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Rotation Matrices. Euler Axis/Angle Representation

in Theme 2. Attitude

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Outline

1 Euler theorem

2 Principal axis/angle rotation

3 Rotation Matrix from axis and angle

4 Inverse mapping

5 Rotation vector

6 Homework

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Euler theorem

Two alternative versions

Theorem A (Euler's theorem on rotations)

When a sphere is moved around its center it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position. This diameter is called axis

Theorem B (Euler's theorem on rotations)

Any motion of a rigid body such that a point, let's say O, on the rigid body remains fixed, is equivalent to a single rotation around some axis that runs through O.

Practical use: If a body is rotating it turns around an axis that passes by points with 0 velocity. Since we are dealing with systems that does not change the origin (rotates maintaining it), the principal axis pass through the origin.

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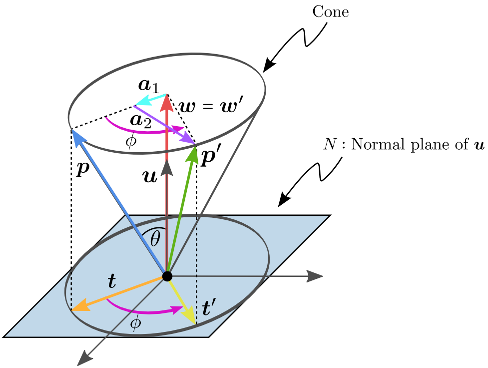
Rotations
Axis/angle

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$



Equivalently, can we find $\mathbf{R}$?

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

2 simple cases:

- vector $\boldsymbol{p}$ is parallel to axis $\boldsymbol{u}$
- vector $\boldsymbol{p}$ is orthogonal/perpendicular to axis $\boldsymbol{u}$

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

Case 1: vector $\boldsymbol{p}$ is parallel to axis $\boldsymbol{u}$

- If $\boldsymbol{p}$ is parallel to $\boldsymbol{u}$, since they share the Origin,

$$\boldsymbol{p} = k\boldsymbol{u}$$

- Since a Rotation is a Linear transformation,

$$\boldsymbol{p}' = T(\boldsymbol{p}) = T(k\boldsymbol{u}) = kT(\boldsymbol{u}) = k\mathbf{R}\boldsymbol{u} = k\boldsymbol{u} = \boldsymbol{p}$$

Therefore vector $\boldsymbol{p}$ does not change of direction (nor size, it is a rotation) when it is parallel to the rotation axis $\boldsymbol{u}$.

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

Case 2: vector $\boldsymbol{p}$ is orthogonal to axis $\boldsymbol{u}$

- If $\boldsymbol{p}$ is orthogonal/perpendicular to $\boldsymbol{u}$, since they share the Origin, $\boldsymbol{p}$ will describe a circumference's arc around $\boldsymbol{u}$ of an angle ϕ.

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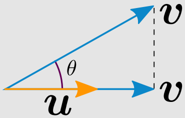
An aside

Vector projection

The vector projection $\boldsymbol{v}'$ of a vector $\boldsymbol{v}$ over a direction $\boldsymbol{u}$



$$\boldsymbol{v}' = \frac{\boldsymbol{v}^\intercal \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} = \frac{\|\boldsymbol{v}\|\,\|\boldsymbol{u}\|}{\|\boldsymbol{u}\|^2}\cos\theta\boldsymbol{u} = \|\boldsymbol{v}\|\cos\theta\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

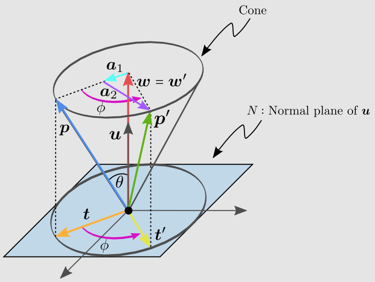
If $\boldsymbol{v}$ and $\boldsymbol{u}$ are unit length vectors ($\|\boldsymbol{v}\| = \|\boldsymbol{u}\| = 1$), then

$$\boldsymbol{v}' = (\boldsymbol{v}^\intercal\boldsymbol{u})\boldsymbol{u} = \cos\theta\boldsymbol{u}$$

