FVS on Cubic Graphs

Marc Abou Serhal
Georges Khater
Hussein Farhat

Motivation and significance

The Feedback Vertex Set problem is an important problem in graph theory and combinatorial optimization. It asks for the smallest set of vertices whose removal destroys all cycles, leaving a forest. This notion is central to many applications where acyclic structure is required, such as circuit design, scheduling, and network reliability. Studying the problem on cubic and subcubic graphs is significant because these graphs represent a clear limit between easy and hard cases, allowing structural results and exact algorithms that are not possible in the general setting.

Problem statement

Given a graph $G = (V, E)$, the Feedback Vertex Set problem asks for a *feedback vertex set* (FVS) of size at most k, where a FVS $S \subseteq V$ is a subset of vertices such that $G - S$ is acyclic. Such a set is called a *minimum feedback vertex set* (MFVS) if it's a FVS of minimum size, and its size is denoted by $\eta(G)$.

Definition 1. *The degree of a vertex u in a graph G, denoted* $\deg_G(u)$*, is the number of edges incident with u. G is called cubic if every vertex has degree* 3*. G is called subcubic if there is no vertex with degree exceeding* 3*.*

The problem is NP-complete for general graphs [1], but becomes polynomial-time solvable when restricted to subcubic graphs via a reduction to matroid matching [4].

Our goal is to investigate whether an alternative combinatorial algorithm can be developed for finding a MFVS in subcubic graphs, without resorting to matroid theory. We aim to understand the structural properties that make the matroid formulation work, and to identify graph transformations or augmenting-path style procedures that achieve the same result in a purely combinatorial way.

Goals

Project focus. Develop a *combinatorial* algorithm for finding a MFVS in subcubic graphs.

- Simplify the working class via reductions [4].
- Identify structural lemmas that enable local progress rules (e.g., safe vertex choices) [4, 5, 3].
- Use equalities and inequalities linking $\eta(G)$, $k(G)$, and $\mu(G)$ as certificates and as guides for local optimality checks [5, 3].
- Analyze running time and implementation details sufficient for a clean polynomial bound.

Literature Review

The typical definition of a cycle considers loops and multiple edges as cycles. If a cycle is defined as a closed walk of length at least 3 with only the first and last vertices being repeated, then we would simply remove or ignore loops and multiple edges in the initial graph.

Definition 2. *A loop is an edge that connects a vertex to itself. A multiple edge is an edge appearing at least twice. A graph G is simple, if it contains no loops or multiple edges.*

Reductions

There are known reductions that reduce the problem from subcubic graphs to simple cubic graphs, such as the ones used in [4]. Below is proof of these reductions, with two added reductions that reduce the problem to simple cubic bridge-less cut-less graphs.

Lemma 3.1. *Let $\{u.v\}$ be a multiple edge in a subcubic graph G. Then either of $\{u, v\}$ must be in a FVS of G and $\eta(G) = 1+\eta(G-\{u\}) = 1+\eta(G-\{v\})$.*

Proof. At least one of $\{u, v\}$ should be in any FVS of G, as otherwise the 2-cycle remains. Let S be a FVS of G containing both u and v, then $S - \{u\}$ is also a FVS of G, as u will not be a part of any cycle because $\deg_{G-(S-\{u\})} \leq 1$. WLOG $S - \{v\}$ is also a FVS of G.

Lemma 3.2. *Let u be a vertex of degree one in a subcubic graph G. Then a MFVS of G does not contain u and $\eta(G) = \eta(G - \{u\})$.*

Proof. Let S is a FVS of G containing u, then $S - \{u\}$ is also a FVS of G, as u will not be a part of any cycle because $\deg_{G-(S-\{u\})}(u) \leq 1$.

Lemma 3.3. *Let u be the vertex of a loop in a subcubic graph G. Then any FVS of G must contain u and $\eta(G) = \eta(G - \{u\}) + 1$.*

Proof. If S doesn't contain u, then $G - S$ still has the loop.

