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Matrices

and their properties

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Outline

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Vector equations

Intro

In the past Lecture we have seen that equations of the type

$$k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 + \ldots + k_n \boldsymbol{v}_n = \boldsymbol{b} \tag{1}$$

arise when studying the linear dependence of a bunch of vectors

Note: Eq. (1) can be written as a system of n linear equations with n unknown variables

$$\begin{aligned} v_{11}k_1 + v_{21}k_2 + \cdots + v_{n1}k_n &= b_1 \\ v_{12}k_1 + v_{22}k_2 + \cdots + v_{n2}k_n &= b_2 \\ &\vdots \\ v_{1n}k_1 + v_{2n}k_2 + \cdots + v_{nn}k_n &= b_n \end{aligned} \tag{2}$$

Linear system of equations and matrix by vector multiplication

The linear system in Eq. (2) is equivalent to:

$$\mathbf{A}\boldsymbol{k} = \boldsymbol{b}$$

where

$$\mathbf{A} = \begin{pmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix}^{\intercal}$$

$$\boldsymbol{b} = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}^{\intercal}$$

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What is a matrix?

Matrices

Matrices

A rectangular matrix with m rows and n columns is a function that for every pair (i, j) designates a real number a_{ij}

$$\mathbf{A} : \{1, 2..., m\} \times \{1, 2..., n\} \rightarrow \mathbb{R}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\mathbf{A}(i, j) = a_{ij} \in \mathbb{R}$$

$$\mathbf{A} \in \mathbb{M}_{m \times n}$$

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Properties
Matrices

Matrix equality

A pair of matrices $\mathbf{A}$ of size $m \times n$ and $\mathbf{B}$ of size $p \times q$ are equal, i.e $\mathbf{A} = \mathbf{B}$ if and only if, $m = p$, $n = q$ and $a_{ij} = b_{ij}\ \forall i = \{1, ..., m\}$, $j = \{1, ..., n\}$

Matrix properties

1. Associativity of the sum: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
2. Commutativity of the sum: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
3. Neutral element of the sum: if $\exists \mathbf{B} \mid \mathbf{A} + \mathbf{B} = \mathbf{A}\,,\ \forall \mathbf{A} \in \mathbb{M}_{m\times n} \Rightarrow$ $b_{ij} = 0\,, \forall\, i = \{1, ..., m\}\,,\ j = \{1, ..., n\}$. $\mathbf{B}$ is noted $\mathbf{0}$
4. Inverse element of the sum: if $\forall \mathbf{A}\, \exists \mathbf{B} \mid \boldsymbol{A} + \boldsymbol{B} = \mathbf{0} \Rightarrow$ $\boldsymbol{B} = -\boldsymbol{A}\,,\ b_{ij} = -a_{ij}\,,\ \forall\, i = \{1, ..., m\}\,,\ j = \{1, ..., n\}$

Properties II

Matrices

Matrix properties (Continuation)

5. Distributivity of the product times the sum of matrices: $k(\mathbf{A}+\mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
6. Distributivity of the product times the sum of scalars: $\mathbf{A}(k_1+k_2) = k_1\mathbf{A} + k_2\mathbf{A}$
7. Associativity of the product: $k_1(k_2\mathbf{A}) = (k_1 k_2)\mathbf{A}$

You will see through this course many times:

- Square matrices, where $m = n$.
- Column matrices, which are equivalent to a vector, $n = 1$.
- Row matrices, which are equivalent to a vector transposed, $m = 1$.

Outline

Matrix multiplication

Mat product

Matrix product

The matrix product of a matrix $\mathbf{A} \in \mathbb{M}_{m\times n}$ by a matrix $\mathbf{B} \in \mathbb{M}_{n\times p}$ results in a matrix $\mathbf{C} \in \mathbb{M}_{m\times p}$ and is defined as:

$$c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$$

Example

$$\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} = ?$$

How to multiply matrices

Mat product

Another example

The product is defined over any rectangular matrix

$$\begin{pmatrix} -1 & 2 & 6 \\ -4 & 3 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = ?$$

More to practice

$$\begin{pmatrix} 5 & -4 & 3 & 1 \\ -2 & 2 & 2 & -2 \\ 1 & 5 & -2 & 3 \\ 5 & 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ -1 & 3 & 5 \\ 4 & -1 & -1 \end{pmatrix} = ?$$

Properties of matrix product

Matrix product properties

- Associativity: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- Distributivity: $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
- Compatibility: $(k_1\mathbf{A})\mathbf{B} = k_1(\mathbf{AB}) = \mathbf{A}(k_1\mathbf{B})$

Note: that special requirements on the size of matrices involved in the product are needed to perform the operation i.e. a matrix with size $(m \times n)$ could only be multiplied by a matrix with size $(n \times p)$
Moreover:

- In general, commutativity does not hold in the matrix product $\mathbf{AB} \neq \mathbf{BA}$
- It is possible to have $\mathbf{AB} = \mathbf{0}$ even if $\mathbf{A}, \mathbf{B} \neq \mathbf{0}$

