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Linear Algebra

Vector Spaces, Linear Dependence and Basis

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Outline

1 Vector Spaces

2 Vector Subspaces

3 Linear Dependence/ Independence

4 Homework

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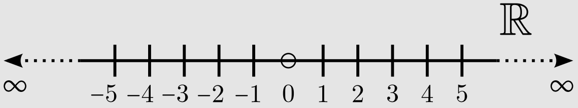
Real Numbers, $\mathbb{R}$

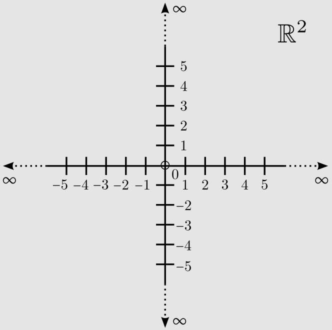


To define a real number we need to set:

- An origin
- Unit of longitude

After that, every number is completely defined.

Plane, $\mathbb{R}^2$



The real plane is obtained by the cartesian product of $\mathbb{R} \times \mathbb{R}$.

Could you imagine what is $\mathbb{R}^3$?

Vectors

A vector is just an ordered array of numbers.
A vector in $\mathbb{R}^n$ is represented by:

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \left(v_1, v_2, \cdots, v_n\right)^T$$

Since every coordinate represents a unique position on a real line, it is clear that

$$\boldsymbol{u} = \boldsymbol{v} \iff v_i = u_i \ , \ \forall i = 1, 2, ..., n$$

Basic operations with vectors

1. The sum operation: Given two vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^n$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \dots \\ u_n + v_n \end{pmatrix}$$

2. The product by a scalar operation: Given a vector $\boldsymbol{u} \in \mathbb{R}^n$ and a scalar number $k \in \mathbb{R}$

$$k\boldsymbol{u} = \begin{pmatrix} ku_1 \\ ku_2 \\ \dots \\ ku_n \end{pmatrix}$$

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Properties of the basic operations

1. Associativity of the sum: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$
2. Commutativity of the sum: $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
3. Neutral element of the sum:
   $\exists \boldsymbol{t} \mid \boldsymbol{u}+\boldsymbol{t}=\boldsymbol{u} \; \forall \boldsymbol{u} \in \mathbb{R}^n \rightarrow \boldsymbol{t}=\mathbf{0}=(0,0,\cdots,0)^\mathsf{T}$
4. Inverse element of the sum:
   $\forall \boldsymbol{u} \, \exists \boldsymbol{t} \mid \boldsymbol{u}+\boldsymbol{t}=\mathbf{0} \rightarrow \boldsymbol{t}=-\boldsymbol{u}=(-u_1,-u_2,\cdots,-u_n)^\mathsf{T}$
5. Distributivity of the product times the sum of vectors:
   $k(\boldsymbol{u}+\boldsymbol{v})=k\boldsymbol{u}+k\boldsymbol{v}$
6. Distributivity of the product times the sum of scalars:
   $\boldsymbol{u}(k_1+k_2)=k_1\boldsymbol{u}+k_2\boldsymbol{u}$
7. Associativity of the product: $k_1(k_2\boldsymbol{u})=(k_1k_2)\boldsymbol{u}$
8. Unity element: $\exists k_1 \mid k_1\boldsymbol{u}=\boldsymbol{u} \; \forall \boldsymbol{u} \in \mathbb{R}^n \rightarrow k_1=1$

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Vector Space

Any set provided with the sum and product operations $(+, \times)$ that accomplish the properties in the previous slide is a Vector Space.

How could you identify a Vector Space?

Let $\mathbb{E}$ be a set defined on M. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two elements $\in \mathbb{E}$.
Therefore:
$\mathbb{E}$ is a vector space iff

$$k_1\boldsymbol{u} + k_2\boldsymbol{v} \in \mathbb{E}\ ,\ \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{E}\ ,\ \forall\, k_1, k_2 \in M$$

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Vector Subspace

A Vector Subspace is a Vector Space embedded in a vector space of higher order. As example

$$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \mathbb{R}^n$$

Vector Subspace

A subset $\mathbb{S} \subset \mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$ iff

- The origin is contained in $\mathbb{S}$
- $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S} \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in \mathbb{S}$
- $\boldsymbol{u} \in \mathbb{S} \Rightarrow k\boldsymbol{u} \in \mathbb{S}\ , \ \forall k \in \mathbb{R}$

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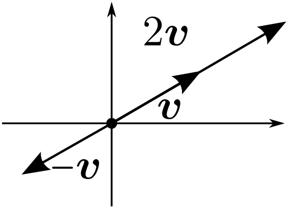
Subspace generators
Vector Subspaces

A Vector

Any non-zero vector $\boldsymbol{u} \in \mathbb{R}^n$ spans a subspace $\langle \boldsymbol{u} \rangle = \{k\boldsymbol{u}\ ,\ \forall k \in \mathbb{R}\}$



Exercise

Let $\boldsymbol{u} = (2, \quad -1, \quad 1)^\intercal$. Demonstrate that $\langle \boldsymbol{u} \rangle = \mathbb{L}$ is a subspace of $\mathbb{R}^3$.

