RLC Parameters of a Two-Wire Line with the Finite Element Method - Formulation Details -

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ABSTRACT

This supplemental document shows the detailed steps used to obtain the weak formulations for the electrostatic, electrokinetic and magnetostatic field regimes.

For static fields in linear media, Maxwell's equations are

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho, \qquad \nabla \times \mathbf{E} = \mathbf{0}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times (\mathbf{B}/\mu) = \mathbf{J}, \tag{2}$$

where ρ and $\mathbf{J}$ are the free charge and current densities respectively, in C/m^3 and A/m^2. The electric scalar potential (V) and magnetic vector potential ($\mathbf{A}$) are related to the fields with

$$\mathbf{E} = -\nabla V, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{3}$$

When substituting these into Maxwell's equations, only two of the four equations are ultimately useful, namely the two Poisson equations for electrostatics and magnetostatics:

$$-\nabla \cdot (\epsilon \nabla V) = \rho, \tag{4}$$

$$\nabla \times (\nabla \times \mathbf{A}/\mu) = \mathbf{J}. \tag{5}$$

The other two Maxwell equations become automatically satisfied and are not needed, since $\nabla \times (\nabla V)$ and $\nabla \cdot (\nabla \times \mathbf{A})$ are always equal to zero.

In conducting media, the current flow (also called electrokinetics) is modeled using the continuity equation for steady currents

$$\nabla \cdot \mathbf{J} = 0, \tag{6}$$

which can be obtained from the second equation appearing in eq. (2). The current density $\mathbf{J}$ is related to the electric field and potential in the conductor according to

$$\mathbf{J} = \sigma \mathbf{E} = -\sigma \nabla V, \tag{7}$$

when neglecting the much smaller ($\sigma \mathbf{v} \times \mathbf{B}$) term. The continuity equation in (6) becomes

$$\nabla \cdot (\sigma \nabla V) = 0. \tag{8}$$

The strong forms of the three foundational differential equations involved are eqs. (4), (5) and (8).

I. ELECTROSTATICS

To be solved in the numerical finite element computation, the differential equations must be given in what is known as a weak formulation. Concerning the electrostatic potential in the nonconducting region (Ω_n), eq. (4) must be specified in this form. This is done by multiplying each side of the equality by a scalar weighting function W (sometimes called test function), and then, by integrating over the related domain as shown below

$$-\int_{\Omega_n} W\left(\nabla \cdot (\epsilon \nabla V)\right) d\Omega_n - \int_{\Omega_n} W\rho \, d\Omega_n = 0. \tag{9}$$

The V symbol now represents the approximate solution for the electric potential in Ω_n. When referring to an approximate solution, the right side of eq. (4) is not necessarily equal to 0, but rather a scalar residual R. It is this residual (hence the left side of the equation) that is multiplied by the weighting function, which leads to eq. (9).

After applying the vector identity

$$f(\nabla \cdot \mathbf{F}) = \nabla \cdot (f\mathbf{F}) - \mathbf{F} \cdot (\nabla f) \tag{10}$$

to the first integral in eq. (9), where $f = W$ and $\mathbf{F} = \epsilon \nabla V$, the formulation becomes

$$\begin{aligned} &-\int_{\Omega_n} \nabla \cdot (W \epsilon \nabla V) \, d\Omega_n \\ &\qquad + \int_{\Omega_n} \epsilon \nabla V \cdot \nabla W \, d\Omega_n - \int_{\Omega_n} \rho W \, d\Omega_n = 0. \end{aligned} \tag{11}$$

Invoking the divergence theorem on the first integral above allows it to be rewritten as the surface integral

$$-\int_{\Gamma_n} \epsilon \nabla V \cdot \mathbf{n} \, W \, d\Gamma_n, \tag{12}$$

where Γ_n is the surface boundary of volume Ω_n and $\mathbf{n}$ is the local unit vector normal to Γ_n (pointing outwards).

