TREE FORM: DEFINITION, INTERPOLATION, EXTRAPOLATION[1]

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ABSTRACT

Definition of tree form requires numerous measurements of height and stem radius or diameter distributed over the entire tree stem. Further definition may involve a graphic plot of stem profile, an analytic expression of radius as a polynomial or rational polynomial function of distance from apex, or the direct numeric evaluation of the major integrals of tree form (length, surface, volume). Linear, quadratic, or harmonic interpolation over short intervals can assume a monotonic one-parameter function without introducing serious error. Extrapolation should employ a two-parameter function passing through the origin and based on three measured pairs of coordinates. Appropriate surface and volume integrals are given for the convex right hyperbola ($XY-QX+PY=O$) *and the concave parabola* ($Y^2+QX-PY=O$).

INTRODUCTION

Most foresters have acquired an intuitive concept of tree form — the geometric shape of the honeycomb-like mass of wood in a single dominant stem of a given tree. This wood mass is bounded by a hypothetical skin that theoretically could be peeled off and unrolled or flattened like the skin of a deer. Furthermore, the mass of wood enlarges annually, growing upwards along a linear axis faster than outwards around a convex periphery. Though distribution of peripheral growth with respect to the vertical linear axis is quite variable, the shape of the accumulating mass suggests that of a solid of revolution, definable in two dimensions by a radial profile along the vertical linear axis. Whatever explicit function might specify that profile, it appears to be monotonic increasing from apex to base, though there may be numerous points of inflection.

Many mensurationists have sought to discover a single simple two-variable function involving only a few parameters which could be used to specify the entire tree profile. Unfortunately, trees seem capable of assuming an infinite variety of shapes, and polynomials (or quotients of polynomials) with degree at least two greater than the observed number of inflections are needed to specify variously inflected forms. Furthermore, coefficients would vary from tree to tree in ways that could only be known after each tree had been completely measured. Thus, explicit analytic definition of tree form requires considerable computational effort, yet lacks generality.

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Analysis of polynomial expressions of tree form into orthogonal components by use of multivariate eigenvalue-eigenvector techniques (Fries and Matérn 1965) may ultimately lead to more general expressions of tree form. But even so, each tree must be regarded as an individual that must be completely measured, or else as a member of a definite population whose average form can only be estimated by complete measurement of other members of the population selected according to a valid sampling plan. Hence, polynomial analysis may rationalize observed variation in form after measurement, but it does not promise more efficient estimation procedures.

Some mensurationists hoped that empirically grouping trees by species, diameter, and height would reduce variation in tree form, but they soon found that upper stem diameters could fluctuate considerably even within a single group. The difference was usually associated with difference in branch growth (past or present) which in turn might depend on accident, early stand density, heredity, etc. — in any event, the factors responsible could rarely be measured or even detected after passage of time.

Hence, additional empirical classifications or indices were used to try to reduce the variability. Most of them relied on an additional measurement of upper stem diameter and height (form class, form point, form factor, form quotient, etc), but some relied on measurement of height to an unmeasured diameter (such as that where live branching begins). As might be expected, employing an additional variable or criterion for stratification *did* reduce variability, but it did not solve the problems of multiple points of inflection or of highly variable distribution of taper in the long sections beyond or between measurements.

The high-degree polynomial deemed appropriate to one of these species-size-form classes was rarely fitted directly. Empirical fitting techniques were usually used, with end results often expressed as relative or absolute taper tables. Such difference tabulations of $(Y_2 - Y_1)$ for constant intervals of $(X_2 - X_1)$ are identical with results obtained by evaluating the implied polynomial at X_2 and X_1, and then subtracting.

Stratification by species, size, and form has also been employed in deriving non-functional regressions for volume. Such regressions make no pretense of being volume integrals, since they ignore the functional relationship between diameter and height. Although they can provide valid estimates of aggregate volume when based on appropriate samples, they cannot specify tree form or volume distribution within a tree. Hence, they will not be discussed here, except to note that foresters tend to apply them far beyond the limits of the tree population defined or implied by the way in which sample data were obtained.

