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Centre de la Imatge i la Tecnologia Multimèdia

Theme 3. Affine Transformations

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Bachelor's Degree in Video Game Design and Development

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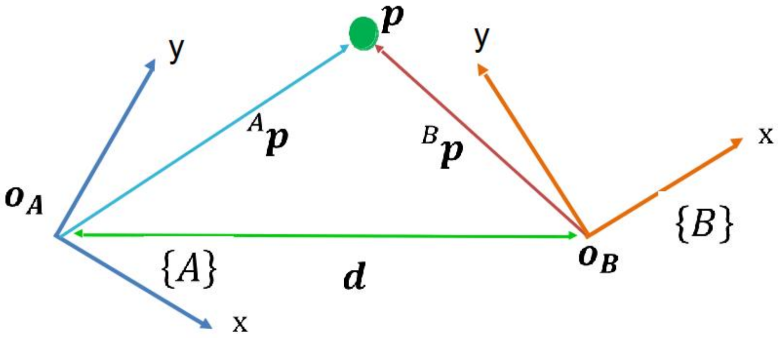
Rotation plus displacement (2D view)



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Definition

$\mathbf{R} + \boldsymbol{d}$

Rotation plus displacement (2D view)

$$ {}^{A}\boldsymbol{p} = {}^{A}\mathbf{R}_{B}\, {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{d}_{A\to B} $$

where

- ${}^{A}\boldsymbol{p}$ is the position of the point $\boldsymbol{p}$ seen from the frame A.
- ${}^{B}\boldsymbol{p}$ is the position of the point $\boldsymbol{p}$ seen from the frame B.
- ${}^{A}\mathbf{R}_{B}$ is the rotation matrix that relates the orientation of both frames.
- ${}^{A}\boldsymbol{d}_{A\to B}$ is the displacement between frames seen from A.

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Rotation plus displacement (2D view)

Equivalently you can define

$$^{B}\boldsymbol{p} = {^{B}\mathbf{R}_{A}}\, {^{A}\boldsymbol{p}} + {^{B}\boldsymbol{d}_{B\to A}}$$

where

- $^{B}\boldsymbol{p}$ is the position of the point $\boldsymbol{p}$ seen from the frame B.
- $^{A}\boldsymbol{p}$ is the position of the point $\boldsymbol{p}$ seen from the frame A.
- $^{B}\mathbf{R}_{A}$ is the rotation matrix that relates the orientation of both frames.
- $^{B}\boldsymbol{d}_{B\to A}$ is the displacement between frames seen from B.

Relating rotations and translations

Let's isolate ${}^B\boldsymbol{p}$ from the first definition

$$
\begin{gathered}
{}^A\boldsymbol{p} = {}^A\mathbf{R}_B\, {}^B\boldsymbol{p} + {}^A\boldsymbol{d}_{A\to B} \implies \\
{}^B\boldsymbol{p} = {}^A\mathbf{R}_B^{-1}\left({}^A\boldsymbol{p} - {}^A\boldsymbol{d}_{A\to B}\right) \implies \\
{}^B\boldsymbol{p} = {}^A\mathbf{R}_B^{-1}\, {}^A\boldsymbol{p} - {}^A\mathbf{R}_B^{-1}\, {}^A\boldsymbol{d}_{A\to B}
\end{gathered}
$$

Hence, from the equivalent definition for ${}^B\boldsymbol{p}$ and rotation matrix properties, we get:

- ${}^A\mathbf{R}_B^{-1} = {}^A\mathbf{R}_B^{\mathsf{T}} = {}^B\mathbf{R}_A$
- $-{}^B\mathbf{R}_A\, {}^A\boldsymbol{d}_{A\to B} = {}^B\boldsymbol{d}_{B\to A}$

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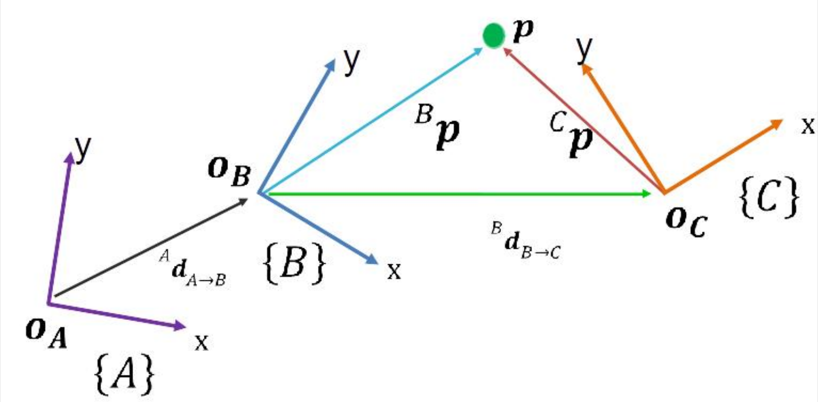
Example



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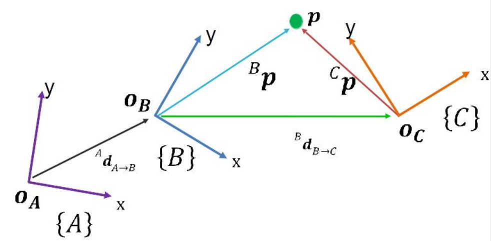
Composition
Affine Transformations

