# Poisson Algebras II, Non-commutative Algebra

Maram Alossaimi (Supervisor: Prof. Vladimir Bavula)

School of Mathematics and Statistics, malossaimi1@sheffield.ac.uk\maram.alosaimi@gmail.com

## 1. Introduction

A commutative algebra $D$ over a field $K$ is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\}: D\times D\to D$, called a *Poisson bracket*, such that

1. $\{a,b\}=-\{b,a\}$ for all $a,b\in D$ (anti-commutative),
2. $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0$ for all $a,b,c\in D$ (Jacobi identity), and
3. $\{ab,c\}=a\{b,c\}+\{a,c\}b$ for all $a,b,c\in D$ (Leibniz rule).

Definition. Let $D$ be a Poisson algebra. An ideal $I$ of the algebra $D$ is a *Poisson ideal* of $D$ if $\{D,I\}\subseteq I$. We denote by $\langle a\rangle$ the Poisson ideal of $D$ generated by the element $a$. Moreover, a Poisson ideal $P$ of the algebra $D$ is a *Poisson prime ideal* of $D$ provided

$$IJ\subseteq P \Rightarrow I\subseteq P \quad\text{or}\quad J\subseteq P$$

where $I$ and $J$ are Poisson ideals of $D$. A set of all Poisson prime ideals of $D$ is called the *Poisson spectrum* of $D$ and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let $D$ be a Poisson algebra over a field $K$. A $K$-linear map $\alpha: D\to D$ is a *Poisson derivation* of $D$ if $\alpha$ is a $K$-derivation of $D$ and

$$\alpha(\{a,b\})=\{\alpha(a),b\}+\{a,\alpha(b)\}\quad\text{for all } a,b\in D.$$

A set of all Poisson derivations of $D$ is denoted by $\mathrm{PDer}_K(D)$.

## 2. How do we get our Poisson algebra class $\mathcal{A}$?

Lemma. [Oh] *Let $D$ be a Poisson algebra over a field $K$, $c\in K$, $u\in D$ and $\alpha,\beta\in\mathrm{PDer}_K(D)$ such that*

$$\alpha\beta=\beta\alpha \quad\textit{and}\quad \{d,u\}=(\alpha+\beta)(d)u \quad\textit{for all } d\in D. \tag{1}$$

*Then the polynomial ring $D[x,y]$ becomes a Poisson algebra with Poisson bracket*

$$\{d,y\}=\alpha(d)y,\quad \{d,x\}=\beta(d)x \quad\textit{and}\quad \{y,x\}=cyx+u \quad\textit{for all } d\in D. \tag{2}$$

*The Poisson algebra $D[x,y]$ with Poisson bracket (2) is denoted by $(D;\alpha,\beta,c,u)$.*

## 3. How do we classify $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A}=(K[t];\alpha,\beta,c,u)$, where $K$ is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a,b\}=0$ for all $a,b\in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0=\{d,u\}=(\alpha+\beta)(d)u \quad\text{for all } d\in K[t],$$

which implies that precisely one of the three cases holds:

(Case I: $\alpha+\beta=0$ and $u=0$), (Case II: $\alpha+\beta=0$ and $u\neq 0$) or (Case III: $\alpha+\beta\neq 0$ and $u=0$) .

## 4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha,\beta\in\mathrm{PDer}_K=\mathrm{Der}_K(K[t])=K[t]\partial_t$ such that $\alpha=f\partial_t$ and $\beta=g\partial_t$, where $f,\ g\in K[t]\backslash\{0\}$, $\partial_t=d/dt$ then*

$$\alpha\beta=\beta\alpha \quad\textit{if and only if}\quad g=\frac{1}{\lambda}f \quad\textit{for some}\quad \lambda\in K^{\times}:=K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha=f\partial_t$, $\beta=\frac{1}{\lambda}f\partial_t$, $c\in K$, $u\in K[t]$, where $f\in K[t]$ and $\lambda\in K^{\times}$. Then we have the class of Poisson algebras $\mathcal{A}=K[t][x,y]=(K[t];\alpha=f\partial_t,\beta=\frac{1}{\lambda}f\partial_t,c,u)$ with Poisson bracket defined by the rule:

$$\{t,y\}=fy,\qquad \{t,x\}=\frac{1}{\lambda}fx \qquad\text{and}\qquad \{y,x\}=cyx+u. \tag{4}$$

### The first case of the classification

The first case (Case I) of the Poisson algebra class $\mathcal{A}$ has two main subcases: Case I.1 and Case I.2. The results were indicated in these six subcases $\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_6,\mathcal{A}_7,\mathcal{A}_9$ and $\mathcal{A}_{10}$. Also, we presented some of their Poisson spectrum in diagrams in the poster called *'Poisson Algebras I'* see the diagram 1.

