1. (i) $f(x) = x^T A x + 2b^T x + c$, $\nabla f(x) = A^T x + Ax + 2b^T = 2Ax + 2b^T$ for $A^T = A$.

$$\|\nabla f(x) - \nabla f(y)\| = \|2Ax + 2b^T - (2Ay + 2b^T)\|$$
$$= \|2A(x-y)\|$$
$$= 2\|A\|\|x-y\|$$

(ii) $g(x) = f(Ax+b)$, $\nabla g(x) = \nabla f(Ax+b) \cdot A$

$$\|\nabla g(x) - \nabla g(y)\| \leq \|A\| \|\nabla f(Ax+b) - \nabla f(Ay+b)\|$$
$$\leq \|A\| L \|Ax+b - Ay - b\| \text{ since } f \in C_L^{1,1}(\mathbb{R}^n)$$
$$\leq \|A\|^2 L \|x-y\|$$

so $g \in C_{\tilde{L}}^{1,1}(\mathbb{R}^n)$ with $\tilde{L} = \|A\|^2 L$.

2. Assume $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$, since multiplying a matrix by a constant doesn't affect it's condition number, we can turn Q into $Q = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$, so $D^{\frac{1}{2}} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 1 \\ 1 & \frac{1}{\sqrt{b}} \end{bmatrix}$

Then $D^{\frac{1}{2}} Q D^{\frac{1}{2}} = A = \begin{bmatrix} 1 & \frac{1}{\sqrt{ab}} \\ \frac{1}{\sqrt{ab}} & 1 \end{bmatrix}$

Let $\alpha_1 \geq \alpha_2$ be eigenvalues of Q and $\gamma_1 \geq \gamma_2$ eigenvalues of A.

$\det(D^{\frac{1}{2}}) = \frac{1}{\sqrt{ab}} < 1$ since $A \succ 0$ & $Q \succ 0$, which gives $ab > 1$ or $\frac{1}{ab} < 1$.

Also, $\kappa(Q) = \frac{\alpha_1}{\alpha_2}$ & $\det(Q) = \alpha_1 \alpha_2$, then $\det(Q)\kappa(Q) = \alpha_1^2$ and $\det(Q)\det(D^{\frac{1}{2}})\det(D^{\frac{1}{2}}) = \gamma_1^2$

We want $\kappa(A) \leq \kappa(Q)$, or $\det(Q)\kappa(A) \leq \det(Q)\kappa(Q)$

Since $\det(Q) > 0$, $\gamma_1^2 / \det(D^{\frac{1}{2}})\det(D^{\frac{1}{2}}) \leq \alpha^2$, or $\gamma_1 \leq \det(D^{\frac{1}{2}})\alpha$, which is equivelent to $\kappa(A) \leq \kappa(Q)$, all terms positive.

Using characteristic polynomial $(t-\alpha_1)(t-\alpha_2)\cdots$ of Q & A, get

$\alpha_1 = \frac{a+b}{2} + \frac{\sqrt{(a-b)^2+4}}{2}$, $\gamma_1 = 1 + \frac{1}{\sqrt{ab}}$

$$\det(D^{\frac{1}{2}})\alpha_1 = \frac{1}{\sqrt{ab}}\left(\frac{a+b}{2} + \frac{\sqrt{(a-b)^2+4}}{2}\right) = \frac{a+b}{2\sqrt{ab}} + \sqrt{\frac{(a-b)^2+4}{4ab}} \geq 1 + \sqrt{\frac{(a-b)^2+4}{4ab}} \geq 1 + \sqrt{\frac{4}{4ab}}$$
$$= 1 + \frac{1}{\sqrt{ab}} = \gamma_1$$

So $\det(D^{\frac{1}{2}})\alpha_1 \geq \gamma_1$, therefore $\kappa(A) = \kappa(D^{\frac{1}{2}} Q D^{\frac{1}{2}}) \leq \kappa(Q)$.

4. (i) $\frac{d}{dx_1} f_1(x) = 1$, $\frac{d}{dx_2} f_1(x) = 10x_2 - 3x_2^2 - 2$

$\frac{d}{dx_1} f_2(x) = 1$, $\frac{d}{dx_2} f_2(x) = 3x_2^2 + 2x_2 - 14$

$\frac{d}{dx_2} f = 2f_1(x) \frac{d}{dx_2} f_1(x) + 2f_2(x) \frac{d}{dx_2} f_2(x) = 2f_1(x)(10x_2 - 3x_2^2 - 2) + 2f_2(x)(3x_2^2 + 2x_2 - 14)$

$\frac{d}{dx_1} f = 0$, $f_1(x) = -f_2(x)$

$\frac{d}{dx_2} f = 0$, $f_1(x)(10x_2 - 3x_2^2 - 2 - 3x_2^2 - 2x_2 + 14) = 0$, $f_1(x)(8x_2 - 6x_2^2 + 12) = 0$

$\frac{d}{dx_1} f = 0$, $x_1 = 21 - 3x_2^2 + 8x_2$

$f_1 = -13 + 21 - 3x_2^2 + 8x_2 + 5x_2^2 - x_2^3 - 2x_2 = 8 + 2x_2^2 - x_2^3 + 6x_2 = (x_2 - 4)(-x^2 6x_2 - 2x_2 - 2)$.

By $\frac{d}{dx_2} f = 0$, $x_2 = \frac{2 \pm \sqrt{22}}{3}$, $x_2 = 4$, or $x_2 \approx 2.23, -0.98, 4$.

plug these back into the equations get stationary points

$(11.41, -0.89)$ $(23.92, 2.23)$ $(5, 4)$.

For $(11.41, -0.89)$, $\nabla^2 f \approx \begin{bmatrix} 4 & -53.36 \\ -53.36 & 781.18 \end{bmatrix}$, $(4-\lambda)(781.18-\lambda) = (-53.36)^2$ gives $\lambda_1, \lambda_2 > 0$.

so $\nabla^2 f(11.41, -0.89) \succ 0$, $(11.41, -0.89)$ local min.

For $(23.92, 2.23)$, $\nabla^2 f \approx \begin{bmatrix} 4 & 21.52 \\ 21.52 & -643.52 \end{bmatrix}$, $(4-\lambda)(-643.52-\lambda) = (21.52)^2$ gives $\lambda_1 < 0$, $\lambda_2 > 0$

so $\nabla^2 f(23.92, 2.23)$ indefinite, $(23.92, 2.23)$ saddle point

For $(5, 4)$, $\nabla^2 f = \begin{bmatrix} 4 & 64 \\ 64 & 3728 \end{bmatrix}$, $(3728-\lambda)(4-\lambda) = 64^2$ gives $\lambda_1, \lambda_2 > 0$

so $\nabla^2 f(5,4) \succ 0$, $(5,4)$ local min

but, $f(4,5) = 0^2 + 0^2 = 0 = \min(f_1^2 + f_2^2)$

$(4,5)$ global min.

5. Multiscale analysis of accelerated gradient methods.

6. Dec-14, 4:00 pm ⇒ 4:25 pm
Dec-16, 4:00 pm ⇒ 4:25 pm
Dec-17, 4:00 pm ⇒ 4:25 pm