The Leech lattice and Conway groups

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J_1 The Janko group with order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ 1, 2, 56

$\mathbb{M}$ The Monster group with order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ 1, 2, 56

$\mathbb{B}$ The Baby monster group with order $2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$ 2, 56

Fi_{24} The Fischer group with order $2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ 2, 56

Fi_{23} The Fischer group with order $2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ 2, 56

Fi_{22} The Fischer group with order $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ 2, 56

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He The Held group with order $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ 2, 56

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J_1 The Janko group with order $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ 2, 56

Introduction

In February 1981, a historic achievement was made by several hundred international mathematicians over a period of thirty years which was completing the classification of finite simple groups. The most interesting of the finite simple groups are the twenty-six sporadic groups. The sporadic groups acquired their name as they do not belong to any infinite family of finite simple groups. The first family of these are the Mathieu groups which consist of M_{11}, M_{12}, M_{22}, M_{23} and M_{24} (see section I.6); which are permutation groups on 11, 12, 22, 23 and 24 points, respectively, and they were discovered by Emile Mathieu in 1861 [Gal]. The second family are the Conway groups which are Co_1, Co_2 and Co_3 (see section II.2 and II.2.2), they may be considered as automorphism groups of the Leech lattice. They were discovered by John Horton Conway around 1968 [Wil]. Despite the fact that before then the Higman-Sims group HS was discovered by D. G. Higman, C. C. Sims (see section II.1.5) and the McLaughlin group McL was discovered by J. McLaughlin (see section II.1.6), they are contained in both Co_2 and Co_3 [Wil]. Meanwhile, the Suzuki group Suz (see section II.2.4) was discovered by Suzuki and the Hall-Janko group $J_2 = HJ$ (see section II.2.3) was discovered by Hall-Janko which may be considered as subgroups of the first Conway group. Although the rest of the twenty-six sporadic groups are not discussed in this paper, we introduce them briefly. A century after finding Mathieu's group, in 1966 Zvonimir Janko found the first Janko group J_1 which has only 175560 elements [Gal]. During the following decade a further twenty sporadic groups were discovered, see table II.20. In 1974, the largest group of the sporadic groups, the Monster group $\mathbb{M}$ or (F_1 Fischer-Griess group) was discovered by Fischer [Gal]. Moerover, in 1980, Griess constructed this group as an automorphism group of 196884 dimensional algebra that has remarkable commutative but non-associative elements [Boy] and the order of $\mathbb{M}$ was computed by Thompson [Gal] which is

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \cdot 10^{54}.$$

Moreover, the sporadic groups may be divided into three levels. The first level consists of the five Mathieu groups which are

$$M_{11}, M_{12}, M_{22}, M_{23} \text{ and } M_{24}.$$

The second level considers as the seven groups related to the Leech lattice

$$Co_1, Co_2, Co_3, HS, McL, J_2 \text{ and } Suz.$$

The third level is that the Monster group and the seven subgroups that are related to the Monster group which are

$$\mathbb{M}, \mathbb{B}, Fi_{24}, Fi_{23}, Fi_{22}, HN, Th \text{ and } He.$$

Finally, the Pariah groups which are

$$Ru, ON, Ly, J_4, J_1 \text{ and } J_1.$$

They are not related to any of the known simple groups except possibly two, McL and M_{24} [Boy], see figure II.2.1. In this paper, we start with general background from group theory and the previous research, followed by presenting the Leech lattice construction on twenty-four dimensions. Then we discuss the three Conway groups together with the Higman-Sims group, the McLaughlin group, the Hall-Janko group and the Suzuki group. To conclude our dissertation we try to classify all the twenty-six sporadic groups in figure II.2.1 and their orders in table II.20. Finally, the main four sources for this paper are (Sphere Packings, Lattices and groups by J. H. Conway and N. J. A. Sloane), (The Finite Simple Groups by Robert A. Wilson), (Finite Simple Groups: Thirty Years of the Atlas and Beyond by M. Bhargava, R. Guralnick, G. Hiss, K. Lux and P. H. Tiep) and (A group of order 8,315,553,613,086,720,000 by J. H. Conway).

§ I Preliminaries

I.1 Background from general group theory

Definition I.1.1. Let v and u be in $\mathbb{R}^n$, then the scalar product of v and u is defined by

$$v \cdot u = \sum_{i=1}^{n} x_i y_i.$$

Definition I.1.2. A basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$ is said to be orthonormal if

1. $v_i \cdot v_j = 0$, whenever $i \neq -j$, that is vectors are mutually orthogonal.

2. $v_i \cdot v_i = 1$ for all i, where $1 \leqslant i \leqslant n$.

Definition I.1.3. A rational orthogonal matrix is a square matrix with rational entries whose columns and rows are orthogonal unit vectors, that is orthonormal vectors.

Definition I.1.4. A group $(G, *)$ is a set G with a binary operation

$$* : G \times G \to G;$$

satisfying the following conditions:

1. For all g_1, g_2 and g_3 in G;
$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3).$$

2. There exists an identity element 1 in G such that
$$g * 1 = 1 * g = g \text{ for all } g \in G.$$

3. For all $g \in G$, there exists an inverse element g^{-1} is in G such that
$$g^{-1} * g = g * g^{-1} = 1.$$

Definition I.1.5. Let S be a finite set, then the group of all bijective maps $\alpha : S \to S$, is called a *symmetric group* $\mathrm{Sym}(S)$ on S and α is called a permutation. The product of two permutations is defined as their composite:
$$\alpha\beta = \alpha \circ \beta.$$

Definition I.1.6. Let G be a group and N a subgroup of G. Then N is a *normal subgroup* if for all g in G we have $N^g = N$.

Definition I.1.7. A non-trivial group G is called a *simple group* if G has no proper non-trivial normal subgroups.

Theorem I.1.8. *Let p be a prime and G be a finite group for which p divides the order of G. There is an element of G which has order p, consequently, G has a cyclic subgroup of order p. This is called Cauchys Theorem.*

Definition I.1.9. Let G and K be groups then a map $\varphi : G \to K$ is a *group homomorphism* if for all g_1 and g_2 in G, we have $(g_1 g_2)\varphi = (g_1)\varphi\,(g_2)\varphi$.

Definition I.1.10. Let G be a group and V be a vector space. Then we say σ is a *representation of group G* if
$$\sigma : G \to GL(V)$$
is a homomorphism, where $GL(V)$ is the general linear group of a vector space V, see definition I.4.1.

Definition I.1.11. *An automorphism group* of a group G, denoted by $\mathrm{Aut}(G)$, is a set whose elements are automorphisms of G, and where multiplication is defined as composition of automorphisms. In other words, it has a subgroup structure that is obtained from $\mathrm{Sym}(G)$, the group of all permutations on G.
$$\mathrm{Aut}(G) = \{\sigma : G \to G \mid \sigma \text{ is a bijection}\}.$$

I.2 Group actions

Definition I.2.1. *An action* of a group G on a non-empty set Ω is a binary operation $* : \Omega \times G \to \Omega$, where for all α in Ω and for all g_1, g_2 in G, we have

1. $\alpha * 1 = \alpha$.
2. $\alpha * (g_1 g_2) = (\alpha * g_1) * g_2$.

Definition I.2.2. Let G be a group acting on a non-empty set Ω and α be in Ω, then we have

1. The set $\alpha^G = \{\alpha g \mid g \in G\} \subseteq \Omega$ is called *the orbit* of α under G.
2. The set $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is called *the stabiliser* of α.
3. We say G *acts transitively* on Ω if G has only one orbit. That is $\alpha^G = \Omega$ for α in Ω.

Definition I.2.3. A group G *acts k-transitively* precisely if for any two sequences of k distinct points from Ω, say $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\beta_1, \beta_2, \ldots, \beta_k)$ there is a group element g in G, where we have $\alpha_i g = \beta_i$ for each i, where $1 \leqslant i \leqslant k$.

Definition I.2.4. Let G be k-transitive, for which k non repeating elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ in Ω, g_1 and g_2 in G such that $\alpha_i g_1 = \alpha_i g_2$ for all $i = 1, \ldots, k$ and $g_1 = g_2$, then G is called *sharply k-transitive* on Ω.

Theorem I.2.5 (Orbit-Stabiliser Theorem). *Let G be a group, Ω be a non-empty set and $G \times \Omega \to \Omega$ be a group action. Then for any α in Ω, we have*

$$|G| = |\alpha^G| \cdot |G_\alpha|.$$

Theorem I.2.6. *Let Ω be a non-empty set, then G acts transitively on Ω if and only if for any two elements α and β in Ω, there exists g in G such that $\beta = \alpha g$.*

Definition I.2.7. Let G be a transitive group acting on a non-empty set Ω. A non-trivial subset Δ of Ω is a block of G if for all g in G, we have $(\Delta)g \cap \Delta = \varnothing$ or Δ, and Δ is called a trivial block if $\Delta = \{\alpha\}$ for some α in Ω or $\Delta = \Omega$, and if G has no non-trivial blocks then G is an imprimitive group, and if G has not have then G is a primitive group.

Theorem I.2.8. *Let G act transitively on a non-empty set Ω and let H be the stabiliser of α in Ω. Then G* acts primitively *on Ω if and only if H is a maximal subgroup of G.*

Theorem I.2.9. *Every doubly transitive group is a primitive group.*

Theorem I.2.10. *Let G be a finite group which acts transitively on a non-empty set Ω, and H be a normal subgroup of G, then the orbits of the induced action of H on Ω all have the same size.*

Lemma I.2.11. *Let G be a finite perfect group which acts primitively on a non-empty set Ω, and the stabiliser H has a normal abelian subgroup say K, whose conjugates generate G, then G is a simple group. This is called Iwasawa's Lemma.*

I.3 Sylow groups

Definition I.3.1. Let G be a group, and let p be a prime number.

1. A group of order p^k for some $k \geqslant 1$ is called a *p-group*. A subgroup of order p^k for some $k \geqslant 1$ is called a *p-subgroup*.
2. If $|G| = p^n m$, where p does not divide m, then a subgroup of order p^n is called a *Sylow p-subgroup* of G.

Corollary I.3.2. *Let G be a finite group and p, q are distict prime divisors of the order of G. If G has only one Sylow p-subgroup, then this group is a normal subgroup and so G is not a simple group.*

Theorem I.3.3. *Let H be a normal subgroup of G, and suppose that P is a Sylow subgroup of H, then $G = N_G(P)H$. This is called the Frattini argument.*

I.4 The general linear group and some of its subgroups

Definition I.4.1. *The general linear group* is the group of all linear automorphisms of an

n-dimensional vector space V over the finite field $GF(q)$ of q elements, where $q = p^n$, p is a prime. If it consists of $n \times n$ invertible matrices with entries from $GF(q)$, then typical notation is $GL_n(q)$, $GL(n, q)$, or simply GL_n if the field is understood. Whereas, if it is the abstract automorphism group, not necessarily written as matrices, it is written as $GL(V)$.

Definition I.4.2. *The special linear group* is the normal subgroup of $GL(n, q)$ consisting of the automorphisms with a determinant of one, written $SL(n, q)$ or $SL_n(q)$.

$$SL_n(q) = \{A \in GL_n(q) \mid \det A = 1\}.$$

Definition I.4.3. *The projective linear group* $PGL(n, q)$ and the projective special linear group $PSL(n, q)$ are the quotients of $GL(n, q)$ and $SL(n, q)$ by their centers. For $n \geqslant 2$ the group $PSL_n(q)$ is simple except for $PSL_2(2) = S_3$ and $PSL_2(3) = A_4$. Therefore, we also call it $L_n(q)$, in conformity with Artin's convention in which single letter names are used for groups that are generally simple, see table I.1.

Definition I.4.4. *The conjugate transpose* of a matrix A, denoted by A^*, is given by

$$A^* = \overline{A^t}.$$

Where the entries of $\overline{A}$ are the conjugates of the corresponding entries of A.

Definition I.4.5. A square non-singular matrix A is *unitary* if

$$A^{-1} = A^*.$$

Definition I.4.6. *The general unitary group* $G\mathrm{U}_n(q)$ is the subgroup of all elements of $GL(q^2)$ that a given non-singular Hermitian form.

$$G\mathrm{U}_n = \{A \in GL_n(q^2) \mid A^*A = I\}.$$

Definition I.4.7. *The special unitary group* $S\mathrm{U}_n(q)$ is the Lie group of $n \times n$ unitary matrices with determinant one.

$$S\mathrm{U}_n(q) = \{A \in G\mathrm{U}_n(q) \mid \det A = 1\}.$$

Definition I.4.8. *The projective general unitary group* $PG\mathrm{U}_n(q)$ and projective special unitary group $PS\mathrm{U}_n(q)$ are the groups obtained from $G\mathrm{U}_n(q)$ and $S\mathrm{U}_n(q)$ on factoring these groups by their centers.

Definition I.4.9. An abelian p-group G is *elementary abelian* if $x^p = 1$ for all x in G and it is denoted by p^n or E.

Group	Isomorphic to	Simple or not	Order
$PSL_2(2) = L_2(2)$	$PS\mathrm{U}_2(2) = \mathrm{U}_2(2) \cong S_3$	Not simple	$2 \cdot 3$
$PSL_2(3) = L_2(3)$	$PS\mathrm{U}_2(3) = \mathrm{U}_2(3) \cong A_4$	Not simple	$2^2 \cdot 3$
$PS\mathrm{U}_3(2)$	$3^2 : Q_8$	Simple	$2^3 \cdot 3^2$
$PSL_2(4) = L_2(4)$	$\mathrm{U}_2(5) \cong L_2(5) \cong A_5$	Simple	$2^2 \cdot 3 \cdot 5$
$L_2(7)$	$\mathrm{U}_2(7) \cong L_3(2) \cong A_1(7)$	Simple	$2^3 \cdot 3 \cdot 7$
$L_2(9)$	$A_6 \cong S_4(2)' \cong \mathrm{U}_2(9)$	Simple	$2^3 \cdot 3^2 \cdot 5$
$L_4(2)$	$A_8 \cong A_3(2) \cong O_6^+(2)$	Simple	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$PS\mathrm{U}_3(5)$	${}^2A_2(5)$	Simple	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$
$PS\mathrm{U}_4(3)$	$O_6^-(3) \cong^2 A_3(3)$	Simple	$2^7 \cdot 3^6 \cdot 5 \cdot 7$
$L_2(11)$	$PS\mathrm{U}_2(11) \cong A_1(11) \cong S_2(11) \cong O_3(11)$	Simple	$2^2 \cdot 3 \cdot 5 \cdot 11$
$L_2(23)$	$PS\mathrm{U}_2(23) \cong A_1(23) \cong S_2(23) \cong O_3(23)$	Simple	$2^3 \cdot 3 \cdot 11 \cdot 23$
$PS\mathrm{U}_6(2)$	${}^2A_5(2)$	Simple	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$

Table I.1: Examples of some of subgroups of the general linear group

I.5 The Steiner system and MOG

Definition I.5.1. A Steiner system $S(t,k,v)$ is a set of k-element subsets of a base set which is a set of v elements, such that any t-element subset of the base set appears in precisely one of the k-element subsets which are called blocks.

Theorem I.5.2. *If there exists an $S(t,k,v)$, then there exists an $S(t-1,k-1,v-1)$.*

Theorem I.5.3. *If there exists an $S(t,k,v)$. Then $\binom{k}{t}$ divides $\binom{v}{t}$, and the number of blocks is $\binom{v}{t} \diagup \binom{k}{t}$.*

Corollary I.5.4. *If there exists an $S(t,k,v)$, then $\binom{k-i}{t-i}$ divides $\binom{v-i}{t-i}$ for each $i = 0,1,2,\dots,$ $t-1$.*

Definition I.5.5. A Steiner system $S(5,8,24)$ is a set of all sets of size eight, which are subsets of a set of size twenty-four elements, say Ω with the property that any subset of size five of Ω appears in only one of the 8-element sets which are called octads. (This means that $S(5,8,24) = \{B \subseteq \Omega \mid \text{for all } X \subseteq \Omega\ \exists!\, B,\ X \subseteq B \ \text{ and } \ |B| = 8,\ \text{where } |X| = 5\}$.)

Theorem I.5.6. *A Steiner system $S(5,8,24)$ exists.*

Theorem I.5.7. *The Steiner system $S(5,8,24)$ is unique.*

Definition I.5.8. Let Ω be

$$\begin{array}{|cc|cc|cc|}
\hline
\infty & 14 & 17 & 11 & 22 & 19 \\
0 & 8 & 4 & 13 & 1 & 9 \\
\hline
3 & 20 & 16 & 7 & 12 & 5 \\
15 & 18 & 10 & 2 & 21 & 6 \\
\hline
\end{array}\ .$$

Definition I.5.9. The twelve dimensional subspace of $\mathcal{P}(\Omega)$ over $GF(2)$ is denoted by $\mathscr{C}$,

$$\mathscr{C} = \{X, Y \in \mathcal{P}(\Omega) \ : \ |X + Y| \geqslant 8\}.$$

Definition I.5.10. The subset of $\mathscr{C}$ that contains all octads is denoted by $\mathscr{C}_8$ which has order 759. Moreover, a triplet of mutually disjoint octads is called a trio.

