1. (a) $\|x-z\|_2 = \|x-y+y-z\|_2 = \|(x-y)+(y-z)\|_2$
$\leq \|x-y\|_2 + \|y-z\|_2$ by triangle inequality.

(b) $\|A+B\|_{a,b}^2 = \|A\|_{a,b}^2 + 2\|AB\|_{a,b} + \|B\|_{a,b}^2$
$\leq \|A\|_{a,b}^2 + 2\cdot\|A\|_{a,b}\|B\|_{a,b} + \|B\|^2_{a,b}$ by Cauchy-Schwarz
$= (\|A\|_{a,b}^2 + \|B\|_{a,b}^2)$

So $\|A+B\|_{a,b} \leq \|A\|_{a,b} + \|B\|_{a,b}$

2. Set $f(x) = a^T x$ s.t. $f: \mathbb{R}^n \to \mathbb{R}$
$\|x^*\| = \|\frac{a}{\|a\|}\| = \frac{\|a\|}{\|a\|} = 1$
$a^T = [a_1, a_2, \cdots, a_n]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$f(x) = a^T x = [a_1, \cdots, a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \langle a, x\rangle$ inner product.

By Cauchy-Schwarez, $a^T x = \langle a, x\rangle \leq \|a\|\|x\|$.
For the above equation to hold, a & x are linearly dependent.
So $a = \alpha x$ where $\alpha \in \mathbb{R}$, then $\max a^T x = \|a\|\|x\|$.
Since $x \in [-1,1]$, $\max a^T x = 1 \cdot \|a\| = \|a\|$ at $\max x = 1$. Thus $\max_{x\in B} a^T x = \|a\|$
Also $a = \alpha x$ imply $x = \frac{a}{\alpha}$, $\|a\| = \|\alpha x\| = |\alpha|\|x\| = |\alpha|\cdot 1 = \alpha$.
So $\alpha = \|a\|$, $x = \frac{a}{\alpha} = \frac{a}{\|a\|}$

3. Set $B = A^* A$ as a Hermit matrix.

Let E be a linear transformation of a Euclidean vector space of B.

So $\exists$ orthonormal basis of E containning eigenvalues of B.

let $\lambda_1 \cdots \lambda_n$ be eigenvalues of B and $\{e_1 \cdots e_n\}$ ~~be~~ be orth. basis of E.

let $x = a_1 e_1 + \cdots + a_n e_n$.

$\|x\| = \sqrt{\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i \rangle} = \sqrt{\sum_{i=1}^{n} a_i^2}$

$Bx = B(\sum_{i=1}^{n} a_i e_i) = \sum_{i=1}^{n} a_i B(e_i) = \sum_{i=1}^{n} \lambda_i a_i e_i$

set $\lambda_{max} = max\{\lambda_1 \cdots \lambda_n\}$. Then

$\|Ax\| = \sqrt{\langle Ax, Ax \rangle} = \sqrt{\langle x, A^* A x \rangle} = \sqrt{\langle x, Bx \rangle} = \sqrt{\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} \lambda_i a_i e_i \rangle}$

$= \sqrt{\sum_{i=1}^{n} a_i \lambda_i a_i} \leq max \sqrt{\lambda_i} \times \|x\|$.

So if $\|A\| = max\{\|Ax\| : \|x\| = 1\}$, $\|A\| \leq max \sqrt{|\lambda_i|}$

Consider $x_{max} = e_{max}$, $\|x_{max}\| = 1$ so that $\|A\|^2 \geq \langle x_{max}, B x_{max} \rangle = \langle e_{max}, B(e_{max}) \rangle$

$= \langle e_{max}, \lambda_{max} e_{max} \rangle = \lambda_{max}$

Thus $\|A\| = max \sqrt{|\lambda_i|}$ where λ_i is eigenvalues of $B = A^* A$.

So $\|A\|_2 = \sqrt{\lambda_{max}(A^* A)} = \sigma_{max}(A)$

4. (i) $\Rightarrow$: Since $A_{n\times n}$ is symmetric, $A^T = A$.

Since $A \succeq$, λ_i of A are non-negative.

Since A is symmetric, it's also diagonalizable as well.

So we have $A = PDP^T$ where P is orthogonal and D is a diagonal matrix with eigenvalue of A as non-negative entries.

Define $C = \sqrt{D}$, then $A = P \cdot C \cdot C \cdot P^T = (PC) \cdot C^T C P^T C^T = (PC) \cdot (PC)^T$

Again since $A \succeq 0$, for $\forall x \in \mathbb{R}^n$, $\exists x^T A x \geq 0$,

so ~~[illegible]~~ $x^T \cdot PC \cdot (PC)^T \cdot x \geq 0$, set $PC = B$, get $x^T B B^T x \geq 0$

Thus $A = BB^T$

$\Leftarrow$: Let $A = BB^T$

$$B = \begin{bmatrix} b_{11} & \cdots & \\ \cdots & \cdots & \\ & \cdots & b_{nn} \end{bmatrix},$$

$$BB^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & \cdots & a_{11}a_{n1} + \cdots + a_{nn}a_{nn} \\ \vdots & & \vdots \\ a_{n1}a_{n1} + \cdots + a_{1n}a_{nn} & \cdots & a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2 \end{bmatrix}$$

Since $|B| \neq 0$

$a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 + \cdots + a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2 \neq 0$

so $|B||B^T| \neq 0$

Thus all of eigenvalues are ~~[illegible]~~ non-negative. So ~~[illegible]~~ $A \succeq 0$.