Lemma 3.4. *Let u be a vertex of degree two, and v and w be the vertices adjacent to v (they might be identical) in a subcubic graph G. Let H be the graph obtained from G by deleting u and adding an edge $\{v, w\}$. Then u need not be in an FVS of G and $\eta(G) = \eta(H)$.*

Proof. Any cycle passing by u must pass by $\{v, w\}$, so cutting either of $\{v, w\}$ is enough to remove any cycle passing by u.

Definition 3.5. *An edge e in a graph G is called a bridge, if its removal from G increases the number of connected components. G is called bridge-less if it contains no such edge.*

Definition 3.6. *A vertex u in a graph G is called a cut, if its removal from G increases the number of connected components. G is called cut-less if it contains no such vertex.*

Lemma 3.7. *Let $\{u, v\}$ be a bridge in a subcubic graph G. Let H be the graph obtained from G by removing the edge $\{u, v\}$. Then $\eta(G) = \eta(H)$.*

Proof. Let S be an FVS of H, then S is also an FVS of G as the addition of $\{u, v\}$ doesn't add any cycle.

Lemma 3.8. *Let u be a cut vertex in a subcubic graph G. Then u is connected to some bridge in G.*

Proof. Let $C_1, ..., C_k$ with $2 \leq k \leq 3$ be the components connected to u. By the Pigeon Hole Principle, we know that there is some component C_i such that there is exactly 1 edge $\{u, v\}$ from u to $v \in C_i$. $\{u, v\}$ is a bridge in G since removing it disconnects C_i from $\{u\} \cup \bigcup_{j \neq i} C_j$.

Theorem 4. *Finding a MFVS for a subcubic graph is polynomial-time reducible to finding a MFVS for a simple cubic cut-less bridge-less graph.*

Proof. By lemma 3.8, removing all bridges leads to the removal of all cut vertices. Applying the operations in lemmas 3.1, 3.2, 3.3, 3.4 and 3.7 until there exists no vertex of degree less than 3, no self-loops, no multi-edges, and no bridges/cuts, the result is a simple cut-less bridge-less cubic graph.

From now on, we'll call reduction(G) the function that given a subcubic graph G, returns a pair (G', S) where G' is the reduced graph of G, and S is the set of removed vertices from G, such that if combined with a minimum FVS of G', forms a minimum FVS of G.

Also note that finding a MFVS for a disconnected graph of k connected components is equivalent to finding the union of MFVSs of each component.

Relation to Maximum Nonseperating Independent Set

Definition 5. *An independent set S of a graph $G = (V, E)$ is a set such that $E \cap \{\{u, v\} : u, v \in S\} = \emptyset$. A nonseperating set S of a graph G is a set such that $G - S$ is connected. A maximum nonseperating independent set (MNSIS) of G is a nonseperating independent set of maximum size. The MNSIS number of a graph G, denoted $k(G)$, is the size of a MNSIS of G.*

Definition 6. *The cyclomatic number or nullity of a graph $G = (V, E)$, denoted $\mu(G)$, is the minimum number of edges that must be removed from the graph to break all its cycles, making it into a tree or forest. It's equal to $|E| - |V| + c$ where c is the number of connected components in G.*

Theorem 7. (E. Speckenmeyer [3]) *For any simple cubic graph G, $\eta(G) + k(G) = \mu(G)$.*

Theorem 8. (E. Speckenmeyer [3]) *For any simple cubic graph G, $\eta(G) \leq \frac{n}{4} + \frac{k(G)+1}{2}$.*

[4] reduces the problem of finding a MNSIS in a simple cubic graph G to finding a maximum matching of the 2-polymatroid $P(G) = (V, f)$, where $f(S) = \mu(G) - \mu(G - S)$. and a matching is a set $M \subseteq V$ such that $f(M) = 2|M|$.

