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Complex Numbers and Quaternions

in Theme 2. Attitude

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Outline

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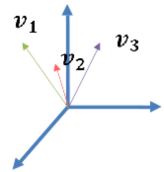
Outline

Composition of rotations / Changes of basis



We know how to relate:

- $\boldsymbol{v_1}$ with $\boldsymbol{v_2}$ (dual: ${}^{F_1}\boldsymbol{v}$ to ${}^{F_2}\boldsymbol{v}$) by using $(\boldsymbol{u_1}, \phi_1)$ or $\boldsymbol{r_1}$ or $(\phi_1, \theta_1, \psi_1)$
- $\boldsymbol{v_2}$ with $\boldsymbol{v_3}$ (dual: $\boldsymbol{v}_{\{F_2\}}$ to $\boldsymbol{v}_{\{F_3\}}$) by using $(\boldsymbol{u_2}, \phi_2)$ or $\boldsymbol{r_2}$ or $(\phi_2, \theta_2, \psi_2)$

Which values takes:
$(\boldsymbol{u_3}, \phi_3)$ or $\boldsymbol{r_3}$ or $(\phi_3, \theta_3, \psi_3)$ to transform $\boldsymbol{v_1}$ to $\boldsymbol{v_3}$ (dual: ${}^{F_1}\boldsymbol{v}$ to ${}^{F_3}\boldsymbol{v}$)?

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Remember
Example

Composition of rotations / Changes of basis

We know:

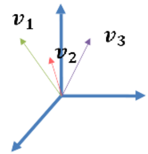
$\mathbf{v_1} = (3, 2, -1)^\intercal$

$\mathbf{v_1}$ to $\mathbf{v_2}$ by $\mathbf{u_1} = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^\intercal$, $\phi_1 = 10^\circ$

$\mathbf{v_2}$ to $\mathbf{v_3}$ by $\mathbf{u_2} = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})^\intercal$, $\phi_2 = 20^\circ$

Which values have:

$(\mathbf{u_3}, \phi_3)$ transforming $\mathbf{v_1}$ to $\mathbf{v_3}$?



Answer

$\mathbf{u_3} = (0.5017, 0.2323, 0.8333)^\intercal$, $\phi_3 = 26.44^\circ$

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Remember

What we know since now

- Rotation matrix, $\mathbf{R} \rightarrow 9$ components. Easy to compose rotations.
- Euler principal axis and angle, $(\boldsymbol{u}, \phi) \rightarrow 4$ components. Compose rotations by transforming to rotation matrices.
- Rotation vector, $\boldsymbol{r} \rightarrow 3$ components. Compose rotations by transforming to rotation matrices
- Euler angles, $(\phi, \theta, \psi) \rightarrow 3$ components. Compose rotations by transforming to rotation matrices

Good for memory storage but... not so good if we have to operate with them

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Outline

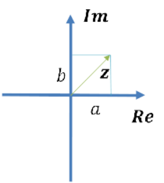
Complex numbers

Intro

Complex numbers

Property. **Unitary complex numbers in 2D retains information about direction.**

$$z = a + bi$$



Norm.

$$\|z\| = \sqrt{z\bar{z}} = \sqrt{a^2 + abi - abi - b^2 i^2} = \sqrt{a^2 + b^2}$$

where $i^2 = -1$

Complex numbers in 2D

Definitions and Properties

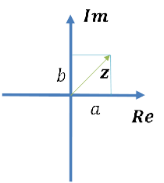
Property. **Unitary complex numbers in 2D retains information about direction.**

$$z = a + bi$$

Norm.

$$\|z\| = \sqrt{z\bar{z}} = \sqrt{a^2 + abi - abi - b^2 i^2} = \sqrt{a^2 + b^2}$$

where $i^2 = -1$



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Product

What happens if we multiply two complex numbers?

$$z_1 = a + bi$$

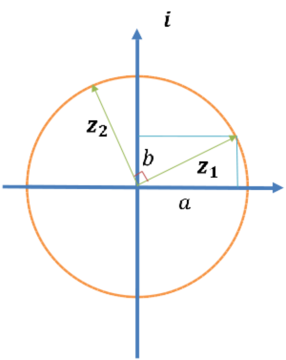
$$z_2 = i$$

$$z_1 z_2 = ai - b = -b + ai$$

Rotates the vector 90°.