Though analytical definitions of tree form have not yet proved very useful, plotted profiles of stem measurements have often suggested less cumbersome models than high-degree polynomials. Such graphic profiles have probably been more practical specifications of tree form than the oversimplified analytic models that they inspired. Unhappily, upper stem measurements were very expensive, even for felled trees (which rarely constituted a valid sample of any standing tree population). In addition, the plotting, smoothing, planimetering, dot-counting, or coordinate-reading required for any quantification of the geometric representation was tedious, subjective, error-prone, and expensive. Often the analog nature of existing profiles discouraged any quantification other than interpolating coordinates at regular intervals to obtain taper tables. It is easy to see why profiling was viewed as a research project, not as a routine portion of the sampling procedure necessary for valid estimates of average tree form in each new tree population of interest.

The foregoing brief discussion of traditional approaches to specification of tree form — analytical, empirical, and graphic — is necessarily sketchy. Much more complete reviews of literature on tree form (from different points of view) can be found in Spurr (1952) and Larson (1963).

Possibly some foresters do not share the author's pessimistic views on the possibility of achieving a simple, accurate, analytic description of tree form. Be that as it may, the last 15 years have seen revolutionary developments in instruments, in computers, and in sampling that encourage a fresh look at the problem.

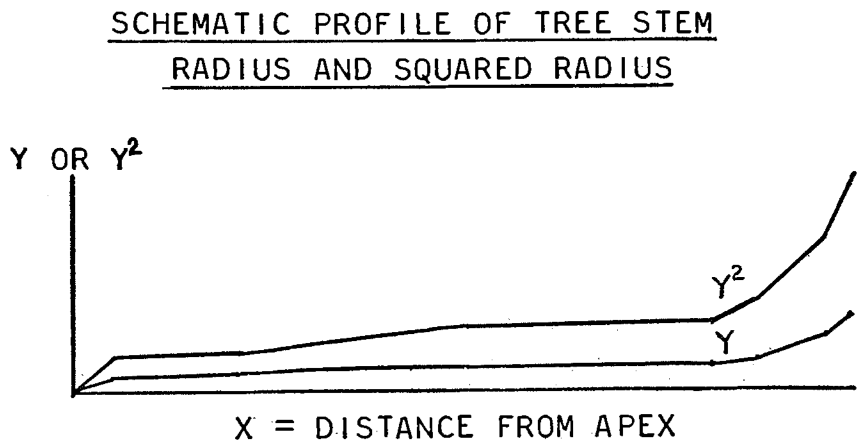
Definition

Whether any simple general expression of tree diameter as a function of height is ever discovered, it is currently feasible to define diameter as a many-termed polynomial function of height with coefficients variable from tree to tree. However, height-diameter measurements must be numerous enough not only to locate all points of inflection, but to damp intervening curves to monotonicity. The same set of measurements, of course, permits adequate graphic representation of stem form.

The recent availability of accurate, practical dendrometers makes it feasible for the first time to obtain all the information needed to specify the form of that portion of a standing tree which is visible from the ground. Now, too, for the first time it is feasible to program on high-speed computers the massive calculations needed either for explicit polynomial specification of tree form, or for smoothly interpolated incremental input needed by specialized plotting devices.

However, a third approach seems to have advantages over either of the foregoing alternatives. A tree can be regarded as the aggregate of a number of short sections related only by sharing a common diameter at junction points and by a requirement for monotonicity of profile. Concavity or convexity becomes relatively unimportant in sections short enough so that the smaller diameter is at least 8/10 the larger diameter. Then the length, surface, and volume of each section can be computed (Grosenbaugh 1964) and accumulated for a single sample tree. This trio of sums embodies the most important quantitative information about tree form that can be obtained either from the explicit polynomial or from the graphic plot. Furthermore, trios for two or more sample trees are just as additive (without weighting) as component elements from a single sample tree, which fact greatly facilitates the definition of average form for a population of trees.

Figure 1 illustrates the graphic profile both of radius and of squared radius in relation to distance from apex of a hypothetical tree. Mechanical determination of areas under these curves gives surface and volume, while length is obtainable simply by scaling. The two profiles can be quantitatively specified by polynomials of the sort indicated at the bottom of the figure.



$$Y = AX + BX^2 + \cdots + GX^N$$

$$Y^2 = (AX + BX^2 + \cdots + GX^N)^2$$

FIGURE 1. Tree stem profile.