They prove that a set S is a NSIS of G if and only if S is a matching of $P(G)$, and then find a linear representation of $P(G)$ which makes finding a maximum matching solvable in polynomial time, by using Lovász' algorithm in [2].

Finding a MNSIS is useful because we can construct a MFVS from it in polynomial-time using the following algorithm:

Algorithm 1: Finding a MFVS S given a MNSIS I

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Input: A simple cubic graph G and a MNSIS I of G
Output: A MFVS S of G
S ← I
for each cycle C in G − I do
    Choose some vertex u in C
    S ← S ∪ {u}
return S
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Definition 9. *A cactus is a graph in which no two cycles have an edge in common. A cactus is nice if no two cycles have a vertex in common (every subcubic cactus is nice since if two cycles share a vertex but not an edge, then that vertex has degree at least 4). A cactus is very nice if every vertex belongs to exactly* *one cycle.*

It's important to note that a nice cactus C contains $\mu(C)$ cycles, and decycling C consists of removing any vertex from each cycle.

Lemma 10.1. *If I is a MNSIS of a simple cubic graph G, then G is a nice cactus.*

Proof. Assume for contradiction that $G - I$ has a 2-vertex connected block B that is not a cycle. That would mean that there is some vertex $u \in B$ whose degree inside B is at least 3, however G is cubic, so $\deg_{G-I}(u) = 3$ and u has no neighbor in I. $I \cup \{u\}$ is therefore independent as well as nonseperating as u has no neighbors outisde of B, meaning $B - \{u\}$ remains connected. This contradicts the fact that I is a MNSIS.

Lemma 10.2 *Let $G \neq K_4$ be a connected subcubic graph. Then there exists a MNSIS I of G such that $G - I$ isn't a cycle.*

Proof. Let I be a MNSIS of G such that $G - I$ is a cycle, and $u \in V(G - I)$. u has some neighbor $v \in I$; then $I' = I \cup \{u\} - \{v\}$ is also an MNSIS of G. If $G - I'$ is also a cycle, then $\{u, v\}$ is an edge shared by two triangles; take w to be an endpoint of one these triangles, then $I'' = I \cup \{w\} - \{v\}$ is a MNSIS of G such that $G - I''$ is not a cycle.

Theorem 11. *The FVS S obtained by Algorithm 1 is a MFVS.*

Proof. I is a NSIS of G and therefore a matching of $P(G)$, then

$$f(I) = \mu(G) - \mu(G - I) = 2|I|$$

By using Theorem 7, we get that

$$\eta(G) + k(G) = \mu(G - I) + 2|I|$$

However, I is a MNSIS, so by definition $k(G) = |I|$, hence

$$\eta(G) = \mu(G - I) + |I|$$

but $|S| = |I| + \#\{\text{cycles in } G - I\} = \eta(G)$ therefore it's a MFVS.

Corollary 12. *$\eta(G) \geq k(G)$ for any simple cubic graph G. Moreover, for every MNSIS I of G, there exists a MFVS S of G such that $I \subseteq S$.*

Another way to construct a MFVS given an algorithm to compute a MNSIS, is by computing $\eta(G)$ using $k(G)$ and Theorem 7, and then checking if the removal of a vertex decreases η.

Bounds

Using Theorem 7, Corollary 12 and the fact that $\mu(G) = \frac{|V|}{2} + 1$ for any connected simple cubic graph $G = (V, E)$, we can obtain the following bound:

$$\frac{|V|}{4} + \frac{1}{2} \leq \eta(G) \leq \frac{|V|}{2} + 1$$

Theorem 13. (E. Speckenmeyer [3]) *For any connected simple cubic graph $G = (V, E)$, $\eta(G) \leq \frac{3}{8}|V| + 1$. This bound is optimal.*

Theorem 14. (Zheng and Lu [8]) *For any connected simple cubic graph $G = (V, E)$ with no triangles, $\eta(G) \leq \frac{|V|}{3}$ except for two specific graphs.*

Other theorems

Theorem 15. (Matthew Johnson *et al.* [7]) *Let $G \neq K_4$ be a connected subcubic graph. Then a minimum size independent FVS of G is also a MFVS of G. Moreover, it's possible to find an independent MFVS of G in polynomial time.*

Proof. Let I be a MNSIS of G such that $G - I$ is not a cycle (which exists by Lemma 10.2). We can alter Algorithm 1 to choose a cut vertex u from each cycle C, so any vertex with degree 3.