Definition I.5.11. The subset of $\mathscr{C}$ that contains all dodecads (umbrals) is denoted by $\mathscr{C}_{12}$, which has order 2576. A dodecad (umbral) is a subset of Ω which has order 12. Moreover, a complementary pair of umbral dodecads is a *duum*.

Definition I.5.12. The set that consists of 0, octads, dodecads, sets of 16 elements, and Ω is called the *$\mathscr{C}$-set* which has order $4096 = 1 + 759 + 2576 + 759 + 1$, respectively.

Theorem I.5.13. *Let $\{a_1, a_2, \dots, a_8\}$ be an octad. Then the number of octads intersecting $\{a_1, \dots, a_i\}$ in $\{a_l, \dots, a_j\}$ (exactly) is the $(j+1)$th entry in the $(i+1)$th line in figure I.5.1.*

Remark I.5.14. The ninth line in figure I.5.1 shows that any two octads intersect in 0, 2, 4 or 8 points.

Definition I.5.15. Let $Y = Y_1 \cup Y_2 \dots \cup Y_s$ be a decomposition of Y into disjoint sets Y_i, and X is a subset of Y, if $|Y_i \bigcap X| = r_i$ points, $1 \leqslant i \leqslant s$, then X cuts this decomposition as $r_1 \cdot r_2 \cdot \dots \cdot r_s$, and $|X| = r_1 + r_2 + \dots + r_s$.

$$\begin{array}{c}
759 \\
506 \quad 253 \\
330 \quad 176 \quad 77 \\
220 \quad 120 \quad 56 \quad 21 \\
130 \quad 80 \quad 40 \quad 16 \quad 5 \\
78 \quad 52 \quad 28 \quad 12 \quad 4 \quad 1 \\
46 \quad 32 \quad 20 \quad 8 \quad 4 \quad 0 \quad 1 \\
30 \quad 16 \quad 16 \quad 4 \quad 4 \quad 0 \quad 0 \quad 1 \\
30 \quad 0 \quad 16 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0 \quad 1
\end{array}$$

Figure I.5.1: The Leech triangle

Corollary I.5.16. *There is a partition of the twenty-four points into six tetrads, which is an correspondence to each four-point of* Ω*, say* Y_i*,* $1 \leqslant i \leqslant 6$*, then* $\Omega = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6$*, with* $|Y_i| = 4, |Y_i + Y_j| = 8$*,* $i \neq j, 1 \leqslant i, j \leqslant 6$ *and* $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ *is called a sextet.*

Lemma I.5.17. *An octad cuts the six tetrads of a sextet* $4^2 \cdot 0^4$*,* $3 \cdot 1^5$ *or* $2^4 \cdot 0^2$*.*

I.5.1 The Miracle Octad Generator (MOG)

The Miracle Octad Generator (MOG) is thirty-six pictures where one of them shows the name of the points, and the other thirty-five pictures contain pairs of brick tetrads and the corresponding group of square bricks. Moreover, taking any one of the pair together with any one of square bricks in the same group is an octad, see figure I.5.2.

I.6 The Mathieu group M_{24} and its subgroups

Definition I.6.1. The 5-transitive group preserving $\mathscr{C}_8$ is called *the Mathieu group* M_{24}, which has order $244,823,040$.

$$M_{24} = \{\sigma \in S_{24} \mid O\sigma \in \mathscr{C}_8 \text{ for all } O \in \mathscr{C}_8\}.$$

Theorem I.6.2. *The Mathieu group* M_{24} *preserves* $\mathscr{C}$*.*

Definition I.6.3. By fixing any point of Ω (say ∞), we obtain the maximal subgroup M_{23} that can be defined as the *stabiliser of the Steiner system* $S(4,7,23)$ and it has 253 blocks. The blocks can be obtained by removing ∞ from all the octads of $S(5,8,24)$ that contain it.

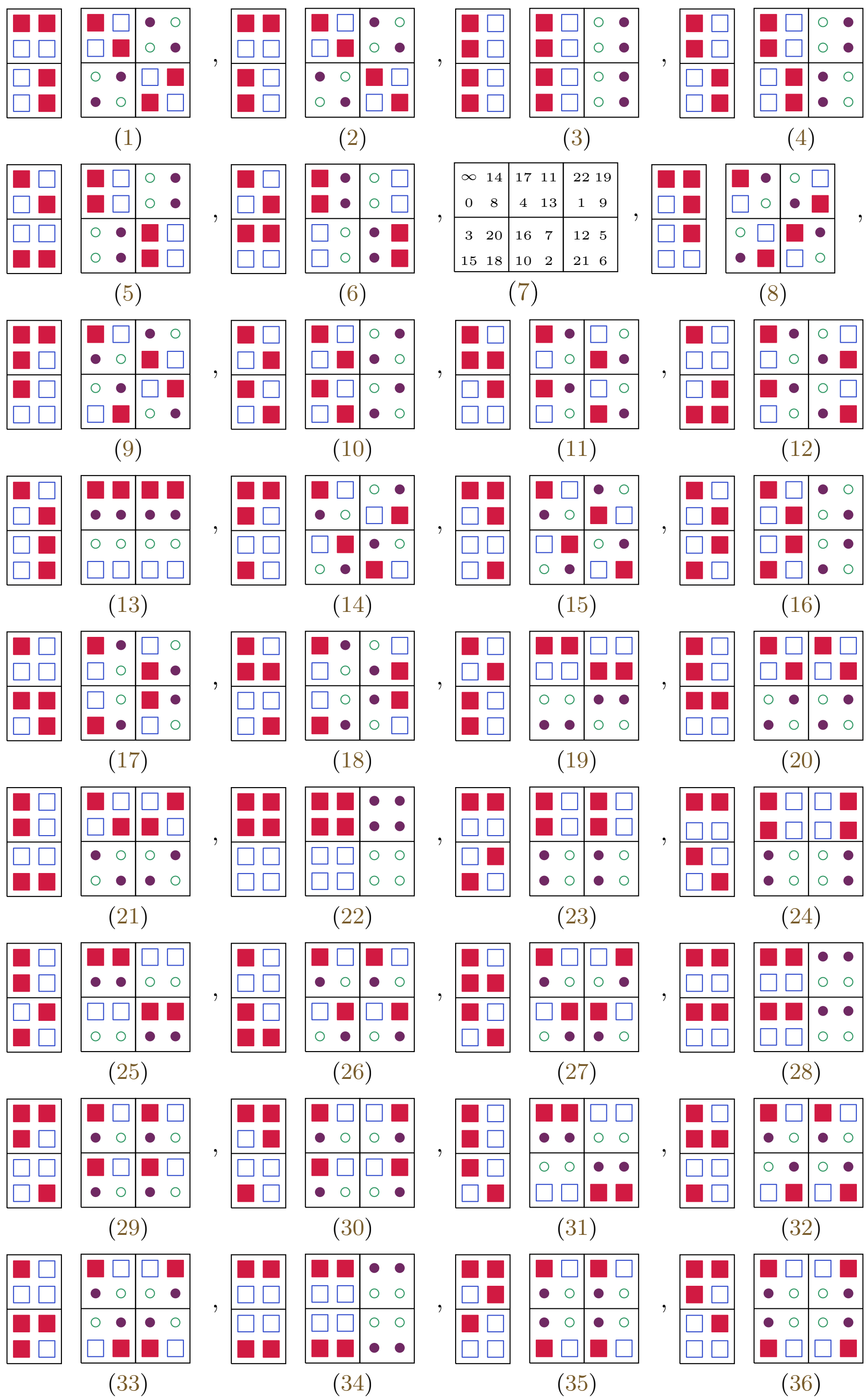


Figure I.5.2: The Miracle Octad Generator (MOG)

Definition I.6.4. By fixing two points of Ω (say ∞ and 0), we obtain the Mathieu group M_{22}, which is not maximal, since we can add to it the permutations which interchange those points, to get $M_{22}:2$. This is *the stabiliser of the Steiner system* $S(3,6,22)$ and it has 77 blocks. The blocks can be obtained by removing ∞ and 0 from all octads of $S(5,8,24)$ that contain them both.

Definition I.6.5. The group that is the sharply 5-transitive permutation group on twelve points and preserves the Steiner system $S(5,6,12)$ is called *the Mathieu group* M_{12}.

Definition I.6.6. The point stabiliser in M_{12} that acts sharply 4-transitively on a set of eleven points, and preserves the Steiner system $S(4,5,11)$ is called *the Mathieu group* M_{11}.

I.6.1 Maximal subgroups of M_{24}

Table I.2 shows all maximal subgroups of M_{24} and gives some descriptions. As an example, the symbol $[1,23]$ indicates a subgroup of M_{24} fixing a point that has one orbit of size 23. The symbol $[2,22]$ indicates a subgroup of M_{24} that has two orbits of size two and 22. Whereas, $[4^6]$ means that the group is transitive, with six imprimitivity sets of size four. (Notice that $[4^6]$ is not the same as $[4,4,4,4,4,4]$.) Again $[8^3]$ represents a transitive subgroup of M_{24}, with three imprimitivity sets of size eight. (Notice that again $[8^3]$ is not the same as $[8,8,8]$.) $[24]$ means that the group is a transitive on Ω with one orbit of size 24. Finally, *monad, dyad* and *triad*, are sets of size one, two and three respectively.

Remark I.6.7. The adjoint permutations of Ω, $\alpha : x \to x+1$, $\beta : x \to 2x$, $\gamma : x \to -x^{-1}$ and $\delta : x \to x^3/9 (x \in Q)$ or $9x^3 (x \notin Q)$, where x is in Ω and $Q = \{0,1,2,3,4,6,8,9,12,13,16,18\}$ are generated M_{24}.

Name	Structure	Order	Orbit	Index
Monad stabiliser	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$[1, 23]$	24
Dyad stabiliser	$M_{22}.2$	$2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$[2, 22]$	276
Triad stabiliser	$M_{21}.S_3$	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	$[3, 21]$	2024
Sextet stabiliser	$2^6.3S_6$	$2^{10} \cdot 3^3 \cdot 5$	$[4^6]$	1771
Octad stabiliser	$2^4.A_8$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	$[8, 16]$	759
Duum stabiliser	$M_{12}.2$	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	$[12^2]$	1288
Trio stabiliser	$2^6(S_3 \times L_2(7))$	$2^{10} \cdot 3^2 \cdot 7$	$[8^3]$	3795
Octern group	$L_2(7)$	$2^3 \cdot 3 \cdot 7$	$[3^8]$	1457280
Projective group	$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	$[24]$	40320

Table I.2: Maximal subgroups of M_{24}

I.7 Maximal subgroups of the Mathieu groups

Table I.3 shows all maximal subgroups of the Mathieu groups M_{23}, M_{22}, M_{12}, M_{11} and gives some descriptions. The left hand column gives the action on the set Ω, with semicolons separating the orbits of the Mathieu group under discussion. For example, the symbol $[1; 1; 7, 15]$ indicates a subgroup of M_{22} (fixing the two 1's) that has orbits of size seven and 15 on the remaining 22 points. In this case there are two conjugacy classes according to which of the two fixed points completes the orbit of size seven to an octad. Moreover, the symbol $[1; 8, 15]$ indicates a subgroup of M_{23} with two orbits of sizes eight and 15, while $[4^3; 4^3]$ means that the group has two orbits, with size three imprimitivity sets of size four. The symbol 6×2 denotes a set of 12 points with at the same time six imprimitivity sets of size two and two of size six, so forming a 6×2. The symbol 4×3 denotes a set of 12 points with at the same time four imprimitivity sets of size three and three of size four, so forming a 4×3. It is similar for the rest.

M_{23}	M_{22}	M_{12}	M_{11}
$[1;23]$ 23.11	$[1;1;1,21]$ M_{21}	$[12;12]$ $L_2(11)$	$[1;11;1,11]$ $L_2(11)$
$[1;1,22]$ M_{22}	$[1;1;2,4^5]$ $2^4.S_5$	$[1,11;12]$ M_{11}	$[1;1,10;6^2]$ M_{10}
$[1;2,21]$ $M_{21}.2$	$[1;1;6,16]$ $2^4.A_6$	$[12;1,11]$ M_{11}	$[1;2,9;3^4]$ $M_9.2$
$[1;3,4^5]$ $2^4(3\times S_5)$	$[1;1;7,15]$ A_7	$[2,10;6^2]$ $M_{10}.2$	$[1;3,8;4,8]$ $M_8.S_3$
$[1;7,16]$ $2^4.A_7$	$[1;1;7,15]$ A_7	$[6^2;2,10]$ $M_{10}.2$	$[1;5,6;2,10]$ S_5
$[1;8,15]$ A_8	$[1;1;8,14]$ $2^3.L_3(2)$	$[3,9;3^4]$ $M_9.S_3$	
$[1;11,12]$ M_{11}	$[1;1;11,11]$ $L_2(11)$	$[3^4;3,9]$ $M_9.S_3$	
	$[1;1;10,6^2]$ M_{10}	$[4,8;4,8]$ $M_8.S_4$	
		$[6\times 2;6\times 2]$ $2\times S_5$	
		$[4\times 3;4\times 3]$ $A_4\times S_3$	
		$[4^3;4^3]$ $4^2.D_{12}$	

Table I.3: Maximal subgroups of the Mathieu groups

§ II The Leech lattice

Definition II.0.1. Let $\mathbb{R}^{24}$ be spanned by the orthonormal basis $\{v_i \mid i \in \Omega\}$ then *the zero Leech lattice* Λ_0 is defined as a subset of $\mathbb{R}^{24}$ and it is spanned by the vectors $2v_C$, where C is in $\mathscr{C}_8$.

Remark II.0.2. We use the notations $v_\infty = (1, 0, 0, 0, 0, \ldots, 0)$, $v_0 = (0, 1, 0, 0, 0, \ldots, 0)$, $v_1 = (0, 0, 1, 0, 0, \ldots, 0)$, $v_2 = (0, 0, 0, 1, 0, \ldots, 0)$, ... , $v_{22} = (0, 0, 0, 0, 0, \ldots, 1)$, and

$$v_\Omega = \sum_{i=\infty}^{22} v_i.$$

Definition II.0.3. The set of all *n-ads* is denoted by Ω_n. As an example, $\Omega_4 = \{T \subset \Omega \mid |T| = 4\}$.

Theorem II.0.4. *The zero Leech lattice set Λ_0 contains all vectors $4v_T$, where T is in Ω_4 and $4v_i - 4v_j$, where i and j are in Ω.*

Proof. Let U, V and T be three tetrads of a sextet. This implies that

$$2v_{T+\mathrm{U}} + 2v_{T+V} - 2v_{\mathrm{U}+V} \in \Lambda_0$$

and

$$2v_{T+\mathrm{U}} + 2v_{T+V} - 2v_{\mathrm{U}+V} = 4v_T.$$

If we assume that T, U and V are the first three tetrads, respectively, then the sum is

$$\begin{array}{cccccc} 2222 & 2222 & 0000 & 0000 & 0000 & 0000 \\ + & & & & & \\ 2222 & 0000 & 2222 & 0000 & 0000 & 0000 \\ - & & & & & \\ 0000 & 2222 & 2222 & 0000 & 0000 & 0000 \\ = & & & & & \\ 4444 & 0000 & 0000 & 0000 & 0000 & 0000. \end{array}$$

Hence, $4v_T$ is in Λ_0. Now, let T be the set $\{i, k, t, q\}$ and T' be the set $\{j, k, t, q\}$, then we have

$$4v_T - 4v_{T'} = 4v_i - 4v_j.$$

From the previous proof we obtain this $4v_i - 4v_j$ is in Λ_0 wher i and j are in Ω. Then the sum is

$$\begin{array}{cccccc} 4444 & 0000 & 0000 & 0000 & 0000 & 0000 \\ - & & & & & \\ 0444 & 4000 & 0000 & 0000 & 0000 & 0000 \\ = & & & & & \\ 4000 & -4000 & 0000 & 0000 & 0000 & 0000. \end{array}$$

□

Theorem II.0.5. *A vector say v belongs to Λ_0 if and only if its coordinate-sum is a multiple of 16 and the coordinates not divisible by 4 fall in the places of a $\mathscr{C}-$set, the coordinates being all even.*

Proof. Suppose v is in Λ_0, then v can be expressed as $2kv_C$, where C is in the $\mathscr{C}_8$ and k is an integer, then

$$v = \sum_{i \in C} 2kv_i$$

$$2k\,2k\,2k\,2k \quad 2k\,2k\,2k\,2k \quad 0000 \quad 0000 \quad 0000 \quad 0000$$

The coordinate-sum is a multiple of 16 and the set of i is an octad so it is in the $\mathscr{C}-$set. Conversely, there are three cases which can be written $v = (x_1, \ldots, x_{24})$ as a vector which satisfies

$$\sum_{i=1}^{24} x_i = 16\, d, \quad \text{where} \;\; d \in \mathbb{Z}$$

and its coordinates not divisble by four fall in the places of a $\mathscr{C}-$set which are $4v_i - 4v_j$, where i, j are in Ω, $4v_T$, where T is in Ω_4 or $2v_C$, where C is in $\mathscr{C}_8$ and they are all in Λ_0. □

Definition II.0.6. The Leech lattice Λ is a set that is spanned by $v_\Omega - 4v_\infty$ and the zero Leech lattice's vectors. Moreover, the vector $v_\Omega - 4v_i$ is in Λ, where i is in Ω. To see this

$$v_\Omega - 4v_i = v_\Omega - 4v_\infty + 4v_\infty - 4v_i$$

and the vector $4v_\infty - 4v_i$ is in Λ_0, which is implies that the vector $v_\Omega - 4v_i$ is also in Λ. (The Leech lattice Λ may be defined as $\mathbb{Z}$-module.)

