Q1

(i) Consider the function

$f(x) = x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 = (x_1+x_2)^2 + x_2^2 + A\cdot x$ for matrix A.

It is convex.

Yet, for max $f(x)$, it's min $-f(x)$.

Since $f(x)$ is convex and is not affine, $-f(x)$ is concave.

So P is not convex.

(ii) Now P is min $f(x) = -x_1^2 - 2x_1x_2 - 2x_2^2 + 3x_1 - x_2$ with

$h(x) = x_1 + x_2 - 1 = 0$, $g_1(x) = -x_1 \le 0$, $g_2(x) = -x_2 \le 0$.

Then $\mathcal{L}(x_1, x_2, u) = -x_1^2 - 2x_1x_2 - 2x_2^2 + 3x_1 - x_2 + u(x_1+x_2-1) - \lambda_1 x_1 - \lambda_2 x_2$

$$\nabla_x \mathcal{L} = \vec{0} = \begin{cases} -2x_1 - 2x_2 + 3 + u - \lambda_1 = 0 \\ -2x_1 - 4x_2 - 1 + u - \lambda_2 = 0 \end{cases}$$

$$\begin{cases} \lambda_1 g_1(x) = 0, & -\lambda_1 x_1 = 0 \\ \lambda_2 g_2(x) = 0, & -\lambda_2 x_2 = 0 \end{cases}$$

① If $x_1 = 0$, $x_2 = 1$, $\lambda_2 = 0$, $\begin{cases} 1 + u - \lambda_1 = 0 \\ -5 + u = 0 \end{cases}$, $\begin{cases} u = 5 \\ x_1 = 6 \end{cases}$

② If $x_2 = 0$, $x_1 = 1$, $\lambda_1 = 0$, $\begin{cases} 1 + u = 0 \\ -3 + u - \lambda_2 = 0 \end{cases}$, $\begin{cases} u = -1 \\ \lambda_2 = -4 \end{cases}$

$\lambda_2 = -4$ does not satisfy $\lambda_i \ge 0$. So $(1,0)$ is not a KKT point. $(0,1)$ is KKT point.

③ If $x_1, x_2 \ne 0$, $\lambda_1 = \lambda_2 = 0$.

$\begin{cases} -2x_1 - 2x_2 + 3 + u = 0 \\ -2x_1 - 4x_2 - 1 + u = 0 \end{cases}$, $\begin{cases} x_1 = \frac{1}{2}(7+u) \\ x_2 = -2 \end{cases}$

does not satisfy $x_1, x_2 \ge 0$. So $(\frac{1}{2}(7+u), -2)$ is not a KKT point.

So KKT point is $(0,1)$.

(iii) At $(0,1)$, $\nabla g_1(x) = [-1\ 0]^T$, $\nabla h(x) = [1, 1]^T$. Since $\lambda_2 = 0$, $\nabla g_2(x)$ is ignored.

Then $[\nabla g_1(x), \nabla h(x)]$ are linearly independent. So $(0,1)$ is regular point.

For all points in feasible set $S: \{x_1 x_2 : x_1 + x_2 = 1, x_1 x_2 \ge 0\}$, either λ_1 or λ_2 or both is 0, so all points are regular. Since KKT condition is necessary for a regular point to be optimal, and since only $(0,1)$ is a KKT point in S, $x = (0,1)$ is a local optimum. Since it is the only local optimum in S, it is the optimal solution of the problem.

So the optimal solution is $x = (0,1)$.

Q2

Consider $f(x)=x_1^2+x_2^2$, $g_1(x)=-2x_1-x_2+10\le 0$, $x_2\ge 0$, $g_2(x)=-\cancel{x_2}x_2\le 0$.

$\mathcal{L}(x_1,x_2,\lambda)=x_1^2+x_2^2+\lambda_1(-2x_1-2x_2+10)+\lambda_2(-x_2)$.

$\nabla_x \mathcal{L}=\vec{0}$, $\begin{cases} \partial/\partial x_1\, \mathcal{L}=2x_1+\lambda_1(-2)=0 \\ \partial/\partial x_2\, \mathcal{L}=2x_2+\lambda_1(-1)+\lambda_2(-1)=0 \end{cases}$

$\lambda_1 g_1(x)=0$, $2x_1+x_2=10$

$\lambda_2 g_2(x)=0$, $\lambda_2=0$ or $x_2=0$.

① If $x_2=0$, $x_1=5$ since $2x_1+x_2=10$, $\lambda_1=5$ & $\lambda_2=-5$

Does not satisfy $\lambda_i\ge 0$. So $(5,0)$ is not KKT point.

② If $\lambda_2=0$, $\begin{cases} \partial/\partial x_1\, \mathcal{L}=2x_1+\lambda_1(-2)=0 \\ \partial/\partial x_2\, \mathcal{L}=2x_2+\lambda_1(-1)=0 \end{cases}$, $\begin{cases} x_1=\lambda_1 \\ x_2=\frac{1}{2}\lambda_1 \end{cases}$

$2x_1+x_2=10$, $2.5\lambda_1=10$, $\lambda_1=4$, $\begin{cases} x_1=4 \\ x_2=2 \end{cases}$

satisfy $\lambda_i\ge 0$. So $(4,2)$ is KKT.

Since $f(x)=x_1^2+x_2^2$ is convex, and constraints are affine, P is convex

So $(4,2)$ is optimal solution of P.

