

Contents

I	Mathematics	1
1	Mathematical Reasoning	3
1.1	Formal Reasoning	4
1.1.1	Sequent and Predicate	4
1.1.2	Rule of Inference	5
1.1.3	Proofs	6
1.1.4	Basic Rules	6
1.2	Propositional Calculus	9
1.2.1	The Notation of Elementary Assertions	9
1.2.2	Inference Rules for Propositional Calculus	11
1.2.3	Some Proofs	14
1.2.4	A Proof Procedure	22
1.2.5	Extending the Notation	26
1.2.6	Some Classical Results	27
1.3	Predicate Calculus	29
1.3.1	The Notation of Quantified Predicates and Substitutions	29
1.3.2	Universal Quantification	32
1.3.3	Non-freeness	32
1.3.4	Substitution	34
1.3.5	Inference Rules for Predicate Calculus	35
1.3.6	Some Proofs	36
1.3.7	Extending the Proof Procedure	38
1.3.8	Existential Quantification	39
1.3.9	Some Classical Results	41
1.4	Equality	42
1.5	Ordered Pairs	47
1.6	Exercises	51

2	Set Notation	55
2.1	Basic Set Constructs	56
2.1.1	Syntax	57
2.1.2	Axioms	60
2.1.3	Properties	62
2.2	Type-checking	64
2.3	Derived Constructs	72
2.3.1	Definitions	72
2.3.2	Examples	72
2.3.3	Type-checking	73
2.3.4	Properties	75
2.4	Binary Relations	77
2.4.1	Binary Relation Constructs: First Series	77
2.4.2	Binary Relation Constructs: Second Series	79
2.4.3	Examples of Binary Relation Constructs	82
2.4.4	Type-checking of Binary Relation Constructs	84
2.5	Functions	85
2.5.1	Function Constructs: First Series	86
2.5.2	Function Constructs: Second Series	89
2.5.3	Examples of Function Constructs	90
2.5.4	Properties of Function Evaluation	90
2.5.5	Type-checking of Function Constructs	93
2.6	Catalogue of Properties	94
2.6.1	Membership Laws	95
2.6.2	Monotonicity Laws	96
2.6.3	Inclusion Laws	97
2.6.4	Equality Laws	99
2.7	Example	115
2.8	Exercises	120
3	Mathematical Objects	123
3.1	Generalized Intersection and Union	123
3.2	Constructing Mathematical Objects	130
3.2.1	Informal Introduction	130
3.2.2	Fixpoints	131
3.2.3	Induction Principle	136
3.3	The Set of Finite Subsets of a Set	141
3.4	Finite and Infinite Sets	144
3.5	Natural Numbers	145
3.5.1	Definition	145
3.5.2	Peano's "Axioms"	148
3.5.3	Minimum	153
3.5.4	Strong Induction Principle	156
3.5.5	Maximum	158

<i>Contents</i>	xxvii
3.5.6 Recursive Functions on Natural Numbers	158
3.5.7 Arithmetic	161
3.5.8 Iterate of a Relation	166
3.5.9 Cardinal of a Finite Set	167
3.5.10 Transitive Closures of a Relation	168
3.6 The Integers	170
3.7 Finite Sequences	174
3.7.1 Inductive Construction	174
3.7.2 Direct Construction	176
3.7.3 Operations on Sequences	177
3.7.4 Sorting and Related Topics	182
3.7.5 Lexicographical Order on Sequences of Integers	187
3.8 Finite Trees	188
3.8.1 Informal Introduction	188
3.8.2 Formal Construction	190
3.8.3 Induction	192
3.8.4 Recursion	194
3.8.5 Operations	197
3.8.6 Representing Trees	199
3.9 Labelled Trees	202
3.9.1 Direct Definition	203
3.9.2 Inductive Definition	203
3.9.3 Induction	205
3.9.4 Recursion	206
3.9.5 Operations Defined Recursively	206
3.9.6 Operations Defined Directly	208
3.10 Binary Trees	208
3.10.1 Direct Operations	209
3.10.2 Induction	209
3.10.3 Recursion	210
3.10.4 Operations Defined Recursively	210
3.11 Well-founded Relations	211
3.11.1 Definition	212
3.11.2 Proof by Induction on a Well-founded Set	213
3.11.3 Recursion on a Well-founded Set	214
3.11.4 Proving Well-foundedness	217
3.11.5 An Example of a Well-founded Relation	219
3.11.6 Other Examples of Non-classical Recursions	219
3.12 Exercises	221
II Abstract Machines	225
4 Introduction to Abstract Machines	227
4.1 Abstract Machines	228
4.2 The Statics: Specifying the State	229