In the numerical implementation of the equation above, Γ_n represents only the surfaces with Neumann boundary conditions, since Dirichlet boundaries have no degrees of freedom (and periodic surfaces do not constitute true outer boundaries). The electric field is tangent to homogeneous Neumann surfaces, hence the dot product with $\mathbf{n}$ is equal to 0. All that needs to be done to enforce this constraint is to simply omit the integral in eq. (12). The electrostatic potential formulation in the nonconducting region is therefore

$$\boxed{\int_{\Omega_n} \epsilon \nabla V \cdot \nabla W \, d\Omega_n - \int_{\Omega_n} \rho W \, d\Omega_n = 0,} \tag{13}$$

subject to the Neumann boundary condition on Γ_n mentioned above (tangent electric field), and Dirichlet constraints (where V is imposed).

II. ELECTROKINETICS

Regarding the electric potential in the conducting region (Ω_c), eq. (8) must be specified in a weak form. Equation (8) is almost identical to eq. (4) when $\rho = 0$. Following the same procedure, the weak formulation for the electric potential in Ω_c is found to be

$$\boxed{\int_{\Omega_c} \sigma \nabla V \cdot \nabla W \, d\Omega_c = 0,} \tag{14}$$

with a similar Neumann boundary condition on Γ_c (tangent electric field), which in this case implies that no current flows through Neumann-type boundaries, and Dirichlet constraints (where V is also imposed).

III. MAGNETOSTATICS

The magnetic vector potential in eq. (5) can be specified in a weak form by performing a dot product of each side of the equation with a vector weighting function $\mathbf{W}$, and then integrating the result over the full domain (Ω) as shown below

$$\int_\Omega \mathbf{W} \cdot (\nabla \times (\nabla \times \mathbf{A}/\mu)) \, d\Omega - \int_\Omega \mathbf{W} \cdot \mathbf{J} \, d\Omega = 0. \tag{15}$$

After applying the vector identity

$$\mathbf{F} \cdot (\nabla \times \mathbf{G}) = \nabla \cdot (\mathbf{G} \times \mathbf{F}) + \mathbf{G} \cdot (\nabla \times \mathbf{F}) \tag{16}$$

to the first integral above, where $\mathbf{F} = \mathbf{W}$ and $\mathbf{G} = \nabla \times \mathbf{A}/\mu$, the formulation becomes

$$\begin{aligned} &\int_\Omega \nabla \cdot ((\nabla \times \mathbf{A}/\mu) \times \mathbf{W}) \, d\Omega \\ &+ \int_\Omega (\nabla \times \mathbf{A}/\mu) \cdot (\nabla \times \mathbf{W}) \, d\Omega - \int_\Omega \mathbf{J} \cdot \mathbf{W} \, d\Omega = 0. \end{aligned} \tag{17}$$

Invoking the divergence theorem on the first integral above allows it to be rewritten as the surface integral

$$\int_\Gamma ((\nabla \times \mathbf{A}/\mu) \times \mathbf{W}) \cdot \mathbf{n} \, d\Gamma, \tag{18}$$

where Γ is the surface boundary of volume Ω and $\mathbf{n}$ is the local unit vector normal to Γ (pointing outwards). With the proper triple product identity, this integral term can be rearranged as

$$\int_\Gamma (\mathbf{n} \times (\nabla \times \mathbf{A}/\mu)) \cdot \mathbf{W} \, d\Gamma. \tag{19}$$

As before, Γ represents only the surfaces with Neumann boundary conditions. The magnetic field is normal to homogeneous Neumann surfaces, hence the cross product with $\mathbf{n}$ is equal to $\mathbf{0}$. All that needs to be done to enforce this constraint is to simply omit the integral in eq. (19).

The third integral in eq. (17) is related to the current density $\mathbf{J}$, which can be explicitly specified when known, or related to the electric potential inside the conductor according to eq. (7). Since the conductivity is equal to zero outside of the conductor, the integral involving $\mathbf{J}$ can be restricted to Ω_c. The magnetic vector potential formulation is therefore

$$\boxed{\int_\Omega \left(\frac{\nabla \times \mathbf{A}}{\mu} \right) \cdot (\nabla \times \mathbf{W}) \, d\Omega - \int_{\Omega_c} \mathbf{J} \cdot \mathbf{W} \, d\Omega_c = 0,} \tag{20}$$

subject to the boundary condition on Γ mentioned above (normal magnetic field), and Dirichlet constraints (imposed potentials).