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Solving

Linear Systems of Equations

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Outline

1 Solving $\mathbf{A}x = b$

2 Gauss Method

3 Pivot's rule for triangulation

4 Gauss-Jordan

5 Additional Relations

6 Homework

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Classification of linear systems of equations

A system of equations is compatible if it has one ore more possible solutions.

- A compatible system is determinate if the system has a unique solution
- A compatible system is indeterminate if it has infinite solutions

In contrast, a system of equations is incompatible if it hasn't solution.

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Elemental operations

Sol. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Equivalent systems of equations

Two systems of equations which share the same unknowns (even with possible different number of equations), are equivalent if they share the same solution.

The next operations transforms a system of equations into an equivalent one

- Sum or subtract one equation to the other
- Multiply or divide one equation by a non-zero number
- Exchange two equations
- Exchange the order of the unknowns

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Elemental operations

Sol. $\mathbf{A}\mathbf{x} = \mathbf{b}$

Example

Verify that the next two systems of equations share the same solution which is $x = 1,\ y = 2,\ z = 3$

$$\left.\begin{array}{rcl} 2x - 4y + z & = & -3 \\ 2x + 3y - 2z & = & 2 \\ -x + 2y + 2z & = & 9 \end{array}\right\} \qquad \left.\begin{array}{rcl} 2x - 11y + 4z & = & -8 \\ 3x + y + z & = & 8 \\ -4x + 8y + 8z & = & 36 \end{array}\right\}$$

Note:

$$\begin{array}{rcl} f_1' & = & 2f_1 - f_2 \\ f_2' & = & f_1 + f_2 + f_3 \\ f_3' & = & 4f_3 \end{array}$$

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Gauss Method

Any system of linear equations can be transformed to an equivalent one in triangular form, i.e. a system where the first non-null coefficient in a row corresponds to an unknown that goes after the unknown with non-null coefficient of the preceding row

Example

$$\left.\begin{array}{rcl} 2x - 4y + z & = & -3 \\ 7y - 3z & = & 5 \\ 5z & = & 15 \end{array}\right\}$$

This system can be solved easily by using backtracking

Gauss Method

How can be applied?

Triangular forms can be designed by using elemental operations. To this end, it is useful to put the system in the augmented matrix form:

$$\left.\begin{array}{r} 2x - 4y + z = -3 \\ 2x + 3y - 2z = 2 \\ -x + 2y + 2z = 9 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 2 & -4 & 1 & -3 \\ 2 & 3 & -2 & 2 \\ -1 & 2 & 2 & 9 \end{array}\right)$$

How can I make appear zeros on the 1st columns of the 2nd and 3rd rows? $f_2' = f_2 - f_1$, and $f_3' = 2f_3 + f1$

$$\left(\begin{array}{ccc|c} 2 & -4 & 1 & -3 \\ 2 & 3 & -2 & 2 \\ -1 & 2 & 2 & 9 \end{array}\right) \cong \left(\begin{array}{ccc|c} 2 & -4 & 1 & -3 \\ 0 & 7 & -3 & 5 \\ 0 & 0 & 5 & -15 \end{array}\right)$$

$$x = 1,\ y = 2,\ z = 3$$

Examples

Gauss

Example

Solve the next systems of equations by using the Gauss method

$$\left.\begin{aligned} x - 2y + z + t &= 8 \\ x + y + 2z - t &= 2 \\ 3x + y - z + t &= 9 \\ 2x + y - 3z + t &= 4 \end{aligned}\right\} \qquad \left.\begin{aligned} 2w - 4z &= -2 \\ -w + 3z &= -4 \end{aligned}\right\}$$

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Classifying LSE

Gauss

Incompatible

A linear system of equations is incompatible, if after applying the Gauss method for triangulate the system, one row is inconsistent

Example

$$\left.\begin{array}{r} x - y + z = 1 \\ x + 2y + 4z = 1 \\ 2x + y + 5z = -1 \end{array}\right\}$$

Classifying LSE

Gauss

indeterminate

A linear system of equations is compatible and indeterminate, if after applying the Gauss method for triangulate the system, one row is redundant.

Example

$$\left.\begin{array}{r} x - y + z = 1 \\ x + 2y + 4z = 1 \\ 2x + y + 5z = 2 \end{array}\right\}$$

In the indeterminate case we end up with more unknowns that equations. This means first, that the system has infinite solutions and that we can free some unknowns and calculate the solution as a function of the value that the free unknown takes

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Definition
Pivoting

The pivot's rule is a systematic procedure to triangulate a matrix

$$\left(\begin{array}{cccccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} & b_1 \\
0 & a_{22} & a_{23} & \cdots & a_{2k} & a_{2,k+1} & \cdots & a_{2n} & b_2 \\
0 & 0 & a_{33} & \cdots & a_{3k} & a_{3,k+1} & \cdots & a_{3n} & b_3 \\
\vdots & \vdots & \vdots & \ddots & & & \cdots & & \vdots \\
0 & 0 & 0 & \cdots & a_{ik} & a_{i,k+1} & \cdots & a_{in} & b_i \\
0 & 0 & 0 & \cdots & a_{i+1,k} & a_{i+1,k+1} & \cdots & a_{i+1,n} & b_{i+1} \\
\vdots & \vdots & \vdots & & \vdots & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & a_{mk} & a_{m,k+1} & \cdots & a_{mn} & b_m
\end{array}\right)$$

