Poisson algebras
Non-commutative algebras

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A nonempty set S' with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set S with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists\, g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. S is an abelian if $g + h = h + g$.

A nonempty set V with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field $\mathbb{C}$ if for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
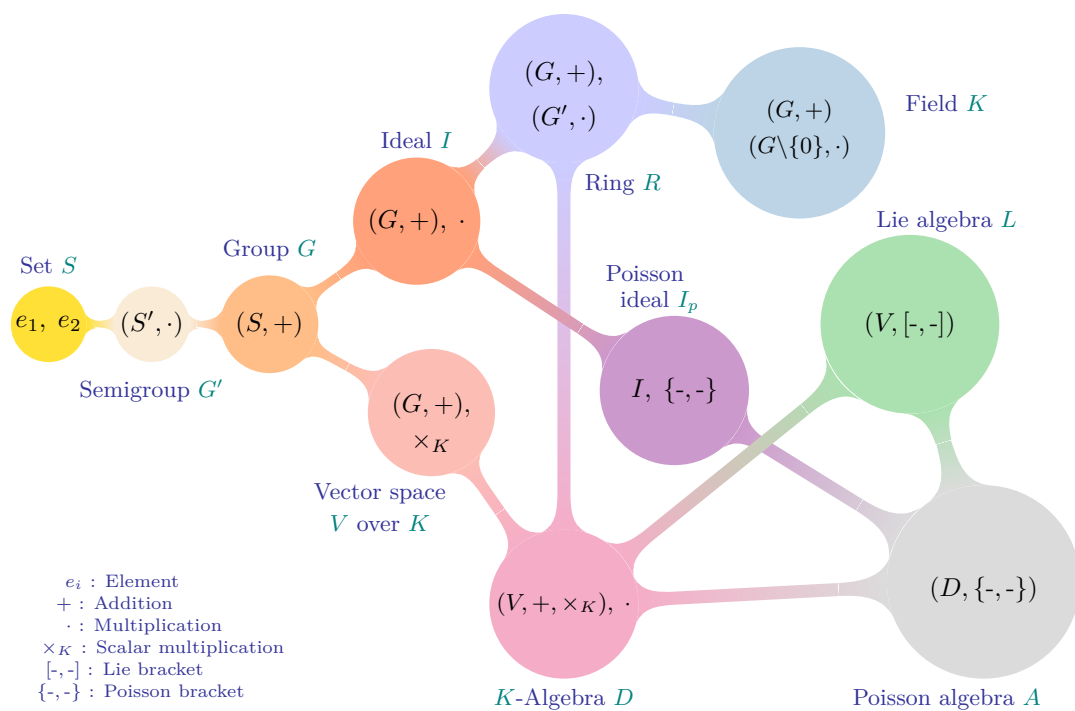6. $1 \times v = v$.

Set S

$e_1,\ e_2$ $(S', \cdot)$ $(S, +)$

Semigroup G'

Group G

Ideal I

$(G, +),\ \cdot$

$(G, +),$
$(G', \cdot)$

Ring R

$(G, +)$
$(G \backslash \{0\}, \cdot)$

Field K

Lie algebra L

$(V, [\text{-},\text{-}])$

Poisson
ideal I_p

$I,\ \{\text{-},\text{-}\}$

$(G, +),$
$\times_K$

Vector space
V over K

e_i : Element
$+$: Addition
$\cdot$: Multiplication
$\times_K$: Scalar multiplication
$[\text{-},\text{-}]$: Lie bracket
$\{\text{-},\text{-}\}$: Poisson bracket

$(V, +, \times_K),\ \cdot$

K-Algebra D

$(D, \{\text{-},\text{-}\})$

Poisson algebra A

Figure 1: Algebraic structure

Outline

Background

Definition 1

A *ring* R is a set on which two binary operations are defined addition $(+)$ and multiplication $(\cdot)$ such that

1. $(R, +)$ is an abelian group,
2. $(R, \cdot)$ is semigroup with an identity element, and
3. The distributive laws hold for all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c \ \text{ and } \ (a + b) \cdot c = a \cdot c + b \cdot c.$$