Note: Vector projection $\neq$ Scalar projection

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

General case: vector $\boldsymbol{p}$ can be described $\boldsymbol{p} = \boldsymbol{t} + \boldsymbol{w}$



$$\boldsymbol{p} = \boldsymbol{t} + \boldsymbol{w}$$

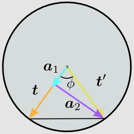
- $\boldsymbol{w}$ is parallel to axis $\boldsymbol{u}$
- $\boldsymbol{t}$ is orthogonal to axis $\boldsymbol{u}$
- $\boldsymbol{w} = (\boldsymbol{p}^\mathsf{T}\boldsymbol{u})\boldsymbol{u}$
- $\boldsymbol{t} = \boldsymbol{p} - (\boldsymbol{p}^\mathsf{T}\boldsymbol{u})\boldsymbol{u}$
- $\|\boldsymbol{t}\| = \|\boldsymbol{p}\| \sin\theta$

$$\boldsymbol{p}' = T(\boldsymbol{p}) = T(\boldsymbol{t} + \boldsymbol{w}) = T(\boldsymbol{t}) + T(\boldsymbol{w}) = \boldsymbol{w} + T(\boldsymbol{t})$$

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

General case: $\boldsymbol{p}' = \boldsymbol{w} + T(\boldsymbol{t})$

So... what is $T(\boldsymbol{t})$?



$$
\begin{aligned}
T(\boldsymbol{t}) &= \boldsymbol{t}' = \boldsymbol{a}_1 + \boldsymbol{a}_2 \\
\boldsymbol{a}_1 &= \|\boldsymbol{t}'\| \cos\phi \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|} = \|\boldsymbol{t}\| \cos\phi \frac{\boldsymbol{t}}{\|\boldsymbol{t}\|} \\
&= \boldsymbol{t} \cos\phi \\
\boldsymbol{a}_2 &= \|\boldsymbol{t}'\| \sin\phi \frac{\boldsymbol{u} \times \boldsymbol{p}}{\|\boldsymbol{u} \times \boldsymbol{p}\|} = \|\boldsymbol{t}\| \sin\phi \frac{\boldsymbol{u} \times \boldsymbol{p}}{\|\boldsymbol{u} \times \boldsymbol{p}\|} \\
&= \|\boldsymbol{p}\| \sin\theta \sin\phi \frac{\boldsymbol{u} \times \boldsymbol{p}}{\|\boldsymbol{p}\| \sin\theta} \\
&= (\boldsymbol{u} \times \boldsymbol{p}) \sin\phi
\end{aligned}
$$

How about a ROTATION $\mathbf{R}$ around the AXIS $\boldsymbol{u}$ of ANGLE ϕ affects a VECTOR $\boldsymbol{p}$? $\boldsymbol{p}' = T(\boldsymbol{p}) = \mathbf{R}\boldsymbol{p}$

General case: $\boldsymbol{p}' = \boldsymbol{w} + T(\boldsymbol{t})$

$$\boldsymbol{p}' = (\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u} + (\boldsymbol{p} - (\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u})\cos\phi + (\boldsymbol{u} \times \boldsymbol{p})\sin\phi$$

$$\boldsymbol{p}' = \boldsymbol{p}\cos\phi + (1 - \cos\phi)(\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u} \times \boldsymbol{p})\sin\phi$$

Remember:

- $\boldsymbol{w} = (\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u}$
- $\boldsymbol{t} = \boldsymbol{p} - (\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u}$

Some numbers
Axis/angle

Example

Using the result in the upper slide, calculate the result of rotating the vectors

- $\boldsymbol{e}_1 = (1,\ 0,\ 0)^\intercal$
- $\boldsymbol{e}_2 = (0,\ 1,\ 0)^\intercal$
- $\boldsymbol{e}_3 = (0,\ 0,\ 1)^\intercal$

an amount of $\frac{\pi}{2}\,\mathrm{rad}$ about the direction of $\boldsymbol{u} = (0,\ 0,\ 1)^\intercal$

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In the past, we had seen that any linear transformation can be written as a matrix by vector multiplication. As consequence it should be possible to write $\boldsymbol{p}' = \mathbf{R}(\boldsymbol{u}, \phi)\boldsymbol{p}$

Rewriting: $\boldsymbol{p}' = T(\boldsymbol{p})$ as $\mathbf{R}(\boldsymbol{u}, \phi)\boldsymbol{p}$