- In fact, product makes

$$\angle z_1 z_2 = \angle z_1 + \angle z_2$$



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Complex numbers
in 2D

Product

What happens if we multiply two complex numbers?

$$z_1 = a + bi = (a, b)^\intercal$$

$$z_2 = \cos\alpha + i\sin\alpha = (\cos\alpha, \sin\alpha) = \mathrm{e}^{i\alpha}$$

$$z_1 z_2 = a\cos\alpha - b\sin\alpha + i(b\cos\alpha + a\sin\alpha)$$

$$z_1 z_2 = (a\cos\alpha - b\sin\alpha, b\cos\alpha + a\sin\alpha)^\intercal$$

Hence

$$z_1 z_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Example:

$$z_1 = 2 + 3i = (2, 3)^\intercal$$

$$z_2 = \cos 30 + i\sin 30 = \mathrm{e}^{i30}$$

$$z_2 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_1 z_2 = \sqrt{3} - \frac{3}{2} + i\left(\frac{3\sqrt{3}}{2} + 1\right)$$

$$z_1 z_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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Outline

Quaternions
Definitions

Definition

Quaternions are a number system that extends the Complex numbers.

$$\mathring{q} = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$$

where i, j, and k are the fundamental quaternion units, accomplishing

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k \ , jk = i \ , ki = j$$

$$ji = -k \ , kj = -i \ , ik = -j$$

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Quaternions
Definition

Product (called Hamilton product)

$$\mathring{q}\mathring{p} = (q_0 + iq_1 + jq_2 + kq_3)(p_0 + ip_1 + jp_2 + kp_3) = \ldots$$

Problem. Do it!

Unlike multiplication of real or complex numbers, multiplication of quaternions is not commutative.

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Product

We change to a new representation of a quaternion,

$$\mathring{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix}$$

where q_o is called its scalar part and $\boldsymbol{q}$ is called its vector part.
Now,

$$\mathring{q}\mathring{p} = \begin{pmatrix} q_0 p_0 - \boldsymbol{q}^\intercal \boldsymbol{p} \\ q_0 \boldsymbol{p} + p_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p} \end{pmatrix}$$

Quaternions

Definition

Product

It can be also written using quaternion matrices,

$$\mathring{q}\mathring{p} = \begin{pmatrix} q_0 p_0 - \boldsymbol{q}^\top \boldsymbol{p} \\ q_0 \boldsymbol{p} + p_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p} \end{pmatrix}$$

$$\mathring{q}\mathring{p} = \underbrace{\begin{pmatrix} q_0 & -\boldsymbol{q}^\top \\ \boldsymbol{q} & q_0 \mathbf{I}_3 + [\boldsymbol{q}]_\times \end{pmatrix}}_{\mathbf{Q}(\mathring{q})} \mathring{p} = \underbrace{\begin{pmatrix} p_0 & -\boldsymbol{p}^\top \\ \boldsymbol{p} & p_0 \mathbf{I}_3 - [\boldsymbol{p}]_\times \end{pmatrix}}_{\tilde{\mathbf{Q}}(\mathring{p})} \mathring{q}$$

$$\mathring{q}\mathring{p} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \mathring{p} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{pmatrix} \mathring{q}$$

Quaternions
Example

Problem

Given

$$\mathring{q} = (1\,,2\,,2\,,1)^{\intercal}$$

$$\mathring{p} = (-1\,,1\,,2\,,-2)^{\intercal}$$

Calculate

$$\mathring{q}\mathring{p} =$$

$$\mathring{p}\mathring{q} =$$

Definitions

- Since quaternions contains imaginary numbers, there exists the conjugate quaternion

$$\tilde{\mathring{q}} = q_0 - iq_1 - jq_2 - kq_3 = \begin{pmatrix} q_0 \\ -\boldsymbol{q} \end{pmatrix}$$

- Quaternions have a norm

$$\|\mathring{q}\|^2 = \mathring{q}\tilde{\mathring{q}} = \tilde{\mathring{q}}\mathring{q} = q_0^2 + \boldsymbol{q}^\intercal \boldsymbol{q} = (q_0\ \boldsymbol{q}^\intercal) \begin{pmatrix} q_0 \\ \boldsymbol{q} \end{pmatrix} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

with $\|\mathring{q}\mathring{p}\| = \|\mathring{q}\|\,\|\mathring{p}\|$ and $\|\alpha\mathring{q}\| = |\alpha|\,\|\mathring{q}\|$

- A quaternion is said to be a pure quaternion if $q_0 = 0$.

