Object-Oriented XVA Analytics:

Practical Counterparty and Funding Risk Modeling under Hull-White Dynamics

Mara Popescu, Mariela Kodzhebasheva, Dilara Korkmaz, Sergei Ponomarev, Tolga Sahin, Jilong Yu
Bocconi University, Master of Quantitative Finance and Risk Management 2025

Abstract

This project addresses the valuation and risk analysis of interest rate derivatives and equity options under counterparty and funding risks. Specifically, it implements the Hull-White short rate model for interest rate instruments and a Geometric Brownian Motion (GBM) model for equity products, incorporating key risk adjustments such as Credit Valuation Adjustment (CVA) and Funding Valuation Adjustment (FVA). The goal is to simulate mark-to-market (MtM) paths, quantify exposures, and compute valuation adjustments under both uncollateralized and collateralized scenarios.

The entire implementation adopts an object-oriented programming (OOP) paradigm to enhance clarity, modularity, and reusability. By structuring the code into well-defined classes such as TermStructure, HullWhiteModel, GeometricBrownianMotion, InterestRateSwap, and DownInPutOption, each with encapsulated data and behavior, the project reflects the natural decomposition of financial models. Additional utils module provides code for analysis and ploting. This design promotes interpretability, clean abstraction, and flexibility, allowing the components to be tested, reused, and extended efficiently in response to evolving modeling or risk management needs.

1. Short Rate Simulation and Zero Curve Consistency (Q1–Q2)

The Hull-White model is a class of no-arbitrage models that are able to fit today's term structure of interest rates. Under the model, the single currency short-rate follows mean-reverting Ornstein–Uhlenbeck process, and its time evolution can be described using SDE:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma(t)dW^{\mathbb{Q}} \qquad \text{eq.1}$$

The model can exactly reproduce the initial zero-coupon curve by matching the time-dependent long-term rate expectation parameter $\theta(t)$ with the instantaneous forward rate *f(0, t)* derived from the initial zero curve:

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \qquad \text{eq.2}$$

The instantaneous forward curve is derived from zero curve with:

$$f(0,t) = -\frac{\partial lnP(0,t)}{\partial t} \qquad \text{eq.3}$$

Volatility σ(t) can also be calibrated daily to match interest rate derivative data such as caplets and swaptions, but given there are no explicit such requirements in the project questions, therefore constant volatility assumption is assumed instead.

With above one-factor Hull-White model and constant volatility preset, the SDE (eq.1) has an exact stochastic solution:

$$r(t) \ = \ r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma\int_s^t e^{-a(t-u)}dW^{\mathbb{Q}}(u) \qquad \text{eq.4}$$

Where $\alpha(t)$ is the integration effect of $\theta(t)$:

$$\alpha(t) \ = \ f(0,t) \ + \ (\frac{\sigma^2}{2a^2}(1-e^{-at})^2) \qquad \text{eq.5}$$

Therefore, to simulate the short rates aligned with a certain zero curve, we can use eq.4 with exact discretization.

$$x(t) = \ x(t-1)e^{-a\Delta t} + \mathcal{N}(0, \ \frac{\sigma^2}{2a}(1-e^{-2a\Delta t})) \qquad \text{eq.6}$$

$$x(t) = \ r(t) \ - \ \alpha(t) \qquad \text{eq.7}$$

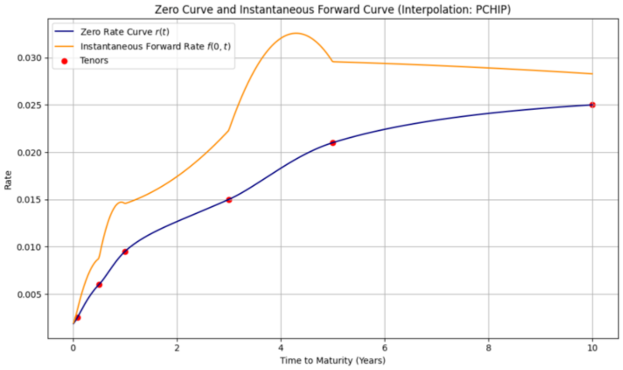
For programming, the following steps were taken:

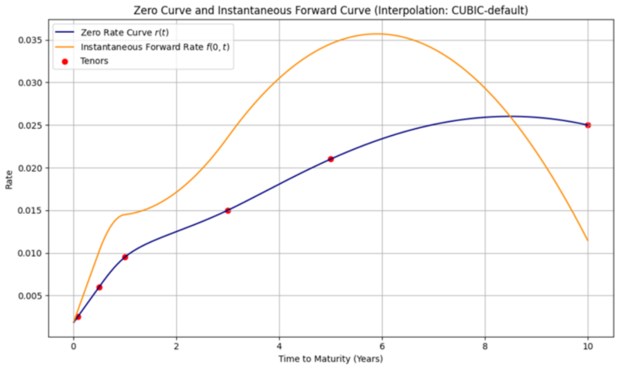
1.Implement a term structure class (TermStructure), which at least

