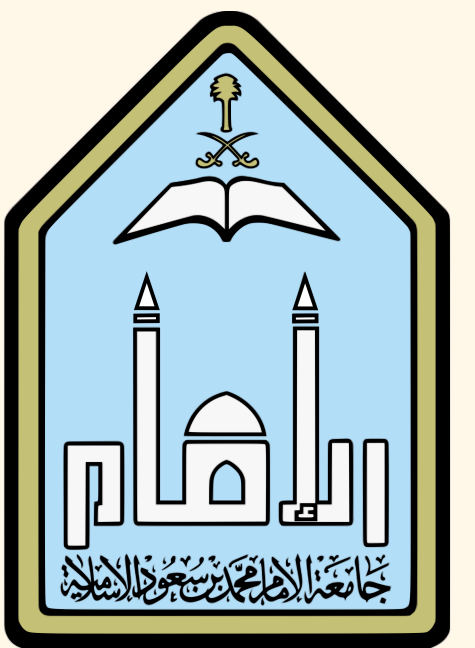




Poisson Algebras I, Non-commutative Algebra

Maram Alossaimi (Supervisor: Prof. Vladimir Bavula)

School of Mathematics and Statistics, malossaimi1@sheffield.ac.uk \ maram.alosaimi@gmail.com



1. Introduction

A commutative algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{ \cdot, \cdot \} : D \times D \rightarrow D$, called a *Poisson bracket*, such that

- $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
- $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
- $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D, I\} \subseteq I$. We denote by $\langle a \rangle$ the Poisson ideal of D generated by the element a . Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P$$

where I and J are Poisson ideals of D . A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\text{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K . A K -linear map $\alpha : D \rightarrow D$ is a *Poisson derivation* of D if α is a K -derivation of D and

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\} \text{ for all } a, b \in D.$$

A set of all Poisson derivations of D is denoted by $\text{PDer}_K(D)$.

2. How do we get our Poisson algebra class \mathcal{A} ?

Lemma. [Oh] Let D be a Poisson algebra over a field K , $c \in K$, $u \in D$ and $\alpha, \beta \in \text{PDer}_K(D)$ such that

$$\alpha\beta = \beta\alpha \text{ and } \{d, u\} = (\alpha + \beta)(d)u \text{ for all } d \in D. \quad (1)$$

Then the polynomial ring $D[x, y]$ becomes a Poisson algebra with Poisson bracket

$$\{d, y\} = \alpha(d)y, \quad \{d, x\} = \beta(d)x \text{ and } \{y, x\} = cyx + u \text{ for all } d \in D. \quad (2)$$

The Poisson algebra $D[x, y]$ with Poisson bracket (2) is denoted by $(D; \alpha, \beta, c, u)$.

3. How do we classify \mathcal{A} ?

We aim to classify all the Poisson algebra's $\mathcal{A} = (K[t]; \alpha, \beta, c, u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a, b\} = 0$ for all $a, b \in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0 = \{d, u\} = (\alpha + \beta)(d)u \text{ for all } d \in K[t],$$

which implies that precisely one of the three cases holds:

(Case I: $\alpha + \beta = 0$ and $u = 0$), (Case II: $\alpha + \beta = 0$ and $u \neq 0$) or (Case III: $\alpha + \beta \neq 0$ and $u = 0$).

4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class \mathcal{A} . This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha, \beta \in \text{PDer}_K = \text{Der}_K(K[t]) = K[t]\partial_t$ such that $\alpha = f\partial_t$ and $\beta = g\partial_t$, where $f, g \in K[t] \setminus \{0\}$, $\partial_t = d/dt$ then

$$\alpha\beta = \beta\alpha \text{ if and only if } g = \frac{1}{\lambda}f \text{ for some } \lambda \in K^\times := K \setminus \{0\}. \quad (3)$$

By using the previous lemma, we can assume that $\alpha = f\partial_t$, $\beta = \frac{1}{\lambda}f\partial_t$, $c \in K$, $u \in K[t]$, where $f \in K[t]$ and $\lambda \in K^\times$. Then we have the class of Poisson algebras $\mathcal{A} = K[t][x, y] = (K[t]; \alpha = f\partial_t, \beta = \frac{1}{\lambda}f\partial_t, c, u)$ with Poisson bracket defined by the rule:

$$\{t, y\} = fy, \quad \{t, x\} = \frac{1}{\lambda}fx \text{ and } \{y, x\} = cyx + u. \quad (4)$$

The next diagram shows the first case (Case I) of Poisson algebra class \mathcal{A} .