Surface can then be obtained from the explicit integral of the first function (with respect to the differential of arc), and volume from the explicit integral of the second.

Figure 2 lists the general integrals for length, surface, and volume applicable to solids of revolution where Y is tree radius and X is distance from apex, both measured in the same units. The radical in the surface integral is very troublesome, but (dy/dx) simplifies to a constant in the case of conoids, and this constant is nearly zero in the case of trees with the relatively slow rate of taper usually prevalent.

A simple analogy illustrates one reason for the utility of the integrals as quantifiers of tree form. If all constants and the radical be ignored, if "dx" be replaced by "f" (an element of frequency in a discrete population), and if each integral sign be replaced by a summation sign, the three integrals become Σf, Σfy, Σfy^2. Well known formulae employ these quantities in expressions for the total number of individuals in a discrete population, the mean size of an individual, and the variance about the mean size. Such a trio of statistics is standard for describing distribution according to size (Y) in a discrete population. The raw sums are actually zeroth, first, and second moments about zero.

Length, surface, and volume have proved to be very usefully correlated with other variables that are affected by size of tree stem, such as lumber or other product yield, defect, weight, density, cost of manufacture, etc. Moreover, this trio of form statistics can be computed from many different

GENERAL INTEGRALS FOR LENGTH, SURFACE, VOLUME OF SOLIDS GENERATED BY REVOLVING Y=F(X) ABOUT LINE Y=0

$$\text{LENGTH} = \int_{X_1}^{X_2} dx = X_2 - X_1$$

$$\text{SURFACE} = 2\pi\int_{X_1}^{X_2} Y\,dx\sqrt{1+(dY/dx)^2} = 2\pi\int_{Y_1}^{Y_2} Y\,dY\sqrt{1+(dx/dY)^2}$$

$$\text{VOLUME} = \pi\int_{X_1}^{X_2} Y^2\,dx = 2\pi\int_{Y_1}^{Y_2} XY\,dY$$

FIGURE 2. Generalized length, surface, and volume integrals.

kinds of raw measurement data — wheel rotation, water displacement, weight, linear distances, angular magnitudes, graphic plots, etc.

Not only are length, surface, and volume directly useful as variables in linear estimators, but comparisons of ratios involving them furnish additional information about tree form. Figure 3 shows three means that would coincide if a tree were cylindrical, without taper. They become increasingly unequal as taper occurs. The fourth ratio measures relative variation in diameter, with zero variation indicated by a ratio value of unity.

Interpolation

Tree form quantification, however achieved, requires ideally that standing trees have their height and diameter dendrometered from stump to tip at reasonably short intervals. Relatively simple interpolation techniques are usually quite adequate for obtaining coordinates at additional intervening points — no analytical specification of the form is necessary.

Interpolation, in its narrow sense, is needed to estimate heights and diameters which will subdivide a tree into desired portions not bounded by pairs of actually measured heights and diameters. Two-point linear, quadratic, or harmonic interpolation is quite adequate for such estimates made on conical, convex, or concave profiles and it matters little which is employed if the interval is relatively short. Four-point or greater interpolation would hardly be worthwhile, since unpredictable inflections frequently intervene between widely separated diameter measurements, although it is often difficult to tell whether the inflection is real or just an artifact of measurement errors.

In a broader sense, computation of surface or volume between two pairs of coordinates can be regarded as a type of implicit interpolation, since

RATIOS INVOLVING LENGTH, SURFACE, VOLUME
OF SOLIDS OF REVOLUTION
(L, S, V SCALED IN FT., FT^2, FT^3)

$13.5406\sqrt{V/L}$ = QUADRATIC MEAN DIAM. (INCHES)

$3.8197\ S/L^*$ = ARITHMETIC MEAN DIAM. (IN.)

$48.0000\ V/S$ = $(\text{QUAD.MEAN})^2/\text{ARITH.MEAN}$ (IN.)