xxviii	<i>Contents</i>	
4.3	The Dynamics: Specifying the Operations	230
4.4	Before-after Predicates as Specifications	231
4.5	Proof Obligation	232
4.6	Substitutions as Specifications	232
4.7	Pre-conditioned Substitution (Termination)	234
4.8	Parameterization and Initialization	236
4.9	Operations with Input Parameters	238
4.10	Operations with Output Parameters	240
4.11	Generous versus Defensive Style of Specification	241
4.12	Multiple Simple Substitution	243
4.13	Conditional Substitution	244
4.14	Bounded Choice Substitution	244
4.15	Guarded Substitution (Feasibility)	246
4.16	A Substitution with no Effect	247
4.17	Contextual Information: Sets and Constants	248
4.18	Unbounded Choice Substitution	252
4.19	Explicit Definitions	256
4.20	Assertions	260
4.21	Concrete Variables and Abstract Constants	261
4.22	Exercises	262
5	Formal Definition of Abstract Machines	265
5.1	Generalized Substitution	265
5.1.1	Syntax	265
5.1.2	Type-checking	270
5.1.3	Axioms	271
5.2	Abstract Machines	272
5.2.1	Syntax	272
5.2.2	Visibility Rules	274
5.2.3	Type-checking	275
5.2.4	On the Constants	278
5.2.5	Proof Obligations	278
5.2.6	About the Given Sets and the Pre-defined Constants	280
6	Theory of Abstract Machines	283
6.1	Normalized Form	283
6.2	Two Useful Properties	287
6.3	Termination, Feasibility and Before-after Predicate	288
6.3.1	Termination	289
6.3.2	Feasibility	290
6.3.3	Before-after Predicate	292
6.4	Set-Theoretic Models	295
6.4.1	First Model: a Set and a Relation	295
6.4.2	Second Model: Set Transformer	298
6.4.3	Set-theoretic Interpretations of the Constructs	301

<i>Contents</i>	xxix
6.5 Exercises	303
7 Constructing Large Abstract Machines	307
7.1 Multiple Generalized Substitution	307
7.1.1 Definition	308
7.1.2 Basic Properties	308
7.1.3 The Main Result	311
7.2 Incremental Specification	312
7.2.1 Informal Introduction	312
7.2.2 Operation Call	314
7.2.3 The INCLUDES Clause	316
7.2.4 Visibility Rules	318
7.2.5 Transitivity	319
7.2.6 Machine Renaming	320
7.2.7 The PROMOTES and the EXTENDS Clauses	320
7.2.8 Example	320
7.3 Incremental Specification and Sharing	322
7.3.1 Informal Introduction	322
7.3.2 The USES Clause	323
7.3.3 Visibility Rules	324
7.3.4 Transitivity	324
7.3.5 Machine Renaming	325
7.4 Formal Definition	325
7.4.1 Syntax	325
7.4.2 Type-checking	326
7.4.3 Proof Obligations for the INCLUDES Clause	331
7.4.4 Proof Obligations for the USES Clause	334
7.5 Exercises	336
8 Examples of Abstract Machines	337
8.1 An Invoice System	338
8.1.1 Informal Specification	338
8.1.2 The <i>Client</i> Machine	339
8.1.3 The <i>Product</i> Machine	341
8.1.4 The <i>Invoice</i> Machine	343
8.1.5 The <i>Invoice_System</i> Machine	348
8.2 A Telephone Exchange	349
8.2.1 Informal specification	349
8.2.2 The <i>Simple_Exchange</i> Machine	352
8.2.3 The <i>Exchange</i> Machine	355
8.3 A Lift Control System	358
8.3.1 Informal Specification	358
8.3.2 The <i>Lift</i> Machine	358
8.3.3 Liveness Proof	364
8.3.4 Expressing Liveness Proof Obligations	366