Theorem 16. (Matthew Johnson *et al.* [7]) *Let $G \neq K_4$ be a connected subcubic graph. There is a minimum size independent FVS of G that contains only vertices of degree* 3 *if and only if G is not a very nice cactus. Moreover, there is a polynomial-time algorithm to find a minimum size independent FVS of G that contains only vertices of degree* 3.

Methodology and possible approaches

Our goal is now to make progress in one of the below approaches, in the hope of finding and proving *useful* theorems that might be restricted to specific classes of graphs, leading to a correct algorithm. There's also the possibility that the ideas are used to develop a Monte-Carlo algorithm for the problem.

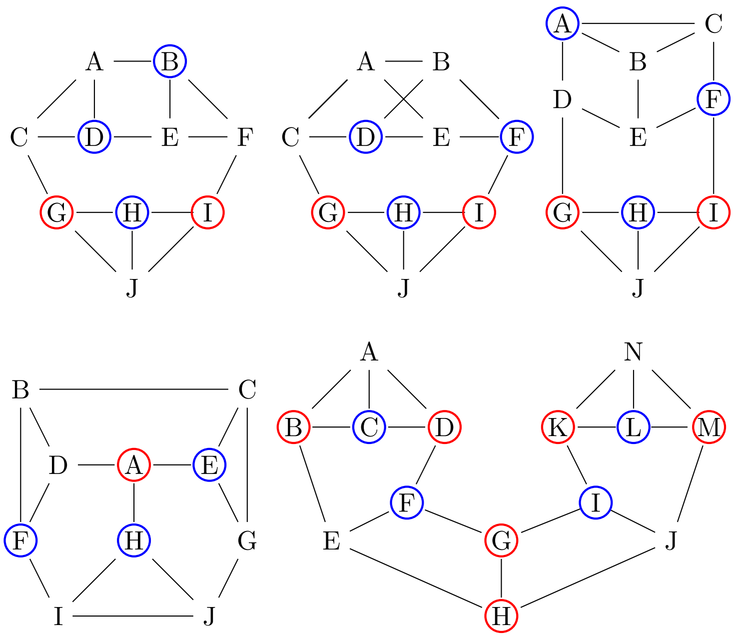
Greedy adding

Given an efficient procedure choose($\cdot$) that given a simple cubic bridge-less cutless graph G, returns a vertex $u \in V(G)$ such that u belongs to some minimum FVS of G, we could run the following algorithm:

Algorithm 2: Finding a MFVS of G

Input: A subcubic graph $G = (V, E)$
Output: A MFVS S of G
$(G', S) \leftarrow \text{reduction}(G)$
while G' *is cyclic* **do**
  $u \leftarrow \text{choose}(G')$
  $(G', S') \leftarrow \text{reduction}(G' - \{u\})$
  $S \leftarrow S \cup \{u\} \cup S'$
return S

It's important to note that there are cases where there is some vertex $u \in V(G)$ where u is not contained in any MFVS of G (even if G is planar), we call these vertices *MFVS-useless*. Examples of such graphs are shown in the below figures.



where a red vertex denotes a MFVS-vertex, and the blue vertices denote a MFVS of the graph with one of the red vertices removed, as well as for the whole graph.

There are some cases that can be easily dealt with, such as the first 3 examples, where it's easy to find a vertex that must be in some FVS, that leads to the eliminition of the MFVS-useless vertices.

