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Centre de la Imatge i la Tecnologia Multimèdia

Euler Angles

in Theme 2. Attitude

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Outline

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Euler angles
Composition

Composition of simple rotations

- We can use the composition of 3 successive (simple) rotations about different axis to generate any possible rotation
- We will use 3,2,1 (*z*-*y*-*x*) global to local notation (ψ, θ, ϕ)

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Euler angles

Composition

Simple rotation about Z axis

$$\boldsymbol{v}_{\{F_1\}} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{v}_{\{F_2\}}$$

Therefore:

$$\boldsymbol{v}_{\{F_2\}} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{v}_{\{F_1\}} = \mathbf{R}_\psi \boldsymbol{v}_{\{F_1\}}$$

Angle ψ is called yaw. Sometimes it is noted as γ.

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Euler angles
Composition

Simple rotation about Y axis

$$\boldsymbol{v}_{\{F_2\}} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \boldsymbol{v}_{\{F_3\}}$$

Therefore:

$$\boldsymbol{v}_{\{F_3\}} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \boldsymbol{v}_{\{F_2\}} = \mathbf{R}_\theta \boldsymbol{v}_{\{F_2\}}$$

Angle θ is called pitch. Sometimes it is noted as β.

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Euler angles
Composition

Simple rotation about X axis

$$\boldsymbol{v}_{\{F_3\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \boldsymbol{v}_{\{F_4\}}$$

Therefore:

$$\boldsymbol{v}_{\{F_4\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \boldsymbol{v}_{\{F_3\}} = \mathbf{R}_\phi \boldsymbol{v}_{\{F_3\}}$$

Angle ϕ is called roll. Sometimes it is noted as α.

Composition of rotations

$$\mathbf{v}_{\{F_2\}} = \mathbf{R}_\psi \mathbf{v}_{\{F_1\}}$$

$$\mathbf{v}_{\{F_3\}} = \mathbf{R}_\theta \mathbf{v}_{\{F_2\}}$$

$$\mathbf{v}_{\{F_4\}} = \mathbf{R}_\phi \mathbf{v}_{\{F_3\}}$$

Hence:

$$\mathbf{v}_{\{F_4\}} = \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi \mathbf{v}_{\{F_1\}}$$

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Euler angles
Composition

Composition of rotations

Rotation matrix $\mathbf{R}$ is equivalent to the change of basis matrix from target frame to initial frame

$$\boldsymbol{v}_{\{F_1\}} = \left(\mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi\right)^\mathsf{T} \boldsymbol{v}_{\{F_4\}} \rightarrow \mathbf{R} = \mathbf{R}_\psi^\mathsf{T} \mathbf{R}_\theta^\mathsf{T} \mathbf{R}_\phi^\mathsf{T}$$

$$\mathbf{R} =$$

$$\begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\cos\psi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi & \sin\psi\sin\psi\cos\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

Different Conventions

It is possible to select other axis permutations rather than the 3-2-1 order presented.
We will go later on this topic. You can check more information at Wikipedia Euler Angles

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Outline

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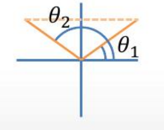
RotMat to Euler
Transformation

From rotation matrix to Euler angles. Step 1

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\cos\psi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi & \sin\psi\sin\psi\cos\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

with

$$r_{31} = -\sin\theta \rightarrow \theta = \arcsin(-r_{31})$$



There are two possibilities, one to be selected.

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RotMat to Euler
Transformation

From rotation matrix to Euler angles. Step 2

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\cos\psi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi & \sin\psi\sin\psi\cos\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

with angle θ (pitch) known:

$$\phi = \text{arctan2}\left(\frac{r_{32}}{\cos\theta}, \frac{r_{33}}{\cos\theta}\right) \rightarrow \text{roll}$$

$$\psi = \text{arctan2}\left(\frac{r_{21}}{\cos\theta}, \frac{r_{11}}{\cos\theta}\right) \rightarrow \text{yaw}$$

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RotMat to Euler

Transformation

From rotation matrix to Euler angles. Exercise

An example

$$\mathbf{R} = \begin{pmatrix} 0.9254 & 0.0180 & 0.3785 \\ 0.1632 & 0.8826 & -0.4410 \\ -0.3420 & 0.4698 & 0.8138 \end{pmatrix}$$

Determine ϕ, θ, and ψ.

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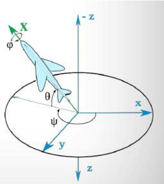
RotMat to Euler
Gimbal lock

When $\theta = \pm k\dfrac{\pi}{2}$ for k odd,

$$\mathbf{R} = \begin{pmatrix} 0 & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\cos\psi\sin\theta + \sin\psi\sin\phi \\ 0 & \sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi & \sin\psi\sin\psi\cos\theta - \cos\psi\sin\phi \\ \pm 1 & 0 & 0 \end{pmatrix}$$

and the past equations are not usable anymore! because the x and z axis are aligned, so the last rotation $\mathbf{R}_\phi$ (roll) cannot be distinguished from the first rotation $\mathbf{R}_\psi$ (yaw).



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RotMat to Euler
Gimbal lock

Therefore we can assign ϕ (roll) and ψ (yaw) a value and determine the other by using the remaining entries of the rotation matrix.

Gimbal lock. Exercise

An example

$$\mathbf{R} = \begin{pmatrix} 0 & 0.3420 & 0.9397 \\ 0 & 0.9397 & -0.3420 \\ -1 & 0 & 0 \end{pmatrix}$$

Determine ϕ, θ, and ψ.

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Composing rotations
Inefficiency

To concatenate rotations with the given parametrizations, we need to pass through the rotation matrices,

$$\mathbf{R}_3 = \mathbf{R}_2\mathbf{R}_1$$

Hence, code is inefficient because we need constantly to relate different 3-4 dimensional parametrizations with the rotation matrices to make operations