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Theme 5. Interpolation over a Series of Rotations

Definitions and Squad

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The interpolation curve between two points can be defined as follows:

Interpolation between two points (rotations)

Given an arbitrary set $\mathcal{M}$ we interpolate between two points $\boldsymbol{x}_0\,,\boldsymbol{x}_1 \in \mathcal{M}$ parametrised by $h \in [0,1]$. The resulting interpolation curve

$$\gamma : \mathcal{M} \times \mathcal{M} \times [0,1] \longrightarrow \mathcal{M}$$

$$\gamma(\boldsymbol{x}_0, \boldsymbol{x}_1, h) = \boldsymbol{x}_h \in \mathcal{M}$$

satisfy the constraints:

$$\begin{aligned} \gamma(\boldsymbol{x}_0, \boldsymbol{x}_1, 0) = &\quad \boldsymbol{x}_0 \\ \gamma(\boldsymbol{x}_0, \boldsymbol{x}_1, 1) = &\quad \boldsymbol{x}_1 \end{aligned}$$

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Recapitulate
SLERP

When interpolating between two rotations SLERP is optimal, but...

SLERP over a series of rotations

Problems emerge:

- The curve is not smooth at the control points,
- The angular velocity is not constant
- The angular velocity is not continuous at the controls points

SLERP over a series of rotations

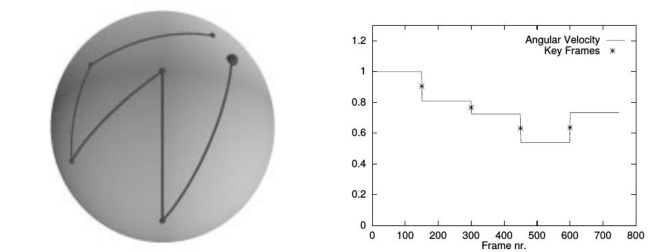

Figure: SLERP is not more optimal when applied ove a series of rotations.

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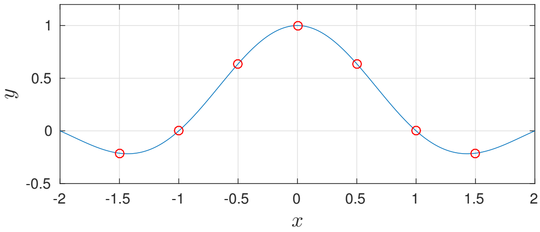
What is interpolation?

Interpolation tries to estimate function values by only knowing information of some discrete samples
The main idea under interpolation is to define interpolant functions $\Phi(x)$. Those functions will take the exact values at the given points and will let us to estimate values at some other points in the domain.



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Polynomial interpolation

Polynomial interpolation tries to find a **unique** interpolant $\Phi(x)$.
This function will be a polynomial of minimum order m (the number of points -1), i.e.

$$\Phi(x) = \sum_{i=0}^{m} a_i x^i = a_0 x^0 +_1 x^1 + a_2 x^2 + \ldots a_{m-1} x^{m-1}$$

It is proven that this polynomial always exist, therefore, how we can calculate a_i coefficients?

Vandermonde matrix

By knowing that the interpolant passes through a set of N points (x_1, y_1), (x_2, y_2), ... (x_N, y_N) it is possible to construct a linear system of equations by imposing that

$$\begin{aligned}
\Phi(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \ldots a_{N-1} x_1^{N-1} = y_1 \\
\Phi(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + \ldots a_{N-1} x_2^{N-1} = y_2 \\
&\vdots \\
\Phi(x_N) &= a_0 + a_1 x_N + a_2 x_N^2 + \ldots a_{N-1} x_N^{N-1} = y_N
\end{aligned}$$

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Polynomial Interpolation

Vandermonde matrix

Vandermonde matrix, continuation

The linear system can be written as

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ & \vdots & & \ddots & \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

This is called the Vandermonde linear system.
The solution of this system is guaranteed since the coefficient matrix will be non-singular always that any point is repeated.
After having calculated the coefficients $(a_0, a_1..., a_{N-1})$ any new point (x, y) can be obtained by using $\Phi(x)$.

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Polynomial Interpolation

Vandermonde matrix

Exercise

- Using the Vandermonde matrix, find the interpolant function from the next set of points:

x	-1.5	-1	-0.5	0	0.5	1	1.5
y	-0.2122	0	0.6366	1	0.6366	0	-0.2122

- Which values you can expect at $x = -0.25$ and $x = 1.25$
- Plot a graph of $\Phi(x)$ with $x \in [-1, 5, 1.5]$

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Polynomial Interpolation
Vandermonde Conclusions

The method presented reduces the problem of interpolating to solve a linear system of equations. However, the Vandermonde method to find the coefficients of the polynomial have two main drawbacks:

- The coefficient matrix of the linear system is dense $\rightarrow$ computational expensive as N grows
- It can become ill-conditioned for non regular spaced points

It has been said (not demonstrated) that there exist a unique polynomial of degree $N-1$ that interpolates N points. So... can we do it better?

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Polynomial Interpolation
Lagrange formulation

Lagrange polynomials

What if we construct the interpolant $\Phi(x)$ as a superposition of other polynomials?
The Lagrange notation takes:

$$\Phi(x) = \sum_{i=0}^{N} l_i(x) y_i$$

with

$$l_i(x) = \prod_{j=1, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}$$

Every $l_i(x)$ is a polynomial of order $N - 1$, and the superposition does not increase the order.