xxx	<i>Contents</i>	
8.4	Exercises	369
III	Programming	371
9	Sequencing and Loop	373
9.1	Sequencing	374
9.1.1	Syntax	374
9.1.2	Axiom	374
9.1.3	Basic Properties	374
9.2	Loop	377
9.2.1	Introduction	377
9.2.2	Definition	378
9.2.3	Interpretation of Loop Termination	382
9.2.4	Interpretation of the Before-after Relation of the Loop	385
9.2.5	Examples of Loop Termination	386
9.2.6	The Invariant Theorem	387
9.2.7	The Variant Theorem	388
9.2.8	Making the Variant and Invariant Theorem Practical	390
9.2.9	The Traditional Loop	392
9.3	Exercises	398
10	Programming Examples	403
10.0	Methodology	403
10.0.1	Re-use of Previous Algorithms	403
10.0.2	Loop Proof Rules	406
10.0.3	Sequencing Proof Rule	407
10.1	Unbounded Search	408
10.1.1	Introduction	408
10.1.2	Comparing two Sequences	411
10.1.3	Computing the Natural Number Inverse of a Function	416
10.1.4	Natural Number Division	420
10.1.5	The Special Case of Recursive Functions	422
10.1.6	Logarithm in a Given Base	424
10.1.7	Integer Square Root	425
10.2	Bounded Search	427
10.2.1	Introduction	427
10.2.2	Linear Search	430
10.2.3	Linear Search in an Array	431
10.2.4	Linear Search in a Matrix	433
10.2.5	Binary Search	435
10.2.6	Monotonic Functions Revisited	437
10.2.7	Binary Search in an Array	442
10.3	Natural Number	446
10.3.1	Basic Scheme	446
10.3.2	Natural Number Exponentiation	447

<i>Contents</i>	xxxi
10.3.3 Extending the Basic Scheme	448
10.3.4 Summing a Sequence	450
10.3.5 Shifting a Sub-sequence	451
10.3.6 Insertion into a Sorted Array	453
10.4 Sequences	455
10.4.1 Introduction	455
10.4.2 Accumulating the Elements of a Sequence	458
10.4.3 Decoding the Based Representation of a Number	461
10.4.4 Transforming a Natural Number into its Based Representation	462
10.4.5 Fast Binary Operation Computations	465
10.4.6 Left and Right Recursion	469
10.4.7 Filters	473
10.5 Trees	482
10.5.1 The Notion of Formula	483
10.5.2 Transforming a Tree into a Formula	484
10.5.3 Transforming a Tree into a Polish String	487
10.5.4 Transforming a Formula into a Polish String	488
10.6 Exercises	496
IV Refinement	499
11 Refinement	501
11.1 Refinement of Generalized Substitutions	501
11.1.1 Informal Approach	501
11.1.2 Definition	503
11.1.3 Equality of Generalized Substitution	503
11.1.4 Monotonicity	504
11.1.5 Refining a Generalized Assignment	506
11.2 Refinement of Abstract Machines	507
11.2.1 Informal Approach	507
11.2.2 Formal Definition	511
11.2.3 Sufficient Conditions	512
11.2.4 Monotonicity	516
11.2.5 Example Revisited	522
11.2.6 The Final Touch	523
11.2.7 An Intuitive Explanation of the Refinement Condition	530
11.2.8 Application to the Little Example	532
11.3 Formal Definition	533
11.3.1 Syntax	533
11.3.2 Type-checking	534
11.3.3 Proof Obligations	537
11.4 Exercises	540
12 Constructing Large Software Systems	551
12.1 Implementing a Refinement	551

