Homework 4

Due: Thursday, Nov. 18
at 11:59 pm EST

Please give complete, well-written solutions to the following exercises and submit via Canvas.

1. (15 points) For each of the folowing stes determine whether they are convex or not (explaining your choice).

   (i). $C_1 = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2 = 1\}$.

   (ii). $C_2 = \{\mathbf{x} \in \mathbb{R}^n : \max_{i=1,2,\cdots,n} x_i \leq 1\}$.

   (iii). $C_3 = \{\mathbf{x} \in \mathbb{R}^n : \min_{i=1,2,\cdots,n} x_i \leq 1\}$.

   (iv.) $C_4 = \{\mathbf{x} \in \mathbb{R}^n_{++} : \Pi_{i=1}^n x_i \geq 1\}$.

   (v). The union of two convex sets.

2. (7 points) Find all the basic feasible solutions of the system

$$\begin{aligned} -4x_2 + x_3 &= 6, \\ 2x_1 - 2x_2 - x_4 &= 1, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

3. (10 points) Let $S = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 \leq 1\}$. Show that

$$\text{ext}(S) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 1\}.$$

4. (6 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex as well as concave function. Show that f is an affine function; that is, there exist $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$ for any $\mathbf{x} \in \mathbb{R}^n$.

5. (a). (6 points) Show that the log-sum-exp function

$$f(\mathbf{x}) = \ln(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$$

   is convex over $\mathbb{R}^n$.

   (b). (6 points) Let $g_1, \cdots, g_m$ be concave functions on $\mathbb{R}^n$, let f be a convex function on $\mathbb{R}^n$, and let μ be a positive constant. Prove that the function

$$\beta(x) = f(x) - \mu \sum_{i=1}^{m} \ln g_i(x)$$

   is convex on the set $S = \{x : g_i(x) > 0, i = 1, \cdots, m\}$.

6. (a). (10 points) Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function, where C is a convex set. Then for any $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in C$ and $\lambda \in \Delta_k$, the following inequality holds:

$$f\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{k} \lambda_i f(\mathbf{x}_i).$$

(b). (10 points) Prove that for any $x_1, x_2, \cdots, x_n \geq 0$ the following inequality holds:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

hint: use the result in (a).