Diagram 1: The 'Poisson Algebras I' poster

The first part of second case (Case II) of the classification is presented in this poster and the next diagram shows the second case structure.

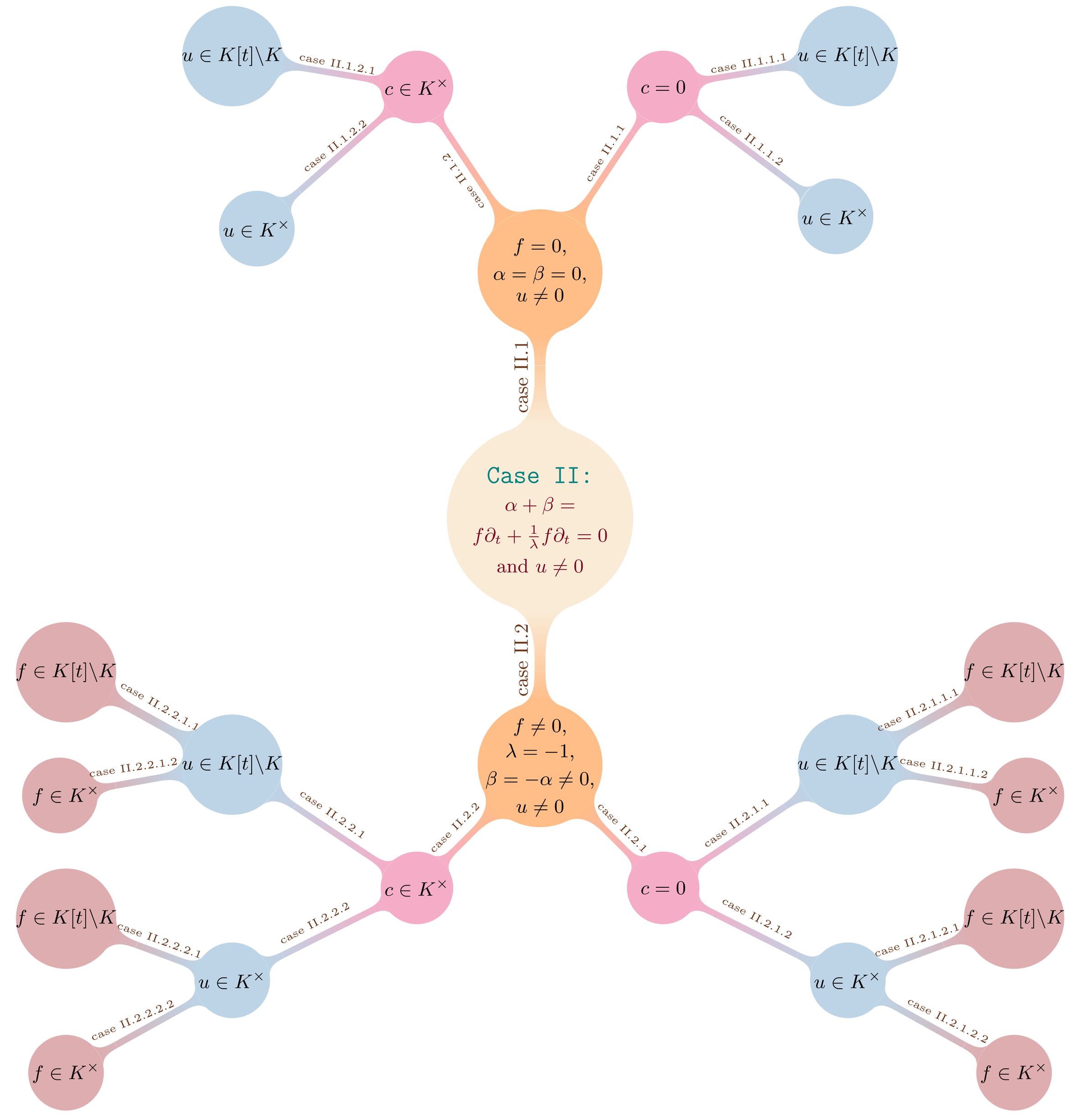


Diagram 2: Structure of the second case of Poisson algebra class $\mathcal{A}$

**Case II:** $\alpha+\beta=f\partial_t+\frac{1}{\lambda}f\partial_t=(1+\frac{1}{\lambda})f\partial_t=0$ and $u\neq 0$

Case II.1:

If $f=0$, i.e. $\alpha=\beta=0$ and $u\in K[t]\backslash\{0\}$ then we have the Poisson algebra $\mathcal{A}_{11}=(K[t];0,0,c,u)$ with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=cyx+u. \tag{5}$$

There are two subcases: $c=0$ and $c\in K^{\times}$.

Case II.1.1: If $c=0$ then we have the Poisson algebra $\mathcal{A}_{12}=(K[t];0,0,0,u)$ with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=u. \tag{6}$$

There are two subcases: $u\in K[t]\backslash K$ and $u\in K^{\times}$.