Make 0's here

It can be achieved by transforming the rows after the row i by

$$f_j' = a_{ik} f_j - a_{jk} f_i \quad j = i + 1 \ldots n$$

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Application
Pivoting

The past row transformation can be expressed element by element as

$$a'_{jl} = a_{ik}a_{jl} - a_{jk}a_{il}, \quad j = i+1 \ldots n,\ l = k+1 \ldots n$$

$$b'_j = a_{ik}b_j - a_{jk}b_i, \quad j = i+1 \ldots n$$

a_{ik} is known as the pivot and
a_{jk} is known as the subpivot

Example

Solve the next system of equations using the Gauss method with the pivot's rule

$$\left.\begin{array}{r} x+4y+3z=1 \\ 2x+5y+4z=4 \\ x-2y-2z=3 \end{array}\right\}$$

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Gauss-Jordan

G-J

Gauss-Jordan method

By using the pivot's rule applied, not only to the posterior rows but, also to the preceding rows of the pivot's row, a diagonal system is obtained. From there it is straightforward to find the system solutions by simply

$$x_i = b_i / a_{ii}, \quad \text{for } i = 1 \ldots n$$

Example

Solve the next system of equations using the Gauss-Jordan method with the pivot's rule

$$\left. \begin{array}{c} 2x + 1y + 1z = 1 \\ 3x - y + z = 2 \\ 5x + y - z = 1 \end{array} \right\}$$

Inverse of a matrix

G-J

If the Gauss-Jordan method is applied over the system

$$\left(\mathbf{A} \mid \mathbf{I} \right)$$

until obtaining the identity on the LHS, then the resulting matrix in the RHS is the matrix $\mathbf{A}^{-1}$

Example

Calculate the inverse of the matrix $\mathbf{A}$ by using the Gauss-Jordan method

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ 5 & 1 & -1 \end{pmatrix}$$

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Part I

Additional Relations

Let $\mathbf{A} \in \mathbb{M}_{m\times n}$

The next assertions are equivalent

- On the triangulation of $\mathbf{A}$ appear k pivots
- k columns of $\mathbf{A}$ are linear independent
- The dimension of the subspace spanned by $\mathrm{C}(\mathbf{A}) = k$
- $\operatorname{rank}(\mathbf{A}) = k$
- The columns, where the pivots appear, are a basis of the subspace spanned by $\mathrm{C}(\mathbf{A})$

Note: $k \leq min(m, n)$

Note: If $m = n$ and $k = n$, the columns of $\mathbf{A}$ are independent $\rightarrow$ $\mathbf{A}$ is regular and has inverse

Implicit equations of the subspace

The implicit equations of the subspace $S = \langle \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \ldots \rangle$ can be obtained by triangulating the next matrix

$$\left(\begin{array}{c|c} & x \\ & y \\ \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \ldots & z \\ & \vdots \end{array} \right)$$

The RHS terms on the rows with no pivots must be zero to make the system solvable. As consequence, these terms equated to 0 are the implicit equations of the subspace

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Example

Calculate the dimension, the implicit equations and a basis for the subspace of $\mathbb{R}^4$

$$S = \langle (1\,2\,-1\,0)^\intercal, (1\,3\,2\,1)^\intercal, (1\,4\,5\,2)^\intercal, (1\,2\,0\,4)^\intercal \rangle$$

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Rouché-Frobenious' theorem

Additional Relations

Let $\mathbf{A} \in \mathbb{M}_{m\times n}$ and $\boldsymbol{b} \in \mathbb{M}_{m\times 1}$. Let in addition $\mathbf{A}_a = (\mathbf{A}|\boldsymbol{b}) \in \mathbb{M}_{m\times n+1}$

Rouché-Frobenious' theorem

- If $\mathrm{rank}\,(A) \neq \mathrm{rank}\,(A)_a \rightarrow$ the system of equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is incompatible
- If $\mathrm{rank}\,(A) = \mathrm{rank}\,(A)_a \rightarrow$ the system of equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is compatible
  - If $\mathrm{rank}\,(A) = n \rightarrow$ the system is determinate
  - else, If $\mathrm{rank}\,(A) < n \rightarrow$ the system is indeterminate

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Book of problems

- Chapter 1: 1, 3, 5, 6 a, 17, 22, 28, 30
- Chapter 3: 3, 16, 19 (a,b), 22

More:

- Chapter 1: 15, 18, 21, 22
- Chapter 3: Redo exercise 9(trying to solve the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$)
- Chapter 2: Exercise 7 d, 8
- Chapter 3: Exercise 18