2 \cdot M_1 \cdot A$$

$$L = M_1^{-1} \cdot M_2^{-1} \cdot \dots \cdot M_{n-1}^{-1}$$

$\boxed{\text{check } A = L \cdot U}$ (at the end)

$$\left.\begin{matrix} LUx = b \\ Ux = z \end{matrix}\right\} \quad \begin{matrix} Lz = b \\ Ux = z \end{matrix}$$

example 4 $\quad A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$ $\quad \begin{cases} 2x_1 + x_2 = 3 \\ 6x_1 + 8x_2 = 9 \end{cases} \Rightarrow \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$

$n = 2$ and $k = \overline{1, n-1} \Rightarrow k = 1$

$M_1 = I_2 - t^{(1)} e_1$

elements from matrix A

$$t^{(1)} = \begin{bmatrix} 0 \\ l_{2,1} \end{bmatrix} \qquad l_{2,1} = \frac{a_{2,1}}{a_{1,1}} = \frac{6}{2} = 3 \Rightarrow t^{(1)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$U = M_1 \cdot A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \leftarrow \text{upper triangular matrix}$$

$$L = M_1^{-1}$$

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} a & b \\ -3a+c & -3b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ -3a+c=0 \Rightarrow c=3 \\ -3b+d=1 \Rightarrow d=1 \end{cases}$$

$$L = M_1^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$\begin{cases} LUx = b \\ Ux = z \end{cases} \Rightarrow \begin{cases} Lz = b \\ Ux = z \end{cases}$$

1. $Lz = b \Rightarrow \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} z_1 \\ 3z_1+z_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \Rightarrow \begin{cases} z_1 = 3 \\ z_2 = 0 \end{cases} \Rightarrow z = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

2. $Ux = z \Rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x_1+x_2 \\ 5x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1+x_2=3 \\ 5x_2=0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 = \frac{3}{2} \end{cases}$

$$\Rightarrow x = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

b) Iterative methods

$$\boxed{Ax = b \Rightarrow x = \tilde{b} - Bx}$$

$x^{(0)} \leftarrow$ a vector with guesses of the solution

$x^{(0)}, x^{(1)}, x^{(2)} \dots x^{(K)}$

$$x^{(K)} = \tilde{b} - Bx^{(K-1)}$$

$$Ax = b$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 & \Rightarrow x_1 = \dots \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n & \Rightarrow x_n = \dots \end{cases}$$

$$\begin{cases} x_1 = u_{12}x_2 + \dots + u_{1n}x_n + c_1 \\ x_2 = u_{21}x_1 + \dots + u_{2n}x_n + c_2 \\ \\ x_n = u_{n1}x_1 + \dots + u_{n,n-1}x_{n-1} + c_n \end{cases}$$

$$u_{ij} = -\frac{a_{ij}}{a_{ii}}$$

$$c_i = \frac{b_i}{a_{ii}}$$

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{pmatrix} \qquad x_i^{(k)} = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}}$$

$x^{(0)}, x^{(1)}, \dots, x^{(k)}, \dots$

$x^{(k)}$ — approx. of the sol

$$\| x^{(k)} - x^{(k-1)} \| < \varepsilon \Rightarrow \frac{\| x^{(k)} - x^{(k-1)} \|}{\| x^{(k)} \|} < \varepsilon$$

example s

$$\begin{aligned} 7x_1 - 2x_2 + x_3 &= 17 \\ x_1 - 9x_2 + 3x_3 - x_4 &= 13 \\ 2x_1 + 10x_3 + x_4 &= 15 \\ x_1 - x_2 + x_3 + 6x_4 &= 10 \end{aligned} \qquad x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = \dfrac{17 + 2x_2^{(0)} - x_3^{(0)}}{7} = \dfrac{17}{7} \\ x_2^{(1)} = \dfrac{13 - x_1^{(0)} - 3x_3^{(0)} + x_4^{(0)}}{-9} = -\dfrac{13}{9} \end{cases} \qquad \begin{cases} x_3^{(1)} = \dfrac{15 - 2x_1^{(0)} - x_4^{(0)}}{10} = \dfrac{15}{10} = \dfrac{3}{2} \\ x_4^{(1)} = \dfrac{10 - x_1^{(0)} + x_2^{(0)} - x_3^{(0)}}{6} = \dfrac{5}{3} \end{cases}$$

Gauss - Seidel iterative method

$$\boxed{x_i^{(k)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}}}$$

Relaxation Method

$\omega \in (0, 2)$

$$\boxed{x_i^{(k)} = \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) + (1 - \omega) x_i^{(k-1)}}$$