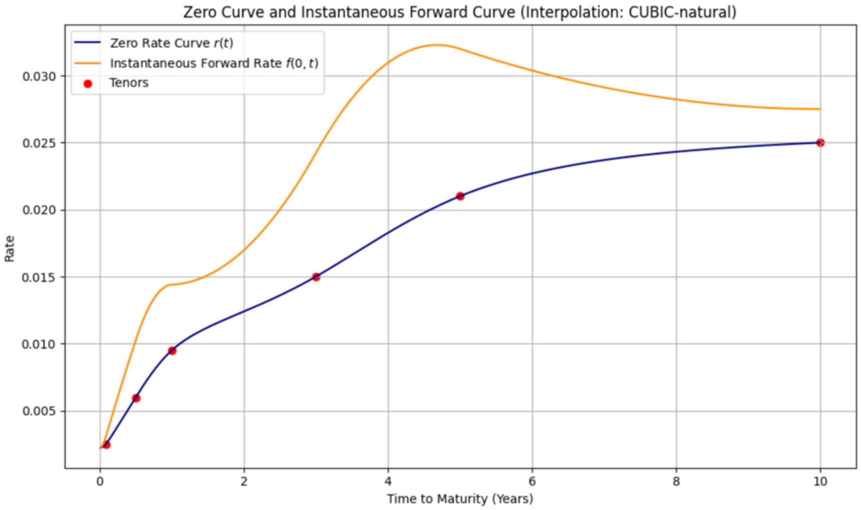
- takes in key rates, short rate at time 0.
- has a method that returns interpolated continuous zero curve with cubic spline interpolation
- has a method that returns discount rate *P(0,t)*.
- has a method that returns instantaneous forward rate *f(0,t)*.

Designing a robust term structure for interest rate modelling involves several nontrivial challenges, particularly when striving for both a smooth and financially realistic forward curve. A central difficulty lies in choosing an interpolation method that balances the need for smoothness with the requirement of monotonic forward rates. While monotonic interpolators such as PCHIP enforce non-negativity and avoid oscillations, they often lack the smoothness needed for accurate pricing and sensitivity analysis. In contrast, cubic spline interpolation offers excellent smoothness but can introduce non-monotonic behavior in forward rates due to its curvature. To mitigate this, we adopted natural cubic splines, which constrain the second derivative at the boundaries to zero. This helps reduce oscillations and limits the extent of monotonicity violations near the curve endpoints, while preserving the overall smoothness necessary for derivative-based modelling.

Additionally, to ensure consistency with continuous compounding, the curve is constructed to satisfy *P(0, 0) = 1*, and the instantaneous rate at t = 0 is explicitly set to zero. This choice improves numerical stability in later applications, particularly during trapezoidal integration for computing expected discount factors.







2.Implement a Hull-White model class (HullWhiteModel), which

- takes in constant parameters a, s and a term-structure object.
- has a method for Monte Carlo Simulation that simulates calibrated short rate paths with given tenor and time steps using exact discretization.
- has a method for path-wise numeraire with Trapezoidal integration.
- has a method for path-wise numeraire with Bond price implementation *P(t, T).*

To simulate the short-rate dynamics and related processes in this project, we employed a Monte Carlo framework. Given the stochastic nature of interest rate models like Hull-White, accurately capturing the evolution of interest rates requires careful treatment of the underlying random processes. To mitigate the sampling error inherent in Monte Carlo simulations, especially the cumulative instability from summing many Gaussian shocks, we implemented a variance-preserving correction. Specifically, after generating standard normal random variables, we adjusted the cumulative Brownian paths to have zero mean and unit variance at each time step by recentralizing and rescaling the distribution. This technique ensures that the simulated paths conform closely to the theoretical properties of Brownian motion, improving both accuracy and numerical stability of the model outputs.

In the initial method for computing the numeraire, we used the trapezoidal integration rule to estimate the expected discount factor from the simulated short-rate paths. The trapezoidal rule offers a strong balance of efficiency and stability, particularly well-suited for regularly spaced time grids in Monte Carlo contexts. While Simpson's rule can deliver higher-order accuracy for smooth deterministic functions, the short-rate paths in our model are inherently noisy due to random fluctuations. Thus, the added complexity of higher-order methods offers minimal benefit, making trapezoidal integration the pragmatic choice in this stochastic setting.

Alternatively, since the numeraire corresponds to the price of a zero-coupon bond *P(t, T)* observed at time t for maturity T, we also implemented a closed-form solution for bond prices under the Hull-White model. This analytical method avoids numerical integration altogether and yields exact results under model assumptions, providing a more accurate and computationally efficient benchmark for validating simulation-based estimates.

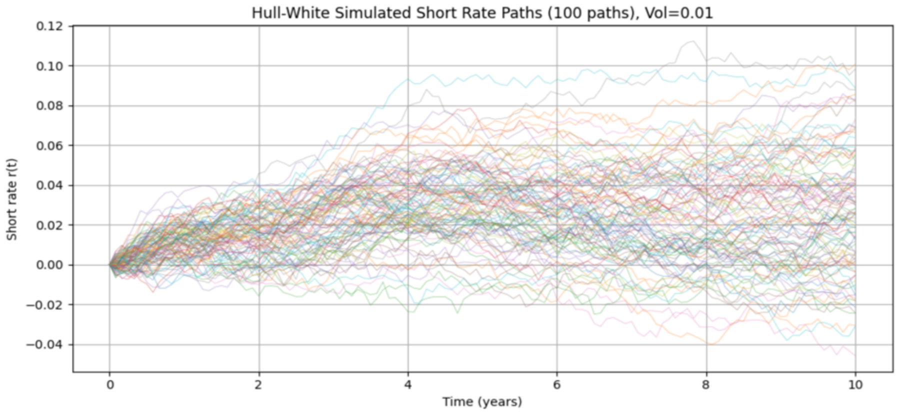
The closed form formula for numeraire is given by:

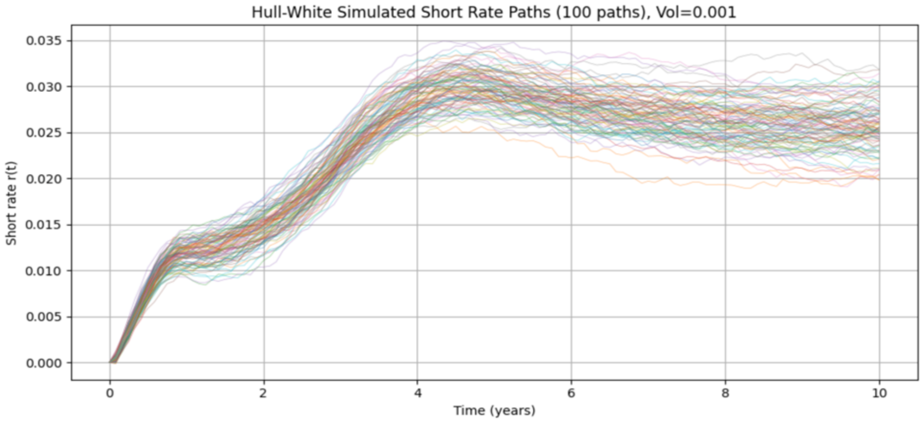
$$P(t,T) \ = \ A(t,T)e^{-B(t,T)\cdot r(t)} \qquad \text{eq.8}$$

$$B(t,T) \ = \ \frac{1}{a}(1-e^{-a(T-t)}) \qquad \text{eq.9}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \ exp\left(B(t,T)f(0,t) \ - \ \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right) \qquad \text{eq.10}$$

By performing Monte Carlo simulations under varying levels of interest rate volatility (specifically comparing σ = 1% and σ = 0.1%), we observe that as volatility decreases, the simulated short-rate paths tend to adhere more closely to the instantaneous forward rate curve implied by the initial term structure. This behavior is intuitive: in the limit of zero volatility, the Hull-White model becomes deterministic, and the short rate effectively tracks the forward rate path exactly. Higher volatility introduces greater random fluctuations around the mean-reverting level, causing paths to deviate from the deterministic trajectory. Conversely, with lower volatility, the stochastic noise diminishes, and the short-rate paths converge toward the forward curve, highlighting the role of volatility in driving uncertainty in interest rate dynamics. This also reinforces the interpretability of the model's parameters, as volatility directly governs the dispersion of possible future rates around the expected path.

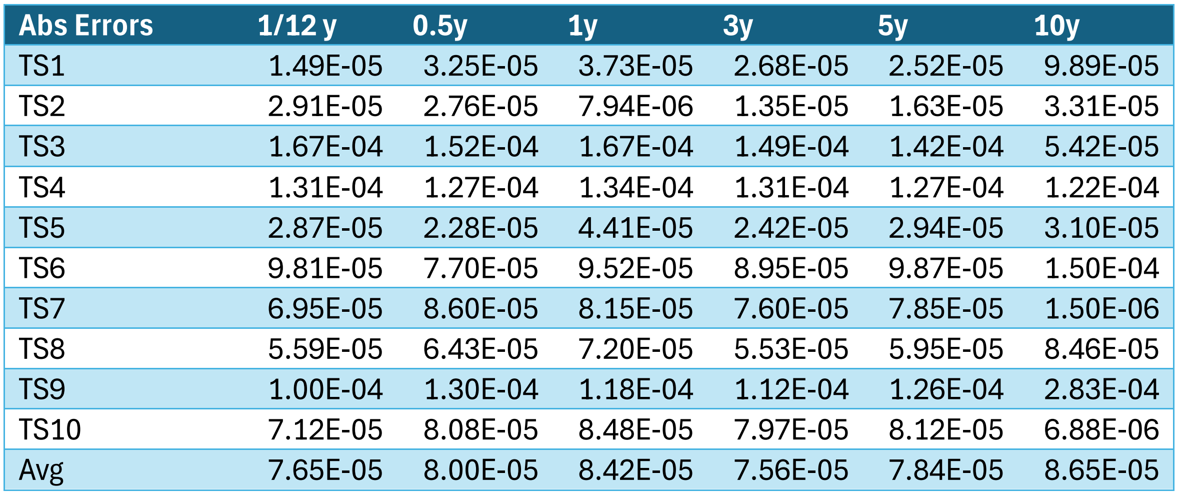




Term Structures	1/12 y	0.5y	1y	3y	5y	10y
TS1	0.0014	0.007	0.0098	0.0135	0.0204	0.0267
TS2	0.0001	0.0056	0.0108	0.0141	0.0203	0.0249
TS3	0.004	0.0054	0.0091	0.0146	0.0232	0.0272
TS4	0.0035	0.0064	0.0102	0.0165	0.0201	0.0262
TS5	0.0012	0.0054	0.0104	0.0136	0.0209	0.0241
TS6	0.0022	0.0032	0.0077	0.0143	0.0219	0.0248
TS7	0.0025	0.0067	0.0086	0.0153	0.0202	0.0233
TS8	0.0021	0.0066	0.0098	0.015	0.0234	0.0254
TS9	0.0035	0.0082	0.0082	0.014	0.0227	0.0242
TS10	0.0025	0.0071	0.0104	0.0168	0.0225	0.0261

To rigorously test the Hull-White short-rate model's ability to replicate today's term structure, we constructed ten synthetic zero rate curves with mild perturbations and simulated 10,000 short-rate paths for each using exact discretization. Each model was calibrated to match its corresponding term structure. We then computed the expected discount factor $E^{\mathbb{Q}}(e^{-\int_0^T r(s)ds})$ and compared it to the market-implied P(0,T) at key maturities.