Definitions

- There exists the identity quaternion

$$\mathring{q}_I = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \rightarrow \mathring{q}\mathring{q}_I = \mathring{q} = \mathring{q}_I\mathring{q}$$

- The inverse (or reciprocal) quaternion of a non-zero quaternion $\mathring{q}$

$$\mathring{q}^{-1} = \frac{\tilde{\mathring{q}}}{\|\mathring{q}\|^2} \rightarrow \mathring{q}\mathring{q}^{-1} = \mathring{q}^{-1}\mathring{q} = \mathring{q}_I$$

- The unit quaternion (or versor) of a non-zero quaternion $\mathring{q}$

$$\boldsymbol{U}_q = \frac{\mathring{q}}{\|\mathring{q}\|}$$

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Outline

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Rotations using Quaternions

Definition

Unit quaternions as Rotations

What happens when we take a unit quaternion $\mathring{q}$ ($\|\mathring{q}\| = 1$) and a pure quaternion $\mathring{v} = (0\,, \boldsymbol{v}^\intercal)^\intercal$, and calculate

$$\mathring{w} = \mathring{q}\mathring{v}\tilde{\mathring{q}}\ ?$$

- $\|\mathring{w}\| = 1 \cdot \|\mathring{v}\| \cdot 1 = \|\mathring{v}\| = \|v\|$
- $Re(\mathring{w}) = Re(\mathring{q}(\mathring{v}\tilde{\mathring{q}})) = Re((\mathring{v}\tilde{\mathring{q}})\mathring{q}) = Re(\mathring{v}(\tilde{\mathring{q}})\mathring{q})) = Re(\mathring{v}) = 0$

So $\mathring{w} = \mathring{q}\mathring{v}\tilde{\mathring{q}}$ is another pure quaternion $\mathring{w} = (0\,, \boldsymbol{w}^\intercal)^\intercal$ (another *vector* $\boldsymbol{w}$) the same length as $\boldsymbol{v}$, but ROTATED from where it was according to the unit quaternion $\mathring{q}$.

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Rotations using Quaternions
How to proceed

Unit quaternions as Rotations

We want to rotate a vector $\boldsymbol{v}$ into a new vector $\boldsymbol{w}$ using a unit quaternion $\mathring{q}$:

- Step 1: Insert the vector $\boldsymbol{v}$ in the shape of a pure quaternion, $\mathring{v} = (0\,, \boldsymbol{v}^\intercal)^\intercal$
- Step 2: Calculate $\mathring{w} = \mathring{q}\mathring{v}\tilde{\mathring{q}} = \mathbf{Q}(\mathring{q})\tilde{\mathbf{Q}}(\tilde{\mathring{q}})\mathring{v}$ with

$$\mathbf{Q}(\mathring{q})\tilde{\mathbf{Q}}(\tilde{\mathring{q}}) = \begin{pmatrix} 1 & \mathbf{0}_3^\intercal \\ \mathbf{0}_3 & \mathbf{R}(\mathring{q}) \end{pmatrix}$$

$$\mathbf{R}(\mathring{q}) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

is an orthonormal matrix.

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Rotations using Quaternions

How to proceed

$\mathbf{R}(\mathring{q})$ is a Rotation matrix

From the past two slides it can be concluded that

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v}$$

with $\mathbf{R}(\mathring{q})$ being a rotation matrix associated to the unit quaternion $\mathring{q}$.

It can be also checked that (Exercise)

$$\mathbf{R}(\mathring{q}) = (q_0^2 - \boldsymbol{q}^\intercal \boldsymbol{q})\mathbf{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^\intercal + 2q_0[\boldsymbol{q}]_\times$$

So, what is the rotation encoded in $\mathbf{R}(\mathring{q})$?

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Rotations using Quaternions
Encoding

What is the rotation encoded in $\mathbf{R}(\mathring{q})$?

