Q1

(i) $C_1 = \{x \in \mathbb{R}^n : \|x^2\| = 1\}$

If choose $C_1 \ni x = (1, 0, 0, \dots, 0)$ & $C_1 \ni y = (0, 1, 0, \dots, 0)$.

$\|x\|^2 = \|(1, 0, 0, \dots, 0)\|^2 = 1^2 + 0^2 + \dots + 0^2 = 1$

$\|y\|^2 = \|(0, 1, 0, \dots, 0)\|^2 = 0^2 + 1^2 + \dots + 0^2 = 1$

Let $\lambda = \frac{1}{2}$, $\frac{1}{2}x + (1 - \frac{1}{2})y = (\frac{1}{2}, 0, 0, \dots, 0) + (0, \frac{1}{2}, 0, \dots, 0) = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$.

$\|\frac{1}{2}x + (1 - \frac{1}{2})y\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + 0^2 + 0^2 + \dots + 0^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$.

or $\lambda x + (1 - \lambda) y \notin C_1$. So C_1 not convex.

(ii) $C_2 = \{x \in \mathbb{R}^n : \max_{i=1,2,\dots,n} x_i \leq 1\}$

Let $C_2 \ni x = x_i$ & $C_2 \ni y = y_i$. $x_i \leq 1$ & $y_i \leq 1$. $\lambda \in [0, 1]$.

$\lambda x_i + (1 - \lambda) y_i$, since $x_i \leq 1$ & $y_i \leq 1$, $\lambda x_i \leq \lambda$ & $(1 - \lambda) y_i \leq 1 - \lambda$.

So $\lambda x_i + (1 - \lambda) y_i \leq \lambda + (1 - \lambda) = 1$. So $\lambda x + (1 - \lambda) y \leq 1 \in C_2$. C_2 is convex.

(iii) $C_3 = \{x \in \mathbb{R}^n, \min_{i=1,2,\dots,n} x_i \leq 1\}$

Let $C_3 \ni x = x_i$ & $C_3 \ni y = y_i$, $x = (4, \frac{1}{2}, 4, \dots, 4)$ where $x_i = \frac{1}{2} \leq 1$ and

$y = (\frac{1}{2}, 4, 4, \dots, 4)$ where $y_i = \frac{1}{2} \leq 1$.

Set $\lambda = 0.75$, $\lambda x + (1 - \lambda) y = 0.75x + 0.25y = 0.75(4, \frac{1}{2}, 4, \dots, 4) + 0.25(\frac{1}{2}, 4, \dots, 4)$

$= (3, 0.375, 3, \dots, 3) + (0.125, 1, \dots, 1)$

$= (3.125, 1.375, 4, \dots, 4)$

$\min\{\lambda x + (1 - \lambda) y\} = 1.375 \geq 1 \notin C_3$. C_3 is not convex.

(iv) $C_4 = \{x \in \mathbb{R}^n_{++} : \prod_{i=1}^n x_i \geq 1\}$

Consider $n = 2$ and $f = \frac{1}{x}$, then

$S = \{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 x_2 \geq 1\} = \{(x, t) \in \mathbb{R}^2_{++} : xt \geq 1\} = \{(x, t) : x \in \mathbb{R}^2_{++}, t \geq f(x)\}$

$=$ epigraph(f). Since f is convex on $\mathbb{R}^2_{++}$, epigraph is convex, so S is convex.

To generalize, consider $\frac{1}{x_1 x_2 \cdots x_n}$, $\forall x \in \mathbb{R}^n_{++}$. Let $\nabla x = [\frac{1}{x_1} \frac{1}{x_2} \cdots \frac{1}{x_n}]^T \in \mathbb{R}^n$

Since $\nabla^2 f = \frac{1}{x}(\nabla x \nabla x^T + \operatorname{diag}(\frac{1}{x_1^2} \frac{1}{x_2^2} \cdots \frac{1}{x_n^2}))$ is positive definite, f is convex on $\mathbb{R}^n$.