Firstly, we use the numerical integration version of numeraire computation. As the results show, the absolute errors remained below $3 \times 10^{-4}$and typically within 1×10^{-4}, even at the 10-year horizon. In addition, by using the nuemraire_closed() method from Hull-White bond pricing, the error almost disappear (smaller than 1×10^{-15}) This confirms that, when correctly calibrated, the Hull-White model can accurately reproduce the initial discount curve over a wide range of scenarios.



Abs Errors	1/12 y	0.5y	1y	3y	5y	10y
TS1	1.49E-05	3.25E-05	3.73E-05	2.68E-05	2.52E-05	9.89E-05
TS2	2.91E-05	2.76E-05	7.94E-06	1.35E-05	1.63E-05	3.31E-05
TS3	1.67E-04	1.52E-04	1.67E-04	1.49E-04	1.42E-04	5.42E-05
TS4	1.31E-04	1.27E-04	1.34E-04	1.31E-04	1.27E-04	1.22E-04
TS5	2.87E-05	2.28E-05	4.41E-05	2.42E-05	2.94E-05	3.10E-05
TS6	9.81E-05	7.70E-05	9.52E-05	8.95E-05	9.87E-05	1.50E-04
TS7	6.95E-05	8.60E-05	8.15E-05	7.60E-05	7.85E-05	1.50E-06
TS8	5.59E-05	6.43E-05	7.20E-05	5.53E-05	5.95E-05	8.46E-05
TS9	1.00E-04	1.30E-04	1.18E-04	1.12E-04	1.26E-04	2.83E-04
TS10	7.12E-05	8.08E-05	8.48E-05	7.97E-05	8.12E-05	6.88E-06
Avg	7.65E-05	8.00E-05	8.42E-05	7.56E-05	7.84E-05	8.65E-05

3. Pricing Interest Rate Swaps under Single-Curve Framework (Q3)

To price fixed-for-floating interest rate swaps under a single-curve framework, we implemented an InterestRateSwap class that accurately computes the net present value of the swap from the payer's perspective. The fixed leg is valued as the sum of discounted fixed payments over the life of the swap, while the floating leg is computed using forward rates derived from the term structure. Both legs are discounted by the bond price directly derived from term structure *P(0,t)*. Specifically, I extended my previously implemented TermStructure class by adding a forward_rate(t0, t1) method, which calculates the forward rate between two dates using the relation

$$F(t0, t1) = \frac{1}{t1 - t0} \cdot \left(\frac{P(0,t0)}{P(0,t1)} - 1\right) \qquad \text{eq.11}$$

This approach reflects how floating leg payments are typically quoted and settled in practice. When tested with the provided zero curve and a fixed rate of 2.488%, the swap's NPV was found to be approximately zero (0.001 due to difference in interpolation method), confirming that this rate corresponds to the par swap rate and validating the implementation.

4. MtM Simulation under Hull-White Dynamics (Q4)

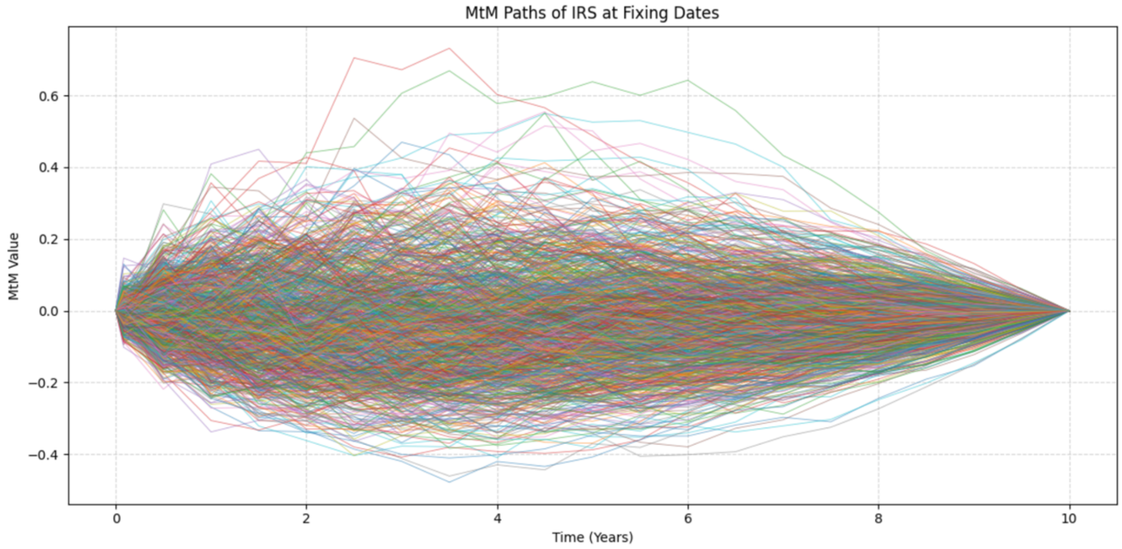
The implementation of the mark-to-market (MtM) valuation and path simulation for interest rate swaps is designed to efficiently capture the evolving value of the contract under the Hull-White short rate model. The MtM method evaluates the swap's value at any arbitrary time t, based on the simulated short rate path and the analytical Hull-White bond pricing formula P(t, T). For each future cash flow date T_k > t, it computes both the discounted fixed leg and the floating leg. Unlike earlier versions that relied on initial term structure forward rates, the current implementation derives the floating leg's forward rate directly from local (path-dependent) bond prices via