Case II.1.1.1:

If $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of $u$ then $\mathcal{A}_{13}=(K[t];0,0,0,u)$ is a Poisson algebra with Poisson bracket (6), we found $\mathrm{PSpec}(\mathcal{A}_{13})$, see diagram 3.

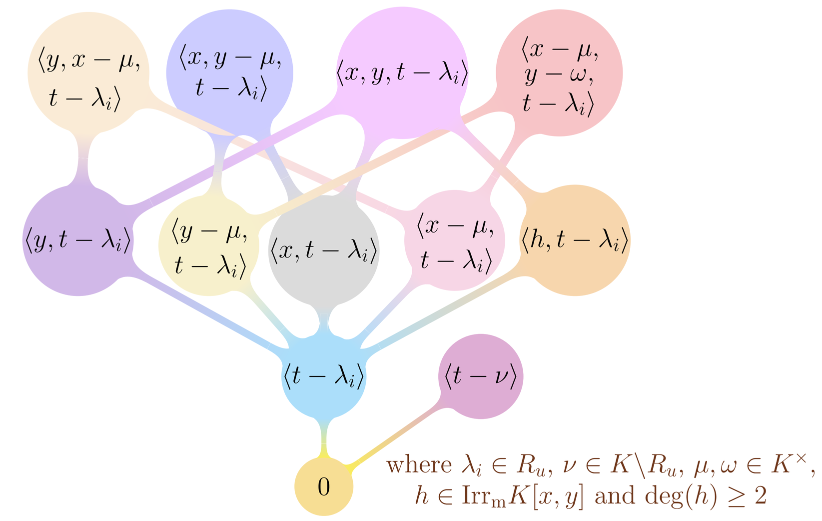


Diagram 3: The containment information between Poisson prime ideals of $\mathcal{A}_{13}$

Case II.1.1.2:

If $u=a\in K^{\times}$, i.e. $R_a=\emptyset$ then we have the Poisson algebra $\mathcal{A}_{14}=(K[t];0,0,0,a)$ with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=a. \tag{7}$$

The $\mathrm{PSpec}(\mathcal{A}_{14})=\{0,(t-\nu)\mid \nu\in K\}\subseteq\mathrm{PSpec}(\mathcal{A}_{13})$.

Case II.1.2: If $c\in K^{\times}$ then we have the Poisson algebra $\mathcal{A}_{15}=(K[t];0,0,c,u)$ with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=cyx+u:=\rho. \tag{8}$$

There are two subcases: $u\in K[t]\backslash K$ and $u\in K^{\times}$.