*DIVISOR SHOULD BE $L\sqrt{1+(D-d)^2/576L^2}$, BUT FOR MOST TREES, RADICAL IS ALMOST UNITY.

$12.5664LV/S^2 = (\text{QUAD.MEAN})^2/(\text{ARITH.MEAN})^2$
= UNITY WHEN CYLINDRICAL

FIGURE 3. Useful ratios of length, surface, and volume.

any integrating device must assume *something* about the intervening curve. The less that need be assumed, the less the possible error involved.

No law compels short sections of tree stem to taper in a linear, convex, or concave manner with cross-sectional areas being a monotonic function of distance from apex or zero-point, but this weak assumption greatly simplifies quantification of tree form and violation of the assumption is infrequent.

The additional assumptions that tree cross-sectional areas are nearly circular or elliptical and that the radial profiles of short sections can be closely approximated by various one-parameter conic sections also can be substantiated by observation in a large majority of cases. Foresters sometimes forget that justification for such assumptions is empirical — there is no magic virtue in one-parameter shapes generated by revolution of one line around another or of a parabola around its bisector. Thus the traditional one-parameter conoid, paraboloid, and neiloid are merely convenient instances in a continuum of short monotonic shapes. More or less convex or concave shapes are common, but differences are imperceptible when frusta are short or when the two terminal diameters differ by 20 percent or less.

Consequently, when tree form or profile has been specified by numerous measurements of diameter and height at relatively short intervals, not only is the problem of interpolating to obtain intervening coordinates simple, but convenient surface and volume integrals are easily evaluated for the one-parameter profiles generated by the series $Y^2=QX$ (convex paraboloid), $Y^2=QX^2$ (conoid), $Y^2=QX^3$ (concave neiloid). Figure 4 shows the integrals for the conoid. These integrals are useful in quantifying tree form where interpolation interval is short or taper is slight. The volume integral in particular is of interest in that the effect of convexity or concavity has been isolated into

CONIC SURFACE AND VOLUME INTEGRALS
(L,S,V SCALED IN FT., FT^2., FT^3., WITH
TERMINAL DIAMETERS D, d SCALED IN INCHES)

$$S = (\pi L/12)(d+(D-d)/2)\sqrt{1+(D-d)^2/576L^2}$$

FOR MOST TREES, TERMINAL RADICAL
MAY BE CONSIDERED UNITY.

$$V = (\pi L/576)(Dd+(D-d)^2/3)$$

SINGLE-PARAMETER PARABOLOID OR
NEILOID VOLUMES CAN BE OBTAINED
IF 2 OR 4 REPLACES DIVISOR 3.

FIGURE 4. Conic integrals.

a single taper-dependent term. Replacing the divisor 3 by the divisor 2 gives exactly the same result as would be obtained for a one-parameter paraboloid by Smalian's formula, while the divisor 4 is appropriate to a one-parameter neiloid.

Simplification of numerical analysis of tree form can be effected by measuring tree stems at regular intervals of height or diameter. Where felled trees are bucked consistently into 5-foot (or 16-foot) lengths, measurement of diameter at such constant intervals of length facilitates office computations. Less well known but more spectacular benefits arise when heights are measured to a series of diameters that decrease in arithmetic progression by a constant difference in diameter. This latter procedure makes possible a numerical integration process called height-accumulation (Grosenbaugh 1954), which will be briefly discussed since it demonstrates how measured pairs of coordinates can be numerically integrated with a minimum of implicit interpolation into more useful specifications of tree form.

Figure 5 shows tree height measured to 4 diameters above the stump, separated by a constant one-inch taper interval. Two simple additions — height, and upward-progressing subtotals of height — permit numeric integration giving length, surface, and volume. Although conic shape is assumed for the short sections, empirical concavity or convexity indices may replace the conic volume index 3. Had the tree extended one more foot to its apex (zero diameter), $H_1=12$ might have been replaced in the formulae by $H_0=13$, but no other terms would be affected. Portions of a tree assigned to different quality classes can be combined with similar grades and diameters of material in other trees by simple addition of height differences, since only lengths are involved. Diameters at both ends of every section are always known