So C_4 is convex.

(v) C_5 = Union of two convex sets.

Consider two circles A & B that do not intersect, which are clearly two convex sets. If one chooses a point from interior of A and a point from interior of B, then at least one point in the line connecting the 2 points is not in either A & B. Thus, Union of two convex sets are not convex.

Q2

$$\begin{cases} -4x_2 + x_3 = 6 \\ 2x_1 - 2x_2 - x_4 = 1 \end{cases}$$

The system is of the form $Ax = b$ where

$A = \begin{pmatrix} 0 & -4 & 1 & 0 \\ 2 & -2 & 0 & -1 \end{pmatrix}$, $x = (x_1, x_2, x_3, x_4)^T$, $b = (6, 1)^T$

The rank of A is 2, so the max number of linearly independent column of A is 2.

$$\begin{array}{cccccc} \begin{pmatrix} 0 & -4 \\ 2 & -2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} & \begin{pmatrix} -4 & 1 \\ -2 & 0 \end{pmatrix} & \begin{pmatrix} -4 & 0 \\ -2 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ x_1 \; x_2 & x_1 \; x_3 & x_1 \; x_4 & x_2 \; x_3 & x_2 \; x_4 & x_3 \; x_4 \end{array}$$

$\begin{pmatrix} 0 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\begin{cases} x_1 = -1 \\ x_2 = -\frac{3}{2} \end{cases}$, Not feasible

$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\begin{cases} x_1 = \frac{1}{2} \\ x_3 = 6 \end{cases}$, Feasible

$\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, No solution

$\begin{pmatrix} -4 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\begin{cases} x_2 = -\frac{1}{2} \\ x_3 = 4 \end{cases}$, Not feasible

$\begin{pmatrix} -4 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\begin{cases} x_2 = -\frac{3}{2} \\ x_4 = 2 \end{cases}$, Not feasible

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\begin{cases} x_3 = 6 \\ x_4 = -1 \end{cases}$, Not feasible.

Thus, the feasible solution is $\begin{cases} x_1 = \frac{1}{2} \\ x_2 = 0 \\ x_3 = 6 \\ x_4 = 0. \end{cases}$

Q3

for $x \in ext(S)$, then $\|x\|_2 \leq 1$.

Case 1: $\|x\|_2 = 0$.

for $a, b \in S$, $b = -a$. For $\lambda = \frac{1}{2}$, $\lambda a + (1-\lambda) b = 0$ and $x = \lambda a + (1-\lambda) b$ for $a \& b \in S$, $a \neq b$, $\lambda \in (0,1)$.

Contradiction to the assumption that x is an extreme point of S.

So $\|x\|_2 \neq 0$.

Case 2: $0 < \|x\|_2 < 1$.

let $a = \frac{x}{\|x\|}$, $b = 0$. Then a line segment between a and b is $\lambda a + (1-\lambda) b = \lambda \frac{x}{\|x\|}$.

Choosing $(0,1) \ni \lambda = \|x\|$, get $\lambda a + (1-\lambda) b = x$ for $a \& b \in S$, $a \neq b$, $\lambda \in (0,1)$.

Contradiction to the assumption that x is an extreme point of S.

So $\|x\|_2 \notin (0,1)$

Case 3: $\|x\|_2 = 1$.

Valid for $\|x\|_2$.

Thus, $ext(S) = \{x \in \mathbb{R}^2 : \|x_2\| = 1\}$.

Q4

Let $y^* = f(0)$ and let $g(x) = f(x) - y^*$, so that $g(0) = 0$. Since f is concave & ~~convex~~ convex, g is also concave & convex.

If $\lambda = 0$ or 1, $g(\lambda a) = \lambda g(a)$ apparently, for $a \in \mathbb{R}^n$ & $\lambda \geq 0$.

If $\lambda \in (0,1)$, $g(\lambda a) = g(\lambda a + (1-\lambda)0) = \lambda g(a) + (1-\lambda) g(0) = \lambda g(a)$.