Theorem II.0.7. *The integral vector $x = (x_\infty, x_0, \ldots, x_{22})$, (that is each x_i is in $\mathbb{Z}$) is in the Leech lattice Λ if and only if*

i. *The coordinates x_i are all congruent modulo 2, to m say,*

$$x_i \equiv m \,(\text{mod } 2), \quad \infty \leqslant i \leqslant 22.$$

We may see this in different way as x is an even vector if $x_i \equiv 0$ (mod 2), and x is an odd vector if $x_i \equiv 1$ (mod 2).

ii. *The set of i for which x_i takes any given modulo 4 is a $\mathscr{C}-$set. That is the set $\{i \mid x_i \equiv k \text{ (mod 4)}\}$ belongs to the $\mathscr{C}-$set, for each k, where k is an integer.*

iii. *The coordinate-sum is congruent to $4m$ modulo 8,*

$$\sum_{i=\infty}^{22} x_i \equiv 4m \,(\text{mod } 8).$$

For any two vectors that are in the Leech lattice Λ the scalar product of them is equal to a multiple of 8 whereas, the scalar product of the vector and itself is equal to a multiple of 16.

Proof. Let x be in the Leech lattice Λ. Then by generating vector of Λ, the vector x can be expressed as $v_\Omega - 4v_\infty$ or $2v_C$, where C is in $\mathscr{C}_8$. If x is $v_\Omega - 4v_\infty$, then its coordinates are odd which could be assumed as

$$\mp 3 \pm 1 \pm 1 \pm 1 \quad \pm 1 \pm 1 \pm 1 \pm 1 \quad \cdots \quad \pm 1 \pm 1 \pm 1 \pm 1$$

since $x_i \equiv 1 \,(\mathrm{mod}\, 2)$, the set of i, where $x_i \equiv 1 (\mathrm{mod}\, 4)$ is Ω which is in the $\mathscr{C}-$set, and

$$\sum x_i = 20 \equiv 4 \pmod 8.$$

Hence, the vector x satisfies (i), (ii) and (iii). Now, if x is $2v_C$ then its coordinates are even which may be assumed as

$$2\,2\,2\,2 \quad 2\,2\,2\,2 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0$$

since $x_i \equiv 0 \,(\mathrm{mod}\ 2)$, the set of i, where $x_i \equiv 2 \,(\mathrm{mod}\, 4)$ is an octad which is in the $\mathscr{C}-$set and

$$\sum x_i = 16 \equiv 0 \pmod 8.$$

Hence, the vector x satisfies (i), (ii) and (iii). In general, it is true for any linear combinations of elements of the Leech lattice Λ. Conversely, from [Con3]. If the vector x satisfies (i), (ii) and (iii) we can find v which has

$$\sum x_i = 16\, d, \quad \text{where} \;\; d \in \mathbb{Z}$$

and its coordinates are even, such that $x - k(v_\Omega - 4v_\infty) = v$ and by using Theorem II.0.5, we get v is in Λ_0, which implies that x is in Λ. There are many cases here and they cannot all be written, but they are all similar calculations. We explain just two cases; one odd case and one even case. For the odd case we choose x as

$$-3\,1\,1\,1 \quad -3 - 3 - 3 - 3 \quad -3 - 3 - 3 - 3 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1.$$

Firstly, each of $x_i \equiv 1 \,(\mathrm{mod}\, 2)$, the set of i, where $x_i \equiv 1 \,(\mathrm{mod}\, 4)$ is Ω which is in the $\mathscr{C}-$set and

$$\sum x_i = -12 \equiv 4 \pmod 8$$

which shows that the vector x satisfies (i), (ii) and (iii). Now,

$$\begin{array}{cccccc}
-3\,1\,1\,1 & -3-3-3-3 & -3-3-3-3 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 \\
- & & & & & \\
1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 \\
+ & & & & & \\
4\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\
= & & & & & \\
0\,0\,0\,0 & -4-4-4-4 & -4-4-4-4 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0
\end{array}$$

we get

$$x-(v_\Omega-4v_\infty)=-4v_C$$

and

$$\sum -4v_C=-32.$$

For the even case we may choose x as

$$0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 2\,2\,2\,2 \quad 2\,2\,2\,2 \quad 2\,2\,2\,2$$

Firstly, each of $x_i \equiv 0 \pmod 2$, the set of i, where $x_i \equiv 2 \pmod 4$ is dodecad which is in the $\mathscr{C}-$set and

$$\sum x_i = 24 \equiv 0 \pmod 8.$$

This shows that the vector x satisfies (i), (ii) and (iii).

$$\begin{array}{cccccc}
0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 2\,2\,2\,2 & 2\,2\,2\,2 & 2\,2\,2\,2 \\
- & & & & & \\
2\,2\,2\,2 & 2\,2\,2\,2 & 2\,2\,2\,2 & 2\,2\,2\,2 & 2\,2\,2\,2 & 2\,2\,2\,2 \\
+ & & & & & \\
8\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\
= & & & & & \\
6-2-2-2 & -2-2-2-2 & -2-2-2-2 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0
\end{array}$$

we get

$$x - 2(v_\Omega - 4v_\infty) = 6v_\infty - 2v_i - 2v_j - 2v_k - 2v_C.$$

We may assume that i, j and k are in the first tetrad, so we have

$$\sum 6v_\infty - 2v_i - 2v_j - 2v_k - 2v_C = -16.$$

$\square$

Definition II.0.8. The subset of Leech lattice Λ that consists of all vectors which have scalar product equal to $16n$, is denoted by Λ_n.

$$\Lambda_n = \{x \in \Lambda \mid x \cdot x = 16n\}.$$

Remark II.0.9. The first Leech lattice set Λ_1 is an empty set, if it were not then there would exist a vector which satisfies $x \cdot x = 16$. From Theorem II.0.7 x is in Λ, then its coordinates must be all even or odd. There is no way to have a vector x with all its coordinates odd and $x \cdot x = 16$. If we assume that all coordinates of x are even, then there are two possibilities which are

$$\pm 2 \pm 2 \pm 2 \pm 2 \quad 0000 \quad 0000 \quad 0000 \quad 0000 \quad 0000.$$

But it is not satisfied the second condition in Theorem II.0.7 since the set of i, where $x_i \equiv 2 \pmod 4$ is a tetrad which is not in the $\mathscr{C}-$set. Another possible way is that,

$$\pm 4000 \quad 0000 \quad 0000 \quad 0000 \quad 0000 \quad 0000.$$

But it does not satisfy the third condition in Theorem II.0.7 since $8 \nmid 4$. Hence, Λ_1 is an empty set. Now, suppose x is in Λ_2, then $x \cdot x = 32$. The only way to have a vector x with its coordinates are all odd is

$$\mp 3 \pm 1 \pm 1 \pm 1 \quad \pm 1 \pm 1 \pm 1 \pm 1 \quad \cdots \quad \pm 1 \pm 1 \pm 1 \pm 1.$$

Notice that x also is in Λ since each $x_i \equiv 1 \pmod 2$, the set of i, say S, where $x_i \equiv 1 \pmod 4$ is Ω which is in the $\mathscr{C}-$set, and

$$\sum x_i = 20 \equiv 4 \pmod 8.$$

Moreover, there are 2^{12} ways for choosing S and 24 positions for (∓ 3), thus, there are $24 \cdot 2^{12}$ vectors of this shape. Whereas, if its coordinates are all even than there are three possibilities which are

$$\pm 4 \pm 2 \pm 2 \pm 2 \quad \pm 2\,0\,0\,0 \quad 0\,0\,0\,0 \quad \cdots \quad 0\,0\,0\,0.$$

But it does not satisfy the second condition in Theorem II.0.7 since the set of i, where $x_i \equiv 2 \pmod 4$ is a tetrad which is not in the $\mathscr{C}-$set. The second possibility is that

$$\pm 2 \pm 2 \pm 2 \pm 2 \quad \pm 2 \pm 2 \pm 2 \pm 2 \quad \cdots \quad 0\,0\,0\,0.$$

It is satisfied that $x_i \equiv 0 \pmod 2$, the set, say O of i, where $x_i \equiv 2 \pmod 4$ is an octad, and the sign $+$ must be evenly to be

$$\sum x_i \equiv 0 \pmod 8$$

Moreover, there are 759 way for choosing O and 2^7 to choose sign $(+)$ evenly. Hence, there are $2^7 \cdot 759$ vectors. The last possibility is that

$$\pm 4 \pm 4\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

It is satisfied that $x_i \equiv 0 \pmod 2$, the coordinates are divided by four and

$$\sum x_i = 0 \quad (\text{mod } 8)$$

Furthermore, there are $\binom{24}{2}$ ways for choosing the position of (± 4) and 2^2 to choose the sign, thus, there are $2^2 \cdot \binom{24}{2}$ vectors.

Remark II.0.10. The order of Λ_2, Λ_3 and Λ_4 are $2^2 \cdot \binom{24}{2} + 2^7 \cdot 759 + 24 \cdot 2^{12} = 196560$, $2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 16773120$, $2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 = 398034000$, respectively, see table II.1.

II.1 The zero Conway group Co_0

II.1.1 The zero Conway group Co_0 and the Leech lattice

Definition II.1.1. The group of all orthogonal transformations of $\mathbb{R}^{24}$ fixing zero vector and preserving the Leech lattice Λ as whole is called *the zero Conway group* Co_0, and it is not a simple group but it has simple subgroups, see table II.7.

Class	Shape	No.
Λ_2^2	$(2^8\ 0^{16})$	$2^7 \cdot 759$
Λ_2^3	$(3\ 1^{23})$	$2^{12} \cdot 24$
Λ_2^4	$(4^2\ 0^{22})$	$2^2 \cdot \binom{24}{2}$
Λ_3^2	$(2^{12}\ 0^{12})$	$2^{11} \cdot 2576$
Λ_3^3	$(3^3\ 1^{21})$	$2^{12} \cdot \binom{24}{3}$
Λ_3^4	$(4\ 2^8\ 0^{15})$	$2^8 \cdot 759 \cdot 16$
Λ_3^5	$(5\ 1^{23})$	$2^{12} \cdot 24$
Λ_4^{2+}	$(2^{16}\ 0^8)$	$2^{11} \cdot 759$
Λ_4^{2-}	$(2^{16}\ 0^8)$	$2^{11} \cdot 759 \cdot 15$
Λ_4^3	$(3^5\ 1^{19})$	$2^{12} \cdot \binom{24}{5}$
Λ_4^4	$(4^4\ 0^{20})$	$2^4 \cdot \binom{24}{4}$
Λ_4^{4+}	$(4^2\ 2^8\ 0^{14})$	$2^9 \cdot 759 \cdot \binom{16}{2}$
Λ_4^{4-}	$(4\ 2^{12}\ 0^{11})$	$2^{12} \cdot 2576 \cdot 12$
Λ_4^5	$(5\ 3^2\ 1^{21})$	$2^{12} \cdot \binom{24}{3} \cdot 3$
Λ_4^6	$(6\ 2^7\ 0^{16})$	$2^7 \cdot 759 \cdot 8$
Λ_4^8	$(8\ 0^{23})$	$2^1 \cdot 24$

Table II.1: The vectors of Λ_2, Λ_3 and Λ_4

Remark II.1.2. A permutation π of Ω, it can be extended to an orthogonal transformation of $\mathbb{R}^{24}$ defined by $v_i\pi = v_{i\pi}$. Moreover, if S is a subset of Ω, then ε_S is an orthogonal transformation of $\mathbb{R}^{24}$ define by

$$v_i\varepsilon_S = v_i\,(i \notin S)\,\text{or}\, - v_i\,(i \in S).$$

Theorem II.1.3. *The following conditions on an element λ of Co_0 are equivalent:*

i. $v_i\lambda = \pm v_j$, *where i, j are in Ω, and some sign ($\pm$).*

ii. $\lambda = \pi\varepsilon_S$, *where π is in M_{24} and a set S is in $\mathscr{C}$.*

These operations form a subgroup $N = 2^{12} : M_{24}$.

Proof. Let λ be $\pi\varepsilon_S$, where π is in M_{24} and S is in $\mathscr{C}$. Then we have

$$v_i\lambda = v_i\pi\varepsilon_S = v_{i\pi}\varepsilon_S = -v_{i\pi}\ (i \in S)\,\text{or}\,v_{i\pi}\ (i \notin S).$$

By assuming $i\pi = j$, where i, j are in Ω and π is a permutation of Ω. Now, we show (i) implies (ii). Let $v_i\lambda$ be $\pm v_j$, then the orthogonal matrix of λ is

$$\begin{bmatrix} & & & \cdots & 0 & \cdots & \\ & & & \cdots & 0 & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \\ 0 & \cdots & 0 & \cdots & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \\ 0 & \cdots & \pm 1 & 0 & & \cdots & 0 \\ & \cdots & 0 & \cdots & 0 & \cdots & \end{bmatrix}$$

Now, the vector $4v_i+4v_k$ under λ is four times the sum of v_i and v_k that is

k^{th}

$$0\,0\,0\,0 \quad \cdots \quad \cdots \quad \pm 4 \cdots 0 \quad \pm 4 \cdots \quad 0\,0\,0\,0.$$

From table II.1, we can see this vector is in Λ_2. This means that λ permutes elements with $(+)$ or $(-)$ so we can write it as $\lambda = \pi\varepsilon_S$, where S is a set and π is a permutation. We need to show that π is in M_{24} that is π preserves $\mathscr{C}$. Let C be in $\mathscr{C}$, then we have

$$2v_C\lambda = 2\sum v_i\lambda = \pm 2\sum v_j = 2v_{C'},$$

where C and C' are in $\mathscr{C}$, this implies that the permutation π preserves $\mathscr{C}$, thus, π is in M_{24}. Now, if a vector v is in the Leech lattice Λ then v can be expressed as $v_\Omega - 4v_\infty$

$$-3\,1\,1\,1 \quad 1\,1\,1\,1 \quad \cdots \quad 1\,1\,1\,1.$$

It can be seen that the set S of i, where $x_i \equiv 3 \pmod 4$ is Ω which is in the $\mathscr{C}-$set.

$$v_\Omega - 4v_\infty\lambda = \pm v_{\Omega\pi} - 4v_{\infty\pi} = (v_\Omega - 4v_\infty)\pi\varepsilon_S,\ \text{where}\ S \in \mathscr{C}.$$

Hence,

$$\lambda = \pi\varepsilon_S,\ \text{where}\ S \in \mathscr{C}.$$

$\square$

II.1.2 The maximal subgroup N of Co_0

Corollary II.1.4. *The group $N = \{\lambda = \pi\varepsilon_S \mid \pi \in M_{24}, S \in \mathscr{C}\}$ is a subgroup of Co_0 and it has order $2^{12} \cdot |M_{24}|$.*

Proof. Let λ_1 be $\pi_1\varepsilon_{C_1}$ and λ_2 be $\pi_2\varepsilon_{C_2} \in N$, where π_1, π_2 are in M_{24} and C_1, C_2 are in $\mathscr{C}$, we need to show that $\lambda_1 \circ \lambda_2$ is in N and λ_1^{-1} is also in N. Now,

$$\lambda_1 \circ \lambda_2 = \pi_1\varepsilon_{C_1}\pi_2\varepsilon_{C_2} = \pi_1 \circ \pi_2\varepsilon_{C_1}\varepsilon_{C_2}$$

since M_{24} is a group, we can assume that $\pi_1 \circ \pi_2 = \pi'$ which is in M_{24}, and $\varepsilon_{C_1}\varepsilon_{C_2} = \varepsilon_{C_1+C_2}$, where $C_1 + C_2$ is in $\mathscr{C}$. Hence, $\lambda_1 \circ \lambda_2$ is in N. Similarly, we can show that λ_1^{-1} is in N. Thus, N is a subgroup of Co_0. Moreover, N can be considered as the group M_{24} acting on the elements of the elementary abelian group E of order 2^{12} (notice that $E = \{\varepsilon_C \mid C \in \mathscr{C}\}$ and $|\mathscr{C}| = 2^{12}$). Therefore, N has order $2^{12} \cdot |M_{24}|$. □

Remark II.1.5. If Ξ is a sextet of T, then we define a map $\eta : \mathbb{R}^{24} \to \mathbb{R}^{24}$ by

$$v_i \to v_i - \frac{1}{2}(v_i + v_j + v_k + v_t), \text{ where} i \in T = \{i, j, k, t\} \in \Xi.$$

From Remark I.5.14, any two octads intersect in 0, 2, 4 or 8 points, and let the sextet be

$$\Xi = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6, \quad \text{where } T_i \subseteq \Omega \text{ and } 1 \leqslant i \leqslant 6.$$