- Note: Prove that $(\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^\intercal)\boldsymbol{p}$
- Note: $\boldsymbol{u} \times \boldsymbol{v}$ can be also written as $[\boldsymbol{u}]_\times \boldsymbol{v}$ with

$$[\boldsymbol{u}]_\times = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

$$\boldsymbol{p}' = \underbrace{\boldsymbol{p}\cos\phi}_{(\mathbf{I}_3\cos(\phi))\boldsymbol{p}} + \underbrace{(1-\cos\phi)(\boldsymbol{p}^\intercal \boldsymbol{u})\boldsymbol{u}}_{((1-\cos(\phi))(\boldsymbol{u}\boldsymbol{u}^\intercal))\boldsymbol{p}} + \underbrace{(\boldsymbol{u}\times\boldsymbol{p})\sin\phi}_{\left([\boldsymbol{u}]_\times \sin(phi)\right)\boldsymbol{p}}$$

Rodrigues' Rotation Formula

$$\mathbf{R}(\boldsymbol{u}, \phi) = \mathbf{I}_3 \cos\phi + (1 - \cos(\phi))(\boldsymbol{u}\boldsymbol{u}^\intercal) + [\boldsymbol{u}]_\times \sin(\phi)$$

Example

What is the rotation matrix that rotates a vector $\frac{\pi}{2}\,\mathrm{rad}$ about the direction of $\boldsymbol{u} = (0,\ 0,\ 1)^\intercal$? Which are the images of the next vectors?

- $\boldsymbol{e}_1 = (1,\ 0,\ 0)^\intercal$
- $\boldsymbol{e}_2 = (0,\ 1,\ 0)^\intercal$
- $\boldsymbol{e}_3 = (0,\ 0,\ 1)^\intercal$

Exercise

Which are the rotation matrices that allows to rotate an angle of θ about the vectors:

- $\boldsymbol{e}_1 = (1,\ 0,\ 0)^\intercal$
- $\boldsymbol{e}_2 = (0,\ 1,\ 0)^\intercal$
- $\boldsymbol{e}_3 = (0,\ 0,\ 1)^\intercal$

Inverse rotation

Remember that for a symmetric matrix

$$\mathbf{A}^\intercal = \mathbf{A}$$

For a antisymmetric matric

$$\mathbf{A}^\intercal = -\mathbf{A}$$

$$\mathbf{R}(\boldsymbol{u}, \phi) = \underbrace{\mathbf{I}_3 \cos\phi}_{symmetric} + \underbrace{(1 - \cos(\phi))(\boldsymbol{u}\boldsymbol{u}^\intercal)}_{symmetric} + \underbrace{[\boldsymbol{u}]_\times \sin(\phi)}_{antisymmetric}$$

As consequence the inverse rotation matrix will be

$$\mathbf{R}^\intercal = \mathbf{I}_3 \cos\phi + (1 - \cos(\phi))(\boldsymbol{u}\boldsymbol{u}^\intercal) - [\boldsymbol{u}]_\times \sin(\phi)$$

Inception

- The rotation matrix that rotates an angle $-\phi$ about $\boldsymbol{u}$

$$\mathbf{R}(-\phi, \boldsymbol{u}) = \mathbf{R}(\phi, \boldsymbol{u})^\intercal$$

- The rotation matrix that rotates an angle ϕ about $-\boldsymbol{u}$

$$\mathbf{R}(\phi, -\boldsymbol{u}) = \mathbf{R}(\phi, \boldsymbol{u})^\intercal$$

Exercise

Which is the rotation matrix that rotates an angle $-\phi$ about $-\boldsymbol{u}$?

Re-writing Rodrigues' Rotation Formula

It can be shown that

$$(\boldsymbol{u}\boldsymbol{u}^\intercal) = (\boldsymbol{u}^\intercal \boldsymbol{u})\, \mathbf{I}_3 + [\boldsymbol{u}]_\times^2$$

Hence $\mathbf{R}(\boldsymbol{u}, \phi)$ can be also expressed as

$$\mathbf{R}(\boldsymbol{u}, \phi) = \mathbf{I}_3 + \sin(\phi)\,[\boldsymbol{u}]_\times + (1 - \cos(\phi))\,[\boldsymbol{u}]_\times^2$$

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Given a rotation matrix, how we can find $\boldsymbol{u}$ and ϕ?

