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Linear Transformations & Rotation Matrices

in Theme 2. Attitude

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Outline

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Outline

Attitude

What is it?

Definition

Attitude is a term used in geometry that refers to the orientation of a body in space.

Express the attitude of a body is not simple. How is it done?

- Define an orthonormal basis. It would be the world reference frame.
- Define an orthonormal basis that moves with the object.

Why you will need it?

- Position of characters in a virtual world
- Virtual Camera views
- VR

Look at that: https://www.youtube.com/watch?v=unxUdhP3bu8
https://www.youtube.com/watch?v=HNOT$_f eL27Y$

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Attitude
Intro

We are going to spend a few of classes talking about:

- Transformations between reference frames
- Attitude representation
- How the several attitude representations relate
- Which is the best way of representing attitude

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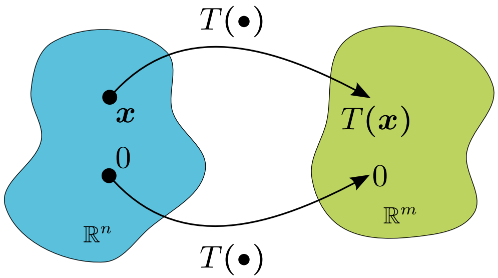
Linear Transformation

What is?

Definition

A Linear Transformation is a Multivariate function that transforms the space maintaining the origin,

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



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Property

If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, T is a Linear Transformation iff

$$\begin{aligned} T(\boldsymbol{a}+\boldsymbol{b}) &= T(\boldsymbol{a}) + T(\boldsymbol{b}) \\ T(\lambda \cdot \boldsymbol{a}) &= \lambda \cdot T(\boldsymbol{a}) \end{aligned}$$

Note: Points on a line remain alienated after the transformation:
$T(\boldsymbol{x} + \lambda \cdot \boldsymbol{v}) = T(\boldsymbol{x}) + \lambda \cdot T(\boldsymbol{v})$

Example

Is $T(\boldsymbol{x}) = T(x_1, x_2, x_3) = \begin{pmatrix} x_1 + 2x_2 \\ 3x_3 \\ -x_1 + x_3 \end{pmatrix}$ a linear transformation?

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Linear Transformation

Some exercises

Exercises 1.1, 1.2, 1.3

Find $\mathbf{A}$, subject to $T(x) = \mathbf{A}x$, for

1. $T(x) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 3x_3 \\ -x_1 + x_2 \end{pmatrix}$ It is ok
2. $T(x) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3x_1}{x_2} + x_3 \\ 2x_1x_3 \end{pmatrix}$ It is not linear
3. $T(x) = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 + 1 \\ -x_1 \end{pmatrix}$ Do not mantain the origin

Linear Transformation

Some exercises

Exercise 2

Find $\boldsymbol{x}$ subject to $T(\boldsymbol{x}) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ if $T(\boldsymbol{x})$ is a linear transformation and

$$T\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$$

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Linear transformation

Matrix form

Proposition

A Transformation $T(\boldsymbol{x})$ is Linear $\iff$ it is possible to find a matrix $\mathbf{A}$ s.t. $T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$

$(\Leftarrow)$ Let $\mathbf{A} \in \mathbb{M}_{m\times n}$ be

$$\mathbf{A} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

Then $\mathbf{A}\boldsymbol{x}$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n \in \mathbb{R}^m$$

Linear transformation

Matrix form

($\Rightarrow$) Is $\mathbf{A}\boldsymbol{x}$ always a linear transformation?
Note that, using the canonical basis:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{I}_n \boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \cdots + x_n \boldsymbol{e}_n$$

Then

$$T(\boldsymbol{x}) = x_1 T(\boldsymbol{e}_1) + x_2 T(\boldsymbol{e}_2) + \cdots + x_n T(\boldsymbol{e}_n) \Rightarrow$$

$$T(\boldsymbol{x}) = \begin{pmatrix} \vdots & \vdots & & \vdots \\ T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \\ \vdots & \vdots & & \vdots \end{pmatrix} \boldsymbol{x} =: \mathbf{A}\boldsymbol{x}$$

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Rotation

What is?