An octad, say O is either the union of two tetrads, O intersects in three points with one tetrad of the sextet and O intersects in one point with the rest, or O intersects in two points with only four of the sextet. In our case, we may assume that the octad O is unoin of the first two tetrads (T_1 and T_2), O intersects in three points with the first tetrad T_1 and one point with the rest or O intersects in two points with the first four tetrads and it is disjoint of the last two tetrads. We get the following

$$x_1 = \;\; 2\,2\,2\,2 \quad 2\,2\,2\,2 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

which means that the vector x_1 cuts the sextet as $\begin{pmatrix}4 & 4 & 0 & 0 & 0 & 0\end{pmatrix}$. Applying η on x_1 we get

$$2 \to 2 - \frac{1}{2}(2 + 2 + 2 + 2) = -2, \quad \text{and} \quad 0 \to 0,$$

which we have

$$x_1\eta = \ \overline{2}\,\overline{2}\,\overline{2}\,\overline{2}\quad \overline{2}\,\overline{2}\,\overline{2}\,\overline{2}\quad 0\,0\,0\,0\quad 0\,0\,0\,0\quad 0\,0\,0\,0\quad 0\,0\,0\,0.$$

In the same way we get

$$x_2 = \ 2\,2\,2\,0\quad 2\,0\,0\,0\quad 2\,0\,0\,0\quad 2\,0\,0\,0\quad 2\,0\,0\,0\quad 2\,0\,0\,0$$

this means that the vector x_2 cuts the sextet as $(3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)$.

$$x_2\eta = \ \overline{1}\,\overline{1}\,\overline{1}\,\overline{3}\quad 1\,\overline{1}\,\overline{1}\,\overline{1}\quad 1\,\overline{1}\,\overline{1}\,\overline{1}\quad 1\,\overline{1}\,\overline{1}\,\overline{1}\quad 1\,\overline{1}\,\overline{1}\,\overline{1}\quad 1\,\overline{1}\,\overline{1}\,\overline{1}$$

$$x_3 = \ 2\,2\,0\,0\quad 2\,2\,0\,0\quad 2\,2\,0\,0\quad 2\,2\,0\,0\quad 0\,0\,0\,0\quad 0\,0\,0\,0$$

which means that the vector x_3 cuts the sextet as $(3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)$.

$$x_3\eta = \ 0\,0\,\overline{2}\,\overline{2}\quad 0\,0\,\overline{2}\,\overline{2}\quad 0\,0\,\overline{2}\,\overline{2}\quad 0\,0\,\overline{2}\,\overline{2}\quad 0\,0\,0\,0\quad 0\,0\,0\,0$$

$$x_4 = \ \overline{3}\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1$$

which means that the vector x_4 cuts the sextet as $(0 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4\,)$.

$$x_4\eta = \ \overline{3}\,1\,1\,1\quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}$$

($\overline{n}$ denotes $-n$). Notice that $x_2\eta$ is not in the Leech lattice Λ since $x_i \equiv 1 = m \pmod 2$, but

$$\sum x_i = -16 \not\equiv 4 = 4m \pmod 8$$

Hence, η does not belong to Co_0. Now, if ξ_T is $\eta\varepsilon_T$, where T is in Ξ (we can assume that T is T_1), then ξ_T can be considered as η with changing the sign of T, (in our case T is T_1).

Then we get

$$
\begin{array}{rl}
x_1 = & 2\,2\,2\,2 \quad 2\,2\,2\,2 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \\[2ex]
x_1\xi_T = & 2\,2\,2\,2 \quad \overline{2}\,\overline{2}\,\overline{2}\,\overline{2} \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \\[2ex]
x_2 = & 2\,2\,2\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0 \\[2ex]
x_2\xi_T = & 1\,1\,1\,3 \quad 1\,\overline{1}\,\overline{1}\,\overline{1} \quad 1\,\overline{1}\,\overline{1}\,\overline{1} \quad 1\,\overline{1}\,\overline{1}\,\overline{1} \quad 1\,\overline{1}\,\overline{1}\,\overline{1} \quad 1\,\overline{1}\,\overline{1}\,\overline{1} \\[2ex]
x_3 = & 2\,2\,0\,0 \quad 2\,2\,0\,0 \quad 2\,2\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 2\,0\,0\,0 \\[2ex]
x_3\xi_T = & 0\,0\,2\,2 \quad 0\,0\,\overline{2}\,\overline{2} \quad 0\,0\,\overline{2}\,\overline{2} \quad 0\,0\,\overline{2}\,\overline{2} \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \\[2ex]
x_4 = & \overline{3}\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \\[2ex]
x_4\xi_T = & 3\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}
\end{array}
$$

since $x_1\xi_T$, $x_2\xi_T$, $x_3\xi_T$ and $x_4\xi_T$ are in the Leech lattice Λ, this implies that ξ_T is in Co_0. However, ξ_T is not in N, since T is not in the $\mathscr{C}-$set which means that N is a proper subgroup of Co_0.

Remark II.1.6. We notice from Remark I.6.7 that is α, β, γ and δ are permutations of Ω that generate M_{24}. Now, we define them from $\mathbb{R}^{24} \to \mathbb{R}^{24}$ by

$$v_i\alpha = v_{i\alpha},$$

where $\alpha = (\infty)$ (0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22).

$$v_i\beta = v_{i\beta} = v_{2i},$$

where $\beta = (\infty)$ (15 7 14 5 10 20 17 11 22 21 19) (0) (3 6 12 1 2 4 8 16 9 18 13).

$$v_i\gamma = v_{i\gamma} = v_{1/-i},$$

where $\gamma = (\infty\ 0)$ (15 3) (7 13) (14 18) (5 9) (10 16) (20 8) (17 4) (11 2) (22 1) (21 12) (19 6).

$$v_i\delta = v_{i\delta} = v_{9i^3}\ (i \notin Q) \ \text{ or } \ v_{i^3/9}\ (i \in Q),$$

where $\delta = (\infty)$(14 17 11 19 22) (15) (20 10 7 5 21) (0) (18 4 2 6 1) (3) (8 16 13 9 12).

$$v_i\varepsilon = v_i\varepsilon_Q = v_{i\varepsilon_Q} = v_i\ (i \notin Q) \ \text{ or } \ -v_i\ (i \in Q),$$

where i is in Ω and $Q = \{0, 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$. Moreover, ξ can be considered as ξ_T, where T can be chosen as $\{0, 3, \infty, 15\}$ which is fixed by δ. Then our sextet is

$$\Xi = \{\{0, 3, \infty, 15\}, \{14, 20, 18, 8\}, \{17, 10, 4, 16\}, \{11, 7, 2, 13\}, \{19, 5, 6, 9\}, \{22, 21, 1, 12\}\}.$$

Remark II.1.7. The group N is generated by $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$, in fact if $\pi\varepsilon_C$ is in N, then π is in M_{24} which means that π can be expressed as linear combination of α, β, γ and δ. Also ε_C can be expressed as linear combination of ε_Q.

Remark II.1.8. The stabiliser of k points in M_{24} is a group M_{24-k} on $24 - k$ points where k is less than or equal to five, so the stabiliser of k vectors v_i in N_{24} is a group

$$N_{24-k} = E_{12-k} : M_{24-k}.$$

A splitting extension of an elementary group E_{12-k} of order 2^{12-k} by the group M_{24-k}, where k is less than or equal to five. Notice that if an element α in Ω then the stabiliser of α of N on Ω is

$$N_\alpha = E_{11} : M_{23}, \quad \text{see table II.2}$$

No. of point	Stabiliser of M_{24}	Order	Stabiliser of $N_{24} = N$	Order
A point	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	N_{23}	$2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
Two	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	N_{22}	$2^{17} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
Three	M_{21}	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	N_{21}	$2^{15} \cdot 3^2 \cdot 5 \cdot 7$
Four	M_{20}	$2^6 \cdot 3 \cdot 5$	N_{20}	$2^{14} \cdot 3 \cdot 5$

Table II.2: Stabiliser groups of N

Remark II.1.9. Table II.3 shows that the orbits of Λ_2, Λ_3 and Λ_4 under group N.

Theorem II.1.10. *The stabilisers of* Λ_2^2, Λ_2^3 *and* Λ_2^4 *in* N *are* $2^5 : 2^4.A_8$, M_{23} *and* $2^{10} : M_{22}.2$, *respectively.*

Proof. Let a vector x be in Λ_2^2, assume our x

2 2 2 2   2 2 2 2   0 0 0 0   0 0 0 0   0 0 0 0   0 0 0 0

No.	Orbits
Three	Λ_2^2, Λ_2^3, Λ_2^4
Four	Λ_3^2, Λ_3^3, Λ_3^4, Λ_3^5
Eight	Λ_4^2, Λ_4^3, Λ_4^4, Λ_4^{4+}, Λ_4^{4-}, Λ_4^5, Λ_4^6, Λ_4^8

Table II.3: Orbits under N

and σ be in $N_x = \{\sigma \in N \mid x\sigma = x\}$, so we can express σ as $\pi\varepsilon_C$, where C is in $\mathscr{C}$ and π is in M_{24}, which fixes an octad. From table I.2, we have π is in $2^4.A_8$, and for the sign we have 2^5. Thus, σ belongs to the group $2^5 : 2^4.A_8 = N_x$. Now, let y be in Λ_2^3, assume our y

$$\bar{3}\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1$$

and λ be in $N_y = \{\lambda \in N \mid y\lambda = y\}$, so we can express λ as $\pi\varepsilon_C$, where C is in $\mathscr{C}$ and π is in M_{24}, which fixes a point. Again from table I.2, we have π is in M_{23}. Thus, λ belongs to the Mathieu group $M_{23} = N_y$. Similarly, if z is in Λ_2^4, we might assume that our z is

$$4\,4\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0$$

and γ is in $N_z = \{\gamma \in N : z\gamma = z\}$, then we can express γ as $\pi\varepsilon_C$, where C is in $\mathscr{C}$ and π belongs to M_{24}, which fixes a dyad. In particular, from table I.2, π belongs to $M_{22}.2$, and for the sign is 2^{10}. Thus, γ is in the group $2^{10} : M_{22}.2 = N_z$. □

Theorem II.1.11. *The group N is not a normal subgroup of Co_0.*

Proof. Let σ be in N, ξ_T be $\eta\varepsilon_T$ and take a vector $8v_\infty$

$$8\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

Then by applying ξ_T^{-1} to $8v_\infty$, we have

$$\bar{4}\,4\,4\,4 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

Now, we may choose σ as $\pi\varepsilon_C$, where $T \cap C = \{0\}$, so applying σ on our vector we get shape $(4^2\ (-4^2)\ 0^{20})$, now by applying ξ_T we have

$$\overline{4}\,\overline{4}\,4\,4 \ \rightarrow\ \overline{4}\,\overline{4}\,4\,4.$$

This means that $\xi_T^{-1}\sigma\xi_T$ is not in N, hence N is not a normal subgroup. □

Remark II.1.12. In 1969, Conway proved that N is a maximal subgroup of Co_0.

Definition II.1.13. The set of all vectors adjacent to the vector x and distinct from x is denoted by $\Lambda(x)$, and $x^{\perp} = \Lambda(x) \cup \{x\}$.

Theorem II.1.14. *The group H that is generated by N and ξ_T is transitive on Λ_2 and H_x is transitive on $\Lambda_2(x)$.*

Proof. We need to prove that H is transitive on Λ_2, that is we need to prove that for any x and y in Λ_2 there exists an element λ which is in H such that $x = y\lambda$. From table I.1, there are three shapes of a vectors in Λ_2. As we can see from above, x_2 has shape $(2^8\ 0^{16})$ which is in Λ_2^2, and ξ_T takes x to a vector which has shape $(1^8\ 3^1\ (-1^{15}))$ which is in Λ_2^3. Moreover, the vector $x = 4v_i - 4v_j$, where i is not equal to j, and they are in two different tetrads

$$x_5 = \quad 4\,0\,0\,0 \quad 4\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0$$

$$x_5\xi_T = \quad \overline{2}\,2\,2\,2 \quad 2\,\overline{2}\,\overline{2}\,\overline{2} \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

Notice that ξ_T takes x_5 to a vector which has shape $((-2^4)\ 2^4\ 0^{16})$ that is in Λ_2^2. Hence, the subgroup H of Co_0 is transitive on Λ_2. Now, we show that H_x is transitive on $\Lambda_2(x)$, where x is in Λ_2. Let the vector x be $v_{-\Omega} - 4v_{\infty}$, then there exists an element α which is in N_x and H_x with order 23 such that $y\alpha \neq y$ for all y in $\Lambda_2(x)$. This means that $y^{H_x} = \Lambda_2(x)$ that is H_x is transitive on $\Lambda_2(x)$. Moreover, $|\Lambda_2(x)| = 93150$. □

Theorem II.1.15. *The zero Conway group Co_0 has order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.*

Proof. By using the Orbit-Stabiliser Theorem, we get $|H| = |x^H||y^{H_x}||H_{x,y}|$ since H is transitive on Λ_2, then we get this $|x^H| = |\Lambda_2|$ since H_x is transitive on $\Lambda_2(x)$ then we get $|y^{H_x}| = |\Lambda_2(x)|$. Thus, $|H| = |\Lambda_2||\Lambda_2(x)||H_{x,y}|$ for any orthogonal pair x and y that are in Λ_2. Let the vector x be $4v_i + 4v_j$, y be $4v_i - 4v_j$ and λ be in $H_{x,y}$, then if λ fixes x and y then v_i is fixed by λ. Hence, λ is in N so, $H_{x,y} = N_{x,y} = N_{22}$ which is a group of order $2^{10} \cdot |M_{22}|$. Thus, $|H| = 196560 \cdot 93150 \cdot 2^{10} \cdot |M_{22}|$, and this shows that N is a proper subgroup of H, which is only possible if $H = Co_0$. $\square$

Theorem II.1.16. *The zero Conway group Co_0 is transitive on the two sets Λ_3 and Λ_4. Moreover, it is generated by N together with any element ξ_T.*

Proof. We need to prove that Co_0 is transitive on Λ_3 and Λ_4, that is we need to prove that for any x and y in Λ_i there exists λ which is in Co_0 such that $x = y\lambda$. From table II.1, there are four shapes of a vectors in Λ_3 which are $(2^{12}\ 0^{12})$, $(3^3\ 1^{21})$, $(4\ 2^8\ 0^{15})$ and $(5\ 1^{23})$, then we have the following

$$x_1 = \quad 2\,2\,2\,0 \quad 2\,2\,2\,0 \quad 2\,2\,2\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0 \quad 2\,0\,0\,0$$

$$x_1\xi_T = \quad 1\,1\,1\,3 \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{3} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{3} \quad 1\,\bar{1}\,\bar{1}\,\bar{1} \quad 1\,\bar{1}\,\bar{1}\,\bar{1} \quad 1\,\bar{1}\,\bar{1}\,\bar{1}.$$

Notice that ξ_T takes x_1 to a vector which has shape $(1^6\ 3\ (-3^2)\ (-1^{15}))$ which is in Λ_3^3.

$$x_2 = \quad \bar{3}\,1\,1\,1 \quad \bar{3}\,1\,1\,1 \quad \bar{3}\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1$$

$$x_2\xi_T = \quad 5\,\bar{1}\,\bar{1}\,1 \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1}$$

we have $x_2\xi_T$ a vector which has shape $(5^1\ (-1^{22})\ 1^1)$ that is in Λ_3^5.

$$x_3 = \quad 4\,0\,0\,0 \quad 2\,2\,0\,0 \quad 2\,2\,0\,0 \quad 2\,2\,0\,0 \quad 2\,2\,0\,0 \quad 0\,0\,0\,0$$

$$x_3\xi_T = \quad \bar{2}\,2\,2\,2 \quad 0\,0\,\bar{2}\,\bar{2} \quad 0\,0\,\bar{2}\,\bar{2} \quad 0\,0\,\bar{2}\,\bar{2} \quad 0\,0\,\bar{2}\,\bar{2} \quad 0\,0\,0\,0$$

we have $x_3\xi_T$ which has shape $((-2^9)\ 2^3\ 0^{12})$ that is in Λ_3^2.

$$x_4 = \quad 5\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1$$

$$x_4\xi_T = \quad \bar{1}\,3\,3\,3 \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} \quad \bar{1}\,\bar{1}\,\bar{1}\,\bar{1}$$

we have $x_4\xi_T$ to vector which has shape $(3^3\ (-1^{21}))$ that is in Λ_3^3. Hence, the zero Conway group Co_0 is transitive on Λ_3. It is similarly with Λ_4. Now, let our vector x be $8v_\infty$

$$8\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0.$$

Then there are 48 images of this vector under N which are $(\pm 8v_i)$, where i is in Ω. If an element σ in Co_0 such that the image of $8v_\infty$ under σ is $(\pm 8v_i)$, then σ belongs to N. Now, for an element λ in Co_0 there exists μ which is in H such that the image of $8v_\infty$ under λ is equal to $8v_\infty\sigma$ which implies that λ is in $N\mu$. $\square$

Corollary II.1.17. *The index of N in Co_0 is the order of Λ_4 divides by 48.*

Proof. The stabiliser of a point of N on Λ_4 is N_x, where x belongs to Λ_4, and notice that $(Co_0)_x = N_x$ then $[Co_0 : N_x] = |x^{Co_0}|$, since Co_0 is transitive on Λ_4 then $|x^{Co_0}| = |\Lambda_4|$. From Remark II.1.8 we have $|N_x| = 2^{11}\cdot|M_{23}|$. Hence, $[Co_0 : N_x] = [Co_0 : N]\cdot[N : N_x]$, this implies that $[Co_0 : N] = \frac{[Co_0:N_x]}{[N:N_x]}$. Thus, $[Co_0 : N] = \frac{|\Lambda_4|}{48}$, where $[N : N_x] = 48$. $\square$

Remark II.1.18. Notice that the shape $(8^1\ 0^{23})$ under η goes to a vector whose shape is $(4^3\ (-4^1)\ 0^{20})$, and the row in the matrix of η have $\frac{1}{2}$ in every position T and zeros elswhere. But four times the sum or difference of this row and any other must be a vector in Λ_2 which has shape $(2^8\ 0^{16})$ or $(4^2\ 0^{22})$. Therefore, the matrix of η is

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

and the matrix of ξ_T is the direct sum of six 4×4 matrices of $\pm\frac{1}{2}$'s in the places of a sextet. Moreover, the generator matrix for the Leech lattice Λ is in figure II.1.1.

$$\frac{1}{\sqrt{8}}\left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc}
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\hline
0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
-3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]$$

Figure II.1.1: Generator matrix for the Leech lattice Λ

II.1.3 The Leech lattice modulo 2

Definition II.1.19. A *frame* or *cross* is a set of 48 vectors that have type four, saying x and y in the same frame if $x \cdot y = 0$ or $x = -y$. (That is, it consists of 24 orthogonal pairs of opposite vectors.)