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Matrix product

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

Exercise

Show that

$$\mathbf{AB} \neq \mathbf{BA}$$

Exercise

Show that

$$\mathbf{AC} = \mathbf{0}$$

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Transpose of a Matrix

Transpose

The transpose of a given matrix $\mathbf{A} \in \mathbb{M}_{m\times n}$, represented by $\mathbf{A}^T$ is a matrix in $\mathbb{M}_{n\times m}$ for which

$$a_{ij}^{\mathsf{T}} = a_{ji}$$

Transpose properties

- $(\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$
- $(k_1\mathbf{A})^{\mathsf{T}} = k_1\mathbf{A}^{\mathsf{T}}$
- $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

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Transpose of a Matrix
Transpose

Exercise

Calculate $(\mathbf{AB})^\intercal$ and $\mathbf{B}^\intercal\mathbf{A}^\intercal$ if

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

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Identity matrix

The identity matrix is a square matrix, usually represented by $\mathbf{I}_n$, which entries fulfill

$$\begin{array}{ll} i_{ij} = 1 & \text{if } i = j \\ i_{ij} = 0 & \text{otherwise} \end{array}$$

Neutral element

The identity matrix represents the neutral element in the matrix product operation. That means that if $\mathbf{A} \in \mathbb{M}_{m\times n}$

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$$

Symmetric and Orthogonal

Square matrices are an special kind of matrices. They arise many times in the practice and we will be focusing manly on them during the course

Symmetric Matrix

A symmetric matrix is a square matrix for which $\mathbf{A}^\intercal = \mathbf{A}$, e.g

$$\begin{pmatrix} 1 & 3 & 5 \\ 3 & -2 & -1 \\ 5 & -1 & 2 \end{pmatrix}, \; a_{ij} = a_{ji} \; \forall i, j$$

Orthogonal Matrix

An orthogonal matrix, is a square matrix for which $\mathbf{A}^\intercal\mathbf{A} = \mathbf{A}\mathbf{A}^\intercal = \mathbf{I}$

Regular matrices

Square Mat

Regular Matrix

A regular (or invertible) matrix is a square matrix that admits inverse, i.e $\exists \mathbf{B} \mid \mathbf{BA} = \mathbf{I} = \mathbf{AB}$
We will name the matrix $\mathbf{B}$ under the name of inverse matrix of $\mathbf{A}$ and will be denoted as $\mathbf{A}^{-1}$

Calculate the inverse of a matrix is not easy specially for high dimensions. However, for the case of 2D-matrices:

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \frac{1}{a_1 a_4 - a_2 a_3} \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$$

Note: that if $a_1 a_4 - a_2 a_3 = 0$, the inverse does not exist and we will say that $\mathbf{A}$ is a singular matrix.

Exercise

Find $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 & 5 \\ -5 & 6 \end{pmatrix}$$

Note: this can be easily used to solve the most simple system of equations

$$\mathbf{A}x = b$$

Inverse of 3×3 Matrix

Matrices of 3×3 admit an explicit representation of the inverse. Let

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \frac{\begin{pmatrix} ei - fh & hc - ib & bf - ce \\ gf - di & ai - gc & dc - af \\ dh - ge & gb - ah & ae - db \end{pmatrix}}{aei + bfg + cdh - gec - hfa - idb}$$

Of course, the inverse is possible only when $aei + bfg + cdh - gec - hfa - idb \neq 0$

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Very often we will be dealing with matrix equations, i.e. equations where the unknown are contained in a whole matrix

Exam exercise

Given the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 7 & 4 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 & -1 & -1 \\ -4 & -1 & 2 \\ 8 & 4 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -1 & 1 \\ -3 & 2 & 1 \end{pmatrix}$$

Expand the next equation and find the value of the matrix $\mathbf{X}$

$$\mathbf{A} + (\mathbf{CB})^{\intercal} = \mathbf{B}^{\intercal}\mathbf{X}$$

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Column Space of a matrix

As seen, matrices arise naturally from vector equations. As a consequence, the columns of a matrix fulfill all the conditions presented on the past lecture for spanning a vector space.

Columnspace of a matrix

The Columnspace of a matrix, denoted by $\mathrm{C}(\mathbf{A})$, is the Vector Space that the columns of the matrix span.

Rank of a matrix

The number of linear independent columns in a matrix represents the dimension of the vector subspace spanned by the columns of the matrix. It is also known as the rank of the matrix.

$$\operatorname{rank}(\mathbf{A}) = \dim(\mathrm{C}(\mathbf{A})) = \#\text{ of independent column vectors}$$

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Open questions
Recap

Since now we have taken some concepts about:

- Vector basics
- Vector spaces
- Vector subspaces
- Dimension of spaces/subspaces
- Linear dependence/independence
- Systems of linear equations
- Matrix basics
- Matrix equations
- Column space of a matrix
- Rank of a matrix

Things related with the solution/solvability of vector equations

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Homework

Homework

Book of problems, chapter 2: 15, 16, 19, 23, 27, 30