$0 < \omega < 1 \rightarrow$ under relaxation $\qquad \omega > 1 \rightarrow$ overrelaxation

The matriceal form of the iterative methods

$A = D + L + U \qquad (D + L + U)x = b$

Jacobi method: $Dx^{(k)} = -(L + U)x^{(k-1)} + b$

Gauss - Seidel method: $(D + L)x^{(k)} = -Ux^{(k-1)} + b$

Relaxation method: $(D + \omega L)x^{(k)} = ((1 - \omega)D - \omega U)x^{(k-1)} + \omega b$

example 13

$$\begin{cases} 4x_1 + x_2 = 3 \\ 2x_1 + 5x_2 = 1 \end{cases} \qquad x^{(0)} = \begin{pmatrix} 3 \\ 11 \end{pmatrix}$$

Jacobi method:

$$x_1^{(1)} = \frac{3 - x_2^{(0)}}{4} = \frac{3-11}{4} = -2$$
$$x_2^{(1)} = \frac{1 - 2x_1^{(0)}}{5} = \frac{1-6}{5} = -1$$
$$\Rightarrow x^{(1)} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$x_1^{(2)} = \frac{3 - x_2^{(1)}}{4} = \frac{3+1}{4} = 1$$
$$x_2^{(2)} = \frac{1 - 2x_1^{(1)}}{5} = \frac{1+4}{5} = 1$$
$$\Rightarrow x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Gauss - Seidel mth:

$$x_1^{(1)} = \frac{3 - x_2^{(0)}}{4} = -2$$
$$x_2^{(1)} = \frac{1 - 2x_1^{(1)}}{5} = \frac{1+4}{5} = 1$$
$$\Rightarrow x^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$x_1^{(2)} = \frac{3 - x_2^{(1)}}{4} = \frac{3-1}{4} = \frac{1}{2}$$
$$x_2^{(2)} = \frac{1 - 2x_1^{(2)}}{5} = \frac{1 - 2\cdot\frac{1}{2}}{5} = 0$$
$$\Rightarrow x^{(2)} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Course 11 - Numerical methods for solving nonlinear eq in $\mathbb{R}$

$f : \Omega \to \mathbb{R}, \ \Omega \subset \mathbb{R}$

$f(x) = 0$

$x_0, x_1, \dots x_{n-1} \to x_n, x_{n+1}, \dots$

$x_0, x_1 \dots x_{n-1}, x_n, x_{n+1}, \dots$

$x_i = F(x_{i-n}, \dots x_{i-1}), \ i = n, n+1 \dots$

$x_0, x_1 \dots x_{n-1} \to$ starting points $\qquad F \to F$ - method

$x_0 \to F$ - method is <u>one - step</u> method

$x_0, \dots x_{n-1} \to F$ - method is a <u>multistep</u> method

$x_0, x_1, \dots x_n, \dots$

$$\lim_{x_i \to \alpha} \frac{\alpha - F(x_{i-n+1}, \dots, x_i)}{(\alpha - x_i)^p} \neq 0 \qquad \alpha = \text{exact solution}$$

$g = f^{-1} \quad f(\alpha) = 0 \Rightarrow \alpha = f^{-1}(0)$

$\alpha = g(0)$ - inverse interpolation

One - step methods

x_0 - given, $x_{i+1} = F(x_i)$

$\alpha = g(0), \ g = f^{-1}$

$g(y) = (T_{m-1}\, g)(y) + (R_{m-1}\, g)(y)$

$$(T_{m-1} g)(y) = \sum_{k=0}^{m-1} \frac{1}{k!}(y-y_i)^k g^{(k)}(y_i) \Rightarrow g \approx (T_{m-1} g)(y)$$

$$\alpha = g(0) \approx (T_{m-1} g)(0) = \sum_{k=0}^{m-1} \frac{(-1)^k (f(x_i))^k}{k!} \cdot g^{(k)}(f(x_i))$$

$$\Rightarrow x_{i+1} = F_m^T(x_i) = x_i + \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} (f(x_i))^{(k)} g^{(k)}(f(x_i))$$

Particular cases:

$\boxed{m=2}$ $\quad F_2^T(x_i) = x_i + \frac{(-1)}{1!} f(x_i) \cdot \frac{1}{f'(x_i)}$

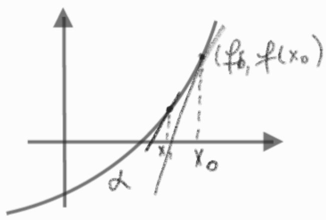
$$x_{i+1} = F_2^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow \text{Newton's method}$$