Exercise

For a set of three point pairs (x_1, y_1), (x_2, y_2) and (x_3, y_3),

- Calculate the three Lagrange polynomials $l_1(x)$, $l_2(x)$ and $l_3(x)$
- Which values they take at $x = x_1$, $x = x_2$ and $x = x_3$?

Lagrange polynomials, continue

At the given points,

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

As consequence:

$$\Phi(x_j) = \sum_{i=1}^{N} l_i(x_j) y_i = l_j(x_j) y_j = y_j$$

It passes for all the points.

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Polynomial Interpolation
Lagrange formulation

Exercise

For a set of three point pairs $(1,2)$, $(2,-3)$ and $(4,0.5)$,

- Calculate the three Lagrange polynomials $l_1(x)$, $l_2(x)$ and $l_3(x)$
- Which is the interpolant polynomial?
- Which value could you expect for y at $x = 3$?

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Polynomial Interpolation

Lagrange formulation & conclusions

Using the Lagrange's polynomials, we don't need to solve a linear system.

However

- The polynomial methods presented have a common drawback:
  - When a new point is added a whole new calculation of the interpolant has to be carried on
  - Runge's phenomenom
- Easiness of the calculation of derivatives, performance of polynomial evaluation etc... are other important aspects
- Other polynomial basis can be used e.g. the newton polynomial basis

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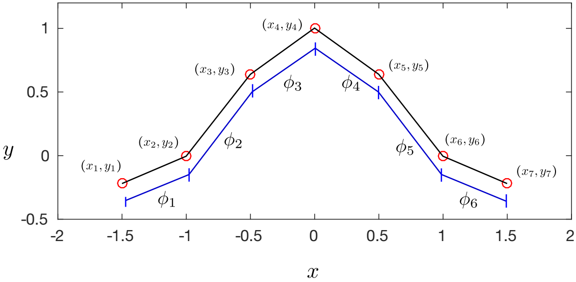
Local Interpolation

Revisiting Linear Interpolation

Why if instead of having a global function we seek for local interpolants?



- Local interpolants can be blended imposing continuity and optionally derivative continuity
- Local interpolation does not suffer from the Runge's phenomena

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Local Interpolation
Linear Technique

Remember that linear technique for interpolation is simple

Linear Interpolation of whatever

Interpolation between $\boldsymbol{x}_0 \in \mathcal{M}$ and $\boldsymbol{x}_1 \in \mathcal{M}$ using $h \in [0,1]$ is defined as:

$$Lin(\boldsymbol{x}_0, \boldsymbol{x}_1, h) = \boldsymbol{x}_0(1-h) + \boldsymbol{x}_1 h$$

but we are looking for something better

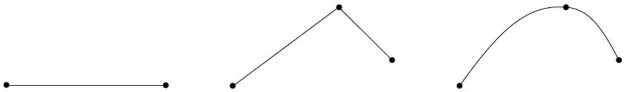

Figure: Looking for differentiability.

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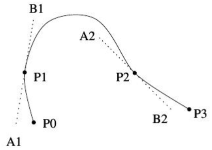
Bézier curve descrption

Interpolation between the points $P1$ and $P2$ with a Bézier curve.



The curve is defined as a third-order curve, where the tangent in the control points is defined by auxiliary points. For example the tangent in $P1$ is defined by the auxiliary points $A1$ and $B1$ (the tangent is $B1 - P1$ or $P1 - A1$. The differentiability is automatically assured since the curve is a third order curve.

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Local Interpolation

Bézier curves

Definition

The Bézier curve (with auxiliary points $B1$ and $A2$) that interpolates between the control points $P1$ and $P2$ can be expressed algorithmically as three steps of linear interpolation:

$$Lin(\boldsymbol{x}_0, \boldsymbol{x}_1, h) = \boldsymbol{x}_0(1-h) + \boldsymbol{x}_1 h$$

$$Bezier(P1, P2, B1, A2, h) = Lin(Lin(P1, P2, h), Lin(B1, A2, h), 2h(1-h))$$

Differentiability in the control points can be ensured by making the tangents coincide in the control points, i.e. ensuring that $B1 - P1 = P1 - A1$.

Taste it at The Bézier Game.

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Local Interpolation
Bézier curves

Main Property

Affine transformations such as translation and rotation can be applied on the curve by applying the respective transform on the control points of the curve.

Exercise

For a set of three point pairs $(1,2)$, $(2,-3)$ and $(4,0.5)$,

- Calculate two different and differentiable Bézier curves
- Which value could you expect for y at $x = 3$?

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Local Interpolation

Squad

A spherical cubic equivalent of a Bézier curve can be formulated.

Definition

This interpolation curve is called Squad (spherical and quadrangle). For $\mathring{q}_i \in \mathbb{H}$ and $h \in [0, 1]$,

$$Squad(\mathring{q}_i, \mathring{q}_{i+1}, \mathring{s}_i, \mathring{s}_{i+1}, h) =$$

$$Slerp(Slerp(\mathring{q}_i, \mathring{q}_{i+1}, h), Slerp(\mathring{s}_i, \mathring{s}_{i+1}, h), 2h(1-h))$$

with

$$\mathring{s}_i = \mathring{q}_i \exp\left(-\frac{\log\left(\mathring{q}_i^{-1}\mathring{q}_{i+1}\right) + \log\left(\mathring{q}_i^{-1}\mathring{q}_{i-1}\right)}{4}\right)$$