1. Set D to be the interior of $B[0,1]$, and ∂D to be the boundary of $B[0,1]$.

For D, $\partial/\partial x\, f(x,y) = 2x+2$, $\partial/\partial y\, f(x,y) = 2y-3$, $\nabla f(x,y) = (2x+2, 2y-3)$

Solve $\nabla f(x,y)=0$ get $(-1, \frac{3}{2})$

For ∂D, $g(x,y) = x^2+y^2$, $\nabla g(x,y) = (2x, 2y)$

Solve $\nabla f(x,y) = \lambda \nabla g(x,y)$, or $\begin{cases} 2x+2 = 2\lambda x \\ 2y-3 = 2\lambda y \\ x^2+y^2=1 \end{cases}$, get $\begin{cases} 2\lambda x - 2x = 2 \\ 2x(\lambda-1)=2 \\ x(\lambda-1)=1 \end{cases}$

$2y-2\lambda y = 3$, $2y(1-\lambda)=3$, $\frac{2}{3}y(1-\lambda)=1$.

$x(\lambda-1) = \frac{2}{3}y(1-\lambda)=1$, so $1-\lambda\neq 0$ & $\lambda-1\neq 0$, or $\lambda\neq 1$.

$x=-\frac{2}{3}y=1$, $x=-\frac{2}{3}y$, $-x=\frac{2}{3}y$, $y=-\frac{3}{2}x$

Thus, $x^2+y^2 = x^2+(-\frac{3}{2}x)^2=1$, $x=\frac{2\sqrt{13}}{13}, -\frac{2\sqrt{13}}{13}$

$y=-\frac{3\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}$

Therefore $(\frac{2\sqrt{13}}{13}, -\frac{3\sqrt{13}}{13})$, $(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13})$

$f(-1,\frac{3}{2}) = -3.25$

$f(\frac{2\sqrt{13}}{13}, -\frac{3\sqrt{13}}{13}) = 1+\sqrt{13} \approx 4.61$

$f(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}) = 1-\sqrt{13} \approx -2.61$

Thus, global min is $(-1,\frac{3}{2})$, global max is $(\frac{2\sqrt{13}}{13}, -\frac{3\sqrt{13}}{13})$.

2. Since $A \in \mathbb{R}^{n\times n}$ & $B \in \mathbb{R}^{m\times m}$ symmetric,

$$\begin{bmatrix} A & O_{n\times m} \\ O_{m\times n} & B \end{bmatrix} = \begin{bmatrix} \begin{matrix} a \cdots a \\ \vdots \\ a \cdots a \end{matrix} & 0 \\ 0 & \begin{matrix} b \cdots b \\ \vdots \\ b \cdots b \end{matrix} \end{bmatrix}$$ where diagonal is non-zero and symmetric.

$(i) \to (ii)$:

Since A & B are PSD, $x_A^T A x_A \geq 0$ & $x_B^T B x_B \geq 0$.

Set $x = \begin{pmatrix} x_A \\ x_B \end{pmatrix}$, then

$$x^T \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} x = (x_A^T, x_B^T) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = x_A^T A x_A + x_B^T B x_B$$

$= \text{non-negative} + \text{non-negative}$

≥ 0

So $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is PSD

$(ii) \to (i)$:

Similarly, since $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is PSD, set $x = \begin{pmatrix} x_A \\ x_B \end{pmatrix}$, then

$$x^T \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} x = (x_A^T, x_B^T) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = x_A^T A x_A + x_B^T B x_B \geq 0.$$

Thus it must be true that A & B are PSD.

3. $\Rightarrow$:

Set x^* to be the unique solution to the problem, that is

$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 + \lambda\|Lx\|^2 = \|Ax^* - b\|^2 + \lambda\|Lx^*\|^2$

Suppose $Null(A) \cap Null(L) \neq 0$, then $\exists \alpha \neq 0$ s.t. $\alpha \in Null(A) \cap Null(L)$.

So $\alpha \in Null(A)$ & $\alpha \in Null(L)$, then $A\alpha = 0$, $L\alpha = 0$.