We say K is a field if K satisfies 1, 3 and $(K \backslash \{0\} = K^{\times}, \cdot)$ is an abelian group.
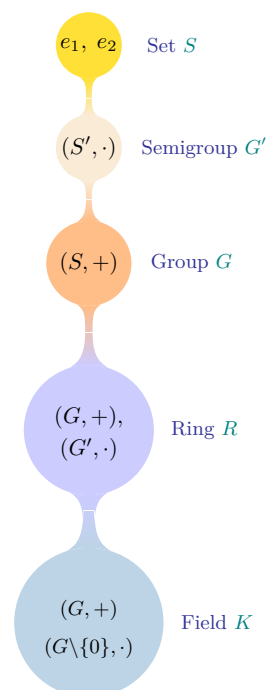
Figure 2: Rings and fields structures

Definition 2

Let K be a field and D be a set on which three operations are defined, addition, multiplication, and multiplication by a scalar such that

1. $(D, +, \cdot)$ is a ring,

2. $(D, +, \times_K)$ is a vector space over K,

3. $(\lambda a)\, b = a\,(\lambda b) = \lambda\,(ab)$ for all $\lambda \in K$ and $a, b \in D$,

then D is called a K-*algebra*

$e_1,\ e_2$ — Set S

$(S', \cdot)$ — Semigroup G'

$(S, +)$ — Group G

$(G, +),\ \times_K$ — Vector space V over K

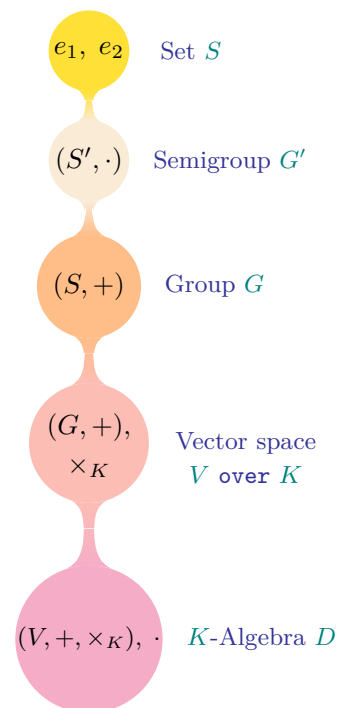$(V, +, \times_K),\ \cdot$ — K-Algebra D

Figure 3: K-algebras structure

Poisson algebras

Definition 3

A (commutative) K-algebra $(D, +, \cdot)$ is called a *Poisson algebra* if there exists bilinear product $\{\text{-},\text{-}\}$ on D, called a Poisson bracket, such that $(D, \{\text{-},\text{-}\})$ is

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),

2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and

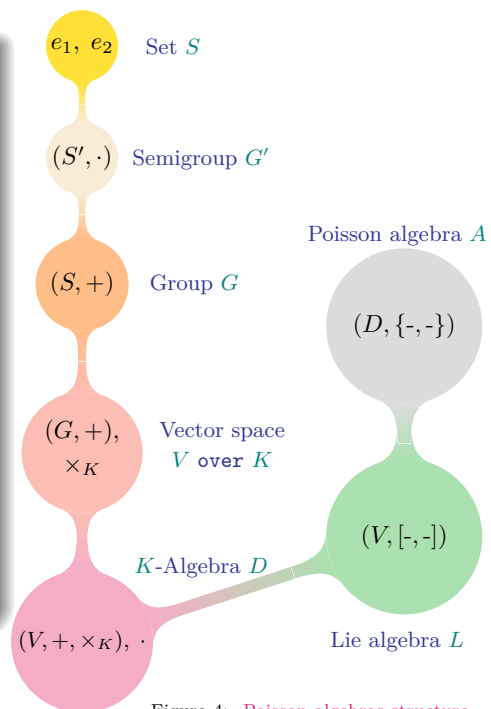3. $\{a \cdot b, c\} = \{a, c\} \cdot b + a \cdot \{b, c\}$ for all $a, b, c \in D$ (Leibniz rule).

e_1, e_2 Set S

$(S', \cdot)$ Semigroup G'

Poisson algebra A

$(S, +)$ Group G

$(D, \{\text{-},\text{-}\})$

$(G, +),$ Vector space
$\times_K$ V over K

$(V, [\text{-},\text{-}])$

$(V, +, \times_K), \cdot$ K-Algebra D Lie algebra L

Figure 4: Poisson algebras structure

Poisson ideals and Poisson prime ideals

Definition 4

Let D be a Poisson algebra. A subset I of D is a *Poisson ideal* of D if

① I is an ideal of algebra D, and

② $\{d, a\} \in I$ for all $d \in D$ and $a \in I$.