Remark II.1.20. In fact one such cross is the standard cross which is this set $\{(8, 0, \ldots, 0)$, $(-8, 0, \ldots, 0)$, $(0, 8, \ldots, 0)$, $(0, -8, \ldots, 0)$, $(0, 0, 8 \ldots, 0), (0, 0, -8, \ldots, 0), \ldots,$ $(0, 0, 0, \ldots, 8)$, $(0, 0, 0, \ldots, -8)\}$ and it is the set Λ_4^8 which has order 48, see tables II.1 and II.4.

Corollary II.1.21. *The stabiliser of the standard cross is N.*

Proof. From table II.4, the standard cross is Λ_4^8, then we need to show

$$N = \{\sigma \in Co_0 \mid x\sigma \in \Lambda_4^8, \quad \text{where } x \in \Lambda_4^8\}$$

Let x be in Λ_4^8, σ be an element in Co_0 and $x\sigma$ be in Λ_4^8, then $x = 8v_i$ or $x = -8v_i$ and $x\sigma = 8v_k$ or $-8v_k$, where i and k are in Ω. Thus, σ is in N. Now, if σ is in N and x is in Λ_4^8 then $x\sigma = \pm 8v_i$, where i is in Ω. This implies that σ belongs to $N_{Co_0}(\Lambda_4^8)$. We can get the rest of the crosses by applying ξ_T for some T to this cross. □

Orbit	Type	4-vectors	$\mid$ Sign changes $\mid = 2^m$	$\mid$ Type $\mid \times 2^{12-m}$	Number
(I)	standard	$(8, 0^{23})$	2^{12}	1	1
(II)	sextet	$(4^4, 0^{20})$	2^{11}	$1771 \cdot 2$	3542
(III)	octad	$(-6, 2^7, 0^{16})$, $(0^8, 2^{16})$	2^6	$759 \cdot 2^6$	48576
(IV)	triad	$(5, -3^2, 1^{21})$, $(1^3, -3^5, 1^{16})$	2	$\binom{24}{3} \cdot 2^{11}$	4145152
(V)	involution	$(0^8, -2^2, 2^{14})$, $(2^8, 4^2, 0^{14})$	2^5	$759 \cdot 15 \cdot 2^7$	1457280
(VI)	duum	$(4, 0^{11}, -2, 2^{11})$	2	$1288 \cdot 2^{11}$	2637824

Table II.4: The six types of cross

Corollary II.1.22. *The permutation representation of the zero Conway group Co_0 on Λ_4 is imprimitive.*

Proof. If Δ is a frame and Δ is a subset of Λ_4 then Δ is a block of Λ_4 since $\Delta\lambda = \Delta$ or $\Delta\lambda \cap \Delta = \varnothing$, where λ is in Co_0. We may assume that our block is Λ_4^8 then let λ be in Co_0, if λ is in N then $\Lambda_4^8\lambda$ is equal to Λ_4^8, and if λ is not in N then $\Lambda_4^8\lambda$ is not equal to Λ_4^8 since the stabiliser of Λ_4^8 on Co_0 is N. Now suppose λ_1, λ_2 do not belong to N then there exists μ_1 and μ_2 which are in H such that λ_1 is in $N\mu_1$ and λ_2 is in $N\mu_2$. Now, if we assume that $8v_\infty\lambda_1$ is equal to $8v_\infty\lambda_2$ which implies that λ_1 is in $N\mu_2$ which is a contradiction so $\Lambda_4^8\lambda_1$ is not equal to $\Lambda_4^8\lambda_2$ and the imprimitive system is

$$\{\Lambda_4^8\,\lambda \mid \lambda \in Co_0\}.$$

$\square$

Theorem II.1.23. *Every vector of* Λ *is congruent modulo* 2Λ *to one of:*

i. *the zero vector;*

ii. *each vector of a unique pair* x, $-x$, *where* x *is in* Λ_2*;*

iii. *each vector of a unique pair* x, $-x$, *where* x *is in* Λ_3*;*

iv. *each of the* 48 *vectors of a coordinate-frame.*

Proof. Let x and y be in $\Lambda_0 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, $y \equiv x \pmod{2\Lambda}$ and y is not equal to either x or $(-x)$, then $y \pm x = 2w$, where w is in Λ, $w \cdot w = 16 \cdot d$ and d is greater than or equal to two.

$$(y \pm x)\cdot(y \pm x) = 4w\cdot w \quad \text{and} \quad 4w\cdot w = 4\cdot 16\cdot d, \qquad \text{where } d \geqslant 2.$$

Hence,

$$16\cdot 8 \leqslant 4w\cdot w = (y \pm x)\cdot(y \pm x) \leqslant (x\cdot x) + (y\cdot y) \leqslant 16\cdot 8.$$

This implies that

$$(y \pm x)\cdot(y \pm x) = 8\cdot 16$$

and $x \cdot y = 0$, so,

$$x\cdot x = y\cdot y = 16\cdot 4.$$

Thus, x and y are in Λ_4 which are both in the same coordinate-frame. Therefore, there are at least

$$1 + \frac{1}{2}|\Lambda_2| + \frac{1}{2}|\Lambda_3| + \frac{1}{48}|\Lambda_4| = 2^{24}$$

distinct classes of $\Lambda/2\Lambda$. □

Definition II.1.24. A vector x has type n, if it is in Λ_n and it has type n_{ab} if x is the sum of two vectors of types a and b.

Theorem II.1.25. *Every vector x of type n has some type n_{ab} in which $a + b = \frac{1}{2}(n + k)$, where $k = 0, 2, 3$ or 4 (corresponding to the cases of the Theorem above), and these possibilities are exclusive. The zero Conway group Co_0 is transitive on vectors of each of the types 2, 3, 5, 6_{22}, 6_{32}, 7, 8_{22}, 8_{32}, 8_{42}, 9_{33}, 9_{42}, 10_{33}, 10_{42}, 11_{43} and 11_{52}, which include all vectors of type n that is less than or equal to twelve.*

Proof. Let x have type n, y be in $\Lambda_0 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$ and $x \equiv y \pmod{2\Lambda}$ this implies that $\frac{1}{2}(x \pm y)$ are in Λ. Then $x = \frac{1}{2}(x+y) + \frac{1}{2}(x-y)$ and we may assume that $\frac{1}{2}(x+y)$ has type a and $\frac{1}{2}(x-y)$ has type b. Hence, the vector x has type n_{ab}. For the second part, we explain only two cases.

Case i: Let $x - y$ has type 6_{32}, x be in Λ_2, y be in Λ_3 and $x \cdot y = -8$. Then

$$(x + y) \cdot (x + y) = x \cdot x + 2x \cdot y + y \cdot y = 4(16)$$

Hence, $x + y$ is in Λ_4. We may assume that

$$x + y = \Big(8,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \ \ldots,\ \ \ldots,\ \ 0,\ 0,\ 0,\ 0\Big),$$

where

$$x = \Big(5,\ 1,\ 1,\ 1,\ \ \ldots,\ \ \ldots,\ \ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\Big)\,\varepsilon_C,$$

and

$$y = \Big(3,\ -1,\ -1,\ -1,\ \ \ldots,\ \ \ldots,\ \ \ldots,\ -1,\ -1,\ -1,\ -1\Big)\,\varepsilon_C,$$

where C is in $\mathscr{C}$ and ∞ does not belong to C. This is the only way to express $x+y$ as $(8, 0, \ldots, 0)$, where x is in Λ_3 and y is in Λ_2. We may assume that C is an empty set, so we have

$$x - y = \Big(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ \ldots,\ \ \ldots,\ \ \ldots, 2,\ 2,\ 2,\ 2\Big).$$

Case ii: Let $x-y$ has type five, x, y are in Λ_2 and $x \cdot y = -8$. Then

$$(x+y)\cdot(x+y) = x\cdot x + 2x\cdot y + y\cdot y = 3(16).$$

Hence, $x+y$ is in Λ_3. We may assume that

$$x + y = \Big(5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ \ldots,\ \ \ldots,\ \ 1,\ 1,\ 1,\ 1\Big),$$

where

$$x = \Big(3,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ 1,\ \ \ldots,\ \ \ldots,\ \ 1,\ 1\Big)$$

and

$$y = \Big(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ \ldots,\ \ \ldots,\ \ 0\Big)$$

we get $x-y$

$$\pm\Big(1,\ -3,\ -3,\ -3,\ -3,\ -3,\ -3,\ -3,\ 1,\ \ \ldots,\ \ \ldots,\ \ 1\Big).$$

Another way we may choose

$$x = \Big(4,\ 4,\ 0,\ 0,\ 0,\ 0,\ \ \ldots,\ \ \ldots,\ \ 0,\ 0\Big)$$

and

$$y = \Big(1,\ -3,\ 1,\ 1,\ \ \ldots,\ \ \ldots,\ \ \ldots, 1\Big)$$

we have $x-y$

$$\pm\Big(-3,\ -7,\ 1,\ 1,\ 1,\ \ \ldots,\ \ \ldots,\ \ \ldots, 1\Big).$$

The coordinates (-3) in the first case being in a special heptad S_7. In the first case we have u_1 as

$$1\,\overline{3}\,\overline{3}\,\overline{3}\quad \overline{3}\,\overline{3}\,\overline{3}\,\overline{3}\quad 1\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1\quad 1\,1\,1\,1.$$

Applying ξ_T and choosing T to be the first tetrad in Ξ, then we get

$$\overline{5}\,\overline{1}\,\overline{1}\,\overline{1} \quad 3\,3\,3\,3 \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}$$

and in the second case we have u_2 as

$$\overline{3}\,1\,1\,1 \quad \overline{7}\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1.$$

Applying ξ_T and choosing T to be the first tetrad in Ξ we get

$$3\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{5}\,3\,3\,3 \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{1}\,\overline{1}\,\overline{1}.$$

Which are plainly equivalent under M_{24} (that is we can find $\pi \in M_{24}$ such that $u_1\pi = u_2$). The other cases are similar calculations, we might use counting methods of Conway (1969) to do it more easily. □

II.1.4 Subgroups of the zero Conway group Co_0

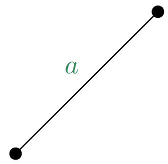
Definition II.1.26. The *infinite group* of all euclidean congruences of the Leech lattice Λ is Co_∞ or $\cdot\infty$, including translations. Any finite subgroup of Co_∞ fixes a point (not neccessarily a lattice point) and so there is a translation t of $\mathbb{R}^{24}$ such that $G^t \subseteq Co_0$.

Remark II.1.27. If S is the name of a simplex, we use $\cdot S$ for the subgroup of these elements of Co_∞ which fix every vertex of S, $^*S$ for a subgroup of elements fixing S as a whole, and $!S$ for a subgroup fixing centroid. It can be seen that $\cdot S$ is a subset of $^*S$ and $^*S$ is a subset of $!S$. Moreover, a simplex is then named by the types of its edges. Thus, the stabiliser of two points whose difference is a vector of type n is written $\cdot n$, the stabiliser of the vertices of a triangle whose sides have types a, b, c is written $\cdot abc$ and so on, see figure II.1.2. The particular groups $\cdot 1$, $\cdot 2$, $\cdot 3$ are also Co_1, Co_2, Co_3, respectively. In figure II.1.2, These groups are identified in table II.7. The group $\cdot 6_{23}$ is the stabiliser of

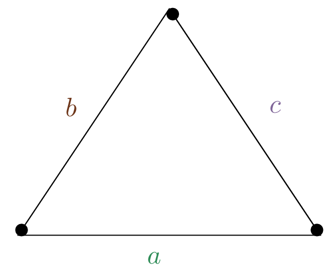
$$\Big(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\quad \ldots,\quad 2\Big)$$

and as we see in Theorem II.1.23 there are 24 ways to write $z = x + y$, where x has type two and y has type three in particular,

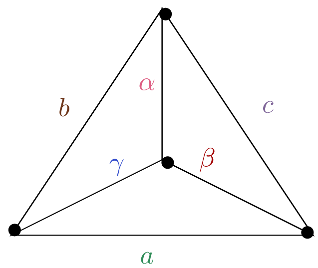
$$x = \Big(-3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big)$$



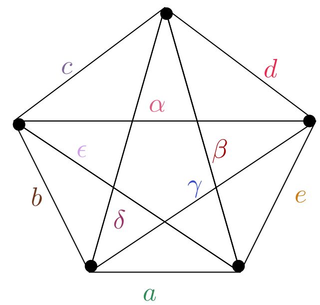
A simplex has type a



A simplex has type abc



A simplex has type $abc\alpha\beta\gamma$



A simplex has type $abcde\alpha\beta\gamma\delta\epsilon$

Figure II.1.2: Types

and

$$y = \Big(5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ \ldots,\ \ 1\Big).$$

Notice that,

$$y - x = \Big(8,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \ \ldots,\ \ 0\Big),$$

which means that $8v_i$ under $\cdot 6_{23}$ goes to $(\pm 8v_j)$ that is $\cdot 6_{23}$ is a subset of N and therefore, $\cdot 6_{23} = M_{24}$. At the same time $\cdot 632$ is the stabiliser of z, y and x. Notice that M_{23} is the stabiliser also of z, y and x, which implies that $M_{23} = \cdot 632$. The groups $\cdot 632$ and $\cdot 432$ are identical, since a parallelogram with sides $\sqrt{2}$ and $\sqrt{3}$ with one diagonal $\sqrt{6}$ might have the other diagonal $\sqrt{4}$, see figure II.1.3. Thus, $\cdot 632 = \cdot 432 = M_{23}$. The group M_{23} is contained in each of M_{24}, Co_2, Co_3 and Co_4. Firstly, the group M_{23} is a subgroup of M_{24}, then M_{23} is a subset of $\cdot 6_{23}$. Moreover, M_{23} fixes

$$z = \Big(5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ \ldots,\ \ 1\Big),$$

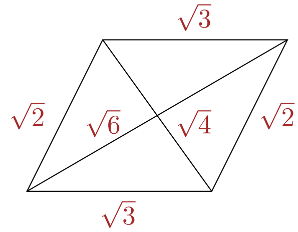


Figure II.1.3: Parallelogram with sides $\sqrt{2}$, $\sqrt{3}$ and diagonals $\sqrt{6}$, $\sqrt{4}$

which has type two

$$y = \Big(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\quad \ldots,\quad 2\Big),$$

which has type six and

$$y - z = \Big(-3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big),$$

which has type two hence M_{23} is a subset of Co_2. Notice that M_{23} fixes also

$$x = \Big(-3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big),$$

which has type two

$$y = \Big(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\quad \ldots,\quad 2\Big),$$

which type six and

$$x - y = \Big(5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big),$$

which has type three. Hence, M_{23} is a subset of Co_3. Similarly, M_{23} fixes

$$x = \Big(-3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big),$$

which has type two

$$z = \Big(5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\quad \ldots,\quad 1\Big),$$

which has type three and

$$z - x = \Big(8,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\quad \ldots,\quad 0\Big),$$

which has type four. Hence, M_{23} is a subset of Co_4.