Claim that $x^* + \alpha$ is also a solution of RLS problem s.t.

$\|A(x^* + \alpha) - b\|^2 + \lambda\|L(x^* + \alpha)\|^2$

$= \|Ax^* + A\alpha - b\|^2 + \lambda\|Lx^* + L\alpha\|^2$

$= \|Ax^* - b\|^2 + \lambda\|Lx^*\|^2$ since $A\alpha = 0$, $L\alpha = 0$

$= \min_{x \in \mathbb{R}^n} \|Ax - b\|^2 + \lambda\|Lx\|^2$

But assumed x^* unique & $x^* + \alpha$ is a solution, contradiction.

So $Null(A) \cap Null(L) = 0$.

$\Leftarrow$:

Given $Null(A) \cap Null(L) = \{0\}$, $(A^T A + \lambda L^T L)x = A^T b$

let $H = A^T A + \lambda L^T L$.

let $x \in Null(H)$, then $Hx = 0$, $(A^T A + \lambda L^T L)x = 0$,

$x^T(A^T A + \lambda L^T L)x = x^T \cdot 0 = 0$, $\|Ax\|^2 + \lambda\|Lx\|^2 = 0$.

So $\|Ax\| = 0$ & $\|Lx\| = 0$, thus $x \in Null(A)$ & $x \in Null(L)$,

or $Null(H) \subseteq Null(A) \cap Null(L)$.

similarly, $Null(A) \cap Null(L) \subseteq Null(H)$.

Thus, $Null(A) \cap Null(L) = Null(H)$.

Since $Null(A) \cap Null(L) = \{0\}$, $Null(H) = \{0\}$

Known that $Hx = (A^T A + \lambda L^T L)x = A^T b$, since $A \in \mathbb{R}^{m \times n}$ & $L \in \mathbb{R}^{p \times n}$,

$A^T A$ & $L^T L \in \mathbb{R}^{n \times n}$,

$Hx = (A^T A + \lambda L^T L)x = A^T b$ is n dimensional with full rank n.

Therefore, H is invertable,

so solution $x = [(A^T A + \lambda L^T L)^{-1}] A^T b$ is unique.

4. (i) $f(x,y) = e^{x^2} + e^{y^2} - x^{200} - y^{200}$

e^{x^2} & e^{y^2} grows ~~the~~ faster than x^{200} & y^{200} as $\|x\| \to \infty$.

Coercive.

(ii) $f(x,y,z) = x^3 + y^3 + z^3$

$\lim_{\|x\| \to \infty} x^3 = \infty$, $\lim_{\|y\| \to \infty} y^3 = \infty$, $\lim_{\|z\| \to \infty} z^3 = \infty$.

All terms are positive and dominant and all approach ∞.

Coercive.

(iii) $f(x,y) = x^2 - 2xy^2 + y^4 = (x+y^2)^2 - 4xy^2$

First term is of order 4 and second term is of order 3.

$(x+y^2)^2$ is dominant over $-4xy^2$ and approach to ∞ as $\|x\|$ & $\|y\| \to \infty$.

Coercive.

(iv) $f(x) = \frac{x^T A x}{\|x\|+1}$ where $A \succ 0$ as $A \in \mathbb{R}^{n \times n}$

Known that if $A^{m \times n} \succ 0$, $f(x) = x^T A x$ is coercive.

If $A = I$, $x^T A x = x^T I x = x^T x = \|x\|^2 > 0$

If $A \neq I$, $x^T A x = \lambda \|x\|^2 > 0$.

So $x^T A x$ is coercive & $\|x\|^2$ is dominant over $\|x\|+1$ as $\|x\| \to \infty$.

So $f(x)$ is coercive.

5. $y = ax^2 + bx + c$

$a = 1.8331$

$b = -2.7886$

$c = -0.0599$

$y = 1.8331x^2 - 2.7886x - 0.0599$

6. center = $(0.5, 0.5417)$

Radius = 0.6783.