Moverover, the algebra D is a *simple Poisson* algebra if the only Poisson ideals of D are D and 0.

Definition 5

Let D be a Poisson algebra. A Poisson ideal P is a *Poisson prime ideal* of D if the following satisfies:

$$IJ \subseteq P \Rightarrow I \subseteq P \ \text{ or } \ J \subseteq P$$

where I and J are Poisson ideals of D. Moreover, a set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

$e_1, \ e_2$ — Set S

$(S', \cdot)$ — Semigroup G'

$(S, +)$ — Group G

$(G, +), \ \cdot$ — Ideal I
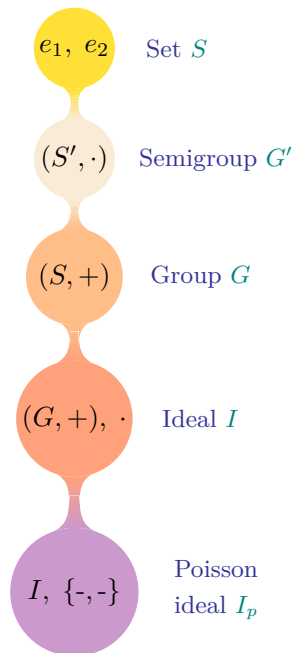
$I, \ \{\text{-},\text{-}\}$ — Poisson ideal I_p

Figure 5: Poisson ideals structure

The Poisson centre, Derivations and Poisson derivations

Definition 6

Let D be a Poisson algebra then

$$\mathrm{PZ}(D) := \{a \in D \mid \{a, d\} = 0 \ \text{ for all } \ d \in D\}$$

is called the *Poisson centre* of D.

Definition 7

Let D be a associative Poisson algebra over K. A K-linear map $\alpha : D \to D$ is called a *derivation* (respectively, *Poisson derivation*) on D if α satisfies 1 (respectively, satisfies 1 and 2) of the following conditions:

1 $\alpha(a \cdot b) = \alpha(a) \cdot b + a \cdot \alpha(b)$ for all $a, b \in D$.

2 $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in D$.

A set of all *derivations* (respectively, *Poisson derivations*) on D denoted by $\mathrm{Der}_K(D)$ (respectively, $\mathrm{PDer}_K(D)$).

The construction

Theorem 8 [Oh2, Theorem 1.1]

Let D be a Poisson algebra over K and α, δ be K-linear maps on D. Then the polynomial ring $D[y]$ becomes a Poisson algebra with Poisson bracket:

$$\{a, y\} = \alpha(a)y + \delta(a) \quad \text{for all } a \in D \tag{1}$$

if and only if α is a Poisson derivation on D and δ is a derivation on D such that

$$\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \delta(a)\alpha(b) - \alpha(a)\delta(b) \text{ for all } a, b \in D. \tag{2}$$

The Poisson algebra $D[y]$ is denoted by $D[y; \alpha, \delta]$ and if δ is zero then it is denoted by $D[y; \alpha]$.

Proof.

$$(D, \{\text{-},\text{-}\}) \xrightarrow[\;(\alpha \in \text{PDer}(D), \delta \in \text{Der}(D))\ (2)\;]{\alpha,\ \delta} (D[y], (1)) \quad \boxed{D[y; \alpha, \delta]}$$

Lemma 9 [Oh2, Lemma 1.3]

Let D be a Poisson algebra over K, $c \in K, u \in D$ and α, β are Poisson derivations such that

$$\alpha\beta = \beta\alpha \quad and \quad \{d, u\} = (\alpha + \beta)(d)u \quad for \ all \ d \in D \tag{3}$$

Then the polynomial ring $D[y, x]$ becomes a Poisson algebra with Poisson bracket

$$\{d, y\} = \alpha(d)y, \quad \{d, x\} = \beta(d)x \quad and \quad \{y, x\} = cyx + u \tag{4}$$

for all $d \in D$. This Poisson algebra is denoted by $A = (D; \alpha, \beta, c, u)$ or
$A = D[y; \alpha, 0][x; \beta, \delta' := u\frac{d}{dy}]$.