Definition

A Rotation is a especial Linear Transformation in the same dimension n

$$R : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

that preserves

- Distances (vector norm)
- Angles (dot product)
- Volumes (triple product)

Definitions

Vector norm Length of a vector

$$\|\boldsymbol{x}\| = \sqrt{x_1 + x_2 + \cdots x_n} = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$$

Dot product Angle between vectors (a.k.a. scalar product, inner product)

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \|\boldsymbol{x}\| \|\boldsymbol{y}\| \cos \alpha$$

Triple product Volume (from vector product / cross product)

$$\boldsymbol{x}^T (\boldsymbol{y} \times \boldsymbol{z})$$

Remember: cross product wikipedia

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DoFs in Rotations

In $\mathbb{R}^3$

Rotations in $\mathbb{R}^3$ can be defined with 3 parameters

Let $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be an orthonormal basis of $\mathbb{R}^3$ and let a Rotation R be defined as $R(\boldsymbol{x}) = \mathbf{R}\boldsymbol{x}$, with

$$\mathbf{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad (9 \text{ DoF})$$

then $\{\mathbf{R}(\boldsymbol{e}_1), \mathbf{R}(\boldsymbol{e}_2), \mathbf{R}(\boldsymbol{e}_3)\}$ is also an orthonormal basis of $\mathbb{R}^3$, so accomplishing 6 restrictions,

$$\mathbf{R}(\boldsymbol{e}_j)^T\mathbf{R}(\boldsymbol{e}_i) = 1 \text{ if } \ j = i \ , \ \mathbf{R}(\boldsymbol{e}_j)^\intercal\mathbf{R}(\boldsymbol{e}_i) = 0 \text{ if } \ j \neq i$$

leading to 3 DoF.

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DoFs in Rotations

In $\mathbb{R}^3$

Equivalent restrictions

The 6 restrictions over matrix $\mathbf{R}$ about orthonormality are equivalent to state that,

- $\det(\mathbf{R}) = 1$, and
- $\mathbf{R}^\intercal = \mathbf{R}^{-1}$

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DoFs in Rotations

An aside (II)

The Grassman rule

The Determinant of a 3×3 marix can be calculated using the Grassman rule

$$\det(\mathbf{M}) = |\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} =$$

$$m_{11}m_{22}m_{33} + m_{21}m_{32}m_{13} + m_{31}m_{12}m_{23} -$$
$$-(m_{11}m_{23}m_{32} + m_{21}m_{12}m_{33} + m_{31}m_{22}m_{13})$$

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DoFs in Rotations

Three Exercises

Geometric meaning

Let $R(\boldsymbol{v}) = \mathbf{R}\boldsymbol{v}$ represent a rotation of 90 degs about the z axis.

1. What are the elements in matrix $\mathbf{R}$ for this case? (Hint: z remains, x goes to y and y goes to $-x$)
2. Which are the images for the vectors, $\boldsymbol{v_1} = (1, 0, 0)^\intercal$, $\boldsymbol{v_2} = (0, 1, 0)^\intercal$, $\boldsymbol{v_3} = (0, 0, 1)^\intercal$, and $\boldsymbol{v_4} = (-1, 2, 2)^\intercal$?
3. Which rotation matrix $\mathbf{R}$ performs the inverse rotation?

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Dualism with change of basis

Rotations & change of basis

Change of basis

$$\mathcal{B}_2 \xrightarrow{\mathbf{C}} \mathcal{B}_1$$

1. $\mathbf{C}$ is a matrix whose columns are the vectors in basis $\mathcal{B}_2$ seen from $\mathcal{B}_1$.
2. $\mathbf{C}\boldsymbol{x}$ means "I take a vector $\boldsymbol{x}$ of $\mathcal{B}_2$ and it goes to $\mathcal{B}_1$"

Exercise

Applying dualism, rotates vector $\boldsymbol{x}$ like in the previous exercise, i.e. rotates 90 degs about the z axis.