Inverse mapping $\mathbf{R} \rightarrow \boldsymbol{u}, \phi$

Given the symmetries and anti-symmetries of the terms in the Rodrigues' rotation formula, we can see that

- $\mathrm{trace}\,(\mathbf{R}) = 3\cos(\phi) + \left(u_1^2 + u_2^2 + u_3^2\right)(1 - \cos(\phi)) = 1 + 2\cos(\phi) \Rightarrow$

$$\phi = \arccos\left(\frac{\mathrm{trace}\,(\mathbf{R}) - 1}{2}\right) \rightarrow \text{Euler's angle } \phi$$

- $\mathbf{R} - \mathbf{R}^\intercal = 2\,[\boldsymbol{u}]_\times \sin(\phi)$

$$[\boldsymbol{u}]_\times = \frac{\mathbf{R} - \mathbf{R}^\intercal}{2\sin(\phi)} \rightarrow \text{Euler's axis } \boldsymbol{u}$$

Rotations

Rotation matrix to Axis/angle

Exercise

Given the next rotation matrix:

$$\mathbf{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate the principal axis of rotation $\boldsymbol{u}$ and the angle ϕ

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Rotation Vector

- We have used 3 parameters to encode a rotation: 3 for the axis $\boldsymbol{u} = (v_1, v_2, v_3)^\intercal$ and 1 for the angle ϕ.
- However rotations are defined by 3 parameters.

Therefore, we can define a 3-dimensional Rotation Vector,

$$\boldsymbol{r} = \phi \boldsymbol{u}$$

such that its direction is the principal axis and its norm is equal to the rotation magnitude:

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \; ; \; \phi = \|\boldsymbol{r}\|$$

Re-writing (again) Rodrigues' Rotation Formula

Using the definition of the rotation vector $\boldsymbol{r}$, the Rodrigues' Rotation Formula can be also expressed as

$$\boldsymbol{p}' = \left[\mathbf{I}_3 \cos(\|\boldsymbol{r}\|) + \frac{\sin(\|\boldsymbol{r}\|)}{\|\boldsymbol{r}\|} [\boldsymbol{r}]_\times + \frac{(1 - \cos(\|\boldsymbol{r}\|)}{\|\boldsymbol{r}\|^2} (\boldsymbol{r}\boldsymbol{r}^\intercal) \right] \boldsymbol{p}$$

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Exercise 1

The matrix $\mathbf{R}$, represents a rotation. Find the axis $\boldsymbol{u}$ and the angle ϕ for which $\mathbf{R}(\boldsymbol{u}, \phi) = \mathbf{R}$

$$\mathbf{R} = \begin{pmatrix} 0.9746 & -.01816 & 0.1309 \\ 0.1309 & 0.9366 & 0.3251 \\ -0.1816 & -0.2998 & 0.9366 \end{pmatrix}$$

Give:

- $\mathbf{R}_1$ the matrix that represents a rotation of $-\phi = 22.5\,\mathrm{deg}$ about $\boldsymbol{u}$.
- $\mathbf{R}_2$ the matrix that represents a rotation of $-\phi = 22.5\,\mathrm{deg}$ about $-\boldsymbol{u}$.
- $\mathbf{R}_3$ the matrix that represents a rotation of $\phi = 22.5\,\mathrm{deg}$ about $-\boldsymbol{u}$.

Exercise 2

It is desired to rotate the vector

$$\boldsymbol{p} = (-1,\ -3,\ 2)^{\mathsf{T}}$$

about an axis $\boldsymbol{u}$ by an amount of $\pi\,\mathrm{rad}$. It is known that the projection of $\boldsymbol{p}$ over $\boldsymbol{u}$ is given by

$$\boldsymbol{w} = (0,\ 0,\ 2)^{\mathsf{T}}$$

What is the image of $\boldsymbol{p}$ after rotating?

Exercise 3

Let the orientation in space of a 3D body with respect to a world frame, to be described by the matrix

$$\mathbf{R}_i = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After some time a new estimation of the orientation arises as:

$$\mathbf{R}_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Find the axis of rotation about which the initial object has to be rotated to achieve the second orientation
- Find the angle rotated to achieve the second orientation