Homework 5

Due: Friday, Dec. 3
at 11:59 pm EST

Please give complete, well-written solutions to the following exercises and submit via Canvas.

1. (8 points) Consider the problem

$$\begin{aligned} & \min \quad f(\mathbf{x}) \\ (\mathrm{P}) \quad & s.t. \quad g(\mathbf{x}) \leq 0 \\ & \qquad \mathbf{x} \in X, \end{aligned}$$

where f and g are convex functions over $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$ is a convex set. Suppose that $\mathbf{x}^*$ is an optimal solution of (P) that staisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{aligned} & \min \quad f(\mathbf{x}) \\ & s.t. \quad \mathbf{x} \in X. \end{aligned}$$

2. (8 points) Let f be strictly convex function over $\mathbb{R}^m$ and let g be a convex function over $\mathbb{R}^n$. Define the function

$$h(\mathbf{x}) = f(A\mathbf{x}) + g(\mathbf{x}),$$

where $A \in \mathbb{R}^{m \times n}$. Assume that $\mathbf{x}^*$ and $\mathbf{y}^*$ optimal solutions of the unconstrained problem of minimizing h. Show that $A\mathbf{x}^* = A\mathbf{y}^*$.

3. (8 points) Consider the Huber function

$$H_\mu(\mathbf{x}) = \begin{cases} \frac{\|\mathbf{x}\|^2}{2\mu}, & \|\mathbf{x}\| \leq \mu, \\ \|\mathbf{x}\| - \frac{\mu}{2}, & \text{else}, \end{cases}$$

where $\mu > 0$ is a given parameter. Show that $H_\mu \in C^{1,1}_{\frac{1}{\mu}}(\mathbb{R}^n)$

4. (20 points) For each of the following optimization problems: (a). show that it is convex; (b). write a CVX code that solves it; and (c).write down the optimal solution (by running CVX).

(i).

$$\begin{aligned} \min \quad & x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 + 3x_1 - 4x_2 \\ \text{s.t.} \quad & \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \leq 6 \\ & x_1, x_2, x_3 \geq 1. \end{aligned}$$

(ii).

$$\begin{aligned}
\min\quad & |2x_1+3x_2+x_3|+x_1^2+x_2^2+x_3^2+\sqrt{2x_1^2+4x_1x_2+7x_2^2+10x_2+6}\\
\text{s.t.}\quad & \frac{x_1^2+1}{x_2}+2x_1^2+5x_2^2+10x_3^2+4x_1x_2+2x_1x_3+2x_2x_3\le 7\\
& x_1\ge 0\\
& x_2\ge 1.
\end{aligned}$$

For this problem also show that the expression inside the square root is always nonnegative, i.e., $2x_1^2+4x_1x_2+7x_2^2+10x_2+6\ge 0$ for all x_1,x_2.

(iii).

$$\begin{aligned}
\min\quad & \frac{x_1^4+2x_1^2x_2^2+x_2^4}{x_1^2+2x_1x_2+x_2^2}+\sqrt{x_3^2+1}\\
\text{s.t.}\quad & x_1^2+x_2^2+2x_3^2+2x_1x_2+2x_1x_3+2x_2x_3\le 100\\
& x_1+x_2+x_3=2\\
& x_1+x_2\ge 1.
\end{aligned}$$

(iv.)

$$\begin{aligned}
\min\quad & \frac{x_1^4}{x_2^2}+\frac{x_2^4}{x_1^2}+2x_1x_2+|x_1+5|+|x_2+5|+|x_3+5|\\
\text{s.t.}\quad & \left((x_1^2+x_2^2+x_3^2+1)^2+1\right)^2+x_1^4+x_2^4+x_3^4\le 200\\
& \max\{x_1^2+4x_1x_2+9x_2^2,x_1,x_2\}\le 40\\
& x_1\ge 1\\
& x_2\ge 1.
\end{aligned}$$

5. (10 points) Consider the minimization problem

$$\text{(P)}\quad \min\{f(\mathbf{x}):\mathbf{a}^T\mathbf{x}=1,\ \mathbf{x}\in\mathbb{R}^n\},$$

where f is a continuously differentiable function over $\mathbb{R}^n$ and $\mathbf{a}\in\mathbb{R}^n_{++}$. Show that $\mathbf{x}^*$ satisfying $\mathbf{a}^T\mathbf{x}^*=1$ is astationary point of (P) if and only if

$$\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{a_1}=\frac{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)}{a_2}=\cdots=\frac{\frac{\partial f}{\partial x_n}(\mathbf{x}^*)}{a_n}.$$

6. (16 points) Consider the minimization problem

$$\text{(Q)}\quad \begin{aligned}
\min\ & 2x_1^2+3x_2^2+4x_3^2+2x_1x_2-2x_1x_3-8x_1-4x_2-2x_3\\
s.t.\ & x_1,x_2,x_3\ge 0.
\end{aligned}$$

(i). (6 points) Show that the vector $(\frac{17}{7},0,\frac{6}{7})$ is an optimal solution of (Q).

(ii). (10 points) Employ the gradient projection method with constant stepsize $\frac{1}{L}$(L being the Lipschitz constant of the gradient of the objective function). Show the function values of the first 100 iterations and the produced solution.