If $\lambda > 1$, then $1/\lambda \in (0,1)$ and hence $g(a) = g((1/\lambda)\lambda a + (1 - 1/\lambda)0)$

$$= (1/\lambda) g(\lambda a) + (1 - 1/\lambda) g(0)$$

$$= (1/\lambda) g(\lambda a).$$

Multiply by λ get $\lambda g(a) = g(\lambda a)$, for $a \in \mathbb{R}^n$ and $\lambda \geq 0$.

Also, $g(a+b) = g((\frac{1}{2})2a + (\frac{1}{2})2b) = \frac{1}{2} g(2a) + \frac{1}{2} g(2b) = g(a) + g(b)$

Thus, $g(a+b) = g(a) + g(b)$ for a & $b \in \mathbb{R}^n$.

Set A to be composed of $a_n = g(e^n)$ where $e^n = (0, \dots, 0, 1, 0, \dots, 0)$ is the coordinate n unit vector. Since $x = \sum_n x_n e^n$,

$g(x) = g(\sum_n x_n e^n) = \sum_n g(x_n e^n) = \sum_n x_n g(e^n) = \sum_n x_n a_n = Ax$.

Thus, $f(x) = g(x) + y^* = Ax + y^* = a^T x + b$ = Affine.

Q5

(a) Need $f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$

$x = e^{x_i}$, $y = e^{y_i}$. So $f(\lambda x + (1-\lambda) y) = \ln\left(\sum_{i=1}^{n} e^{\lambda x_i + (1-\lambda) y_i}\right) = \ln\left(\sum_{i=1}^{n} x^{\lambda} y^{(1-\lambda)}\right)$

By Holder's ~~inequ~~ inequality, $\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned}
\ln\left(\sum_{i=1}^{n} x^{\lambda} y^{(1-\lambda)}\right) &\le \ln\left[\left(\sum_{i=1}^{n} x^{\lambda \cdot \frac{1}{\lambda}}\right)^{\lambda} \cdot \left(\sum_{i=1}^{n} y^{(1-\lambda)\frac{1}{1-\lambda}}\right)^{1-\lambda}\right] \\
&= \lambda \ln\left(\sum_{i=1}^{n} x\right) + (1-\lambda) \ln\left(\sum_{i=1}^{n} y\right) \\
&= \lambda \ln\left(\sum_{i=1}^{n} e^{x_i}\right) + (1-\lambda) \ln\left(\sum_{i=1}^{n} e^{y_i}\right) \\
&= \ln\left(\sum_{i=1}^{n} e^{\lambda x_i + (1-\lambda) y_i}\right)
\end{aligned}$$

Thus $f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$, convex.

(b) Let x & $y \in S$, $\beta(\lambda x + (1-\lambda) y) = f(\lambda x + (1-\lambda) y) - \mu \sum_{i=1}^{n} \ln g(\lambda x + (1-\lambda) y)$.

Since f is convex, $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$.

Since g_i is concave, $g(\lambda x + (1-\lambda) y) \geq \lambda g(x) + (1-\lambda) g(y)$.

According to weighted AM-GM inequality,

$\lambda g(x) + (1-\lambda) g(y) \geq g(x)^{\lambda} g(y)^{1-\lambda}$

Since ln functions are increasing and $g(x) > 0$ & $g(y) > 0$.

$\ln[g(\lambda x + (1-\lambda) y)] \geq \ln[g(x)^{\lambda} g(y)^{1-\lambda}]$, or

$\ln[g(\lambda x + (1-\lambda) y)] \geq \lambda \ln[g(x)] + (1-\lambda) \ln[g(y)]$.

