Due Date : January 12th (11pm), 2020

Instructions

- *This assignment serves as a warm-up for the following assignments. You are not obliged to finish this assignment, but some of the results here might be useful for the upcoming assignments. Unless otherwise specified, you may use the results in this assignment directly in your answer in the future.*
- *Use LaTeX and the template we provide when writing your answers. You may reuse most of the notation shorthands, equations and/or tables. See the assignment policy on the course website for more details.*
- *You will be using Gradescope, you should have received an invitation email (sent to the email address you use for StudiuM). Otherwise reply to the thread named "Gradescope Sign Up" on Piazza to let the TA know your "name", "email" and "student ID" (matricule) as shown on StudiuM to ask the TA to add you on Gradescope.*
- *Submit this test submission on Gradescope (you don't need to complete it, but are recommended to do it for self evaluation).*

Question 1. Let $\sigma(x) = \frac{1}{1+e^{-x}}$. Find $\frac{d\sigma}{dx}$ using the definition of the derivative (i.e. taking the limit of difference quotients).

Question 2. Compute the gradients of $||\boldsymbol{x}||_2^2$, $||\boldsymbol{x}-\boldsymbol{a}||_2$, $||\boldsymbol{x}||_F^2$, and $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}$ w.r.t the input vector $\boldsymbol{x}$.

Question 3. Recall the variance of X is $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

1. Let X be a random variable with finite mean. Show $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
2. Let X and Z be random variables on the same probability space. Show that $\text{Var}(X) = \mathbb{E}_Z[\text{Var}(X|Z)] + \text{Var}_Z(\mathbb{E}[X|Z])$. (Hint : $\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}[X|Y]]$.)

Question 4. Recall the density function of the uniform distribution on $[a, b]$ for $a < b$ is equal to $\frac{1}{b-a}$ for $x \in [a, b]$ and 0 elsewhere.

1. Use the density function to compute the mean and variance of a uniform distribution on $[a, b]$.
2. For integer $n > 0$, derive a formula to compute the moment $\mathbb{E}[X^n]$ for X uniformly distributed between a and b.

Question 5. Let $X \in \mathcal{X}$ be a random variable with density function f_X, and $g : \mathcal{X} \to \mathcal{Y}$ be continuously differentiable, where $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathbb{R}$. Let $Y := g(X)$, which is continuously distributed with density function f_Y.

1. Show that if g is monotonic, $f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right|$.
2. Let $f_X(x) = \mathbf{1}_{x\in[0,1]}(x)$ and $f_Y(y) = \mathbf{1}_{y\in[0,2]}(y) \cdot \frac{y}{2}$. Find a monotonic mapping g that translates f_X and f_Y.

Question 6. Let Q and P be univariate normal distributions with mean and variance μ, σ^2 and m, s^2, respectively. Derive the entropy $H(Q)$, the cross-entropy $H(Q, P)$, and the KL divergence $D_{\text{KL}}(Q||P)$.