Q3

(i) Consider min $f(x) = x_1 - 4x_2 + x_3$

$$g(x) = x_1^2 + x_2^2 + x_3^2 - 1 \le 0$$

$$h(x) = x_1 + 2x_2 + 2x_3 + 2 = 0$$

Since $f(x) = A \cdot x$ for some matrix A, $f(x)$ is affine and thus convex.
Since $g(x)$ consists of sum of squares and a constant, it's convex.
Since $h(x) = B \cdot x + 2$ for matrix B, $h(x)$ is affine.
So P is convex. Also $f(x)$ & $g(x)$ & $h(x)$ are continuously differentiable.
So a KKT point must be an optimal ~~[illegible]~~ solution.

(ii) $\mathcal{L}(x_1, x_2, x_3, \lambda, \mu) = x_1 - 4x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1) + \mu(x_1 + 2x_2 + 2x_3 + 2)$.

$$\nabla_{\vec{x}} \mathcal{L} = \vec{0} = \begin{cases} 1 + 2\lambda x_1 + \mu = 0 \\ -4 + 2\lambda x_2 + 2\mu = 0 \\ 1 + 2\lambda x_3 + 2\mu = 0 \end{cases} \qquad \begin{cases} x_1 = \frac{-1-\mu}{2\lambda} \\ x_2 = 4 - 2\mu / 2\lambda \\ x_3 = -1 - 2\mu / 2\lambda \end{cases} \qquad \lambda \neq 0.$$

$h(x) = x_1 + 2x_2 + 2x_3 + 2 = 0$, $\frac{1}{2\lambda}(-1 - \mu + 8 - 4\mu - 2 - 4\mu + 4\lambda / 2\lambda = 0$, $\lambda = 9\mu - 5 / 4$

$\lambda g(x) = 0$, if $\lambda = 0$, no solution for μ. Thus $\lambda > 0$.

So $\lambda g(x) = 0$, $x_1^2 + x_2^2 + x_3^2 = 1$, $(1+\mu)^2 + (4 - 2\mu)^2 + (1 + 2\mu)^2 = 4\lambda^2$,

$\mu^2 + 2\mu + 1 + 4\mu^2 - 16\mu + 16 + 4\mu^2 + 4\mu + 1 = (9\mu - 5)^2 / 4$, solve get

$\mu_1 = -0.61$, $\mu_2 = 1.72$.

① $\mu = -0.61$, $\lambda = -2.01$, $\lambda_i \ge 0$ not satisfied. So $\mu \neq -0.61$, $\lambda \neq -2.01$.

② $\mu = 1.72$, $\lambda = 2.62$, $x_1 = -0.519$, $x_2 = 0.107$, $x_3 = -0.847$.

So $x = (-0.519, 0.107, -0.847)$ is KKT point, also optimal solution of P.

Q4

(i) $\nabla^2 f(x) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \not\succeq 0.$

So $f(x)$ not convex. So P not convex.

(ii) $g(x) = x_1^4 + x_2^4 + x_3^4 - 1 \le 0$

$\mathcal{L}(x_1, x_2, x_3, \lambda) = x_1^2 - x_2^2 - x_3^2 + \lambda(x_1^4 + x_2^4 + x_3^4 - 1)$

Since $\nabla g(x) = \begin{bmatrix} 4x_1^3 \\ 4x_2^3 \\ 4x_3^3 \end{bmatrix}$, only irregular point is $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\nabla_{\vec{x}} \mathcal{L} = \vec{0} = \begin{cases} 2x_1 + \lambda 4x_1^3 = 0 \\ -2x_2 + \lambda 4x_2^3 = 0 \\ -2x_3 + \lambda 4x_3^3 = 0 \end{cases}, \quad \begin{cases} x_1 = 0 & \forall \lambda \ge 0 \\ x_2 = 0, \pm\frac{1}{\sqrt{2\lambda}} & \lambda > 0 \\ x_3 = 0, \pm\frac{1}{\sqrt{2\lambda}} & \lambda > 0. \end{cases}$

$\lambda g(x) = 0$, if $\lambda = 0$, $x_1 = x_2 = x_3 = 0$, $(0,0,0)$ is a KKT point.

if $\lambda \ne 0$, $x_1^4 + x_2^4 + x_3^4 - 1 = 0$.

Since $\lambda_i \ge 0$, $\lambda > 0$, $x_1 = 0$, x_2 & x_3 not both 0.

① $x_2 = 0$, $x_3 = \frac{1}{\sqrt{2\lambda}}$, $x_3^4 = \frac{1}{4\lambda^2} = 1$, $\lambda = \pm\frac{1}{2}$.

Since $\lambda_i \ge 0$, $\lambda = \frac{1}{2}$, $(0,0,1)$ is KKT point

② $x_3 = 0$, similar to ①, $\lambda = \frac{1}{2}$, $(0,1,0)$ is KKT point.

③ $x_2 x_3 \ne 0$, $x_2 = x_3 = \frac{1}{\sqrt{2\lambda}}$. $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$, $\frac{1}{4\lambda^2} = \frac{1}{2}$, $\lambda = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

$x_2 = x_3 = \frac{1}{\sqrt[4]{2}} = 2^{-\frac{1}{4}}$, $(0, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}})$ is a KKT point.

Thus, KKT point: $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}})$

(iii) Consider $(0,0,0)$, $f(0,0,0) = 0$

Since all other feasible points are regular, KKT condition is necessary for optimal solution.

$f(0,0,0) = f(0,1,0) = -1$.

$f(0, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2\sqrt{2}}{2} < -1$.

So $(0, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}})$ is optimal solution.