Theorem 17. *Let $\{u, v\}$ be a shared edge between two distinct triangles in a simple subcubic graph G. Then either of $\{u, v\}$ must be in some MFVS of G.*

Proof. Let S be a MFVS of G not containing either of $\{u, v\}$, and let w_1 and w_2 be the respective ends of the two triangles sharing the edge $\{u, v\}$. Both

w_1 and w_2 must be in S, as otherwise, $G - S$ would still contain one of the triangles $\{u, v, w_1\}$ and $\{u, v, w_2\}$. Consider $S' = S - \{w_1, w_2\} \cup \{u, v\}$ which is still a FVS of G as $\deg_{G-S'}(w_1), \deg_{G-S'}(w_2) \leq 1$, and has the same size as S.

There are some other scenarios that don't seem easy to deal with. All graphs with MFVS-useless vertices up to size 16 have been computed and displayed in the notebooks **here**. It could be useful to take advantage of Corollary 12, which implies that any vertex u that is in some MNSIS, is not MFVS-useless.

Greedy removing

We initialize S to V, and then as long S is not minimal (contains a FVS as a proper subset), we remove a vertex (or vertices) according to *some* criteria.

It turns out that in a simple cubcubic graph, there is no vertex that is *MFVS-essential*, meaning there is no vertex that is forced to be all in MFVSs.

Lemma 18. *Let T be a tree and $x, y, z \in V(T)$. There exists a vertex $u \in V(T)$ such that x, y and z lie in different connected component of $T - \{u\}$.*

Proof. If x, y and z lie on the same simple path, we can remove the vertex that lies in the simple path between the two other vertices. Otherwise, there is a unique vertex u of degree at least 3, that lies in the (x, y)-path, (x, z)-path and (y, z)-path, whose removal disconnects all 3 vertices.

Theorem 19. *No simple subcubic graph $G = (V, E)$ has a MFVS-essential vertex. In other words, for every vertex $u \in V$, there exists a MFVS S of G not containing u, i.e. $u \notin S$.*

Proof. Let S be a MFVS of G containing u. If u has 3 neighbors in $G - S$ lying in the same tree, then by Lemma 19, there is some vertex $v \in G - S$ such that $S \cup \{v\} - \{u\}$ is a MFVS of G. Otherwise, $G - (S - \{u\})$ contains a single cycle C, $S \cup \{v\} - \{u\}$ is a MFVS of G for any $v \in C - \{u\}$,

This guarantees that any initial removal is valid. With more properties related to MFVSs between G and $G - \{u\}$ for any $u \in V(G)$, this could lead to a correct algorithm. There could be a way to *augment* a MFVS of $G - \{u\}$ into a MFVS of G; the simplest would be a sort of exchange theorem.

Conjecture 20. *There exists an integer k such that: for every simple cubic graph $G = (V, E)$, for every MFVS S of $G - \{u\}$ where $u \in V$, there exists a MFVS S' of G such that $|S \oplus S'| \leq k$.*

If Conjecture 20 was true, the following recursive algorithm would run in polynomial time:

Algorithm 3: MFVS(G)

Input: A subcubic graph G
Output: A MFVS S of G
$(G, S') \leftarrow \text{reduction}(G)$
if G *is empty* **then**
 return S'
Let u be a random vertex in $V(G)$
$S' \leftarrow S' \cup \text{MFVS}(G - \{u\})$
$S \leftarrow S' \cup \{u\}$
for $T \subseteq V, |T \oplus S'| \leq k$ **do**
 if $G - T$ *is acyclic and* $|T| < |S|$ **then**
  $S \leftarrow T$
return S

Augmenting paths

Inspired by Lovász' matroid matching algorithm from [2] which uses augmenting paths to improve a matching, it might be possible to find a purely combinatorial algorithm that doesn't depend on matroids and uses augmenting paths.

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