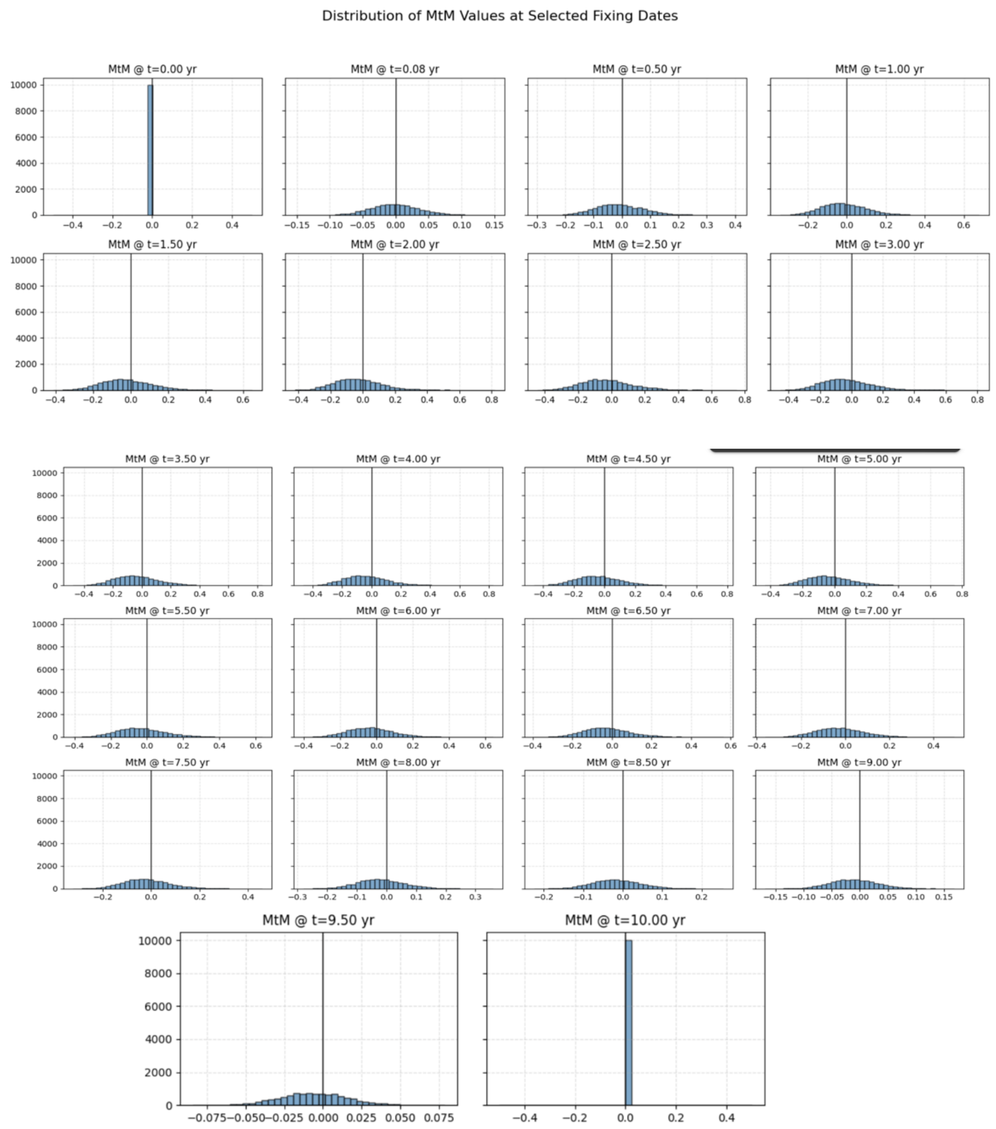
$$F(t, T_{k-1}, T_k) = \frac{1}{dt} \cdot \left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right) \qquad \text{eq.12}$$

ensuring that all inputs are observed at time t and consistent with the simulated dynamics.

The simulate_mtm_paths method extends this to full Monte Carlo simulation. It generates short rate paths and, for each scenario and time step, evaluates MtM by summing discounted future cash flows using Hull-White bond prices P(t, T_k). Forward rates are computed on-the-fly from simulated discount factors rather than precomputed from static curves, improving temporal consistency. This approach yields a robust and efficient framework for scenario-based valuation, enabling accurate exposure and credit risk analysis across time.

Below are graphs of the simulated MtM paths and distritbution MtM distributions on different fixing dates.