We know that

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v}$$

$$\mathbf{R}(\mathring{q}) = (q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q})\mathbf{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^\intercal + 2q_0[\boldsymbol{q}]_\times$$

Let's take $\boldsymbol{q}||\boldsymbol{v} \to \boldsymbol{v} = \lambda\boldsymbol{q}$

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v} = (q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q})\boldsymbol{v} + 2(\boldsymbol{v}^\intercal\boldsymbol{q})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{v})$$

$$\boldsymbol{w} = (q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q})\lambda\boldsymbol{q} + 2(\lambda\boldsymbol{q}^\intercal\boldsymbol{q})\boldsymbol{q} = \lambda(q_0^2 + \boldsymbol{q}^\intercal\boldsymbol{q})\boldsymbol{q} = \boldsymbol{v}$$

According to the Rodrigues' rotation formula, the component parallel to the axis $\boldsymbol{u}$ will not change magnitude nor direction under the rotation. Hence

$$\boldsymbol{q} = \beta\boldsymbol{u}$$

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Rotations using Quaternions
Encoding

What is the rotation encoded in $\mathbf{R}(\mathring{q})$?

We know that

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v}$$

$$\mathbf{R}(\mathring{q}) = (q_0^2 - \boldsymbol{q}^\top\boldsymbol{q})\mathbf{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^\top + 2q_0[\boldsymbol{q}]_\times$$

Let's take $\boldsymbol{q} \perp \boldsymbol{v}$

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v} = (q_0^2 - \boldsymbol{q}^\top\boldsymbol{q})\boldsymbol{v} + 2(\boldsymbol{v}^\top\boldsymbol{q})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v})$$

$$\boldsymbol{w} = (q_0^2 - \boldsymbol{q}^\top\boldsymbol{q})\boldsymbol{v} + 2q_0(\boldsymbol{q} \times \boldsymbol{v})$$

$$\boldsymbol{w} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\boldsymbol{v} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\boldsymbol{v}_\perp = \cos\theta\boldsymbol{v} + \sin\theta\boldsymbol{v}_\perp$$

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Rotations using Quaternions
Encoding

What is the rotation encoded in $\mathbf{R}(\mathring{q})$?

Using both results, it can be demonstrated, for $\|\boldsymbol{u}\| = 1$, that,

$$\mathring{q} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \boldsymbol{u}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Exercise: Calculate $\|\mathring{q}\|$

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Rotations using Quaternions

Composing

Composing rotations

If we can rotate a vector $\boldsymbol{v}$ by using a unit quaternion as

$$\mathring{w} = \mathring{q}\mathring{v}\tilde{\mathring{q}}$$

what happens if we rotate this image by using another quaternion $\mathring{p}$?

$$\mathring{t} = \mathring{p}\mathring{w}\tilde{\mathring{p}} = \mathring{p}\mathring{q}\mathring{v}\tilde{\mathring{q}}\tilde{\mathring{p}}$$

Hence, the quaternion that rotate $\boldsymbol{v}$ first by using the quaternion $\mathring{q}$ and second $\mathring{p}$ is $\mathring{p}\mathring{q}$

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Exercises

- $\mathbf{R}(\mathring{q})$ is a rotation matrix, so demonstrate that

$$\mathbf{R}^\intercal(\mathring{q}) = \mathbf{R}^{-1}(\mathring{q})$$

. Hence, it can be derivated from

$$\boldsymbol{w} = \mathbf{R}(\mathring{q})\boldsymbol{v}$$

that

$$\boldsymbol{v} = \mathbf{R}^\intercal(\mathring{q})\boldsymbol{w}$$

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Exercises

- Demonstrate that

$$\mathbf{Q}(\tilde{\mathring{q}}) = \mathbf{Q}(\mathring{q})^\intercal$$

$$\tilde{\mathbf{Q}}(\tilde{\mathring{q}}) = \tilde{\mathbf{Q}}(\mathring{q})^\intercal$$

- Demonstrate that

$$\frac{\partial \mathring{q}\mathring{p}}{\partial \mathring{q}} = \tilde{\mathbf{Q}}(\mathring{p})$$

$$\frac{\partial \mathring{q}\mathring{p}}{\partial \mathring{p}} = \mathbf{Q}(\mathring{q})$$

Exercises

Homeworks

- Quaternion's questions from exams in the `Exams` folder