Homework 3

Due: Monday, Nov 1
at 8:30 pm EST

Please give complete, well-written solutions to the following exercises and submit via Canvas.

1. (i). (5 points) Let $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, where A is a symmetric $n \times n$ matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that the smallest Lipchitz constant of ∇f is $2\|A\|$.

   (ii). (5 points) Let $f \in C_L^{1,1}(\mathbb{R}^m)$, and let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$. Show that the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(\mathbf{x}) = f(A\mathbf{x} + \mathbf{b})$ satisfies $g \in C_{\tilde{L}}^{1,1}(\mathbb{R}^n)$, where $\tilde{L} = \|A\|^2 L$.

2. (10 points) Consider the minimization problem

$$\min\{\mathbf{x}^T Q\mathbf{x} : \mathbf{x} \in \mathbb{R}^2\},$$

   where Q is a positive definite 2×2 matrix. Suppose we use the diagonal scaling matrix

$$\begin{pmatrix} Q_{11}^{-1} & 0 \\ 0 & Q_{22}^{-1} \end{pmatrix}.$$

   Show that the above scaling matrix improves the condition number of Q in the sense that

$$\kappa(D^{1/2}QD^{1/2}) \le \kappa(Q).$$

3. (20 points) Consider the quadratic minimization problem

$$\min\{\mathbf{x}^T A\mathbf{x} : \mathbf{x} \in \mathbb{R}^5\},$$

   where A is 5×5 Hilbert matrix defined by

$$A_{i,j} = \frac{1}{i+j-1}, \quad i,j = 1,2,3,4,5.$$

   Write codes to implement the following methods and compare the number of iterations required by each of the methods when the initial vectors is $\mathbf{x}_0 = (1,2,3,4,5)^T$ to obtain a solution $\mathbf{x}$ with $\|\nabla f(\mathbf{x})\| \le 10^{-4}$:

   (i). gradient method with backtracking stepsize rule and parameters $\alpha = 0.5$, $\beta = 0.5$, $s = 1$;

   (ii). gradient method with backtracking stepsize rule and parameters $\alpha = 0.1$, $\beta = 0.5$, $s = 1$;

   (iii). gradient method with exact line search;

(iv). diagonally scaled gradient method with diagonal elements $D_{ii} = \frac{1}{A_{ii}}, i = 1, 2, 3, 4, 5$ and exact line search

(iv). diagonally scaled gradient method with diagonal elements $D_{ii} = \frac{1}{A_{ii}}, i = 1, 2, 3, 4, 5$ and backtracking line search with parameters $\alpha = 0.1$, $\beta = 0.5$, $s = 1$.

4. (30 points) Consider the Freudenstein and Roth test function

$$f(\mathbf{x}) = f_1^2(\mathbf{x}) + f_2^2(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

where

$$f_1(\mathbf{x}) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2,$$

and

$$f_2(\mathbf{x}) = -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2.$$

(i). Show that the function f has three stationary points. Find them and prove that one is a global minimizer, one is a strict local minimum and the third is a saddle point.

(ii) Write codes to employ the following three methods on the problem of minimizing f:

1. the gradient method with backtracking and parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.
2. the hybrid Newton's method with parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.
3. damped Gauss-Newton's method with a backtracking line search strategy with parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.

All the algorithms should use the stopping criteria $\|\nabla f(\mathbf{x})\| \leq 10^{-5}$. Each algorithm should be employed four times on the following four starting points: $(-50, 7)^T, (20, 7)^T, (20, -18)^T, (5, -10)^T$. For each of the four starting points, compare the number of iterations and the points to which each method converged. If a method did not converge, explain why.

5. Write down the reference you are going to read and do a presentation for the final project.

6. Write down three available time slots (25 minutes = 20 minutes for presentation + 5 minutes for Q&A) from between. 13rd and Dec 23rd (Please use EST, I will be available from 8 am to 8 pm).