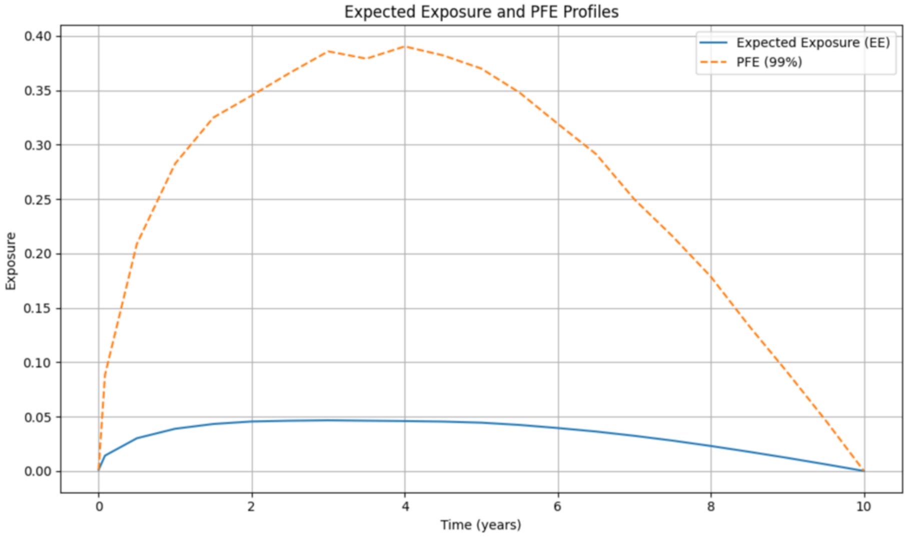
From the MtM distribution graph, we can see that throughout the lifetime of the IRS, the mean exposures are negative for banks because we used a positively slopped rate curve. The forward rate (floating payments) will increase on average, which is against the bank's interst.

4. Exposure Profiles: EE and PFE (Q5)

The compute_EE_PFE method is designed to calculate two key counterparty credit risk metrics: Expected Exposure (EE) and Potential Future Exposure (PFE), based on a matrix of simulated mark-to-market (MtM) paths. Given a set of fixing dates embedded within a larger time grid, the function first aligns these fixing dates with their corresponding indices in the full simulation timeline. For each fixing date, it extracts the MtM values across all scenarios, computes the positive exposure (i.e., max(MtM, 0)), and calculates the average exposure (EE) and the specified quantile (99% PFE) to reflect extreme but plausible future exposures. This structure allows for an efficient and intuitive assessment of counterparty risk, especially under Monte Carlo simulation frameworks, and is crucial for modeling future credit exposure and informing capital requirements.

$$EE(t) = E^{\mathbb{Q}}(max\ (MtM(t),\ 0))$$

$$PFE_{99}\ (t) = inf\{x \in \mathbb{R}:\ \mathbb{P}(max(MtM(t),\ 0) \leq x) \geq 0.99\}$$



In the context of Questions 6 to 9, calculating XVA metrics, such as Credit Valuation Adjustment (CVA) and Funding Valuation Adjustment (FVA) at time zero (t = 0) is sufficient for pricing and valuation purposes. These adjustments are defined as the present value of expected future losses or funding costs, based on projected exposure profiles. Although the exposure evolves over time, all future values are discounted back to time zero, making a single-point calculation appropriate for most applications including pricing, P&L impact, and risk reporting. Even in the case of collateralized trades with a margin period of risk (MPOR), or exotic payoffs such as barrier options under stochastic rates, the adjustments are computed based on scenarios simulated at inception. Dynamic XVA profiling (i.e., XVA(t) for t > 0) is only necessary for applications such as real-time risk monitoring, regulatory capital analysis, or pathwise sensitivity analysis, which go beyond the scope of this project.

5. Credit Valuation Adjustment (CVA) and Sensitivity (Q6)

The calculation of unilateral CVA (Credit Valuation Adjustment) is theoretically given by

$$UCVA_t = \mathbb{E}_t\left\{(1 - R_c) \cdot \mathbb{1}_{t<\tau_c<T} \cdot D(t,\tau_c) \cdot MtM(\tau_c)^+\right\} \qquad \text{eq.13}$$

While this expression involves taking the expectation over the counterparty's random default time τ_c with more valid theoretical ground, this formulation is not directly practical for implementation. It requires continuous-time modeling of default events and mark-to-market values, which is computationally intensive and infeasible when exposures (MtM) are only known at discrete points. Instead, the industry adopts a discretized approximation using piecewise constant hazard rates derived from market CDS spreads, applying them over a grid of fixing dates. This method assumes deterministic survival probabilities and avoids the need to simulate default times, enabling efficient and robust estimation of CVA while remaining consistent with market data and risk-neutral valuation principles.

In our CVA implementation, the price of counterparty credit risk is calculated using a discrete approximation of the integral formulation. At each future time point ti, we compute the product of the expected exposure EE(ti) , which is already done in MtM simluations and last question. the risk-free discount factor D(0, ti) is derived from interpolated zero-curve, and the marginal default probabilityDQ(ti). This last term DQ(ti) represents the probability that the counterparty defaults between ti-1 and ti, and is approximated using a piecewise-constant hazard rate model derived from market CDS spreads via $h(t) = \frac{CDS\ spread}{LGD}$. The CVA formula then takes the form:

$$CVA \approx\ LGD \cdot \sum_i EE(t_i) \cdot D(0, t_i) \cdot \Delta Q(t_i) \qquad \text{eq.14}$$

The CVA calculated using the original CDS spreads is 0.003317, and with a parallel +100bps shift in spreads, the CVA increases to 0.005508. The sensitivity to a +100bps shift is 0.002191.