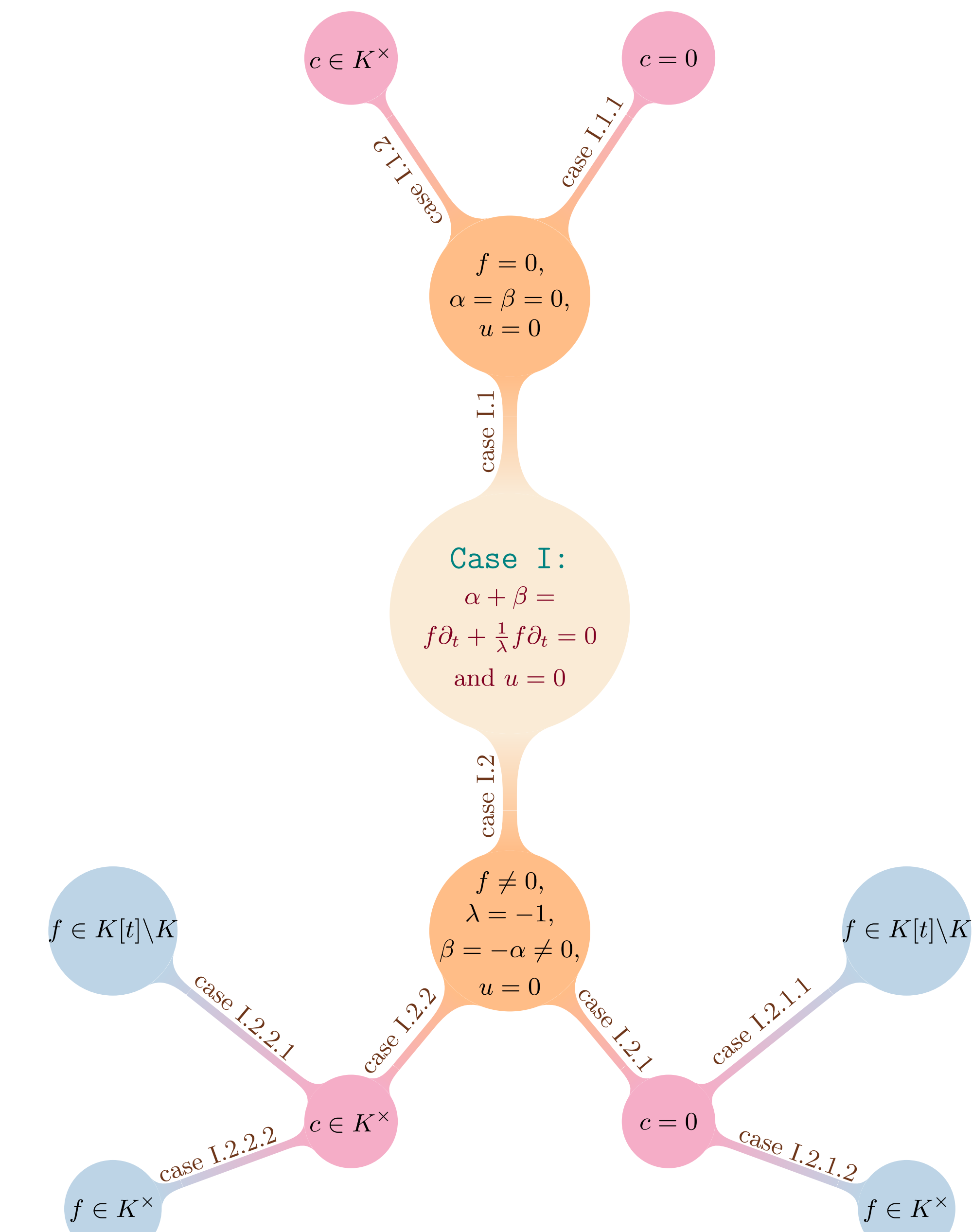


Diagram 1: Structure of the first case of Poisson algebra class \mathcal{A}

Case I: $\alpha + \beta = f\partial_t + \frac{1}{\lambda}f\partial_t = (1 + \frac{1}{\lambda})f\partial_t = 0$ and $u = 0$

