Homework 1

Due: Monday, Oct. 4
at 2:25 pm EST

Please give complete, well-written solutions to the following exercises.

1. (a). (5 points) Show that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$

$$\|\mathbf{x} - \mathbf{z}\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 + \|\mathbf{y} - \mathbf{z}\|_2.$$

(b). (5 points) Suppose that $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with norms $\|\cdot\|_b$ and $\|\cdot\|_a$, respectively. Show that the induced matrix norm $\|\cdot\|_{a,b}$ satisfies the triangle inequality. That is, for any $A, B \in \mathbb{R}^{m\times n}$ the inequality

$$\|A + B\|_{a,b} \leq \|A\|_{a,b} + \|B\|_{a,b}$$

holds.

2. (5 points) Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector. Show that the maximum of $\mathbf{a}^T\mathbf{x}$ over $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ is attained at $\mathbf{x}^* = \frac{\mathbf{a}}{\|\mathbf{a}\|}$ and that the maximal value is $\|\mathbf{a}\|$.

3. (10 points) If $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_2$, show that the induced norm of a matrix $A \in \mathbb{R}^{m\times n}$ is the maximum singular value of A, that is,

$$\|A\|_2 = \|A\|_{2,2} = \sqrt{\lambda_{\max}(A^TA)} = \sigma_{\max}(A),$$

where λ represents eigenvalue and σ represents sigular value.

4. Let A be an $n \times n$ symmetric matrix.

   (i). (5 points) Show that A is positive semidefinite if and only if there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $A = BB^T$.

   (ii). (5 points) Prove that A is positive definite if and only if B has a full row rank.

   (iii). (5 points) Let $\mathbf{x} \in \mathbb{R}^n$ and let A be defined as

$$A_{ij} = x_i x_j, \quad i, j = 1, 2, \cdots, n.$$

   Show that A is positive semidefinite and that is not a positive definite matrix when $n > 1$.

5. For each of the following matrices determine wether they are positive/negative semidefinite/definite or indefinite:

(i). (2.5 points) $A = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$,

(ii). (2.5 points) $B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}$,

(iii). (2.5 points) $C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$,

(iv). (2.5 points) $D = \begin{pmatrix} -5 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & -5 \end{pmatrix}$.

6. For each of the following functions: (1). find all the stationary points and classify them according to whether they are saddle points, strict/nonstrict local/global minnimum/maximum points; (2). draw the contour and surface plots and mark all stationary points on the contour plots.

   (i). (4 points) $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$ on $\mathbb{R}^2$,

   (ii). (4 points) $f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + x_1 - x_2$ on $\mathbb{R}^2$,

   (iii). (4 points) $f(x_1, x_2) = (x_1 - 2x_2)^4 + 64x_1x_2$ on $\mathbb{R}^2$,

   (iv). (4 points) $f(x_1, x_2) = x_1^4 + 2x_1^2x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$ on $\mathbb{R}^2$.