6. Funding Valuation Adjustment (FVA) (Q7)

Funding Valuation Adjustment (FVA) accounts for the cost of funding uncollateralized positive exposures that the bank faces over the life of a derivative. It reflects the additional cost above the risk-free rate to finance future expected exposures. The FVA is calculated by discounting the expected exposures multiplied by the bank's funding spread over time. In continuous form, it is given by:

$$FVA_0 = \int_0^T \mathbb{E}[max(MtM(t), 0)] \cdot D(0, t) \cdot S_F(t) \cdot dt \qquad \text{eq.15}$$

where D(0, t) is the discount factor to time t, and $S_F(t)$ is the funding spread (expressed in decimal, i.e., a 40 bps spread is 0.004, thus divide by 10,000 if given in bps). In practice, we use a discrete version:

$$FVA_0 \approx\ \sum_{i=1}^{N} \mathbb{E}[max(MtM(t_i), 0)] \cdot D(0, t_i) \cdot S_F(t_i) \cdot \Delta t_i \qquad \text{eq.16}$$

This discrete approach enables implementation over simulated MtM paths and uses piecewise constant funding spreads to reflect real market practices.

For coding, similar to compute_cva(), we added the compute_fva() method under InterestRateSwap class. The FVA calculated with 40 bps funding spread is 0.001221.

7. Collateralized CVA and FVA with MPOR (Q8)

In the collateralized setting with daily margining and immediate collateral transfer, the counterparty exposure is effectively neutralized each day through the prompt posting of variation margin. This means that, under normal conditions, the MtM of the trade is fully collateralized on each revaluation date, leaving no residual exposure at that moment. However, the bank still faces counterparty credit and funding risks over the Margin Period of Risk (MPOR) of 10 days, during which collateral cannot be adjusted due to potential default or operational delays. As a result, CVA and FVA calculations shift focus from the full trade horizon to the MPOR following each margining date. At each revaluation point, the bank computes the expected exposure over the next 10 days and uses this short-window exposure to assess credit and funding adjustments. This approach substantially reduces CVA and FVA compared to the uncollateralized case, as exposure is limited both in size and duration.

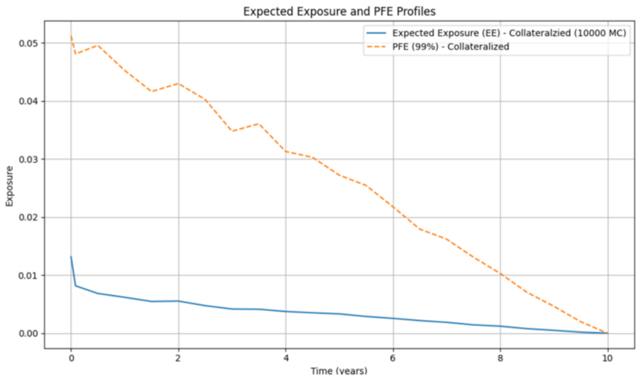
In codes, we added a compute_EE_collateralized() method under InterestRateSwap class. In this method, daily MtM paths are generated by Monte Carlo simulation. The EE is computed on a 10 day rolling window starting on each repricing date.

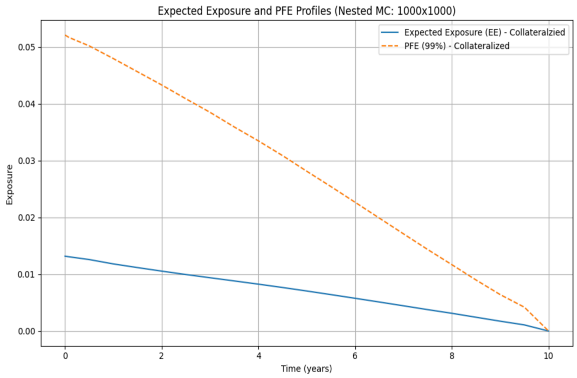
$$EE(t) = E^{\mathbb{Q}}[\max_{h\in[0,\Delta]}(MtM(t+h) - collateral(t))^{+}]$$

$$PFE_{99}\,(t) = inf\left\{x \in \mathbb{R}\colon\ \mathbb{P}(\max_{h\in[0,\Delta]}(MtM(t+h) - collateral(t))^{+} \leq x) \geq 0.99\right\}$$

The exposure definition of collateralized positions is slightly different from the exposure of uncollateralized positions, in terms of time horizon. The latter one is measured on each day. For collateralized positions, there is always a spike in initial exposure if no collateral is posed in the beginning.

Specifically, we also implemented a compute_EE_collateralized_nested() method, in this method, up to repricing date we only need monthly frequency, and a nested MC simulation on daily frequency is applied on the MPOR period. This method reduced the noise of valuation based on 1 single future rate path in MPOR period for each path up to time t.





The CVA and FVA based on collateralized EE are 0.000622 and 0.000244 respectively, both reduced significantly.

8. Equity Option Modeling with Barrier Features (Q9)

The Geometric Brownian Motion (GBM) with a stochastic interest rate r(t) as the drift is described by the following stochastic differential equation (SDE):

$$dS_t = r(t)S_tdt + \sigma(t)S_tdW_t^{\mathbb{Q}} \qquad \text{eq.17}$$

In our implementation the r(t) is time dependent while volatility σ is constact.