Case I.1:

If $f = 0$, i.e. $\alpha = \beta = 0$ and $u = 0$ then $\mathcal{A}_1 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = 0, \quad \{t, x\} = 0 \text{ and } \{y, x\} = cyx. \quad (5)$$

Case I.1.1: If $c = 0$ then the polynomial Poisson algebra $\mathcal{A}_2 = (K[t]; 0, 0, 0, 0)$ has trivial Poisson structure and $\text{PSpec}(\mathcal{A}_2)$ is the spectrum of the polynomial ring in three variables, i.e. $\text{Spec}(K[t, x, y])$.

Case I.1.2: If $c \in K^\times$ then $\mathcal{A}_3 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket (5), we found $\text{PSpec}(\mathcal{A}_3)$, see diagram 2.

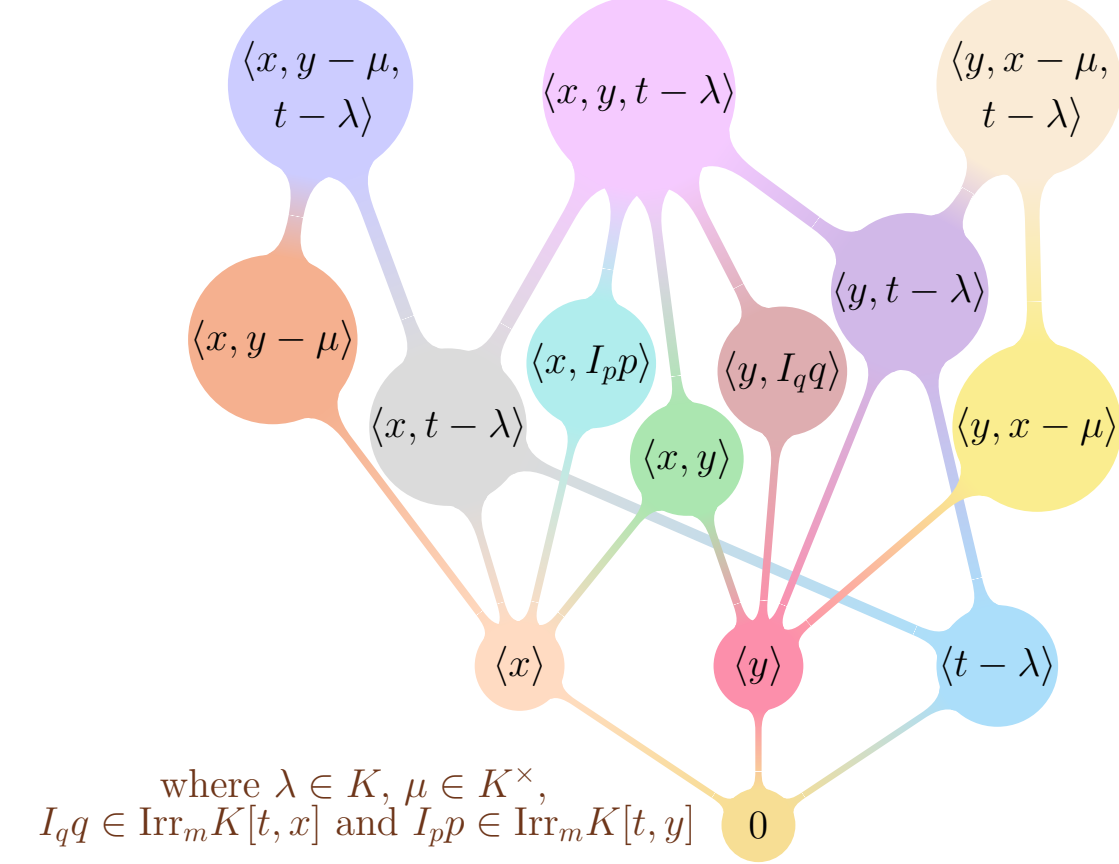


Diagram 2: The containment information between Poisson prime ideals of \mathcal{A}_3