HEIGHT-ACCUMULATION USED TO COMPUTE TREE LENGTH, SURFACE, VOLUME

D	H	H′
1	H_1=12	29
2	10	17
3	6	7
4	1	1
5	0	0
	ΣH=29	54=$\Sigma H'$

$$V = .01091\Sigma H' + .00182 H_1$$
$$S = .262\Sigma H + .131 H_1$$
$$L = H_1$$

$$V = .611 \text{ FT}^3.$$
$$S = 9.170 \text{ FT}^2.$$
$$L = 12.000 \text{ FT.}$$

D IS DIAM. IN ARITH. PROGRESSION (INCHES).
H IS CORRESPONDING HEIGHT IN FEET.
H′ IS UPWARD-PROGRESSIVE SUBTOTALS OF H.

FIGURE 5. Numerical example of height accumulation.

even when lengths from numerous different trees are added together. Range-finder dendrometers are unfortunately not well adapted to field identification of this progression of diameters. Such dendrometers can, however, accurately locate and measure points where quality or taper changes abruptly, and modern computers are not at all burdened by extra calculations necessitated by irregular lengths and taper intervals.

Where tree profile is well defined by pairs of coordinates at short intervals, or by pairs of coordinates at control points such as points of inflection or points of radical change in quality or rate of taper, tree form can be readily quantified into length, surface and volume by the one-parameter integrals illustrated in figures 4 and 5.

Extrapolation

Most traditional volume tables are applied to sample trees in a manner which is primarily extrapolative. Foresters optimistically hope that a given sample tree will extend beyond measured d.b.h. (or 16-foot or 32-foot d.o.b.) along a profile similar to that assumed by their volume table (derived from quite a different tree population). Where total height to zero diameter is actually measured, the process becomes nominally interpolative. However, the behavior of taper in variously branching live crowns is so erratic that the actual location of zero diameter is often of little help in characterizing the form of the lower and more important portion of the stem.

Extrapolation is inherently arbitrary and always incurs the risk of serious bias when it extends far beyond the last measurement, or when the range of values that can be assumed by unmeasured diameters is great. Least squares theory, minimax principles, and maximum likelihood do not provide any valid guide for testing or evaluating the merits of arbitrary projections. Extrapolation should never be employed for greater lengths than is absolutely necessary.

The extrapolation procedures to be discussed presume availability of four or more pairs of measurements for the lower, larger, more valuable portions of the tree. For this reason alone, these procedures should provide better estimates of tree form than those based on fewer measurements and more extrapolation.

Although the one-parameter conoid, paraboloid, or neiloid assumptions permit satisfactory interpolation where intervals are short, such assumptions could lead to sizable mistakes when extrapolated over considerable distances. At least two parameters seem necessary to specify the shape of longer (but still monotonic) profiles. Five relatively simple two-parameter functions passing through the origin come to mind immediately. They are shown in figure 6.

The first relationship (fully logarithmic) has real appeal when P=1 (a paraboloid), P=2 (a conoid), and P= 3 (a neiloid). However, this first function turns out to be analytically undesirable, since X itself is not available in the analytic stage (only the differences between several X's), and since the surface integral is obtainable only when P is an integer. A pair of exponentials (semi-logarithmic) are also possible, but suffer from similar analytic disadvantages.

SIMPLEST 2-PARAMETER MONOTONIC FUNCTIONS PASSING THROUGH ORIGIN

$Y^2-QX^P=0$ (CONCAVE OR CONVEX LOGARITHMIC)
TWO SEMILOG FUNCTIONS ARE ALSO POSSIBLE.

$Y^2-QX\pm PY=0$ (CONVEX PARABOLIC)
$Y^2+QX-PY=0$ (CONCAVE PARABOLIC)*

$XY-QX+PY=0$ (CONVEX HYPERBOLIC)*
$XY+QX-PY=0$ (CONCAVE HYPERBOLIC)

*ASTERISKS INDICATE PREFERRED FUNCTIONS.

FIGURE 6. Possible extrapolative functions.

