Poisson Algebras I, Non-commutative Algebra

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1. Introduction

A commutative algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\}: D \times D \to D$, called a *Poisson bracket*, such that

1. $\{a,b\} = -\{b,a\}$ for all $a,b \in D$ (anti-commutative),
2. $\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0$ for all $a,b,c \in D$ (Jacobi identity), and
3. $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all $a,b,c \in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D,I\} \subseteq I$. We denote by $\langle a \rangle$ the Poisson ideal of D generated by the element a. Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ \subseteq P \Rightarrow I \subseteq P \quad \text{or} \quad J \subseteq P$$

where I and J are Poisson ideals of D. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K. A K-linear map $\alpha: D \to D$ is a *Poisson derivation* of D if α is a K-derivation of D and

$$\alpha(\{a,b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\} \quad \text{for all } a,b \in D.$$

A set of all Poisson derivations of D is denoted by $\mathrm{PDer}_K(D)$.

2. How do we get our Poisson algebra class $\mathcal{A}$?

Lemma. [3] *Let D be a Poisson algebra over a field K, $c \in K$, $u \in D$ and $\alpha, \beta \in \mathrm{PDer}_K(D)$ such that*

$$\alpha\beta = \beta\alpha \quad \text{and} \quad \{d,u\} = (\alpha+\beta)(d)u \quad \text{for all } d \in D. \tag{1}$$

Then the polynomial ring $D[x,y]$ becomes a Poisson algebra with Poisson bracket

$$\{d,y\} = \alpha(d)y, \quad \{d,x\} = \beta(d)x \quad \text{and} \quad \{y,x\} = cyx + u \quad \text{for all } d \in D. \tag{2}$$

The Poisson algebra $D[x,y]$ with Poisson bracket (2) is denoted by $(D;\alpha,\beta,c,u)$.

3. How do we classify $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A} = (K[t];\alpha,\beta,c,u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a,b\} = 0$ for all $a,b \in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0 = \{d,u\} = (\alpha+\beta)(d)u \quad \text{for all } d \in K[t],$$

which implies that precisely one of the three cases holds:

$$(\text{Case I: } \alpha+\beta = 0 \text{ and } u = 0), \quad (\text{Case II: } \alpha+\beta = 0 \text{ and } u \neq 0) \quad \text{or} \quad (\text{Case III: } \alpha+\beta \neq 0 \text{ and } u = 0).$$

4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha, \beta \in \mathrm{PDer}_K = \mathrm{Der}_K(K[t]) = K[t]\partial_t$ such that $\alpha = f\partial_t$ and $\beta = g\partial_t$, where $f,\ g \in K[t]\backslash\{0\}$, $\partial_t = d/dt$ then*

$$\alpha\beta = \beta\alpha \quad \text{if and only if} \quad g = \frac{1}{\lambda}f \quad \text{for some} \quad \lambda \in K^\times := K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha = f\partial_t$, $\beta = \frac{1}{\lambda}f\partial_t$, $c \in K$, $u \in K[t]$, where $f \in K[t]$ and $\lambda \in K^\times$. Then we have the class of Poisson algebras $\mathcal{A} = K[t][x,y] = (K[t];\alpha = f\partial_t, \beta = \frac{1}{\lambda}f\partial_t, c, u)$ with Poisson bracket defined by the rule:

$$\{t,y\} = fy, \qquad \{t,x\} = \frac{1}{\lambda}fx \quad \text{and} \quad \{y,x\} = cyx + u. \tag{4}$$

The next diagram shows the first case (Case I) of Poisson algebra class $\mathcal{A}$.

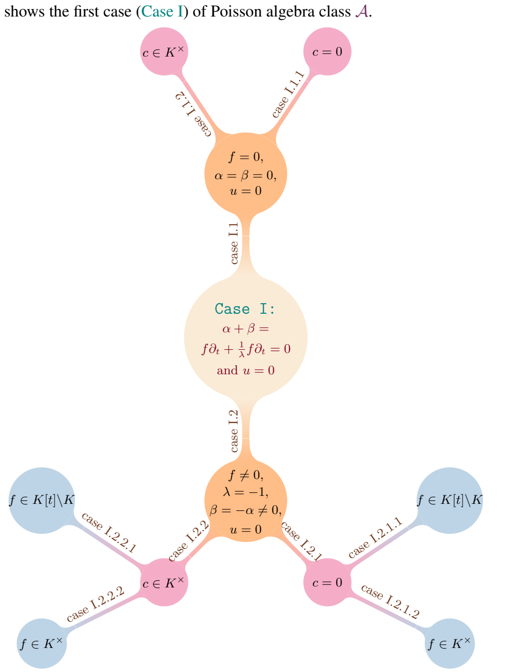

Diagram 1: Structure of the first case of Poisson algebra class $\mathcal{A}$

$$\text{Case I: } \alpha + \beta = f\partial_t + \tfrac{1}{\lambda}f\partial_t = (1+\tfrac{1}{\lambda})f\partial_t = 0 \text{ and } u = 0$$