Case I.2:

If $\lambda = -1$, i.e. $\beta = -\alpha = -f\partial_t$ for some $f \in K[t] \setminus \{0\}$ and $u = 0$ then $\mathcal{A}_4 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = fy, \quad \{t, x\} = -fx \text{ and } \{y, x\} = cyx. \quad (6)$$

Case I.2.1: If $c = 0$ then $\mathcal{A}_5 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = fy, \quad \{t, x\} = -fx \text{ and } \{y, x\} = 0. \quad (7)$$

Case I.2.1.1:

If $f \in K[t] \setminus K$ and $R_f = \{\lambda_1, \dots, \lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_6 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket (7), we found $\text{PSpec}(\mathcal{A}_6)$, see diagram 3.

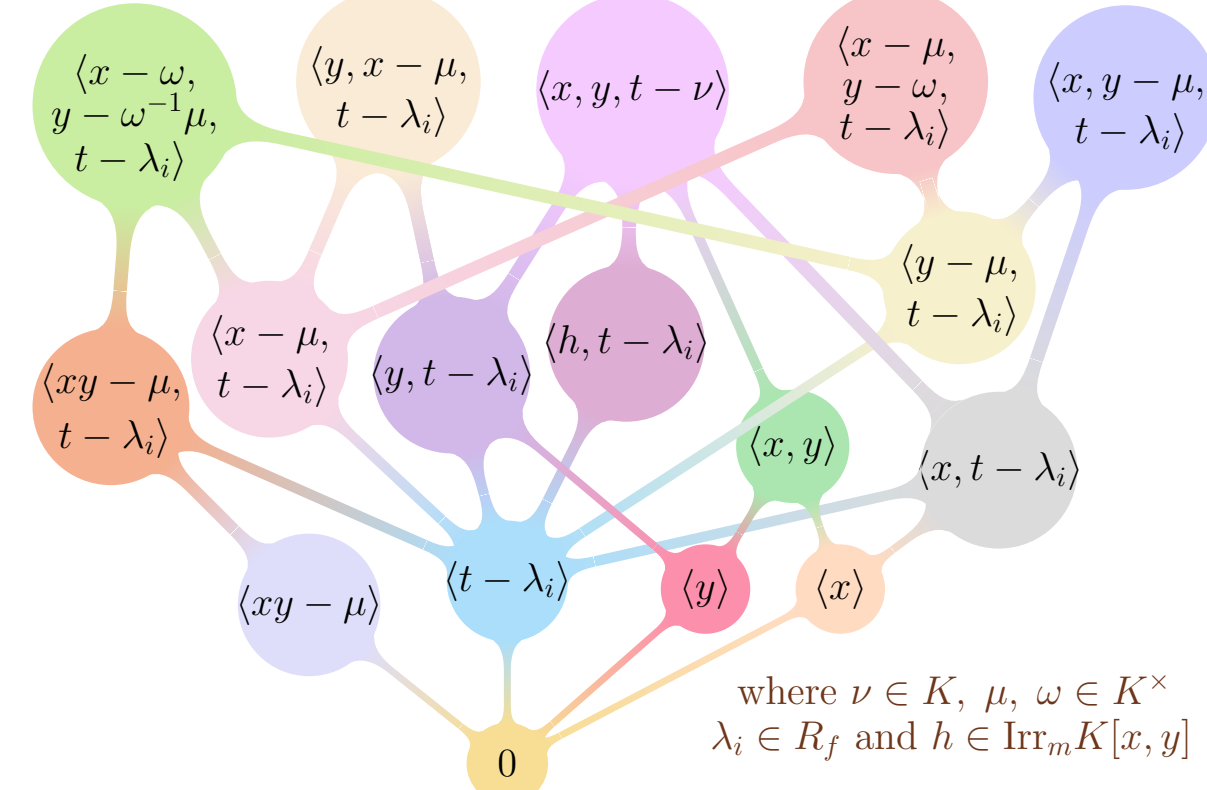


Diagram 3: The containment information between Poisson prime ideals of \mathcal{A}_6