The two remaining pairs of functions are the horizontal parabola with origin translated to a point lying on the curve, and the equilateral hyperbola rotated 45° with origin translated to a point lying on the curve. The ellipse does not qualify because it would require 3 or more parameters to pass through the origin unless origin and vertex coincide — a limitation not found in the equilateral hyperbola or parabola. It is of interest to note that either the two-parameter horizontal parabola or the two-parameter equilateral hyperbola can be fitted to any three points whose successive ordinate differences are alike in sign, yet differ from each other and from zero.

There are reasons having to do with extrapolation, however, for preferring the equilateral hyperbola when form is convex, and the horizontal parabola when form is concave. This arbitrary choice further simplifies the algebraic problem of sign (only positive values of X and Y now need be considered), and thus automatically eliminates the dilemma caused by two-valued functions.

The one-parameter horizontal parabola has been frequently exploited by mensurationists, but they have overlooked the greater utility of the two-parameter parabola. Furthermore, although a form of the two-parameter equilateral hyperbola was nominally employed by several early mensurationists, either one or both parameters were constrained in such ways that the hyperbola could not be used to extrapolate unless total height or merchantable height of a tree were known in addition to the height of some basal diameter (Behre 1927; Girard and Bruce 1948). Hence, the discussion that follows appears to involve the first forestry use of these 2 two-parameter functions for extrapolation where location of X-origin is an unknown and must be analytically established from taper behavior.

Briefly, the recommended extrapolation technique depends on the availability of at least three sets of height-diameter coordinates in addition to the stump measurements. Taper analysis of these three sets determines whether the projection must be arbitrarily conic (zero taper, reverse taper, non-monotonic taper), empirically conic (constant taper), convex hyperbolic (taper increasing towards tip), or concave parabolic (taper decreasing towards tip). Each projection terminates at the distance where diameter becomes zero.

Figures 7 — 9 outline the taper analysis and the derivation of parameters P and Q appropriate to each extrapolative function. Figures 10 and 11 give exact surface and volume integrals for each function, established *ab initio* by the author after failing to find any published integrals appropriate to the particular functions. Although manual evaluation, especially in the case of the surface integrals, is prohibitively time consuming, the modern computer completely eliminates this bugaboo. Furthermore, tabular reference to tables of incomplete elliptic integrals is unnecessary since the iterative computer process of the arithmetic-geometric mean can be used with descending Landen transformation (Milne-Thomson 1964).

Alternatively, a progression of Y-values at close intervals can be established for which corresponding X-values can be computed by means of the formulae in figures 8 or 9, and then conic integrals in figure 4 can be used.

Several refinements can be incorporated into the extrapolative process. Total height can be used as an outer bound beyond which extrapolation is forbidden. If felled-tree measurements of a small sample of extrapolated

TREE STEM TAPER ANALYSIS AND
APPROPRIATE CONIC PROJECTION

X=FT. FROM APEX, Y=STEM RADIUS IN FT.

$$T=(X_3-X_2)/(Y_3-Y_2)$$
$$t=(X_2-X_1)/(Y_2-Y_1)$$

IF T OR $t \leqq 0$, SET $T=t$=ARBITRARY CONIC TAPER
IF $T=t$, USE OBSERVED EMPIRICAL CONIC TAPER

$X=TY$
$Y=X/T$
$Y-X/T=0$ (SEE FIG. 4 FOR CONIC INTEGRALS)

ELSE $A=T-t \neq 0$, AND SEE FIGS. 8 OR 9

FIGURE 7. Analysis of tree taper and conic projection.

APPROPRIATE HYPERBOLIC PROJECTION
(CONTINUED FROM FIG. 7)

IF $A>0$, STEM FORM IS CONVEX AND

$M=(X_3-X_1)/(Y_3-Y_1)$
$Q=(TY_3-tY_1)/A$
$P=(M)(Q-Y_3)(Q-Y_1)/Q$

$X=PY/(Q-Y)$
$Y=QX/(X+P)$
$XY-QX+PY=0$ (SEE FIG. 10 FOR INTEGRALS)

ELSE, SEE FIG. 9

FIGURE 8. Analysis of tree taper and convex projection.