Case I.1:
If $f = 0$, i.e. $\alpha = \beta = 0$ and $u = 0$ then $\mathcal{A}_1 = (K[t];0,0,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\} = 0, \quad \{t,x\} = 0 \quad \text{and} \quad \{y,x\} = cyx. \tag{5}$$

Case I.1.1: If $c = 0$ then the polynomial Poisson algebra $\mathcal{A}_2 = (K[t];0,0,0,0)$ has trivial Poisson structure and $\mathrm{PSpec}(\mathcal{A}_2)$ is the spectrum of the polynomial ring in three variables, i.e. $\mathrm{Spec}(K[t,x,y])$.

Case I.1.2: If $c \in K^\times$ then $\mathcal{A}_3 = (K[t];0,0,c,0)$ is a Poisson algebra with Poisson bracket (5), we found $\mathrm{PSpec}(\mathcal{A}_3)$, see diagram 2.

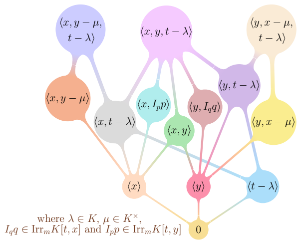

Diagram 2: The containment information between Poisson prime ideals of $\mathcal{A}_3$

Case I.2:
If $\lambda = -1$, i.e. $\beta = -\alpha = -f\partial_t$ for some $f \in K[t]\backslash\{0\}$ and $u = 0$ then $\mathcal{A}_4 = (K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\} = fy, \quad \{t,x\} = -fx \quad \text{and} \quad \{y,x\} = cyx. \tag{6}$$

Case I.2.1: If $c = 0$ then $\mathcal{A}_5 = (K[t];f\partial_t,-f\partial_t,0,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\} = fy, \quad \{t,x\} = -fx \quad \text{and} \quad \{y,x\} = 0. \tag{7}$$

Case I.2.1.1:
If $f \in K[t]\backslash K$ and $R_f = \{\lambda_1,\ldots,\lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_6 = (K[t];f\partial_t,-f\partial_t,0,0)$ is a Poisson algebra with Poisson bracket (7), we found $\mathrm{PSpec}(\mathcal{A}_6)$, see diagram 3.

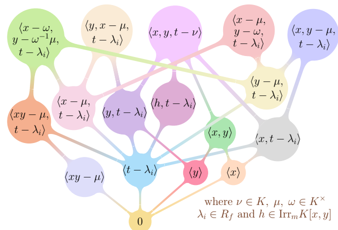

Diagram 3: The containment information between Poisson prime ideals of $\mathcal{A}_6$

Case I.2.1.2:
If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_7 = (K[t];a\partial_t,-a\partial_t,0,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\} = ay, \quad \{t,x\} = -ax \quad \text{and} \quad \{y,x\} = 0. \tag{8}$$

The Poisson spectrum of $\mathcal{A}_7$ is a subset of $\mathrm{PSpec}(\mathcal{A}_6)$.

Case I.2.2: If $c \in K^\times$ then $\mathcal{A}_8 = (K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket (6).

Case I.2.2.1:
If $f \in K[t]\backslash K$ and $R_f = \{\lambda_1,\ldots,\lambda_s\}$ is the set of distinct roots of f then $\mathcal{A}_9 = (K[t];f\partial_t,-f\partial_t,c,0)$ is a Poisson algebra with Poisson bracket (6), we found $\mathrm{PSpec}(\mathcal{A}_9)$, see diagram 4.

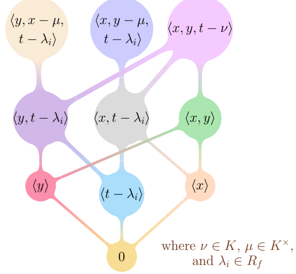

Diagram 4: The containment information between Poisson prime ideals of $\mathcal{A}_9$

Case I.2.2.2:
If $f = a \in K^\times$, i.e. $R_a = \emptyset$ then $\mathcal{A}_{10} = (K[t];a\partial_t,-a\partial_t,c,0)$ is a Poisson algebra with Poisson bracket

$$\{t,y\} = ay, \quad \{t,x\} = -ax \quad \text{and} \quad \{y,x\} = cyx. \tag{9}$$

The Poisson spectrum of $\mathcal{A}_{10}$ is a subset of $\mathrm{PSpec}(\mathcal{A}_9)$.

5. Conclusion / Future research

A classification of Poisson prime ideals of $\mathcal{A}$ was obtained in 10 cases out of 22. We will complete the classification of $\mathcal{A}$. Then we aim to classify some simple finite dimension modules over the class $\mathcal{A}$.

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