II.1.5 The Higman-Sims group (HS)

The Higman-Sims group is a simple group which was discovered by D. G. Higman and C. C. Sims in 1968 [Gal]. It is the group of even permutations of a certain graph on 100 vertices, in particular, it is $\cdot 332$ group. Now, let X be $4v_i + v_\Omega$, Y be $4v_j + v_\Omega$ and Z be 0, where i, j are in two different tetrads. Then XYZ is a triangle of type 332. There are 100 points T such that $XYZT$ is of type 332222, which are $P = 4v_i + 4v_j$, 22 points of the form $Q_k = v_\Omega - v_k$, where k is in $\Omega\backslash\{i, j\}$, and 77 points the form $R_K = 2v_K$, where K is in $\mathscr{C}_8$. We say that two of these points are incident when their difference has type three, so we have

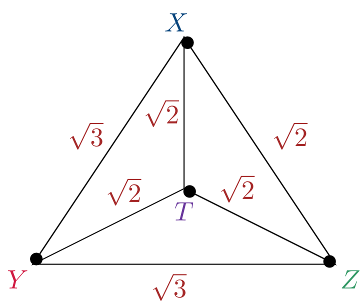


Figure II.1.4: Tetrahedra of type 332222

$$P - Q_k = 4v_i + 4v_j - v_\Omega + 4v_k = -v_\Omega + 4v_i + 4v_j + 4v_k.$$

It has shape $(3^3\ (-1^{21}))$, which has type three.

$$R_K - Q_k = 2v_K - v_{\Omega_4} + 4v_k, \quad \text{where } (k \in K),$$

$$\begin{array}{cccccc} 2\,2\,2\,2 & 2\,2\,2\,2 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ - & & & & & \\ 1\,\bar{3}\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 \\ = & & & & & \\ \bar{1}\,5\,\bar{1}\,\bar{1} & \bar{1}\,\bar{1}\,\bar{1}\,\bar{1} & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1. \end{array}$$

It has shape $((-1^7)\ (-5^1)\ 1^{16})$, which has type three, and

$$R_K - R_{K'}, \quad \text{where } (K \cap K' = \{i, j\}),$$

$$\begin{array}{cccccc} 2\,2\,2\,2 & 2\,2\,2\,2 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ - \\ 0\,0\,0\,0 & 0\,0\,2\,2 & 2\,2\,2\,2 & 2\,2\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ = \\ 2\,2\,2\,2 & 2\,2\,0\,0 & \overline{2}\,\overline{2}\,\overline{2}\,\overline{2} & \overline{2}\,\overline{2}\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0. \end{array}$$

It has shape $(2^6\ (-2^6)\ 0^{12})$, which has type three. Thus, incidences are (P, Q_k), (Q_k, R_K) where k is in K, and $(R_K, R_{K'})$ where K and K' intersect only in i and j. The incidence graph is visibly identical with the Higman-Sims graph in figure II.1.5.

Corollary II.1.28. *The stabiliser of a point of the Higman-Sims group HS on Λ is M_{22}.*

Proof. Let σ be in HS, then $X\sigma = X$, $Y\sigma = Y$ and $Z\sigma = Z$, where X, Y, Z are as above. Now, σ belongs to HS_P this implies that $P\sigma = P$ so we have

$$\begin{array}{rcccccc} X = & 1\,1\,1\,1 & 1\,1\,1\,1 & 5\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 \\ - \\ Y = & 1\,1\,5\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,1\,1\,1 \\ - \\ Z = & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ + \\ P = & 0\,0\,4\,0 & 0\,0\,0\,0 & 4\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ = \\ & 0\,0\,0\,0 & 0\,0\,0\,0 & 8\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0. \end{array}$$

We have $X - Y - Z + P = 8v_i$, where i is in Ω which means that σ is in N. In particular, HS_P is M_{22} which is the group of permutations of Ω and fixes two points say i and j. $\square$

Corollary II.1.29. *The Higman-Sims group HS has order $|M_{22}| \cdot 100$.*

Proof. By using the Orbit-Stabiliser Theorem, we have

$$|HS| = |HS_P| \cdot |P^{HS}| = |M_{22}| \cdot 100.$$

$\square$

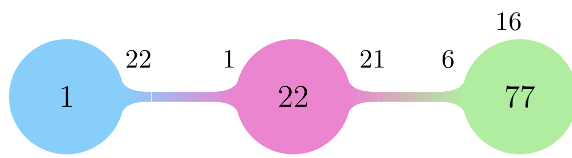


Figure II.1.5: The Higman-Sims graph

Corollary II.1.30. *The infinite Conway group Co_∞ is transitive on triangles of type* 332.

Proof. Let λ be in Co_∞ such that λ fixes X, Y and Z, but it does not fix P. Let $X\lambda$ be $2v_C$, $Y\lambda$ be $2v_D$, and $Z\lambda$ be 0 where C, D are in $\mathscr{C}_{12}$ and $C + D$ is in $\mathscr{C}_8$. Notice that λ exists, in particular, λ is in the group $\cdot 332$. Now suppose K the subgroup of N and σ is in K, then σ fixes X, Y and Z, but σ does not fix any of the 100 points $T\lambda$. Notice that each of $T\lambda - X\lambda$, $T\lambda - Y\lambda$, $T\lambda - Z\lambda$ has type two so $\lambda^{-1}H\lambda$ fixes each of X, Y and Z but not P. Hence, Co_∞ is transitive on tetrahedra of type 332222 and therefore it is transitive on triangles of type 332. $\square$

II.1.6 The McLaughlin group McL

The McLaughlin group is a simple group which was discovered by J. McLaughlin in 1969 [Gal]. It is a group of automorphisms of a graph on 275 vertices, in particular, it is $\cdot 322$ group. Also, it has order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. Now, let X be 0, Y be $4v_i + v_\Omega$ and Z be $-4v_j + v_\Omega$ where i is not equal to j. Then, YXZ has triangle of type 322. There are 275 points T such that $YXZT$ is of type 322222, which fall into three sets: 22 points in $\mathrm{U}_k = \{v_\Omega - 4v_k : k \in \Omega\backslash\{i,j\}\}$, 77 points in $V_K = \{2v_K : \{i,j\} \subseteq K \in \mathscr{C}_8\}$, and 179 points in $W_{K'} = \{2v_{K'} : K' \cap \{i,j\} = \{i\} \text{ and } K' \in \mathscr{C}_8\}$.

We say that two of these points are incident when their difference has type three, we have

$$W_{K'} - V_K = 2v_{K'} - 2v_K \quad \text{where} K \cap K' = \{i,t\} \text{ and } t \in \Omega\backslash\{j\}$$

Subgroup	Order	Index
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	100
$PSU_3(5) : 2$ (two)	$2^5 \cdot 3^2 \cdot 5^3 \cdot 7$	176
$L_3(4) : 2_1$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	1100
S_8	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	1100
$2^4 \cdot S_6$	$2^8 \cdot 3^2 \cdot 5$	3850
$4^3 : L_3(2)$	$2^9 \cdot 3 \cdot 7$	4125
M_{11} (two)	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	5600
$4 \cdot 2^4 : S_5$	$2^9 \cdot 3 \cdot 5$	5775
$2 \times A_6 \cdot 2^2$	$2^6 \cdot 3^2 \cdot 5$	15400
$5 : 4 \times A_5$	$2^4 \cdot 3 \cdot 5^2$	36960

Table II.5: Maximal subgroups of the Higman-Sims group HS

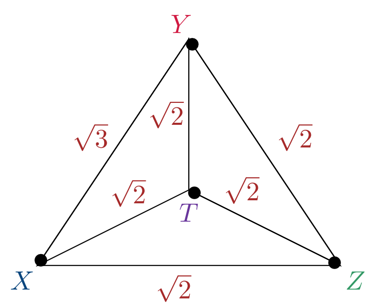


Figure II.1.6: Tetrahedra of type 322222

$$
\begin{array}{cccccc}
2\,2\,2\,2 & 2\,2\,2\,2 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\
- & & & & & \\
0\,0\,0\,0 & 0\,2\,2\,0 & 2\,2\,2\,2 & 2\,2\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\
= & & & & & \\
2\,2\,2\,2 & 2\,0\,0\,2 & \overline{2}\,\overline{2}\,\overline{2}\,\overline{2} & \overline{2}\,\overline{2}\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0.
\end{array}
$$

It has shape $(-2^6\ 2^6\ 0^{16})$, which has type three.

$$
\mathrm{U}_k - W_{K'}, \qquad \text{where } k \in K',
$$

$$\begin{array}{c} 1\,1\,1\,1 \quad 1\,\overline{3}\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \\ - \\ 2\,2\,2\,2 \quad 2\,2\,2\,2 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \\ = \\ \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{5}\,\overline{1}\,\overline{1} \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1. \end{array}$$

It has shape $(1^{16}\ (-5^1)\ (-1^7))$, which has type three, and

$$\mathrm{U}_k - V_K, \quad \text{where} \quad k \in K,$$

$$\begin{array}{c} 1\,1\,1\,1 \quad 1\,\overline{3}\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \\ - \\ 0\,0\,0\,0 \quad 0\,2\,2\,0 \quad 2\,2\,2\,2 \quad 2\,2\,0\,0 \quad 0\,0\,0\,0 \quad 0\,0\,0\,0 \\ = \\ \overline{1}\,\overline{1}\,\overline{1}\,\overline{1} \quad \overline{1}\,\overline{5}\,\overline{1}\,\overline{1} \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1 \quad 1\,1\,1\,1. \end{array}$$

It has shape $(1^{16}\ (-5^1)\ (-1^7))$, which has type three. So the incidences are (U_k, V_K), where k is in K, $(\mathrm{U}_k, W_{K'})$, where k is in K' and $(W_{K'}, V_K)$, where $(K \cap K' = \{i,t\}, t \in \Omega\backslash\{j\})$, see figure II.1.7.

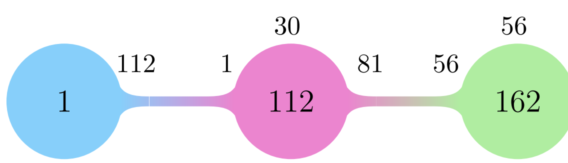


Figure II.1.7: The McLaughlins graph

II.1.7 The group $PS\mathrm{U}_6(2)$

The group $PS\mathrm{U}_6(2)$ is the projective special unitary group of a vector space V with six dimensions over a finite field $GF(4)$. In particular, it is ·222 group which is a simple group. Notice that if K is a normal subgroup of $6 \cdot M_{22}$ which has order two, then $6 \cdot M_{22}/H = 3 \cdot M_{22}$ which can be generated by some 6×6 unitary matrices over $GF(4)$, this implies that $3 \cdot M_{22}$ is a subgroup of $PS\mathrm{U}_6$.

Subgroup	Order	Index
$PSU_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	275
M_{22} (two)	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2025
$PSU_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	7128
$3^{1+4} : 2S_5$	$2^4 \cdot 3^6 \cdot 5$	15400
$3^4 : M_{10}$	$2^4 \cdot 3^6 \cdot 5$	15400
$L_3(4) : 2$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	22275
$2 \cdot A_8$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	22275
$2^4 : A_7$ (two)	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	22275
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	113400
$5^{1+2} : 3 : 8$	$2^3 \cdot 3 \cdot 5^3$	299376

Table II.6: Maximal subgroups of the McLaughlin group McL

II.2 The first Conway group Co_1

II.2.1 The first Conway group Co_1 and Leech groups

Definition II.2.1. The largest simple Conway group is Co_1 which is defined as the quotient of Co_0 by its center, $< \varepsilon_\Omega >= \{1, -1\}$ which define as

$$v_i \varepsilon_\Omega = -v_i (i \in \Omega) \quad \text{or} \quad v_i (i \notin \Omega).$$

Hence, $Co_0 / < \varepsilon_\Omega > \cong Co_1$. Moreover, the first Conway group Co_1 has order $|Co_0|/| < \varepsilon_\Omega > | = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

Theorem II.2.2. *The zero Conway group Co_0 acts transitively on ordered pairs of vectors of Λ_2 with any given scalar product.*

Proof. In order to prove this we need to show that the number of y that are in Λ_2 is 1, 4600, 47104, 93150, 47104, 4600 or 1 given that x is a given vector in Λ_2 and $x \cdot y$ equals 32, 16, 8, 0, -8, -16, or -32, respectively. Let the vector x be $4v_i + 4v_j$ where i, j are in Ω, then we have seven cases

Name	Order	Structure
·0	$2^{22}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$	$2.Co_1$
·1	$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$	Co_1
·2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$	Co_2
·3	$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$	Co_3
·4	$2^{18}\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	$2^{11}M_{23}$
·5	$2^8\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	$McL.2$
$\cdot 6_{22}$	$2^{16}\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	$PS\mathrm{U}_6(2).2$
$\cdot 6_{32}$	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	M_{24}
·7	$2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	HS
$\cdot 8_{22}$	$2^{18}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$	Co_2
$\cdot 8_{32}$	$2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	McL
$\cdot 8_{42}$	$2^{15}\cdot 3^2\cdot 5\cdot 7$	$2^{1+8}.A_8$
$\cdot 9_{33}$	$2^5\cdot 3^7\cdot 5\cdot 11$	$3^5.M_{11}.2$
$\cdot 9_{42}$	$2^7\cdot 3^2\cdot 5^3\cdot 7\cdot 11\cdot 23$	M_{23}
$\cdot 10_{33}$	$2^{10}\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	$HS.2$
$\cdot 10_{42}$	$2^{17}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$2^{10}.M_{22}$
$\cdot 11_{43}$	$2^{10}\cdot 3^2\cdot 5\cdot 7$	$2^4.A_8$
$\cdot 11_{52}$	$2^8\cdot 3^6\cdot 5\cdot 7$	$PS\mathrm{U}_4(3).2$
·222	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	$PS\mathrm{U}_6(2)$
·322	$2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	McL
·332	$2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	HS
·333	$2^4\cdot 3^7\cdot 5\cdot 11$	$3^5.M_{11}$
·422	$2^{17}\cdot 3^2\cdot 5\cdot 7\cdot 11$	$2^{10}.M_{22}$
·432	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	M_{23}
·433	$2^{10}\cdot 3^2\cdot 5\cdot 7$	$2^4.A_8$
·442	$2^{12}\cdot 3^2\cdot 5\cdot 7$	$2^{1+8}.A_7$
·443	$2^7\cdot 3^2\cdot 5\cdot 7$	$M_{21}.2$
·522	$2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	McL
·532	$2^8\cdot 3^6\cdot 5\cdot 7$	$PS\mathrm{U}_4(3).2$
·533	$2^4\cdot 3^2\cdot 5^3\cdot 7$	$PS\mathrm{U}_3(5)$
·542	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$	M_{22}
·633	$2^6\cdot 3^3\cdot 5\cdot 11$	M_{12}
*2 =!2	$2^{19}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$	$(\cdot 2)\times 2$
*3 =!3	$2^{11}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$	$(\cdot 3)\times 2$
*4	$2^{19}\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	$(\cdot 4)\times 2$
!4	$2^{22}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	$2^{12}.M_{24}$
!333	$2^7\cdot 3^9\cdot 5\cdot 11$	$3^6.2.M_{12}$
!442	$2^{15}\cdot 3^4\cdot 5\cdot 7$	$2^{1+8}.A_9$

Table II.7: Subgroups of the zero Conway group Co_0

- In the first case $x \cdot y = 32$, and y is in Λ_2. Then, the only way for choosing y which is in Λ_2 is x itself, then

$$y = 4v_i + 4v_j.$$

- In the second case $x \cdot y = 16$, and y is in Λ_2. Then, there are three possible shapes for y. If y has shape $(4^2\ 0^{22})$, then y can be expressed as $4v_j \pm 4v_k$, where k is not equal to j. There are 22 places in which k can fall and two possible sign changes. Therefore, there are $22 \cdot 2$ vectors. Or, $\pm 4v_h + 4v_j$, where h is not equal to i so there are also 22 places in which h can fall and two possible sign changes. Therefore, $22 \cdot 2$ vectors. If y has shape $(2^8\ 0^{16})$, then y can be expressed as $2v_K\varepsilon_C$, where $(\{i,j\} \subset K \in \mathscr{C}_8)$ and $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, from figure I.5.1, there are 77 octads that intersect in two points say i, j, and 32 possible sign changes. Hence, there are $77 \cdot 32$ vectors. If y has shape $(3^1\ (-1^{23}))$, then y can be expressed as $(4v_i - v_\Omega)\varepsilon_C$ where j is in C that is in $\mathscr{C}$, there are two possible positions for 3 and 2^{10} possible sign changes. Thus, there are $2 \cdot 2^{10}$ vectors, see table II.8.

Orbit	Length
$(4^2\ 0^{22})$	$2 \cdot 2 \cdot 22$
$(2^8\ 0^{16})$	$32 \cdot 77$
$(3^1\ (-1^{23}))$	$2 \cdot 2^{10}$

Table II.8: Orbit of Λ_2 with scalar product 16 under Co_0

- In the third case $x \cdot y = 8$, and y is in Λ_2. Then, there are two possible shapes for y. If y has shape $(2^8\ 0^{16})$, then y can be expressed as $2v_K\varepsilon_C$ where $(K \in \mathscr{C}_8$ $|K \cap \{i,j\}| = 1)$ and $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, from figure I.5.1, there are 176 octads that contain i, but not j and 2^7 possible sign changes. Similarly for $2v_{K'}$ where $|K' \cap \{i,j\}| = 1$, there are 176 octads that contain j, but not i and 2^7 possible sign changes. Now, if y has shape $(3^1\ (-1^{23}))$ then y can be expressed as $(4v_i - v_\Omega)\varepsilon_C$ where $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, there are two possible positions for 3 and 2^{10} possible sign changes. Therefore, there are $2 \cdot 2^{10}$ vectors, see table II.9.