APPROPRIATE PARABOLIC PROJECTION
(CONTINUED FROM FIGS. 7,8)

IF $A<0$, STEM FORM IS CONCAVE AND

$$M=(X_3-X_1)/(Y_3-Y_1)$$

$$Q=\text{ABSOLUTE VALUE } (Y_3-Y_1)/A$$

$$P=MQ+(Y_3+Y_1)$$

$$X=(PY-Y^2)/Q$$

$$Y=(P-\sqrt{P^2-4QX})/2$$

$$Y^2+QX-PY=0$$ (SEE FIG. 11 FOR INTEGRALS)

FIGURE 9. Analysis of tree taper and concave projection.

SURFACE, VOLUME INTEGRALS OF CONVEX HYPERBOLOID
(XY−QX+PY=0 REVOLVED ABOUT LINE Y=0)

$$R=Q-Y_1 \quad \text{AND} \quad \Delta_R=\sqrt{R^4+(QP)^2}$$

$$r=Q-Y_2 \quad \text{AND} \quad \Delta_r=\sqrt{r^4+(QP)^2}$$

$$\phi=\text{ARCTAN}\left[(2PQ)^{.5}(P-Q+Y_2+PY_2/r)\right]$$

$$\theta=\text{ARCTAN}\left[(2PQ)^{.5}(P-Q+Y_1+PY_1/R)\right]$$

$$\$(\phi)=F(\phi,45^\circ)-2E(\phi,45^\circ)+2\sqrt{1-.5\sin^2\phi}\,\tan\phi$$

WITH F, E BEING LEGENDRE INCOMPLETE ELLIPTIC INTEGRALS OF FIRST, SECOND KIND.

$$S=2\pi Q\sqrt{PQ}\left[\$(\phi)-\$(\theta)\right]-\pi\left[\Delta_R-\Delta_r+(QP)\ln_e\left(\frac{(\Delta_R-QP)r^2}{(\Delta_r-QP)R^2}\right)\right]$$

$$V=\pi P\left[Y_2^3/r-Y_1^3/R+(Y_2-Y_1)(Y_2+Y_1+2Q)-(2Q^2)\ln_e\left(\frac{R}{r}\right)\right]$$

FIGURE 10. Convex hyperbolic integrals.

SURFACE, VOLUME INTEGRALS OF CONCAVE PARABOLOID
($Y^2+QX-PY=0$ REVOLVED ABOUT LINE $Y=0$)

$R=P/2-Y_1$ AND $\triangle_R=\sqrt{R^2+Q^2/4}$

$r=P/2-Y_2$ AND $\triangle_r=\sqrt{r^2+Q^2/4}$

$$S=(\pi P/Q)\left[R\triangle_R-r\triangle_r+(Q^2/4)\ln_e\left(\frac{R+\triangle_R}{r+\triangle_r}\right)\right]-(4\pi/3Q)\left[\triangle_R^3-\triangle_r^3\right]$$

$$V=(\pi/Q)\left[P(Y_2^3-Y_1^3)/3-(Y_2^4-Y_1^4)/2\right]$$

FIGURE 11. Concave parabolic integrals.

surfaces and volumes are obtained, appropriate corrections to the aggregate extrapolative predictions can be made (analogous to sampling non-respondents in mail surveys).

CONCLUSION

To sum up, quantification of tree form requires measurement of tree heights and diameters along as much of the stem as possible. No single pair of coordinates is of much value in assessment of tree form. It is possible to derive polynomials or ratios of polynomials from individual tree coordinates as an expression of individual tree form, but this step at present serves no useful purpose that cannot be more simply accomplished by direct calculation of length, surface, and volume. Interpolation between two measured sets of height-diameter coordinates where interval is short can be harmonic, linear, or quadratic; though a one-parameter concave, conic, or convex function must be assumed, the magnitude of its coefficient is unimportant. At least two parameters must be assumed where extrapolation is involved, but the parameters should be based upon or consistent with taper behavior immediately adjacent to the point where extrapolation commences. The convex equilateral hyperbola and the concave horizontal parabola are the simplest suitable two-parameter extrapolative functions that allow quantitative expression of convexity or concavity. Surface and volume integrals have been derived for the solids generated by revolving these curves around axes parallel to but not coincident with the transverse axis of the horizontal parabola and the transverse asymptote of the equilateral hyperbola (rotated 45°).

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