Proof.

$$\begin{array}{ccc}
\text{By Theorem 8} & & \text{By Theorem 8} \\[4pt]
(D, \{\text{-},\text{-}\}) \xrightarrow{\ \alpha, \ \ \delta = 0\ } (D[y], (1)) & \xrightarrow[\delta' = u\frac{d}{dy}]{\beta, \ \ \beta(y) = cy} (D[y][x], (4)) & \boxed{(D; \alpha, \beta, c, u)} \\[4pt]
D[y; \alpha] & D[y; \alpha][x; \beta, \delta']
\end{array}$$

The new Poisson algebra class $A = (K[t]; \alpha, \beta, c, u)$

We assume that

- K is an algebraically closed field with $\mathrm{char}(K) = 0$,
- D is a polynomial ring in one variable $K[t]$ with trivial Poisson bracket, i.e. $\{a, b\} = 0$ for all $a, b \in K[t]$, and
- $c \in K$.

If K-linear maps α and β are Poisson derivations on $K[t]$, i.e. $\alpha, \beta \in \mathrm{PDer}_K(K[t]) = \mathrm{Der}_K(K[t]) = K[t]\partial_t$ such that

$$\alpha = f\partial_t, \quad \beta = g\partial_t, \quad \text{where } \partial_t = \frac{d}{dt}, \quad \text{for some } f, g \in K[t].$$

Then by $\textcolor{magenta}{(3)}$ in Lemma $\textcolor{magenta}{9}$

$$u \in \mathrm{PZ}(K[t]) = K[t].$$

$$0 = \{d, u\} = (\alpha + \beta)(d)u \text{ for all } d \in D$$
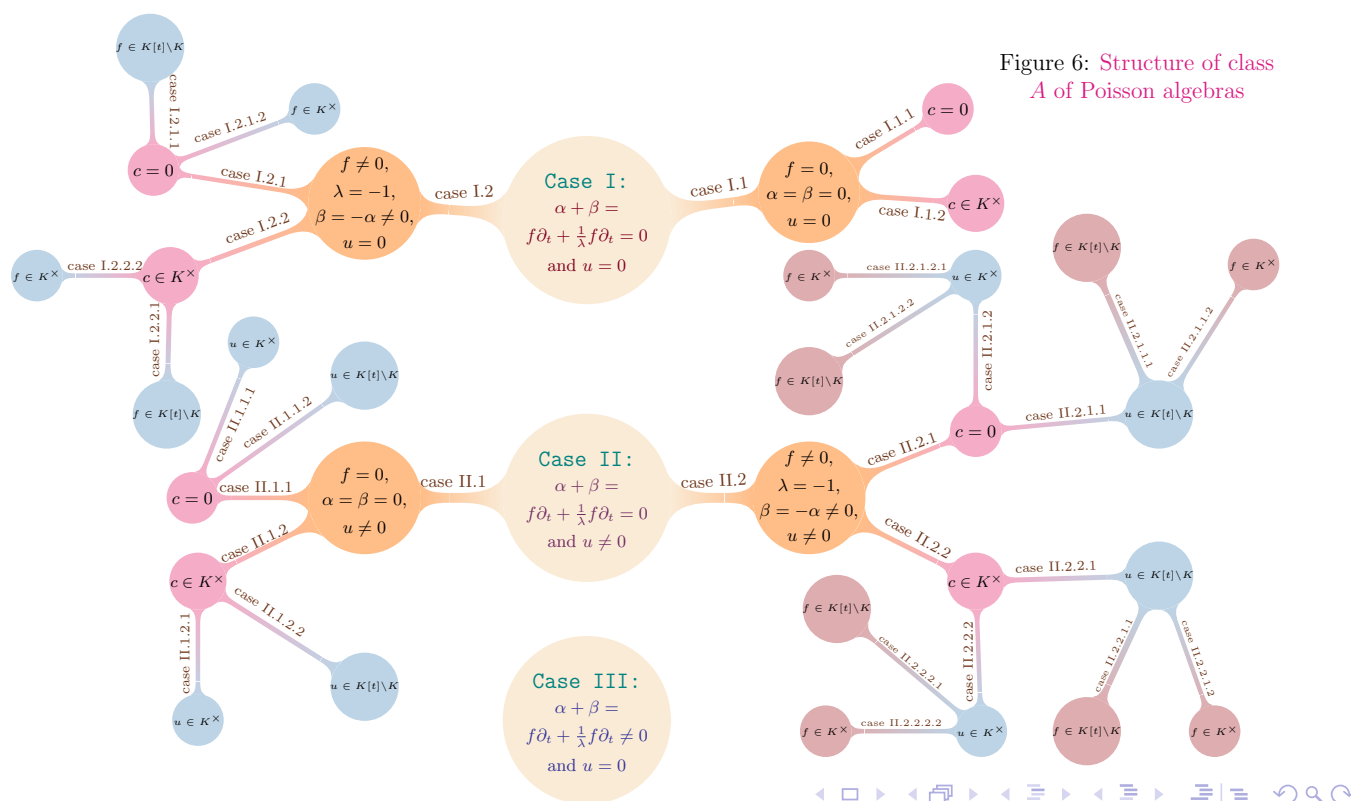
$$(\alpha + \beta)u = 0.$$