Orbit	Length
$(2^8\ 0^{16})$	$2 \cdot 176 \cdot 2^7$
$(3^1\ (-1^{23}))$	$2 \cdot 2^{10}$

Table II.9: Orbit of Λ_2 with scalar product 8 under Co_0

- In the fourth case $x \cdot y = 0$, and y is in Λ_2. Then, there are three possibile shapes for y. If y has $(4^2 0^{22})$, then y can be expressed as $\pm(4v_i - 4v_j)$ which are two vectors. Or, $\pm(v_k + v_h)$, where k, h are not equal to either i or j, there are 22 places in which k can fall and 21 places in which h can fall, and four possible sign changes. Therefore, there are $22 \cdot 21 \cdot 4$. If y has shape $(2^8\ 0^{16})$, then y can be expressed as $2v_K\varepsilon_C$ where $(K \in \mathscr{C}_8,\ \{i,j\} \cap K = \varnothing)$ and $(C \in \mathscr{C})$ from figure I.5.1 there are 330 octads that do not contain i and j and 2^7 possible sign changes. Hence, there are $330 \cdot 2^7$ vectors. Or, $2v_{K'}\varepsilon_C$ where $(K' \in \mathscr{C}_8\ \{i,j\} \subset K')$ and $(|\{i,j\} \cap C| = 1)$. Again from figure I.5.1, there are 77 octads that contain i, j and 2^6 possible sign changes. Therefore, there are $77 \cdot 2^6$ vectors. Now, if y has shape $((-3^1)\ 1^{23})$ then y can be expressed as $(4v_k - v_\Omega)\varepsilon_C$ where k is not equal to either i or j and $(|\{i,j\} \cap C| = 1)$, there are 22 way for choosing k and 2^{11} possible sign changes. Thus, there are $22 \cdot 2^{11}$ vectors, see table II.10.

Orbit	Length
$(4^2\ 0^{22})$	$2 + 22 \cdot 21 \cdot 4$
$(2^8\ 0^{16})$	$330 \cdot 2^7 + 77 \cdot 2^6$
$((-3^1)\ 1^{23})$	$22 \cdot 2^{11}$

Table II.10: Orbit of Λ_2 with scalar product 0 under Co_0

- In the fifth case $x \cdot y = -8$, and y is in Λ_2. Then, there are two possible shapes for y. If y has shape $((-2^8)\ 0^{16})$, then y can be expressed as $-2v_K\varepsilon_C$ where $(K \in \mathscr{C}_8,\ |K \cap \{i,j\}| = 1)$ and $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, from figure I.5.1 there are 176 octads that contain i, but not j and 2^7 possible sign changes. Similarly for $-2v_K$ where $|K \cap \{i,j\}| = 1$, there are 176 octads that contain j, but not i and 2^7 possible sign changes. Now, if y has shape $((-3^1)\ 1^{23})$ then y can be expressed as $(v_\Omega - 4v_i)\varepsilon_C$ where $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, there are two possible positions for 3 and 2^{10} possible sign changes. Therefore, there are $2 \cdot 2^{10}$ vectors, see table II.11.

Orbit	Length
$((-2^8)\ 0^{16})$	$2 \cdot 176 \cdot 2^7$
$((-3^1)\ 1^{23})$	$2 \cdot 2^{10}$

Table II.11: Orbit of Λ_2 with scalar product (-8) under Co_0

- In the sixth case $x \cdot y = -16$, and y is in Λ_2. Then, there are three possible shapes for y. If y has shape $((-4^2)\ 0^{22})$, then y can be expressed as $-4v_j \pm 4v_k$, where k is not equal to j so there are 22 places which k can fall and two possible sign changes. Therefore, there are $22 \cdot 2$ vectors. Or, $\pm 4v_h - 4v_j$, where h is not equal to i, there are 22 places which h can fall and two possible sign changes. Therefore, there are $22 \cdot 2$ vectors. If y has shape $((-2^8)\ 0^{16})$, then y can be expressed as $-2v_K\varepsilon_C$, where $(\{i,j\} \subset K \in \mathscr{C}_8)$ and $(\{i,j\} \cap C = \varnothing, C \in \mathscr{C})$, from figure I.5.1 there are 77 octads that intersect in two points say i, j, and 32 possible sign changes. Hence, there are $77 \cdot 32$ vectors. If y has shape $((-3^1)\ 1^{23})$, then y can be expressed as $(v_\Omega - 4v_i)\varepsilon_C$ where j is in C that is in $\mathscr{C}$, there are two possible positions for 3 and 2^{10} possible sign changes. Therefore, there are $2 \cdot 2^{10}$ vectors, see table II.12.

- In the seventh case $x \cdot y = -32$, and y is in Λ_2. Then, the only way for choosing

Orbit	Length
$((-4^2)\ 0^{22})$	$2 \cdot 2 \cdot 22$
$((-2^8)\ 0^{16})$	$32 \cdot 77$
$((-3^1)\ 1^{23})$	$2 \cdot 2^{10}$

Table II.12: Orbit of Λ_2 with scalar product (-16) under Co_0

y which is in Λ_2 is $(-x)$

$$y = -4v_i - 4v_j.$$

Notice that these are the only possible scalar products for any two vectors x and y that are in Λ_2. Thus, the zero Conway group Co_0 is a transitive on ordered pairs of vectors in Λ_2. □

Definition II.2.3. The set of diameters of Λ_2, which has order $|\Lambda_2|/2 = 98280$ diameters, is denoted by $\overline{\Lambda}_2$. A diameter is a pair $\{x, -x\}$ where x is in Λ_2.

Corollary II.2.4. *The first Conway group Co_1 acts on the 98280 diameters in such a way that the stabiliser of any diameter has orbit of order 1, 4600, 47104 or 46575.*

Proof. We need to show that the length of orbit of an element $\{x, -x\}$ in $\overline{\Lambda}_2$ is 1, 4600, 47104 or 46575 by using that Co_0 is transitive on ordered pairs of vectors of Λ_2 with any given scalar product ± 32, ± 16, ± 8 or 0, respectively. Let us assume that $x = 4v_i + 4v_j$ and $G = Co_1$, if $x \cdot y = 32$, then we have $y = x$ and $\{x, -x\}\varepsilon_\Omega = \{-x, x\}$, thus

$$\{x, -x\}^G = \{4v_i + 4v_j, -4v_i - 4v_j\}.$$

If $x \cdot y = 16$ then by using Theorem II.2.2 there are four possibilties for y which are $4v_i \pm 4v_h$ where $(h \neq j)$, $\pm 4v_k + 4v_j$ where $(k \neq i)$, $2v_k\varepsilon_C$ where $(\{i, j\} \subset K,\ K \in \mathscr{C}_8)$, $(\{i, j\} \cap C = \varnothing,\ C \in \mathscr{C})$ and $(4v_i - v_\Omega)\varepsilon_C$ where j is in C that is in $\mathscr{C}$). Thus, $\{x, -x\}^G = \{\{4v_i + 4v_h, -4v_i - 4v_h\}, \{4v_i - 4v_h, -4v_i + 4v_h\}, \{4v_k + 4v_j, -4v_k - 4v_j\}, \{-4v_k + 4v_j, 4v_k - 4v_j\}, \{2v_K\varepsilon_C, -2v_K\varepsilon_C\}, \{(4v_i - v_\Omega)\varepsilon_C, (-4v_i + v_\Omega)\varepsilon_C\}\}$. From tables

II.8 and II.12 these are 4600 pairs. If $x \cdot y = 8$ then by using Theorem II.2.2 there are two possibilties for y which are $2v_K\varepsilon_C$ where ($K \in \mathscr{C}_8$, $|K \cap \{i,j\}| = 1$), ($\{i,j\} \cap C = \varnothing$, $C \in \mathscr{C}$), and $(4v_i - v_\Omega)\varepsilon_C$ where ($\{i,j\} \cap C = \varnothing$, $C \in \mathscr{C}$). Hence,

$$\{x, -x\}^G = \{\{2v_K\varepsilon_C, -2v_K\varepsilon_C\}, \{(4v_i - v_\Omega)\varepsilon_C, (-4v_i + v_\Omega)\varepsilon_C\}\}.$$

From tables II.9 and II.11 these are 47104 pairs.

If $x \cdot y = 0$ then by using Theorem II.2.2 again there are four possibilties for choosing y which are $\pm(4v_i - 4v_j)$, $\pm(4v_k + 4v_h)$ where $(k, h \neq i, j)$, $2v_K\varepsilon_C$ where ($\{i,j\} \subset K, K \in \mathscr{C}_8$), ($C \in \mathscr{C}$), $2v_{K'}\varepsilon_C$ where ($\{i,j\} \cap K' = \varnothing$, $K' \in \mathscr{C}_8$), ($|\{i,j\} \cap C| = 1$) and $(4v_k - v_\Omega)\varepsilon_C$ where $(k \neq i,j)$, ($|\{i,j\} \cap C| = 1$). Therefore, $\{x, -x\}^G = \{\{\pm(4v_i - 4v_j), \mp(4v_i - 4v_j)\}, \{2v_K\varepsilon_C, -2v_K\varepsilon_C\}, \{2v_{K'}\varepsilon_C, -2v_{K'}\varepsilon_C\}, \{(4v_i - v_\Omega)\varepsilon_C, (-4v_i + v_\Omega)\varepsilon_C\}$. From table II.10 these are $93150/2 = 46575$ pairs. □

Corollary II.2.5. *The first Conway group Co_1 acts primitively on $\overline{\Lambda}_2$.*

Proof. By transitivity of Co_0 on Λ_2 in Theorem II.1.15, for any non-trivial subset Δ of $\overline{\Lambda}_2$, we can find an element λ which is in Co_0, such that $\Delta \cap \Delta\lambda < \varepsilon_\Omega > \neq \varnothing$. Thus, Co_1 acts primitively on $\overline{\Lambda}_2$. □

Theorem II.2.6. *There is no λ in Co_0 such that λ has prime order greater then* 23.

Corollary II.2.7. *There is no λ in Co_0 such that λ has order* $13 \cdot 23$.

Theorem II.2.8. *The first Conway group Co_1 is a simple group.*

Proof. We need to show that Co_1 has no normal subgroups, if Co_1 is not a simple group, then Co_1 would have a normal subgroup say K, then $\{1, -1\} \subset K \subset Co_0$, K must act transitively on $\overline{\Lambda}_2$. Otherewise, its orbits would be imprimitive sets for Co_1 which is a contradiction with Corollary II.2.5, this implies that the order of K is divisible by 13. By using the Frattini argument, we have $Co_0 = N_{Co_0}(P)\ K$, where P is a Sylow 13-subgroup of K. Now, either K or $N_{Co_0}(P)$ has an element of order 23, say λ, if λ is in $N_{Co_0}(P)$, then Co_0 has a cyclic subgroup of order $13 \cdot 23$ which is a contradiction with Corollary II.2.7. This means that λ must be in K, so $K \cap N$ is a normal subgroup of N with order divisible by 23, hence this group is N but N is a maximal subgroup, so $K = N$. But, this is a contradiction with Theorem II.1.11 which is N is not a normal subgroup of Co_0. Thus, Co_1 is a simple group. □

II.2.2 The small Conway groups

Definition II.2.9. The second Conway group Co_2 is one of the twenty-six sporadic groups and it was discovered by John H. Conway around 1968 [Wil] as the group of automorphisms of the Leech lattice Λ. In particular, it is the stabiliser group of a vector which has type two and it has order $|Co_2| = |Co_1|/98280 = 2^{17} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23$.

Theorem II.2.10. *There are* 46575 *vectors in* Λ_2 *such that they have scalar product 0 with* $x = 4v_\infty - 4v_0$.

Proof. Let y be in Λ_2 and $y \cdot x = 0$ then, there are three possible shapes for y.

- If y has $(4^2\ 0^{22})$, then y can be expressed as $\pm(4v_\infty \pm 4v_0)$ which is a vector. Or, as $\pm(4v_i \pm 4v_j)$ where i and j are not equal to either ∞ or 0, there are 22 places in which i can fall and 21 places in which j can fall. Hence, there are $22 \cdot 21$ vectors.

- If y has $(2^8\ 0^{16})$, then y can be expressed as $2v_K\varepsilon_C$ where $(\{i,j\} \subseteq K, K \in \mathscr{C}_8)$ and $(C \cap \{i,j\} = \varnothing, C \in \mathscr{C})$, from figure I.5.1 there are 77 octads which intersect in two points say i, j and 2^5 possible sign changes. Thus, there are $77 \cdot 2^5$ vectors. Or, as $y = 2v_K\varepsilon_C$ where $(K \cap \{i,j\} = \varnothing, K \in \mathscr{C}_8)$, and C is in $\mathscr{C}$. Again from figure I.5.1, there are 330 octads which do not intersect in two points say i, j and 2^6 possible sign changes.

- If y has $((-3^1)\ 1^{23})$, then y can be expressed as $(v_\Omega - 4v_k)\varepsilon_C$ where k is not equal to either 0 or ∞. There are 22 places in which k can fall and 2^{10} possible sign changes. Hence, there are $22 \cdot 2^{10}$ vectors. Thus, we get table II.13.

Orbit	Length
$(4^2\ 0^{22})$	$1 + 21 \cdot 22$
$(2^8\ 0^{16})$	$77 \cdot 2^5 + 330 \cdot 2^6$
$((-3^1)\ 1^{23})$	$22 \cdot 2^{10}$

Table II.13: The 46575 vectors that are in Λ_2 and orthogonal with $4v_\infty - 4v_0$

□

Remark II.2.11. Notice that from Theorem II.1.10 the stabiliser of $4v_\infty - 4v_0$ is $2^{10} : M_{22} : 2$ which has index 46575 in Co_2. It is a maximal subgroup of Co_2 and generated by conjugates of abelian group 2^{10}. Hence, by using Iwasawa's Lemma we have that the second Conway group Co_2 is a simple group.

Subgroup	Order	Index
$PSU_6(2) : 2$	$2^{16} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	2300
$2^{10} : M_{22} : 2$	$2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	46575
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	47104
$2^{1+8} : S_6(2)$	$2^{18} \cdot 3^4 \cdot 5 \cdot 7$	56925
$HS : 2$	$2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	476928
$(2^4 \times 2^{(1+6)})A_8$	$2^{17} \cdot 3^2 \cdot 5 \cdot 7$	1024650
$PSU_4(3).D_8$	$2^{10} \cdot 3^6 \cdot 5 \cdot 7$	1619200
$2^{4+10}(S_5 \times S_3)$	$2^{18} \cdot 3^2 \cdot 5$	3586275
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	4147200
$3^{1+4} \cdot 2^{1+4} S_5$	$2^8 \cdot 3^6 \cdot 5$	45337600
$5^{1+2} : 4S_4$	$2^5 \cdot 3 \cdot 5^3$	3525451776

Table II.14: Maximal subgroups of the second Conway group Co_2

Definition II.2.12. The third Conway group Co_1 is one of the twenty-six sporadic groups and it was discovered by John H. Conway around 1968 [Wil] as the group of automorphisms of the Leech lattice Λ. In particular, it is the stabiliser group of a vector which has type three, and it has order$|Co_3| = |Co_1|/8386560 = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$.

Theorem II.2.13. *The third Conway group Co_3 is a doubly transitive on 276 letters.*

Proof. Let x be in Λ_3 which has shape $(5^1\ 1^{23})$, y, z be in Λ_2, and $\{z, y\}$ be a pair such that $z + y = x$, then there are 23 vectors, say y of shape $(4^2\ 0^{22})$ such that the first entry say y_∞ is not zero, and for shape $(2^8\ 0^{16})$ where y_∞ is not zero, there are 253 vectors, since that from figure I.5.1, the number of octads intersect in one point, say y_∞ is 253 octads. Now, let $\{y', z'\}$ be one of these pairs such that either $y - y'$ or $z - y'$ for each other pairs $\{z, y\}$. By assuming transitivity of Co_0 on type 322 and tetrahedra 322222, we get that Co_3 is a doubly transitive on 276 letters. □

Theorem II.2.14. *The stabiliser of a point of Co_3 on 276 letters is* $\cdot 322$.

