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Basis and Change of basis

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Definitions
Basis

Generators

A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is a generator of the vector space E iff

$$E = \mathrm{span}\left(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\right)$$

Which means that any vector of E can be described as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$

Basis

A basis $\mathfrak{B} = \{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \ldots,\ \boldsymbol{e}_n\}$ for the subspace E is a set of linear independent vectors generators of E

Since the elements of $\mathfrak{B}$ are independent, n coincides with the dimension of E and any vector $\boldsymbol{u} \in E$ can be written in a unique form as

$$\boldsymbol{u} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \ldots + x_n\boldsymbol{e}_n$$

As consequence, the coefficients x_i are the components of the vector $\boldsymbol{u}$ in the basis $\mathfrak{B}$

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Example I

Basis

Exercise

Let

$$S = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \right\rangle$$

- Calculate the dimension of S
- Give two different basis for S
- Is the vector $\boldsymbol{u} = (-4, 11, -3)^\intercal$ contained in S?
- What are the components of $\boldsymbol{u} = (11, 13, 0)^\intercal$ on the given basis?

Canonical basis

Basis

The most used basis is the canonical basis. The vectors that conforms this basis are

$$\boldsymbol{e}_1 = (1,\ 0,\ \ldots,\ 0)^\intercal,\ \boldsymbol{e}_2 = (0,\ 1,\ \ldots,\ 0)^\intercal, \ldots, \boldsymbol{e}_n = (0,\ 0,\ \ldots,\ n)^\intercal$$

If nothing different is said, it must be assumed that any vector given is expressed on this basis. The canonical basis is the default basis. I.e $\boldsymbol{u} = (5,\ 3,\ -1)^\intercal = (5\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - \boldsymbol{e}_3)$

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Same vector in different basis

Change of Basis

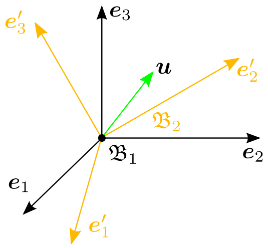
Let $\mathfrak{B}_1 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_k\}$ and $\mathfrak{B}_2 = \{\boldsymbol{e}'_1, \boldsymbol{e}'_2, ..., \boldsymbol{e}'_k\}$ be two different basis of the same subspace E with $dim(E) = k$.

Therefore a vector $\boldsymbol{u} \in E$ can be written as

$$\boldsymbol{u} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + ... + x_k\boldsymbol{e}_k = \sum_{i=1}^{k} x_i\boldsymbol{e}_i$$

and equivalently

$$\boldsymbol{u} = x'_1\boldsymbol{e}'_1 + x'_2\boldsymbol{e}'_2 + ... + x'_k\boldsymbol{e}'_k = \sum_{j=1}^{k} x'_j\boldsymbol{e}'_j$$



Same vector in different basis

Change of Basis

Let the vectors of the $\mathfrak{B}_2$ to be known in the basis $\mathfrak{B}_1$, i.e

$$\mathbf{e}'_j = \sum_{i=1}^{k} a_{ij}\mathbf{e}_i$$

And since

$$\sum_{i=1}^{k} x_i\mathbf{e}_i = \sum_{j=1}^{k} x'_j \boxed{\mathbf{e}'_j} = \sum_{j=1}^{k} x'_j \sum_{i=1}^{k} a_{ij}\mathbf{e}_i = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}x'_j\mathbf{e}_i$$

Hence

$$x_i = \sum_{j=1}^{k} a_{ij}x'_j$$

Which is a matrix multiplication

Change of basis

$$(\mathbb{R}^k, \mathfrak{B}_2) \xrightarrow{\mathbf{C}} (\mathbb{R}^k, \mathfrak{B}_1)$$

Given the components of a vector on $\mathfrak{B}_2$ as $\boldsymbol{u}_{\mathfrak{B}_2} = (x'_1, x'_2, ..., x'_k)$, the components of the same vector in the basis $\mathfrak{B}_1$, can be calculated as $\boldsymbol{u}_{\mathfrak{B}_1} = \mathbf{C}\boldsymbol{u}_{\mathfrak{B}_2}$

Note: that $\mathbf{C}$ is composed by the row concatenation of the column vectors of the basis $\mathfrak{B}_2$ as seen from $\mathfrak{B}_1$

$$\mathbf{C} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & \ddots & & a_{2k} \\ \vdots & & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

(Re)-Change of basis

Since $\mathbf{C}$ is composed by linearly independent vectors, $\mathbf{C}$ is regular $\rightarrow \exists \mathbf{C}^{-1}$ hence $\boldsymbol{u}_{\mathfrak{B}_2} = \mathbf{C}^{-1}\boldsymbol{u}_{\mathfrak{B}_1}$

$$(\mathbb{R}^k, \mathfrak{B}_1) \xrightarrow{\mathbf{C}^{-1}} (\mathbb{R}^k, \mathfrak{B}_2)$$

A direct consequence is that $\mathbf{C}^{-1}$ represents the components of the basis $\mathfrak{B}_1$ seen from $\mathfrak{B}_2$ as columns

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Examples II

Change of Basis

Example

- Verify that $\boldsymbol{e}'_1 = (1, 1, 1)^\intercal$, $\boldsymbol{e}'_2 = (1, 1, 2)^\intercal$, $\boldsymbol{e}'_3 = (0, 1, 1)^\intercal$ forms a basis of the $\mathbb{R}^3$.
- Give the components of the vector $\boldsymbol{u}' = (-1, 2, 1)^\intercal$ in the canonical basis
- Find the components of the vectors of the canonical basis on the given basis

Multiple basis

Change of Basis

Many times we will need to transform between two basis for which we don't know the vectors of one basis seen from the other, but we know the components of all the vectors in a third base.

$$
\begin{tikzcd}
(\mathbb{R}^k, \mathfrak{B}_1) \arrow[rr, shift left, "\mathbf{Q}^{-1}\mathbf{P}"] \arrow[dr, "\mathbf{P}"'] & & (\mathbb{R}^k, \mathfrak{B}_2) \arrow[ll, shift left, "\mathbf{P}^{-1}\mathbf{Q}"] \arrow[dl, "\mathbf{Q}"] \\
& (\mathbb{R}^k, \mathfrak{B}_3) &
\end{tikzcd}
$$

Note: The order of the matrix multiplications

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Examples III

Change of Basis

Example

- Verify that $\boldsymbol{e}'_1 = (1,1,1)^\intercal$, $\boldsymbol{e}'_2 = (1,1,2)^\intercal$, $\boldsymbol{e}'_3 = (0,1,1)^\intercal$ and that $\boldsymbol{e}''_1 = (1,1,2)^\intercal$, $\boldsymbol{e}''_2 = (0,0,1)^\intercal$, $\boldsymbol{e}''_3 = (-1,1,2)^\intercal$ are valid basis basis of $\mathbb{R}^3$.
- Knowing $\boldsymbol{u}' = (-1,2,1)^\intercal$, calculate $\boldsymbol{u}''$
- Knowing $\boldsymbol{u}'' = (-1,2,1)^\intercal$, calculate $\boldsymbol{u}'$

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Homework

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- Chapter 3: 39, 40, 42, 46, 47, 52