Homework 2

Due: Monday, Oct. 18
at 2:25 pm EST

Please give complete, well-written solutions to the following exercises and submit via Canvas.

1. (5 points) Find the global minimum and maximum points of the function $f(x, y) = x^2 + y^2 + 2x - 3y$ over the unit ball $S = B[0, 1] = \{(x, y) : x^2 + y^2 \leq 1\}$.

2. (10 points) Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be two symmetric matrices. Prove that the following two claims are equivalent:

   (i). A and B are positive semidefinite

   (ii). $\begin{pmatrix} A & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & B \end{pmatrix}$ is positive semidefinite.

3. (10 points) Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $L \in \mathbb{R}^{p\times n}$, and $\lambda > 0$. Consider the regularized least squares problem
$$\min_{\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|^2 + \lambda\|L\mathbf{x}\|^2.$$
Show that the problem has a unique solution if and only if
$$\text{Null}(A) \cap \text{Null}(L) = \{\mathbf{0}\}.$$

4. (20 points) For each of the following functions, determine wether it is coercive or not:

   (i). $f(x_1, x_2) = e^{x_1^2} + e^{x_2^2} - x_1^{200} - x_2^{200}$,

   (ii). $f(x_1, x_2) = x_1^3 + x_2^3 + x_3^3$,

   (iii). $f(x_1, x_2) = x_1^2 - 2x_1x_2^2 + x_2^4$,

   (iv). $f(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{\|\mathbf{x}\|+1}$, where $A \in \mathbb{R}^{n\times n}$ is positive definite.

5. (10 points) Consider thirty points $(x_i, y_i), i = 1, 2, \cdots, 30$ with $x_i = (i - 1)/29$, $y_i = 2x^2 - 3x + \epsilon_i$, where ϵ_i is randomly generated from a standard normal distribution $\mathcal{N}(0, (0.05)^2)$. Find the quadratic function $y = ax^2 + bx + c$ that best fits the points in the least square sense. Indicate what are the parameters a, b, c found by the least squares solution, and plot the points along with the derived quadratic function.

6. (15 points) Write a function circle_fit whose input is an $n \times m$ matrix A; the columns of A are the m vectors in $\mathbb{R}^n$ to which a circle should be fitted. The call to the function will be of the form:

[$\mathbf{x}$,r] = circle_fit(A)

The output $(\mathbf{x}, r)$ is the optimal solution of

$$\min_{\mathbf{x}\in\mathbb{R}^n, r\in\mathbb{R}_+} \sum_{i=1}^{m} (\|\mathbf{x}-\mathbf{a}_i\|^2 - r^2)^2. \tag{1}$$

Use the code in order to find the best circle fit in the sense of (1) of the 5 points

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{a}_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

And plot the points along with the derived circle.