(ii) $\Leftarrow$: If B has a full row rank, B is non-singular,

so $X^TAX = X^TBB^TX = (B^TX)^T(B^TX) = \|B^TX\|^2 > 0$.

So $X^TAX > 0$, thus A is $\succ 0$.

$\Rightarrow$: Since $A \succ 0$, $X^TAX > 0$, $X \neq 0$.

Take $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$X^TAX = [x_1 \cdots x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1^2 a_{11} + \cdots + x_n^2 a_{nn} > 0$$

Then A has a full rank.

5. (i) $\vec{x}^T A \vec{x} = [x_1\ x_2\ x_3\ x_4] \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$= 2a^2 + 4ab + 2b^2 + 3c^2 + 2cd + 3d^2$

$= a^2 + 2ab + b^2 + a^2 + 2ab + b^2 + 2c^2 + c^2 + 2cd + d^2 + 2d^2$

$= (a+b)^2 + (a+b)^2 + 2c^2 + (c+d)^2 + 2d^2 \geq 0.$

so $A \succeq 0$.

(ii) $\vec{x}^T B \vec{x} = [x_1\ x_2\ x_3] \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 2a^2 + 4ab + 4ac + 3b^2 + 3c^2 + 6bc$

$= 2a^2 + 4a(b+c) + 3(b^2 + 2bc + c^2)$

$= 2a^2 + 3(b+c)^2 + 4a(b+c)$

$\begin{cases} tr(B) = 8 \geq 0 \\ det(B) = 0 \geq 0 \end{cases}$

indefinite

(iii) $\vec{x}^T C \vec{x} = [x_1\ x_2\ x_3] \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 2a^2 + 2ab + 6ac + 2b^2 + 2c^2 + 2bc$

$= a^2 + b^2 + (a+b)^2 + 2c^2 + 2c(3a+b)$

$\begin{cases} tr(C) = 6 \geq 0 \\ det(C) = -8 \leq 0 \end{cases}$

indefinite.

(iv) $\vec{x}^T D \vec{x} = [x_1 \ x_2 \ x_3] \begin{bmatrix} -5 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= -5a^2 + 2ab + 2ac - 7b^2 - 5c^2 + 2bc$

$= -6a^2 + a^2 + 2ab + b^2 - 8b^2 - 5c^2 + 2ac + 2bc$

$= (a+b)^2 - 6a^2 - 8b^2 - 5c^2 + 2ac + 2bc$

$= (a+b)^2 - 7a^2 + a^2 + 2ac + c^2 - 6c^2 - 8b^2 + 2bc$

$= (a+b)^2 + (a+c)^2 - 7a^2 - 6c^2 - 8b^2 + 2bc$

$= (a+b)^2 + (a+c)^2 - 7a^2 - 9b^2 + b^2 + 2bc + c^2 - 7c^2$

$= (a+b)^2 + (a+c)^2 + (b+c)^2 - 7a^2 - 9b^2 - 7c^2$

$\begin{cases} tr(D) = -17 \\ det(D) = -156 \end{cases}$

indefinite

6. (i) $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$

$d/dx_1 f = 6x_2 x_1$ ①

$d/dx_2 f = 6x_2^2 - 12x_2 + 3x_1$ ②

solve $\begin{cases} ① = 0 \\ ② = 0 \end{cases}$ get $(0,0)$ $(0,2)$

according to the graphs, $(0,0)$ is local max

$(0,2)$ is local min.

(ii) $f(x_1, x_2) = x_1^2 + 4x_1 x_2 + x_2^2 + x_1 - x_2$

$d/dx_1 f = 2x_1 + 4x_2 + 1$ ①

$d/dx_2 f = 4x_1 + 2x_2 - 1$ ②

solve $\begin{cases} ① = 0 \\ ② = 0 \end{cases}$ get $(\frac{1}{2}, -\frac{1}{2})$

according to the graphs, $(\frac{1}{2}, -\frac{1}{2})$ is local max.

(iii) $f(x_1, x_2) = (x_1 - 2x_2)^4 + 64x_1 x_2$

$d/dx_1 f = 4(x_1 - 2x_2)^3 + 64x_2$ ①

$d/dx_2 f = -8(x_1 - 2x_2)^3 + 64x_1$ ②

solve $\begin{cases} ① = 0 \\ ② = 0 \end{cases}$ ~~get~~ get $(-1, \frac{1}{2})$ $(0,0)$ $(1, -\frac{1}{2})$

according to the graphs, $(-1, \frac{1}{2})$ is a local min

$(0,0)$ is a local min

$(1, -\frac{1}{2})$ is a local min.

(iv) $f(x_1, x_2) = x_1^4 + 2x_1^2 x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$

$d/dx_1 f = 4x_1^3 + 4x_2 x_1 - 8x_1 - 8$ ①

$d/dx_2 f = 2x_1^2 + 2x_2 - 8$ ②

solve $\begin{cases} ① = 0 \\ ② = 0 \end{cases}$ get $(1,3)$

according to the graphs, $(1,3)$ is a local max.