Case I.2.1.2:

If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_7 = (K[t]; a\partial_t, -a\partial_t, 0, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = ay, \quad \{t, x\} = -ax \text{ and } \{y, x\} = 0. \quad (8)$$

We found $\text{PSpec}(\mathcal{A}_7)$, see diagram 4.

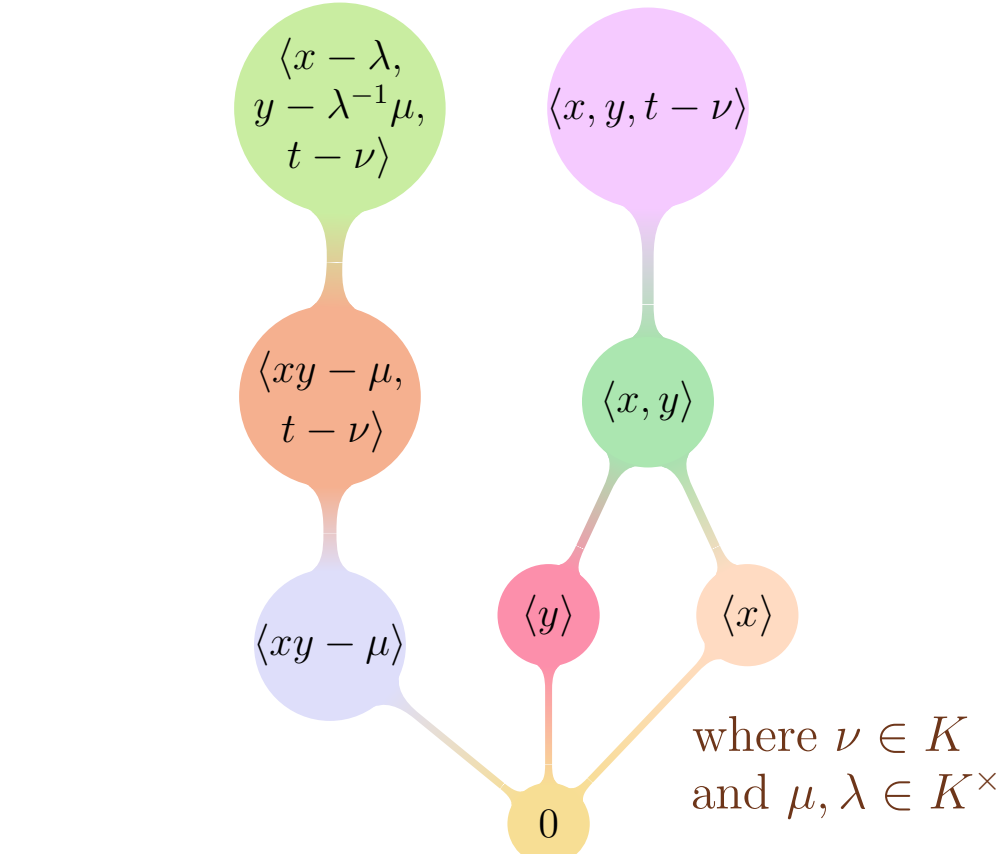


Diagram 4: The containment information between Poisson prime ideals of \mathcal{A}_7

Case I.2.2: If $c \in K^\times$ then $\mathcal{A}_8 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket (6).

Case I.2.2.1:

If $f \in K[t] \setminus K$ and $R_f = \{\lambda_1, \dots, \lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_9 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket (6), we found $\text{PSpec}(\mathcal{A}_9)$, see diagram 5.