The analytical solution of S_t is given by:

$$S_t = S_s \times exp(\int_s^t r(u)du - \frac{1}{2}\sigma^2(t-s) + \sigma(W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}})) \qquad \text{eq.18}$$

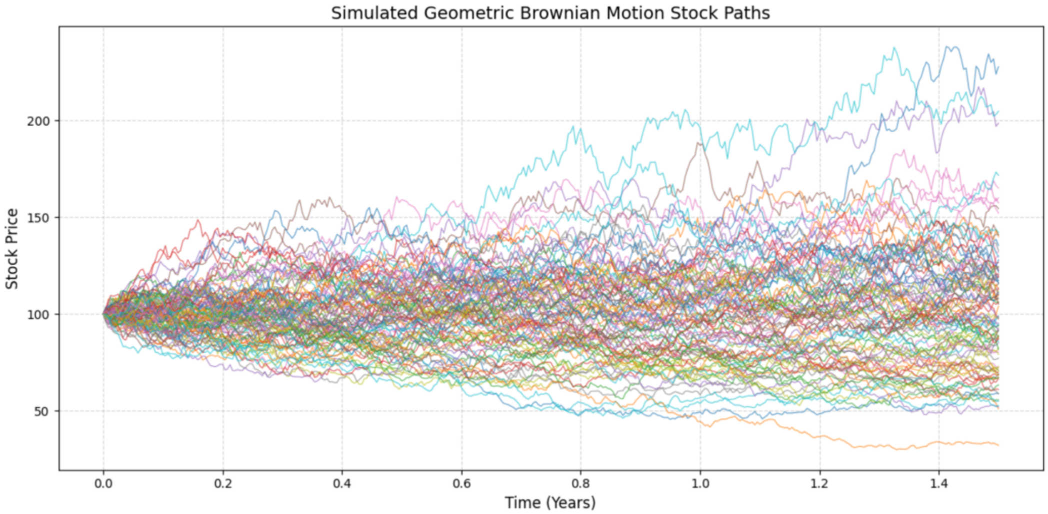
With eq.6, the exact dicretization can be used for Monte Carlo simulation of the price paths.

To minimize the numerical error that could arise from numerical integration, we use numeraire closed form described in question 1 to substitue the integration term in eq.6, therefore we have:

$$S_t = \frac{S_{t-1}}{P(t-1,t)} \times exp(-\frac{1}{2}\sigma^2\Delta t + \mathcal{N}(0, \sigma^2\Delta t)) \qquad \text{eq.19}$$

Note that the two separate Wienner processes in rate and price evolution might be slightly negatively correlated, in our example, we used a correlation of -0.3.

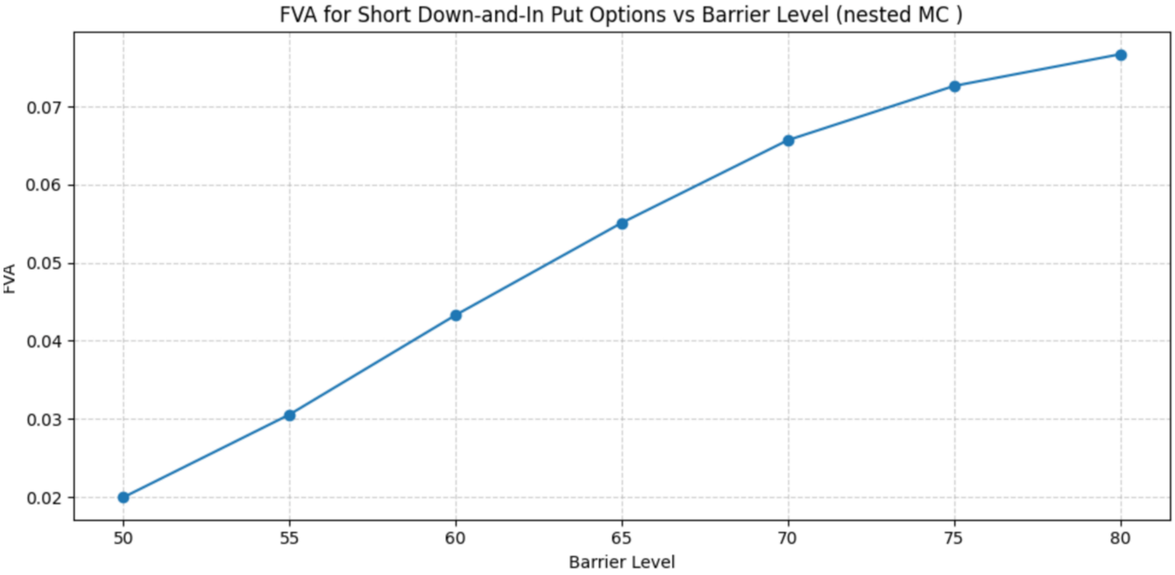
For coding, a class of GeometricBrownianMotion is created which takes in a HullWhiteModel object for rate paths generation and numeraire calculation, initial price and constant volatility as parameter. With a simulate() method, it can generate paths based on eq.7



For the specific problem of pricing barrier option, the class of the DownInPutOption is implemented. It can give a price based on Monte Carlo simulation, and also caclulate FVA by discounting and averaging path-dependent exposures.

To compute the Funding Valuation Adjustment (FVA) for the European down-and-in put option, we adopt nested Monte Carlo simulation of two layers. Specifically, we simulate the underlying stock outer paths with 5-day time intervals using a Geometric Brownian Motion with Hull-White stochastic interest rates. In addition, on the valuation dates (repricing dates), for each outer path a nested MC simulation is applied with the same time interval and last up to maturity date. The outer path up to valuation date in combination with all possible inner paths up to maturity are used to determine the barrier hit condition and therefore exposures.

For each barrier level, the Down-In-Put FVA is computed by summing up discounted funding requirements (EE) during each time interval assuming the requirement will be constant for 5-days. The result is as followed:



As we can see, the FVA is monotonously increasing with barrier level given the increased chance of being exercised thus increased expected exposure.