Proof. Let our point be $y = 4v_\infty + 4v_0$, then it fixes under M_{22} by fixing ∞ and 0, hence there are four orbits for 276 pairs under M_{22} which has size 1, 22, 77 and 176 which is contained in M_{23} which fall into $1 + 22$, and $77 + 176$ orbits, McLaughlin group ($\cdot 322$) with 1, and $22 + 77 + 176$ orbits, and Higman-Sims group ($\cdot 332$) into $1 + 22 + 77$ and 176 orbits. □

Theorem II.2.15. *The third Conway group Co_3 is a simple group.*

Proof. Let S be a normal subgroup of Co_3, since $McL : 2$ is the stabiliser of Co_3 then either $S \cap McL = S$ or $H \cap McL = \{1\}$. If S is a subset of McL then, S has order $276 = 23 \cdot 3 \cdot 2$, this implies that S has a 23-sylow subgroup and it is abelian, say K. So K is a normal subgroup of M_{23}, but M_{23} is a simple group which is a contradiction. Now, for the second case S would have index two in Co_3, this implies that the 552 vectors as above in Theorem II.2.13. and we may assume our point x has shape $(5^1\ 1^{23})$ then the 552 vectors are as follows: 23 have shape $(4^2\ 0^{22})$ where the first entry, say y_∞ is four, 23 have shape $((-3)\ 1^{23})$ where y_∞ is one, 253 vectors have shape $(2^8\ 0^{16})$ where y_∞ is two, and 253 have shape $(3\ (-1^7)\ 1^{16})$ where y_∞ is three, they fall into two different sets with size 276 under S but this is impossible since than the scalar product would have a fixed value for any two vectors in the same set. □

Subgroup	Order	Index
$McL:2$	$2^8\cdot3^6\cdot5^3\cdot7\cdot11$	276
HS	$2^9\cdot3^2\cdot5^3\cdot7\cdot11$	11178
$PSU_4(3):2^2$	$2^9\cdot3^6\cdot5\cdot7$	37950
M_{23}	$2^7\cdot3^2\cdot5\cdot7\cdot11\cdot23$	48600
$3^5:(2\times M_{11})$	$2^5\cdot3^7\cdot5\cdot11$	128800
$2\cdot S_6(2)$	$2^{10}\cdot3^4\cdot5\cdot7$	170775
$PSU_3(5):S_3$	$2^5\cdot3^3\cdot5^3\cdot7$	655776
$3^{1+4}:4S_6$	$2^6\cdot3^7\cdot5$	708400
$2^4\cdot A_8$	$2^{10}\cdot3^2\cdot5\cdot7$	1536975
$L_3(4)\cdot D_{12}$	$2^8\cdot3^3\cdot5\cdot7$	2049300
$2\times M_{12}$	$2^7\cdot3^3\cdot5\cdot11$	2608200
$2^2.[2^7.3^2].S_3$	$2^{10}\cdot3^3$	17931375
$S_3\times L_2(8):3$	$2^4\cdot3^4\cdot7$	54648000
$A_4\times S_5$	$2^5\cdot3^2\cdot5$	344282400

Table II.15: Maximal subgroups of the third Conway group Co_3

Definition II.2.16. The number of orbits of the point stabiliser is called *the rank*.

II.2.3 The Hall-Janko group J_2

The Hall-Janko group (J_2) is a simple group which is one of the twenty-six sporadic groups, and it was discovered by Hall-Janko in 1967 [Gal] as a rank three permutation group on 100 points . The stabiliser of J_2 is $PSU_3(3)$, and J_2 has order $2^7\cdot3^3\cdot5^2\cdot7$.

Subgroup	Order	Index
$PSU_3(3)$	$2^5 \cdot 3^3 \cdot 7$	100
$3 \cdot PGL_2(9)$	$2^4 \cdot 3^3 \cdot 5$	280
$2^{1+4} : A_5$	$2^7 \cdot 3 \cdot 5$	315
$2^{1+4} : (3 \times S_3)$	$2^7 \cdot 3^2$	525
$A_4 \times A_5$	$2^4 \cdot 3^2 \cdot 5$	840
$A_5 \times D_{10}$	$2^3 \cdot 3 \cdot 5^2$	1008
$L_3(2) : 2$	$2^4 \cdot 3 \cdot 7$	1800
$5^2 : D_{12}$	$2^2 \cdot 3 \cdot 5^2$	2016
A_5	$2^2 \cdot 3 \cdot 5$	10080

Table II.16: Maximal subgroups of the Janko group J_2

II.2.4 The Suzuki group Suz

The Suzuki group (Suz) is a simple group which is one of the twenty-six sporadic groups and it was discovered by Suzuki in 1969 [Gal] as a rank three permutation group on 1782 points. The stabiliser of Suz is $G_2(4)$, and Suz has order $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Notice that the Suzuki group (Suz) does not relate to the Suzuki groups of Lie type.

Subgroup	Order	Index
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	1782
$3_2 \cdot PSU_4(3) : 2$	$2^8 \cdot 3^7 \cdot 5 \cdot 7$	22880
$PSU_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	32760
$2^{1+6} \cdot PSU_4(2)$	$2^{13} \cdot 3^4 \cdot 5$	135135
$3^5 : M_{11}$	$2^4 \cdot 3^7 \cdot 5 \cdot 11$	232960
$J_2 : 2$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	370656
$2^{4+6} : 3A_6$	$2^{13} \cdot 3^3 \cdot 5$	405405
$(A_4 \times L_3(4)) : 2$	$2^9 \cdot 3^3 \cdot 5 \cdot 7$	926640
$2^{2+8} : (A_5 \times S_3)$	$2^{13} \cdot 3^2 \cdot 5$	1216215
$M_{12} : 2$	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	2358720
$3^{2+4} : 2(A_{-4} \times 2^2).2$	$2^6 \cdot 3^7$	3203200
$(A_6 \times A_5) \cdot 2$	$2^6 \cdot 3^3 \cdot 5^2$	10378368
$(3^2 : 4 \times A_6) \cdot 2$	$2^6 \cdot 3^4 \cdot 5$	17297280
$L_3(3) : 2$ (two)	$2^5 \cdot 3^3 \cdot 13$	39916800
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	57480192
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	177914880

Table II.17: Maximal subgroups of the Suzuki group Suz

Subgroup	Order	Index
Co_2	$2^{17} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23$	98280
$3 \cdot Suz : 2$	$2^{14} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1545600
$2^{11} : M_{24}$	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	8282375
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	8386560
$2^{1+8} \cdot O_8^+(2)$	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$	46621575
$PSU_6(2) : S_3$	$2^{16} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11$	75348000
$(A_4 \times G_2(4)) : 2$	$2^{15} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13$	688564800
$2^{2+12} : (A_8 \times S_3)$	$2^{21} \cdot 3^3 \cdot 5 \cdot 7$	2097970875
$2^{4+12} \cdot (S_3 \times 3S_6)$	$2^{21} \cdot 3^4 \cdot 5$	4895265375
$3^2 \cdot PSU_4(3).D_8$	$2^{10} \cdot 3^8 \cdot 5 \cdot 7$	17681664000
$3^6 : 2M_{12}$	$2^7 \cdot 3^9 \cdot 5 \cdot 11$	30005248000
$(A_5 \times J_2) : 2$	$2^{10} \cdot 3^4 \cdot 5^3 \cdot 7$	57288591360
$3^{1+4} : 2PSU_4(2) : 2 \cdot 2$	$2^8 \cdot 3^9 \cdot 5$	165028864000
$(A_6 \times PSU_3(3)) \cdot 2$	$2^9 \cdot 3^5 \cdot 5 \cdot 7$	954809856000
$3^{3+4} : 2(S_4 \times S_4)$	$2^7 \cdot 3^9$	1650288640000
$A_9 \times S_3$	$2^7 \cdot 3^5 \cdot 5 \cdot 7$	3819239424000
$(A_7 \times L_2(7)) : 2$	$2^7 \cdot 3^3 \cdot 5 \cdot 7^2$	4910450688000
$(D_{10} \times (A_5 \times A_5).2).2$	$2^7 \cdot 3^2 \cdot 5^3$	28873450045440
$5^{1+2} : GL_2(5)$	$2^5 \cdot 3 \cdot 5^4$	69296280109056
$5^3 : (4 \times A_5).2$	$2^5 \cdot 3 \cdot 5^4$	69296280109056
$7^2 : (3 \times 2A_4)$	$2^3 \cdot 3^2 \cdot 7^2$	1178508165120000
$5^2 : 2A_5$	$2^3 \cdot 3 \cdot 5^3$	1385925602181120

Table II.18: Maximal subgroups of the first Conway group Co_1

Group	Describe
$GL_n(q)$	The general linear group with n-dimensional over $GF(q)$
$PGL_n(q)$	The projective linear group with n-dimensional over $GF(q)$
$SL_n(q)$	The special linear group with n-dimensional over $GF(q)$
$PSL_n(q) = L_n\|(q)$	The projective special linear group with n-dimensional over $GF(q)$
$G\mathrm{U}_n(q^2)$	The general unitary group with n-dimensional over $GF(q)$
$S\mathrm{U}_n(q^2)$	The special unitary group with n-dimensional over $GF(q)$
$PS\mathrm{U}_n(q^2) = \mathrm{U}_n(q^2)$	The projective special unitary group with n-dimensional over $GF(q)$
$GO_n(q)$	The orthogonal group with n-dimensional over $GF(q)$
$SO_n(q)$	The special orthogonal group with n-dimensional over $GF(q)$
$PSO_n(q)$	The projective special orthogonal group, n-dimensional over $GF(q)$
P^n or E	An elementary of order p^n
P^{1+2n}	An extraspecial group of order p^{1+2n}
Q_8	The quaternion group
D_n	A dihedral group of order $2n$
A_n	An alternating group of degree n
S_n	A symmetric group of dearee n
$A_n(q)$	The Chevalley group of degree n over $GF(q)$, $n \geqslant 1$
$B_n(q)$	The Chevalley group of degree n over $GF(q)$, $n \geqslant 2$
$C_n(q)$	The Chevalley group of degree n over $GF(q)$, $n \geqslant 3$
$D_n(q)$	The Chevalley group of degree n over $GF(q)$, $n \geqslant 4$
$G_2(q)$	The Chevalley group of degree 2 over $GF(q)$
$S_n(q)$	The symplectic group with 2n-dimensional over $GF(q)$
$S!$	The stabiliser of centroid of a set S
$^*S$	The stabiliser of a set S
$\cdot S$	The stabiliser of vertex of a set S
$\cdot n$	The stabiliser of vector of type n
$\cdot n_{ab}$	The stabiliser of any vectors whose difference is a vector of type n
$\cdot abc$	The stabiliser of a triangle of type a, b, c
$A \cdot B$	An (split or non-split) extension of the group A by the group B
$A : B$	A split extension of the group A by the group B
$A \cdot B$	A non-split extension of the group A by the group B
$\mathscr{C}$	
O	An octad

Table II.19: List of group notations

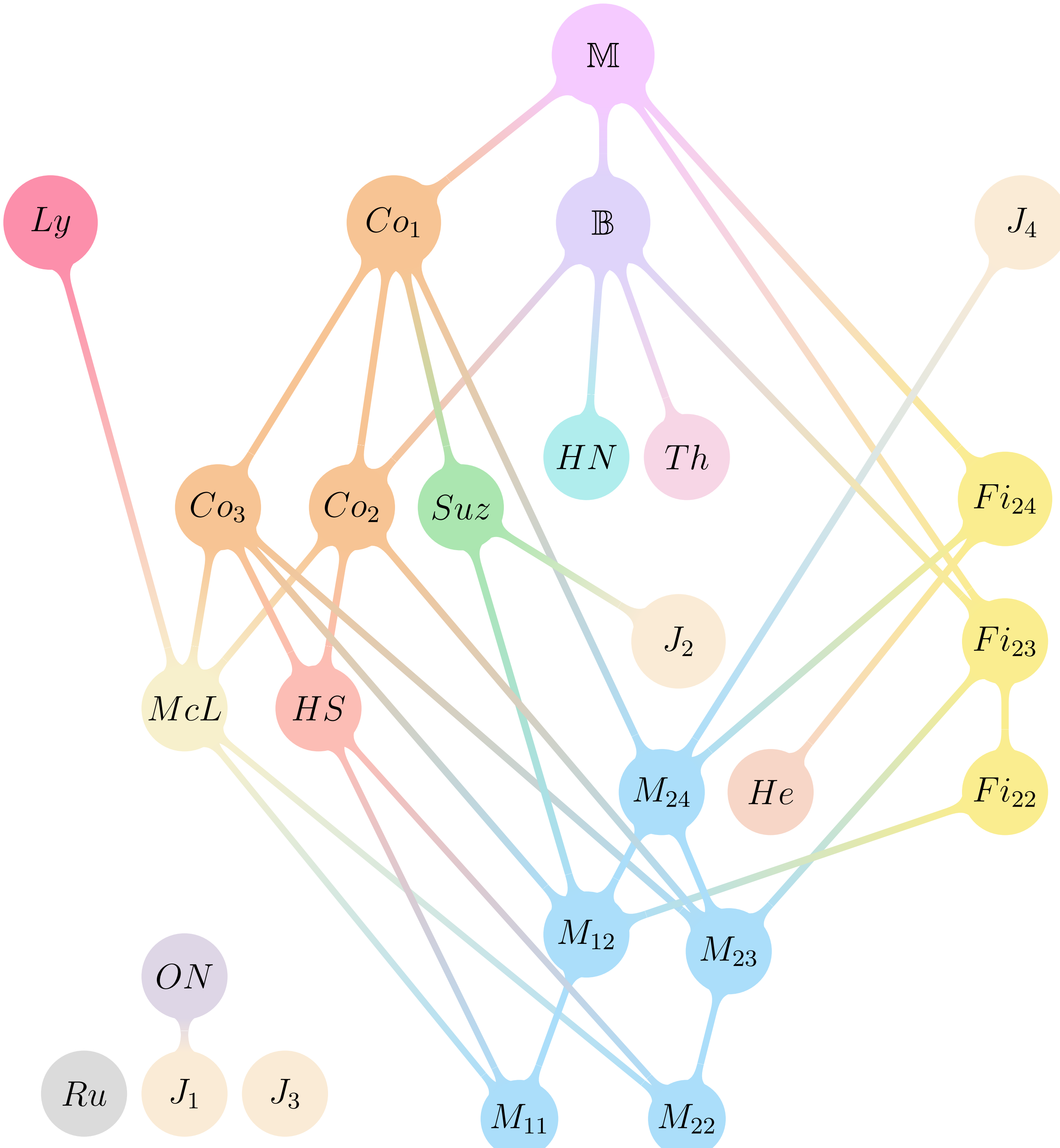


Figure II.2.1: Genetic relation between sporadic finite simple groups

Group	Describe	Order
M_{11}	The Mathieu group	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	The Mathieu group	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
J_1	The Janko group	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
M_{22}	The Mathieu group	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$J_2 = HJ$	The Hall-Janko group	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
M_{23}	The Mathieu group	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
HS	The Higman-Sims group	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
J_1	The Janko group	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
M_{24}	The Mathieu group	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
He	The Held group	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
McL	The McLaughlin group	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Ru	The Rudvalis group	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
Suz	The Suzuki group	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
ON	The O'Nan group	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_3	The Conway group	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_2	The Conway group	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	The Fischer group	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
HN	The Harada-Norton group	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Fi_{23}	The Fischer group	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Co_1	The Conway group	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Th	The Thompson group	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
Ly	The Lyons group	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
Fi_{24}	The Fischer group	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
J_4	The Janko group	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
$\mathbb{B}$	The Baby monster group	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$\mathbb{M}$	The Monster group	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Table II.20: The twenty-six sporadic groups

Bibliography

[BSW] B. Bollobas, P. Sarnak and C. T. C. Wall. *S*poradic groups. (Cambridge University Press, Cambridge, 1994) 250–299.

[Gor] D. Gorenstein. *F*inite Simple Groups. (Springer Science, New York, 1982).

[Gal] J. A. Gallian. "The Search for Finite Simple Groups". *M*athematics Magazine 49(4) (1976), 163–180.

[Con1] J. H. Conway, "A group of order 8,315,553,613,086,720,000", *B*ulletin of the London Mathematics Society 1 (1969), 79–88.

[Con2] J. H. Conway. "A characterisation of Leech's Lattice". *I*nventiones Mathematics 7 (1969), 137–142.

[Con3] J. H. Conway, "Three lectures on exceptional groups", *F*inite Simple Groups (ed. M. B. Powell and G. Higman). (Academic Press, New York, 1971), 215–247.

[CCNPW] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *A*tlas of Finite group (Oxford University. Press, London, 1985).

[CoSl] J. H. Conway and N. J. A Sloane. *S*phere packings, Lattices and groups, 2nd edn. (Springer, New York, 1993).

[Mck] J. Mckay. *F*inite Groups coming of Age. (The American Mathematical Society, 1985).

[Boy] L. Boya. "Introduction to Sporadic Group". *S*ymmetry, Integrability and Geometry: Methods and Applications 009 (2011), 14–15.

[BGHLH] M. Bhargava, R. Guralnick, G. Hiss, K. Lux and P. Huu Tiep. *Finite Simple Groups: Thirty Years of the Atlas and Beyond*. (Princeton University, Princeton, 2015), 66–69.

[GrGr] M. Grannell and T. Griggs. "An introduction to Steiner systems". *Mathematical Spectrum* 26 no.3 (1994), 74–80.

[Row] P. J. Rowley. "Cross Orbits". *LMS Journal of Computation and Mathematics* 7 (2004), 219–223.

[Wil] R. A. Wilson. *The Finite Simple Groups*. (Springer, London, 2009).

[Cur1] R. T. Curtis. "A New Combinatorial Approach to M_{24}". *Proceedings of the Cambridge Philosophical Society* 79 (1976), 25-42.

[Cur2] R. T. Curtis. "On Subgroup of Co_0, I: Lattice Stabilisers". *Journal of algebra* 27 (1973), 549–573.

[CuFa] R. T. Curtis, and B. Fairbairn. "Symmetric representation of the elements of the Conway group Co_0". *Symbolic Computation* 44 (2009), 1044–1067.

[Gri] R. L. Jr. Griess. *Twelve Sporadic Groups*. (Springer, New York, 1998) 95–111.