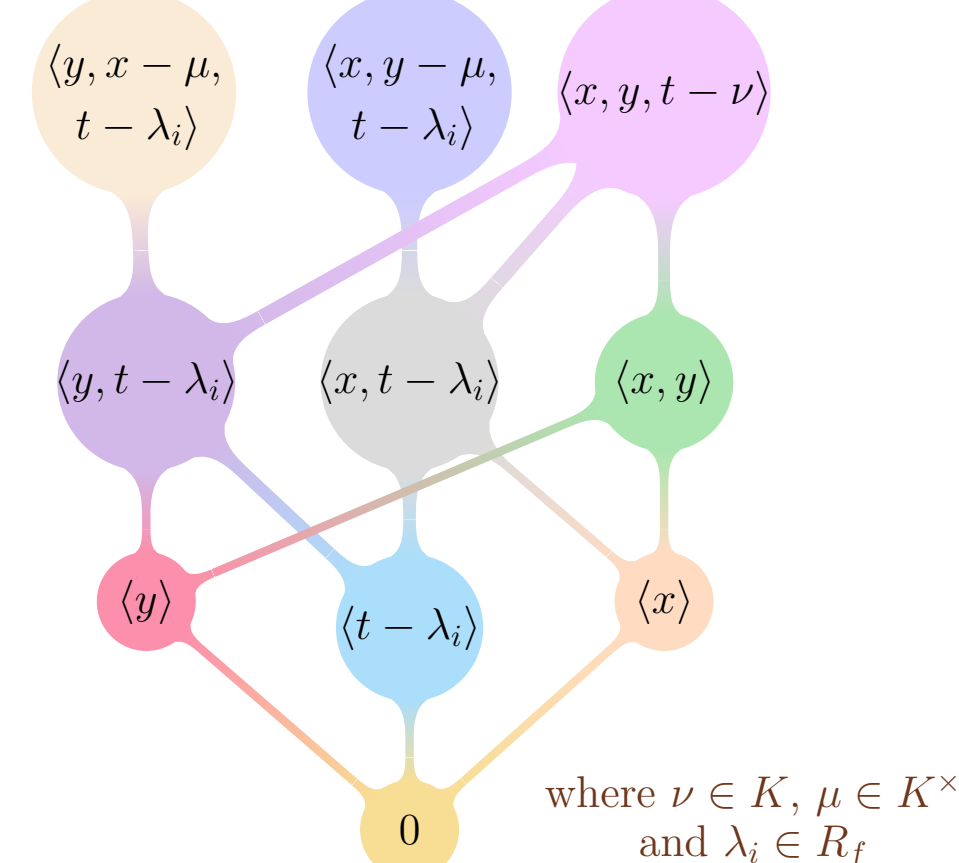


Diagram 5: The containment information between Poisson prime ideals of \mathcal{A}_9

Case I.2.2.2:

If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_{10} = (K[t]; a\partial_t, -a\partial_t, c, 0)$ is a Poisson algebra with Poisson bracket

$$\{t, y\} = ay, \quad \{t, x\} = -ax \text{ and } \{y, x\} = cyx. \quad (9)$$

The Poisson spectrum of \mathcal{A}_{10} is a subset of $\text{PSpec}(\mathcal{A}_9)$.

5. Conclusion / Future research

A classification of Poisson prime ideals of \mathcal{A} was obtained in 10 cases out of 22. We will complete the classification of \mathcal{A} . Then we aim to classify some simple finite dimension modules over the class \mathcal{A} .

Acknowledgements

I would like to thank my supervisor Vladimir for providing guidance and feedback throughout this research. Also, I would like to thank my sponsor the University of Imam Mohammad Ibn Saud Islamic.

References

- [Bav] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105 – 119.
- [GoWa] K. R. Goodearl and R. B. Warfield. *An introduction to noncommutative noetherian rings*. 2nd ed. New York: Cambridge University Press. (2004), pages 1 – 85, 105 – 122 and 166 – 186.
- [Oh] Sei-Qwon Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265 – 1277.