Newton's method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad x_0 \text{ - given}$$

$$|\alpha - x_n| \le \frac{1}{m_1} |f(x_n)|, \quad m_1 \le m_1 f = \min_{x \in [a,b]} |f'(x)|$$

$$|\alpha - x_n| \le |x_n - x_{n-1}|, \quad n \ge n_0$$



(x_0, y_0) $\quad \boxed{y - y_0 = f'(x_0)(x - x_0)}$

$y_0 = f(x_0) \Rightarrow y - f(x_0) = f'(x_0)(x - x_0)$

$-f(x_0) = f'(x_0)(x_1 - x_0) \quad |: f'(x_0)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow$ Newton's method (tangent method)

example 13 $\quad x^3 - x^2 - 1 = 0 \qquad f' = 3x^2 - 2x$

$x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{1} = 2$

Multistep method

$y_k = f(x_k), \quad k = \overline{0,n} \qquad g = f^{-1}$

$g = L_n g + R_n g$

$$g \approx L_n g(y) = \sum_{k=0}^{n} \frac{(y-y_0)\cdots(y-y_{k-1})(y-y_{k+1})\cdots(y-y_n)}{(y_k-y_0)\cdots(y_k-y_{k-1})(y_k-y_{k+1})\cdots(y_k-y_n)} g(y_k)$$

$F_n^L(x_0, x_1, \dots, x_n) = (L_n g)(0)$

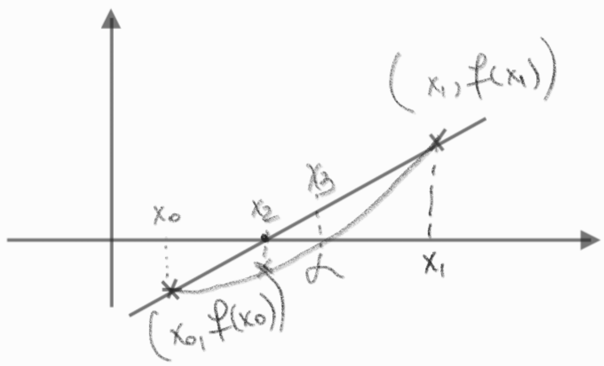
$$\alpha = g(0) \approx (L_n g)(0) = \sum_{k=0}^{n} \frac{(-1)^n y_0 y_1 \cdots y_{k-1} y_{k+1} \cdots y_n}{(y_k-y_0)\cdots(y_k-y_{k-1})(y_k-y_{k+1})\cdots(y_k-y_n)} x_k$$

ord (F_n^L) is the positive sol of the eq.

$$t^{n+1} - t^n - \dots - t - 1 = 0$$

$n = 1 \Rightarrow F_1^L(x_0, x_1) \Rightarrow x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)}$ → secant method

$$\boxed{x_{i+1} = x_i - \frac{(x_i - x_{i-1}) f(x_i)}{f(x_i) - f(x_{i-1})}}$$



$(x_{n-1}, f(x_{n-1})), (x_n, f(x_n))$

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

$$\Rightarrow \frac{-f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x_{n+1} - x_n}{x_{n-1} - x_n} \quad \Big| \cdot (x_{n-1} - x_n)$$

$$\Rightarrow x_{n+1} = x_n - \frac{(x_{n-1} - x_n) f(x_n)}{f(x_{n-1}) - f(x_n)}$$

example 17

$x_0 = 1 \quad x_1 = 2$

$x^3 - x^2 - 1 = 0$

$$x_2 = x_1 - \frac{(x_0 - x_1) f(x_1)}{f(0) - f(x_1)} = 2 - \frac{(-1)(8 - 4 - 1)}{(-1) - 3} = 2 - \frac{-3}{-4}$$

$$= 2 - \frac{3}{4} = \frac{8 - 3}{4} = \frac{5}{4}$$

Bisection method

$f(a) \cdot f(b) < 0$

$c = \frac{a+b}{2}$

$f(a) \cdot f(c) < 0 \Rightarrow a_1 = a, \ b_1 = c$

$f(c) f(b) < 0 \Rightarrow a_1 = c, \ b_1 = b$

example $f = x^3 - x^2 - 1$ $\qquad [a, c] \Rightarrow c_1 = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4}$

$a = 1, \ b = 2, \ c = \frac{1+2}{2} = \frac{3}{2}$

False position method

$f(a) f(b) < 0$

$c = b - \frac{(b-a) f(b)}{f(b) - f(a)}$

$f(a) f(c) < 0 \Rightarrow a_1 = a, \ b_1 = c$

$> 0 \Rightarrow a_1 = c, \ b_1 = b$