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Take care I

Rotations & change of basis

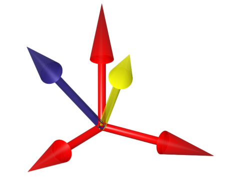
Take the rotation matrix

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

and the vector $\boldsymbol{p} = (1,\, 0,\, 1)^{\mathsf{T}}$, then: $\boldsymbol{p}' = \mathbf{R}\boldsymbol{p} = \frac{1}{\sqrt{2}}\left(1,\, 1,\, \sqrt{2}\right)^{\mathsf{T}}$

Rotations

If we are talking about rotations, we meant



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Take care II

Rotations & change of basis

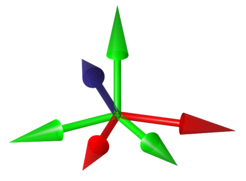
Take the rotation matrix

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

and the vector ${}^{A}\boldsymbol{p} = (1,\, 0,\, 1)^{\intercal}$, then: ${}^{B}\boldsymbol{p} = \mathbf{R}\,{}^{A}\boldsymbol{p} = \frac{1}{\sqrt{2}}\left(1,\, 1,\, \sqrt{2}\right)^{\intercal}$

Change of basis

If we are talking about change of basis, we meant



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Concatenation

Linear transforms

Definition

Let $T_1(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}$ and $T_2(\boldsymbol{x}) = \mathbf{B}\boldsymbol{x}$ be linear transformations, then the concatenation

$$T_2(T_1(\boldsymbol{x})) = \mathbf{BA}\boldsymbol{x}$$

is also a linear transformation.

- Note the order!!!, $\mathbf{BA} \neq \mathbf{AB}$
- The same definition applies to rotations.

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Exercise 1
Homework

Exercise 1

Given the next two transformations, find which of them are linear transformations and which of them are rotations and argument why.

1. $T_1(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3$

$$T_1(\boldsymbol{x}) = \begin{pmatrix} \frac{\sqrt{3}}{2}x_1 + x_2 \\ 3 - x_3 + \frac{1}{\sqrt{3}} \\ x_1 \end{pmatrix}$$

2. $T_2(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3$

$$T_2(\boldsymbol{x}) = \begin{pmatrix} \frac{1}{2}\left(x_1 + x_3\right) - \frac{\sqrt{2}}{2}x_2 \\ \frac{\sqrt{2}}{2}\left(x_3 - x_2\right) \\ \frac{1}{2}\left(x_1 + x_3\right) + \frac{1}{\sqrt{2}}x_2 \end{pmatrix}$$

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Exercise 2

Homework

Exercise 2

Two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ have known components on two orthonormal basis $\{A\}$ and $\{B\}$

$$
\begin{aligned}
{}^{A}\boldsymbol{v}_1 &= (1,\,0,\,0)^{\mathsf{T}} & \qquad {}^{B}\boldsymbol{v}_1 &= \tfrac{1}{2}\left(\sqrt{3},\,1,\,0\right)^{\mathsf{T}} \\
{}^{A}\boldsymbol{v}_2 &= (1,\,1,\,0)^{\mathsf{T}} & \qquad {}^{B}\boldsymbol{v}_2 &= \tfrac{1}{2}\left(\sqrt{3}-1,\,1+\sqrt{3},\,0\right)^{\mathsf{T}}
\end{aligned}
$$

- Calculate the components of the vector $\boldsymbol{p}$ in the frame $\{A\}$, that is ${}^{A}\boldsymbol{p}$, if it is known that in the frame $\{B\}$ the vector is given by:

$$
{}^{B}\boldsymbol{p} = \left(3,\,-2,\,\frac{1}{2}\right)^{\mathsf{T}}
$$

Exercise 3

Homework

Exercise 3

The matrix

$$\mathbf{R} = \begin{pmatrix} a_1 & -0.2655 & -0.2113 \\ a_2 & 0.9640 & -0.0726 \\ a_3 & a_4 & 0.9747 \end{pmatrix}$$

represents a rotation. Which are the values of a_1, a_2, a_3 and a_4?