Lemma 10

Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and α, β are in $\mathbf{Der}_K(K[t])$ such that $\alpha = f\partial_t \neq 0$ and $\beta = g\partial_t \neq 0$ then

$$\alpha\beta = \beta\alpha \ \ \text{if and only if} \ \ g = \frac{1}{\lambda}f \ \ \text{for some} \ \ \lambda \in K^{\times}.$$

This implies that precisely one of the three cases holds:

(Case I: $\alpha + \beta = 0$ and $u = 0$),   (Case II: $\alpha + \beta = 0$ and $u \neq 0$)   or   (Case III: $\alpha + \beta \neq 0$ and $u = 0$).

The classification of Poisson algebra class A, so far

Figure 6: Structure of class A of Poisson algebras

The first case

Case I: $\alpha + \beta = f\partial_t + \frac{1}{\lambda}f\partial_t = (1 + \frac{1}{\lambda})f\partial_t = 0$ and $u = 0$

Case I:

If $\alpha + \beta = f\partial_t + \frac{1}{\lambda}f\partial_t = 0$ for some $\lambda \in K^\times$,
$f \in K[t]$, $u = 0$ and $c \in K$. Notice that,
$f\partial_t + \frac{1}{\lambda}f\partial_t = 0$, implies that there are two
subcases: $f = 0$ and $\lambda = -1$, see figure 7.

Case I.1:

If $f = 0$, i.e. $\alpha = \beta = 0$ and $u = 0$
then $A_1 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra
with Poisson bracket

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx. \qquad (5)$$

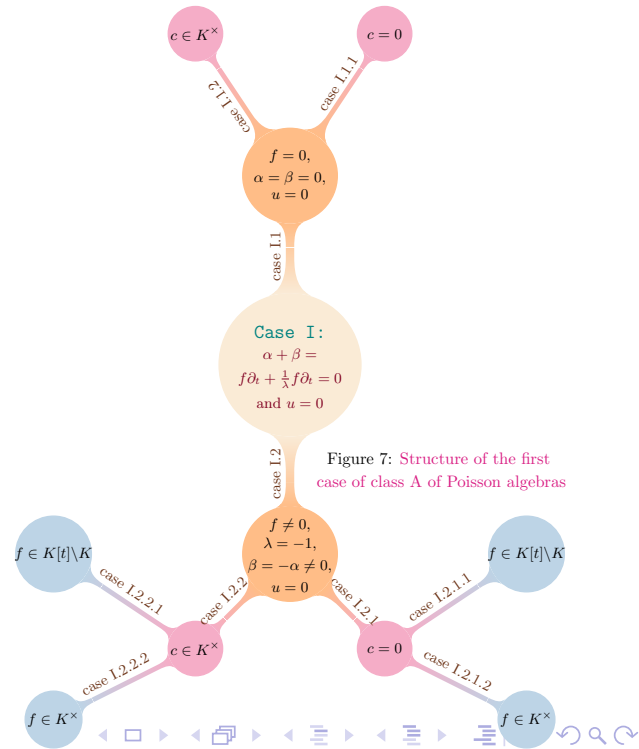There are two subcases: $c = 0$ and $c \in K^\times$.