xxxii	<i>Contents</i>	
	12.1.1 Introduction	551
	12.1.2 The Practice of Importation	556
	12.1.3 The IMPLEMENTATION Construct	559
	12.1.4 The IMPORTS Clause	561
	12.1.5 Visibility Rules	561
	12.1.6 Machine Renaming	563
	12.1.7 The VALUES Clause	563
	12.1.8 Comparing the IMPORTS and the INCLUDES Clauses	565
	12.1.9 The PROMOTES and EXTENDS Clauses	565
	12.1.10 Concrete Constants and Concrete Variables Revisited	566
	12.1.11 Allowed Constructs in an Implementation	566
12.2	Sharing	574
	12.2.1 Introduction	574
	12.2.2 The SEES Clause	579
	12.2.3 Visibility Rules	579
	12.2.4 Transitivity and Circularity	583
	12.2.5 Machine Renaming	583
	12.2.6 Comparing the USES and the SEES Clauses	583
12.3	Loops Revisited	584
12.4	Multiple Refinement and Implementation	584
12.5	Recursively Defined Operations	587
	12.5.1 Introduction	588
	12.5.2 Syntax	591
	12.5.3 Proof Rule	591
12.6	Formal Definition	594
	12.6.1 Syntax of an IMPLEMENTATION	594
	12.6.2 Type-checking with an IMPORTS Clause	595
	12.6.3 Type-checking with a SEES Clause	596
	12.6.4 Proof Obligations of an IMPLEMENTATION	597
	12.6.5 Proof Obligation for a SEES Clause	601
13	Examples of Refinements	603
	13.1 A Library of Basic Machines	603
	13.1.1 The BASIC_CONSTANTS Machine	604
	13.1.2 The BASIC_IO Machine	604
	13.1.3 The BASIC_BOOL Machine	605
	13.1.4 The BASIC_enum Machine for Enumerated Sets	606
	13.1.5 The BASIC_FILE_VAR Machine	607
13.2	Case Study: Data-base System	608
	13.2.1 Machines for Files	611
	13.2.2 Machines for Objects	623
	13.2.3 A Data-base	630
	13.2.4 Interfaces	637
13.3	A Library of Useful Abstract Machines	647
	13.3.1 The ARRAY_VAR Machine	647

<i>Contents</i>	xxxiii
13.3.2 The <i>SEQUENCE_VAR</i> Machine	647
13.3.3 The <i>SET_VAR</i> Machine	647
13.3.4 The <i>ARRAY_COLLECTION</i> Machine	648
13.3.5 The <i>SEQUENCE_COLLECTION</i> Machine	648
13.3.6 The <i>SET_COLLECTION</i> Machine	650
13.3.7 The <i>TREE_VAR</i> Machine	650
13.4 Case Study: Boiler Control System	655
13.4.1 Introduction	655
13.4.2 Informal Specification	656
13.4.3 System Analysis	661
13.4.4 System Synthesis	673
13.4.5 Formal Specification and Design	676
13.4.6 Final Architecture	693
13.4.7 Modifying the Initial Specification	694
Appendix A Summary of Notations	701
A.1 Propositional Calculus	701
A.2 Predicate Calculus	702
A.3 Equality and Ordered Pairs	702
A.4 Basic and Derived Set Constructs	702
A.5 Binary Relations	703
A.6 Functions	705
A.7 Generalized Intersection and Union	706
A.8 Finiteness	706
A.9 Natural Numbers	707
A.10 Integers	709
A.11 Finite Sequences	711
A.12 Finite Trees	713
Appendix B Syntax	715
B.1 Predicate	715
B.2 Expression	716
B.3 Substitution	716
B.4 Machine	717
B.5 Refinement	719
B.6 Implementation	720
B.7 Statement	721
Appendix C Definitions	725
C.1 Logic Definitions	725
C.2 Basic Set-theoretic Definitions	726
C.3 Binary Relation Definitions	726
C.4 Function Definitions	728
C.5 Fixpoint Definitions	728
C.6 Finiteness Definitions	729
C.7 Natural Number Definitions	730
C.8 Integer Extensions	732
C.9 Finite Sequence Definitions	734

xxxiv	<i>Contents</i>	
	C.10 Finite Tree Definitions	736
	C.11 Well-founded Relation Definition	738
	C.12 Generalized Substitution Definitions	738
	C.13 Set-theoretic Models	741
	C.14 Refinement Conditions	742
	<i>Appendix D Visibility Rules</i>	743
	D.1 Visibility of a Machine	743
	D.2 Visibility of a Refinement	747
	D.3 Visibility of an Implementation	750
	<i>Appendix E Rules and Axioms</i>	753
	E.1 Non-freeness Rules	753
	E.2 Substitution Rules	754
	E.3 Basic Inference Rules	756
	E.4 Derived Inference Rules	758
	E.5 Set Axioms	760
	E.6 Generalized Substitution Axioms	761
	E.7 Loop Proof Rules	761
	E.8 Sequencing Proof Rule	762
	<i>Appendix F Proof Obligations</i>	763
	F.1 Machine Proof Obligations	763
	F.2 INCLUDES Proof Obligations	765
	F.3 USES Proof Obligations	767
	F.4 Refinement Proof Obligations	769
	F.5 Implementation Proof Obligations	771
	Index	775