Poisson Algebras I, Non-commutative Algebra

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1. Introduction

A commutative algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\} : D \times D \to D$, called a *Poisson bracket*, such that

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D, I\} \subseteq I$. Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ \subseteq P \Rightarrow I \subseteq P \quad \text{or} \quad J \subseteq P$$

where I and J are Poisson ideals of D. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K. A K-linear map $\alpha : D \to D$ is a *Poisson derivation* of D if α is a K-derivation of D and

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\} \quad \text{for all } a, b \in D.$$

A set of all Poisson derivations of D is denoted by $\mathrm{PDer}_K(D)$.

2. How do we get our Poisson algebra class $\mathcal{A}$?

Lemma. [Oh3] *Let D be a Poisson algebra over a field K, $c \in K$, $u \in D$ and $\alpha, \beta \in \mathrm{PDer}_K(D)$ such that*

$$\alpha\beta = \beta\alpha \quad \text{and} \quad \{d, u\} = (\alpha + \beta)(d)u \quad \text{for all } d \in D. \tag{1}$$

Then the polynomial ring $D[x, y]$ becomes a Poisson algebra with Poisson bracket

$$\{d, y\} = \alpha(d)y, \quad \{d, x\} = \beta(d)x \quad \text{and} \quad \{y, x\} = cyx + u \quad \text{for all } d \in D. \tag{2}$$

The Poisson algebra $D[x, y]$ with Poisson bracket (2) is denoted by $(D; \alpha, \beta, c, u)$.

3. How do we classify $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A} = (K[t]; \alpha, \beta, c, u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a, b\} = 0$ for all $a, b \in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0 = \{d, u\} = (\alpha + \beta)(d)u \quad \text{for all } d \in K[t],$$

which implies that precisely one of the three cases holds:

(Case I: $\alpha + \beta = 0$ and $u = 0$), (Case II: $\alpha + \beta = 0$ and $u \neq 0$) or (Case III: $\alpha + \beta \neq 0$ and $u = 0$).

4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha, \beta \in \mathrm{PDer}_K = \mathrm{Der}_K(K[t]) = K[t]\partial_t$ such that $\alpha = f\partial_t$ and $\beta = g\partial_t$, where $f, g \in K[t]\backslash\{0\}$, $\partial_t = d/dt$ then*

$$\alpha\beta = \beta\alpha \quad \text{if and only if} \quad g = \frac{1}{\lambda}f \quad \text{for some} \quad \lambda \in K^\times := K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha = f\partial_t$, $\beta = \frac{1}{\lambda}f\partial_t$, $c \in K$, $u \in K[t]$, where $f \in K[t]$ and $\lambda \in K^\times$. Then we have the class of Poisson algebras $\mathcal{A} = K[t][x, y] = (K[t]; \alpha = f\partial_t, \beta = \frac{1}{\lambda}f\partial_t, c, u)$ with Poisson bracket defined by the rule:

$$\{t, y\} = fy, \quad \{t, x\} = \frac{1}{\lambda}fx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{4}$$

The next diagram shows the first case (Case I) of Poisson algebra class $\mathcal{A}$.

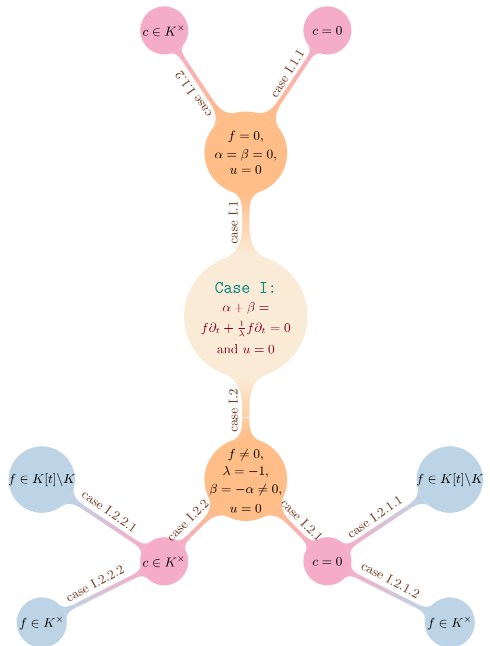

Diagram 1: Structure of the first case of Poisson algebra class $\mathcal{A}$

Case I: $\alpha + \beta = f\partial_t + \frac{1}{\lambda}f\partial_t = (1 + \frac{1}{\lambda})f\partial_t = 0$ and $u = 0$

Case I.1:
If $f = 0$, i.e. $\alpha = \beta = 0$ and $u = 0$ then $\mathcal{A}_1 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx. \tag{5}$$

There are two subcases: $c = 0$ and $c \in K^\times$.