Figure 7: Structure of the first case of class A of Poisson algebras

Case I.1.1:

If in addition, $c = 0$ then the polynomial Poisson algebra $A_2 = (K[t]; 0, 0, 0, 0)$ has trivial Poisson structure and the Poisson spectrum of A_2 is the spectrum of a polynomial ring in three variables, i.e.

$$\text{PSpec}(A_2) = \text{Spec}(K[t, x, y]).$$

Case I.1.2:

If in addition, $c \in K^\times$ then $A_3 = (K[t]; 0, 0, c, 0)$ is a Poisson algebra with Poisson bracket (5). In particular,

Figure 8: The containment information between Poisson prime ideals of A_3

$$A_3 = K[t] \otimes K[x, y]$$

is a tensor product of two Poisson algebras:

$$(K[t], \{-, -\} = 0) \quad \text{and} \quad (K[x, y], \{y, x\} = cyx).$$
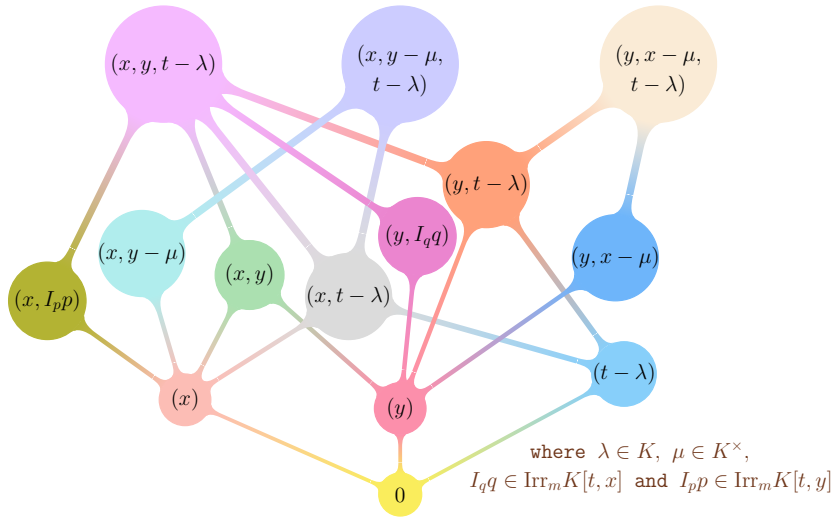
Then we found $\text{PSpec}(A_3)$, see figure 8.

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[Bav] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

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[Sta] T. Stafford. *Noncommutative algebra*, Lecture Notes in MATH 62041 at the University of Manchester, Manchester in Autumn 2016, 1–87.

Thank you for listening

Further resources

Poisson algebra applications :

- Skew polynomial rings.

[MyOh] N.-H. Myung and S.-Q. Oh, A construction of an Iterated Ore extension. arXiv:1707.05160v2 (2018).

- Poisson polynomial ring.

[Oh2] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), 1265–1277.

- Poisson modules and Poisson enveloping algebras.

[Oh1] S.-Q. Oh, Poisson enveloping algebras. *Communications in Algebra*, **27** (1999), 2181–2186.

- Weyl algebras and generalised algebras.

[Bav] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), 105–119.

Further resources

- Quantization and deformation algebras.

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- Semiclassical limits.

[ChOh] E.-H. Cho and S.-Q. Oh, Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra. *Lett. Math. Phys.* **106** (2016), 997–1009.

- Star product and Poisson algebras.

[EtSt] P. Etingof and D. Stryker, Short star-products for filtered quantizations, I. arXiv:1909.13588v5 (2020).

- Poisson manifolds.

[Kon] M. Kontsevich, Deformation quantization of Poisson manifolds. arXiv:q-alg/9709040v1 (1997).

- Hamiltonian systems.