Case II.1.2.1:

If $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of $u$ then $\mathcal{A}_{16}=(K[t];0,0,c,u)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=\rho. \tag{9}$$

It follows that the element $\rho=cyx+u$ is an irreducible polynomial in $\mathcal{A}_{16}$. Moreover, we found $\mathrm{PSpec}(\mathcal{A}_{16})$, see diagram 4

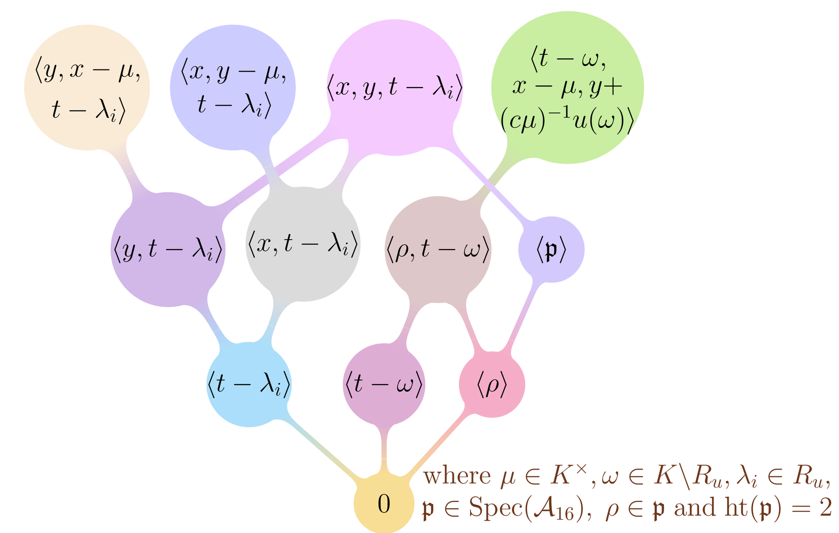


Diagram 4: The containment information between Poisson prime ideals of $\mathcal{A}_{16}$

Case II.1.2.2:

If $u=a\in K^{\times}$, i.e. $R_a=\emptyset$ then we have the Poisson algebra $\mathcal{A}_{17}=(K[t];0,0,c,a)$ with Poisson bracket

$$\{t,y\}=0,\qquad \{t,x\}=0 \qquad\text{and}\qquad \{y,x\}=\rho. \tag{10}$$

It follows that $\mathcal{A}_{17}=K[t]\otimes K[x,y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the Poisson algebra $K[x,y]$ with $\{y,x\}=\rho$. The element $\rho=cyx+a$ is an irreducible polynomial in $\mathcal{A}_{17}$. Moreover, we found $\mathrm{PSpec}(\mathcal{A}_{17})$, see diagram 5.

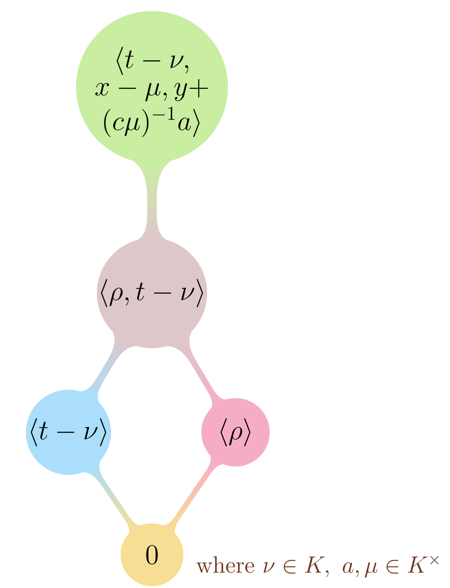


Diagram 5: The containment information between Poisson prime ideals of $\mathcal{A}_{17}$

## 5. Conclusion / Future research

A classification of Poisson prime ideals of $\mathcal{A}$ was obtained in 10 cases out of 22. We will complete the classification of $\mathcal{A}$. Then we aim to classify some simple finite dimension modules over the class $\mathcal{A}$.

## Acknowledgements

I would like to thank my supervisor Vladimir for providing guidance and feedback throughout this research. Also, I would like to thank my sponsor the University of Imam Mohammad Ibn Saud Islamic.

## References

[Bav1] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105 − 119.

[Bav2] V. V. Bavula, The PBW Theorem and simplicity criteria for the Poisson enveloping algebra and the algebra of Poisson differential operators, submitted, arxiv.2107.00321.

[GoWa] K. R. Goodearl and R. B. Warfield. *An introduction to noncommutative noetherian rings*. 2nd ed. New York: Cambridge University Press. (2004), pages 1 − 85, 105 − 122 and 166 − 186.

[Oh] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265 − 1277.