Example

Known

- ${}^{A}\mathbf{R}_{B}$, ${}^{B}\mathbf{R}_{C}$
- ${}^{A}\boldsymbol{d}_{A\to B}$, ${}^{B}\boldsymbol{d}_{B\to C}$
- ${}^{C}\boldsymbol{p}$

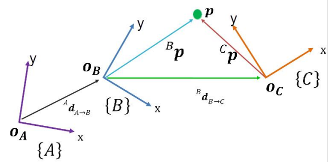


Composition
Affine Transformations

Example with numbers

Known

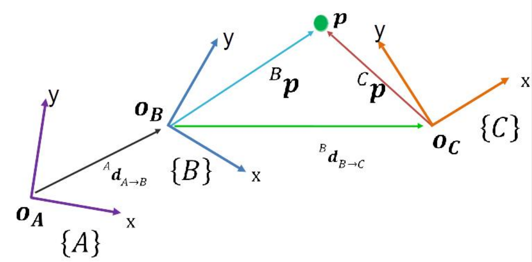
- $${}^{A}\mathbf{R}_B = \begin{pmatrix} 0.9397 & 0.3420 \\ -0.3420 & 0.9397 \end{pmatrix},$$
  $${}^{B}\mathbf{R}_C = \begin{pmatrix} 0.7660 & -0.6428 \\ 0.6428 & 0.7660 \end{pmatrix}$$
- $${}^{A}\boldsymbol{d}_{A\rightarrow B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
  $${}^{B}\boldsymbol{d}_{B\rightarrow C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
- $${}^{C}\boldsymbol{p} = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$$



Example with numbers

Could you find

- ${}^{A}\mathbf{R}_{C}$?
- ${}^{A}\boldsymbol{d}_{A\rightarrow C}$?



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Homogeneous coordinates

Affine Transformations

- 2D vector: $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- 2D homogeneous vector: $\hat{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$
- 3D vector: $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
- 3D homogeneous vector: $\hat{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$

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Homogeneous coordinates
Affine Transformations

- Translate a vector:

$$\hat{\boldsymbol{x}}' = \begin{pmatrix} \mathbf{I}_{3\times3} & \boldsymbol{t} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \hat{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} + \boldsymbol{t} \\ 1 \end{pmatrix}$$

- Rotate a vector:

$$\hat{\boldsymbol{x}}' = \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \hat{\boldsymbol{x}} = \begin{pmatrix} \mathbf{R}\boldsymbol{x} \\ 1 \end{pmatrix}$$

Augmented matrix – Affine transformation matrix

Using an augmented matrix $\mathbf{A}$ and an augmented vector $\hat{\boldsymbol{x}}$, it is possible to represent both the translation and the rotation using a single matrix multiplication.

$$\hat{\boldsymbol{x}}' = \mathbf{A}\hat{\boldsymbol{x}} = \begin{pmatrix} \mathbf{R} & \boldsymbol{t} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \hat{\boldsymbol{x}} \implies \boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t}$$

- A has an inverse, $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^\intercal & -\mathbf{R}^\intercal \boldsymbol{t} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}$$

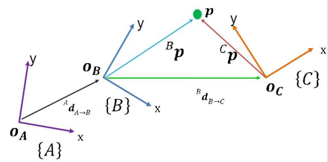
Check it!

Composition
Affine Transformations

Example with numbers

Known

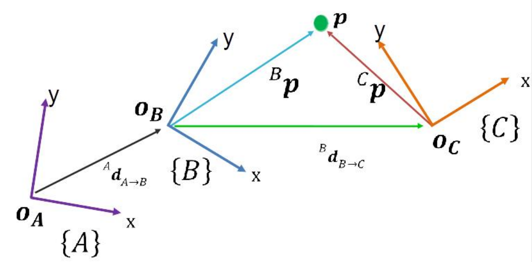
- $${}^{A}\mathbf{R}_{B} = \begin{pmatrix} 0.9397 & 0.3420 \\ -0.3420 & 0.9397 \end{pmatrix},$$
  $${}^{B}\mathbf{R}_{C} = \begin{pmatrix} 0.7660 & 0.6428 \\ -0.6428 & 0.7660 \end{pmatrix}$$
- $${}^{A}\boldsymbol{d}_{A\to B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
  $${}^{B}\boldsymbol{d}_{B\to C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
- $${}^{A}\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



Example with numbers

Could you find

- ${}^{A}\mathbf{R}_{C}$?
- ${}^{C}\boldsymbol{d}_{C\to A}$?



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Homework
Affine Transformations

Exercises

- Midterm exam: Exercise 5
- Final exam: Exercise 2
- Additional exercise: Origin $\boldsymbol{O_B}$ is known from the reference frame $\{A\}$ through the vector $\boldsymbol{t}$. Orientation of the reference frame $\{A\}$ with respect to the reference frame $\{B\}$ is codified in rotation matrix $\mathbf{R}$, so ${}^{B}\boldsymbol{p} = \mathbf{R}\,{}^{A}\boldsymbol{p}$. If $\boldsymbol{O_A} = \boldsymbol{O_B}$, a common origin for both reference frames, which is the affine transformation matrix $\mathbf{A}$ that allow to represent a vector known in $\{B\}$ in the reference frame $\{A\}$?