$\mu \ln[g(\lambda x + (1-\lambda) y)] \geq \mu \lambda \ln[g(x)] + \mu (1-\lambda) \ln[g(y)]$ since $\mu > 0$

$-\mu \ln[g(\lambda x + (1-\lambda) y)] \leq -\mu \lambda \ln[g(x)] - \mu (1-\lambda) \ln[g(y)]$

So $-\mu \ln g(x)$ is convex. By lecture note, sum of convex is convex, so

$\sum_{i=1}^{n} -\mu \ln g_i(x) = -\mu \sum_{i=1}^{\infty} \ln g_i(x)$ is convex, so

$\sum_{i=1}^{n} -\mu \ln g_i(\lambda x + (1-\lambda) y) \leq \lambda \sum_{i=1}^{\infty} -\mu \ln g_i(x) + (1-\lambda) \sum_{i=1}^{\infty} -\mu \ln g_i(y)$.

Thus, putting above inequalities together, we get

$$\begin{aligned}\beta(\lambda x + (1-\lambda) y) &\leq \lambda f(x) + (1-\lambda) f(x) - \lambda \mu \sum_{i=1}^{n} \ln g_i(x) - (1-\lambda) \mu \sum_{i=1}^{n} \ln g_i(y) \\ &= \lambda [f(x) - \mu \sum_{i=1}^{n} \ln g_i(x)] + (1-\lambda)[f(y) - \mu \sum_{i=1}^{n} \ln g_i(y)] \\ &= \lambda \beta(x) + (1-\lambda) \beta(y)\end{aligned}$$

or $\beta(\lambda x + (1-\lambda) y) \leq \lambda \beta(x) + (1-\lambda) \beta(y)$

So β convex.

Q6

(a) For $k=2$, $f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2)$.

For $k=n$, $f(\sum_{i=1}^{n} \lambda_i x_i) \le \sum_{i=1}^{n} \lambda_i f(x_i)$

For $k=n+1$, $f(\sum_{i=1}^{n+1} \lambda_i x_i) = f(\lambda_{n+1} x_{n+1} + \sum_{i=1}^{n} \lambda_i x_i)$

$$= f(\lambda_{n+1} x_{n+1} + (1-\lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i)$$

$$\le \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) f(\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i)$$

$$= \lambda_{n+1} f(x_{n+1}) + f(\sum_{i=1}^{n} \lambda_i x_i)$$

$$\le \lambda_{n+1} f(x_{n+1}) + \sum_{i=1}^{n} \lambda_i f(x_i)$$

$$= \sum_{i=1}^{n+1} \lambda_i f(x_i)$$

So for $k=n$ & $k=n+1$, $f(\sum_{i=1}^{k} \lambda_i x_i) \le \sum_{i=1}^{k} \lambda_i f(x_i)$

By mathematical induction,

$f(\sum_{i=1}^{k} \lambda_i x_i) \le \sum_{i=1}^{k} \lambda_i f(x_i)$

(b) Let $f(x) = -\ln x$ be a convex function. Then, considering $\frac{1}{n}\sum_{i=1}^{n} x_i \geq (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$

$-\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq -\ln((\prod_{i=1}^{n} x_i)^{\frac{1}{n}})$

$-\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq -\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)$

Let $\lambda_i = \frac{1}{n}$ for all $i = 1 \cdots n$, which satisfies $\lambda \in \Delta n$.

Then $-\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) = f(\sum_{i=1}^{n} \lambda_i x_i)$ with $\lambda_i = \frac{1}{n}$, $f(x) = -\ln(x)$

$-\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) = \sum_{i=1}^{n} \frac{1}{n}(-\ln(x_i)) = \sum_{i=1}^{n} \lambda_i f(x_i)$

Since $f(x) = -\ln(x)$ is convex, from part (a), $-\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq -\ln((\prod_{i=1}^{n} x_i)^{\frac{1}{n}})$

Thus $-\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq -\ln((\prod_{i=1}^{n} x_i)^{\frac{1}{n}})$ is true.

So $\ln(\frac{1}{n}\sum_{i=1}^{n} x_i) \geq \ln((\prod_{i=1}^{n} x_i)^{\frac{1}{n}})$, or

$\frac{1}{n}\sum_{i=1}^{n} x_i \geq (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ since $\ln(x)$ is increasing for $x \geq 0$, $x_i \geq 0$ for $\forall i$.