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Subspace generators

Vector Subspaces

And more general

A bunch of vectors

Any linear combination of vectors spans into a subspace

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n \rangle = \{k_1 \boldsymbol{u}_1 + k_2 \boldsymbol{u}_2 + \ldots + k_n \boldsymbol{u}_n \ , \ \forall k_1, k_2, \ldots, k_n \in \mathbb{R}\}$$

The questions here are:

- How many vectors we need to define a given vector space, as example $\mathbb{R}^l$?
- Any vector is valid?

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Linear Independence

A set of l vectors is linearly independent if

$$k_1\boldsymbol{u}_1 + k_2\boldsymbol{u}_2 + \ldots + k_l\boldsymbol{u}_l = 0 \Rightarrow k_1 = k_2 = \ldots = k_l = 0$$

Or otherwise:

Linear Dependence

A set of l vectors is linearly dependent if it exists a linear combination

$$k_1\boldsymbol{u}_1 + k_2\boldsymbol{u}_2 + \ldots + k_l\boldsymbol{u}_l = 0$$

with at least one $k_i \neq 0$

If a set of vectors is linearly dependent it will be always possible to express one of the vectors as linear combination of the others.

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Some exercises

Linear Dependence/ Independence

Exercise 1

Are the following three vectors, (linearly) dependent or (linearly) independent? If linearly dependent, find the maximum independent set of vectors.

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}^\intercal \quad \boldsymbol{u}_2 = \begin{pmatrix} -1 & -1 & 2 \end{pmatrix}^\intercal \quad \boldsymbol{u}_3 = \begin{pmatrix} 5 & 4 & -5 \end{pmatrix}^\intercal$$

Exercise 2

Are the following four vectors, (linearly) dependent or (linearly) independent? If linearly dependent, find the maximum independent set of vectors.

$$\begin{array}{ll} \boldsymbol{u}_1 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^\intercal & \boldsymbol{u}_2 = \begin{pmatrix} 3 & -1 & 1 \end{pmatrix}^\intercal \\ \boldsymbol{u}_3 = \begin{pmatrix} 5 & 3 & 3 \end{pmatrix}^\intercal & \boldsymbol{u}_4 = \begin{pmatrix} 9 & -5 & 2 \end{pmatrix}^\intercal \end{array}$$

Basis of a subspace

Linear Dependence/ Independence

Basis

A Basis for a Vector Subspace is a minimum set of vectors that spans that vector subspace.

Since the set is required to be minimum:

$$\boldsymbol{v}_1, \boldsymbol{v}_2 \dots \boldsymbol{v}_p \text{ spans a vector space}$$

$$\iff \boldsymbol{v}_1, \boldsymbol{v}_2 \dots \boldsymbol{v}_p \text{ are linearly independent}$$

Note: By this definition, a Basis is not unique. In fact we can replace any of the previous vectors by a linear combination of themselves and both, the former and the new set will be a valid basis of the same vector space.

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More Exercises

Linear Dependence/ Independence

Exercise 3

Give three different basis for the vector space that spans the next set of vectors.

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^\intercal \quad \boldsymbol{u}_2 = \begin{pmatrix} 3 & -1 & 1 \end{pmatrix}^\intercal$$
$$\boldsymbol{u}_3 = \begin{pmatrix} 5 & 3 & 3 \end{pmatrix}^\intercal \quad \boldsymbol{u}_4 = \begin{pmatrix} 9 & -5 & 2 \end{pmatrix}^\intercal$$

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Basis of a subspace

Linear Dependence/ Independence

Dimension

The minimum number of vectors needed to span a vector space/subspace is known as the Dimension of the space/subspace.
Equivalently
The Dimension of a vector space/subspace is the maximum number of independent vectors that can live in that space/subspace.

Note that this number is constant and of course will not depend on the selected basis

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Exercises for the next day:

- Problems book, chapter 3: 1, 5, 7, 9, 12.