Case I.1.1: If $c = 0$ then the polynomial Poisson algebra $\mathcal{A}_2 = (K[t]; 0, 0, 0, 0)$ has trivial Poisson structure and $\mathrm{PSpec}(\mathcal{A}_2)$ is the spectrum of the polynomial ring in three variables, i.e. $\mathrm{Spec}(K[t, x, y])$.

Case I.1.2: If $c \in K^\times$ then $\mathcal{A}_3 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket (5), we found $\mathrm{PSpec}(\mathcal{A}_3)$, see diagram 2.

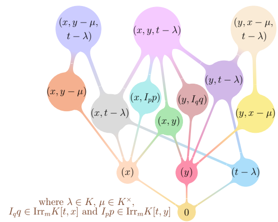

Diagram 2: The containment information between Poisson prime ideals of $\mathcal{A}_3$

Case I.2:
If $\lambda = -1$, i.e. $\beta = -\alpha = -f\partial_t$ for some $f \in K[t]\backslash\{0\}$ and $u = 0$ then $\mathcal{A}_4 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx. \tag{6}$$

There are two subcases: $c = 0$ and $c \in K^\times$.

Case I.2.1: If $c = 0$ then $\mathcal{A}_5 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = 0. \tag{7}$$

There are two subcases: $f \in K[t]\backslash K$ and $f \in K^\times$.

Case I.2.1.1:
If $f \in K[t]\backslash K$ and $R_f = \{\lambda_1, \dots, \lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_6 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket (7), we found $\mathrm{PSpec}(\mathcal{A}_6)$, see diagram 3.

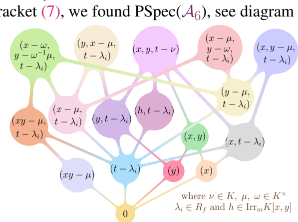

Diagram 3: The containment information between Poisson prime ideals of $\mathcal{A}_6$

Case I.2.1.2:
If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_7 = (K[t]; a\partial_t, -a\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = ay, \quad \{t, x\} = -ax \quad \text{and} \quad \{y, x\} = 0. \tag{8}$$

The Poisson spectrum of $\mathcal{A}_7$ is a subset of $\mathrm{PSpec}(\mathcal{A}_6)$.

Case I.2.2: If $c \in K^\times$ then $\mathcal{A}_8 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket (6). There are two subcases: $f \in K[t]\backslash K$ and $f \in K^\times$.

Case I.2.2.1:
If $f \in K[t]\backslash K$ and $R_f = \{\lambda_1, \dots, \lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_9 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket (6), we found $\mathrm{PSpec}(\mathcal{A}_9)$, see diagram 4.

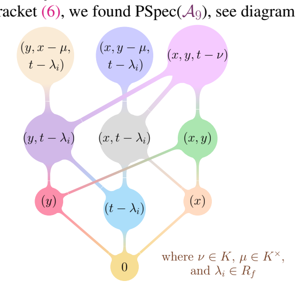

Diagram 4: The containment information between Poisson prime ideals of $\mathcal{A}_9$

Case I.2.2.2:
If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_{10} = (K[t]; a\partial_t, -a\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = ay, \quad \{t, x\} = -ax \quad \text{and} \quad \{y, x\} = cyx. \tag{9}$$

The Poisson spectrum of $\mathcal{A}_{10}$ is a subset of $\mathrm{PSpec}(\mathcal{A}_9)$.

5. Conclusion / Future research

A classification of Poisson prime ideals of $\mathcal{A}$ was obtained in 10 cases out of 22. We will complete the classification of $\mathcal{A}$. Then we aim to classify some simple finite dimension modules over the class $\mathcal{A}$.

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